Due: Wednesday, Dec 1, 2021

Math 202 / Problem Set 11 (two pages)

Basics about connectedness, compactness, continuity

Be sure that you know/review the facts from the class:

- A subset $Y \subset \mathbb{R}$ is connected iff Y is an interval.
- **Heine–Borel**: A subset $Y \neq \emptyset$, of $\mathbb{R}^e$ or $\mathbb{C}^e$ is compact iff Y is closed and bounded.
- **Heine–Borel for metric spaces** X: *A subspace* $\emptyset \neq Y \subset X$ *is compact iff every sequence* $(x_n)_n \in \mathcal{S}(Y)$ *has a convergent subsequence* $x_{n_k} \to x \in Y$.

Be sure that you know/review the facts from the class (here, $\mathcal{A}_\bullet$ denotes the closes subsets):

- For a map of topological spaces $f : X \to Y$ the following hold:
  (i) f is continuous at $x \in X$ iff $f^{-1}(V) \in \tau_{X,x}$ for all $V \in \tau_{Y,y}$, where $y = f(x)$.
  (ii) f is continuos on X iff $f^{-1}(\tau_Y) \subset \tau_X$ iff $f^{-1}(\mathcal{A}_Y) \subset \mathcal{A}_X$.
- For a map of topological spaces $f : X \to Y$ the following hold:
  (i) If $Z \subset X$ is connected, then $f(Z) \subset Y$ is connected.
  (ii) If $Z \subset X$ is compact, then $f(Z) \subset Y$ is compact.
- For a map $f : X \to Y$ of metric spaces, $y = f(x)$. TFAE: (i) f is continuous at $x \in X$; (ii) $\forall\, \epsilon > 0\ \exists\, \delta > 0$ s.t. $f\Big(B_\delta(x)\Big) \subset B_\epsilon(y)$; (iii) $\forall\ x_n \to x$ in X one has $f(x_n) \to f(x)$ in Y.

Be sure that you know/review the facts from the class (here $F \subset \mathbb{C}$ is asubring):

- If $f : X \to Y$ and $g : Y \to Z$ are continuous maps of arbitrary topological spaces. Then the composition $g \circ f : X \to Z$ is continuous.
- $\mathcal{C}(X, F) := \{f : X \to F \,|\, f \text{ continuous}\} \subset \operatorname{Maps}(X, F)$ is an F-subalgebra, i.e.:
  $\mathcal{C}(X, F)$ *is closed w.r.t. addition, multiplication, and multiplication by* $a \in F$.

1) Let X, Y be metric spaces, $I \subset \mathbb{R}$ be an interval. Check all the details of the proofs of:

a) Let $f : I \to \mathbb{R}$ be continuous. Then f is injective iff f is strictly monotone.
b) Let $f : X \to \mathbb{R}$ be continuous, and X compact. Then f has points of absolute minimum and maximum, i.e., $\exists\, x_m, x_M \in X$ s.t. $f(x_m) \leqslant f(x) \leqslant f(x_M)\ \forall\, x \in X$.
c) Let X be compact, $f : X \to Y$ be continuous bijective. Then $f^{-1} : Y \to X$ is continuous.

Continuity of f_α and $\exp_a$, $\log_a$

2) Prove in all detail that the power-α maps $f_\alpha : (0, \infty) \to (0, \infty)$ and the exponential functions $\exp_a : \mathbb{R} \to (0, \infty)$ are continuous.

Conclude: f_α, $\alpha \neq 0$, *and* $\exp_a$, $a > 0$, $a \neq 1$ *are continuous bijective. Moreover,* $f_\alpha^{-1} = f_{\frac{1}{\alpha}}$.

[**Hint**: Use HW # 9, Problem 7), etc...]

Terminology: The inverse map of $\exp_a$ is $\log_a : (0, \infty) \to \mathbb{R}$, the *logarithm in basis* a.

3) Prove that $\log_a : (0, \infty) \to \mathbb{R}$ is continuous, surjective and satisfies:

a) $\log_a(x_1 x_2) = \log_a(x_1) + \log_a(x_2)$ and $\log_a(x^\alpha) = \alpha \log_a(x)\ \forall\, x, x_1, x_2 \in (0, \infty)$, $\alpha \in \mathbb{R}$.
b) $\log_a$ is strictly monotone. For which bases a is $\log_a$ increasing/decreasing?

More about analytic functions. Let $F = \mathbb{R}$, $\mathbb{C}$, and $f(t) = \sum_n a_n t^n \in F[[t]]$.

4) Let $\rho_f := \sup\{ r \in \mathbb{R}_{\geqslant 0} \mid (a_n r^n)_n$ is bounded in $\mathbb{C}\} \in [0, \infty]$. Check all the details of the proof that $\sum_n a_n z^n$ is absolutely convergent for $z \in \mathbb{C}$ iff $|z| < \rho_f$.

Recall: In the above notation, $D_f := \left\{ z \in F \,\middle|\, |z| < \rho_f \right\}$ is the *domain of absolute convergence* of $f(t)$, and $f : D_f \to F$, $z \mapsto f(z) :\overset{\text{def}}{=\!=} \sum_n a_n z^n$ is the *analytic function* defined by $f(t)$.

5) Suppose that $\rho_f > 0$. Check all the details of proofs of the assertions from the class:

a) $f_1(t) := \sum_{n>0} n a_n t^{n-1} \in F[[t]]$ has $\rho_{f_1} = \rho_f$, hence $D_{f_1} = D_f$.

b) $f : D_f \to F$ is continuous on D_f, hence $f_1 : D_f \to F$ is continuous as well.

c) If $z, z+h \in D_f$, then $\lim\limits_{h\to 0} \frac{f(z+h)-f(z))}{h}$ exists and equals $f_1(z)$.

Famous/useful (formal) power series.

For a given commutative ring R, define $R_1 :\overset{\text{def}}{=\!=} R[[t_1]]$, $R_2 :\overset{\text{def}}{=\!=} R_1[[t_2]] = R[[t_1, t_2]], \dots$

For $\mathbb{Q} \subset R$, recall the following very *famous/useful* (formal) power series in $R[[t]]$.

- (formal) exponential: $\exp(t) = \sum_n \frac{t^n}{n!} = 1 + \frac{t}{1!} + \dots + \frac{t^n}{n!} + \dots$
- (formal) cosinus: $\cos(t) = \sum_n (-1)^n \frac{t^{2n}}{(2n)!} = 1 - \frac{t^2}{2!} + \dots + (-1)^n \frac{t^{2n}}{(2n)!} + \dots$
- (formal) sinus: $\sin(x) = \sum_n (-1)^n \frac{t^{2n+1}}{(2n+1)!} = \frac{t}{1!} - \frac{t^3}{3!} + \dots + (-1)^n \frac{t^{2n+1}}{(2n+1)!} + \dots$
- (formal) logarithm around 1: $\log(1+t) = \sum_n (-1)^n \frac{t^{n+1}}{n+1} = t - \frac{t^2}{2} + \dots + (-1)^n \frac{t^{n+1}}{n+1} + \dots$
- (formal) power-α around 1: $(1+t)^\alpha = \sum_n \binom{\alpha}{n} t^n = 1 + \frac{\alpha}{1!} t + \dots + \frac{\alpha \dots (\alpha - n)}{(n+1)!} t^{n+1} + \dots$

6) Prove that in $R[[x, y]]$ one has the identities of power series:

a) $\exp(x+y) = \exp(x) \cdot \exp(y)$

b) $\exp(-x) = \sum_n (-1)^n \frac{x^n}{n!}$ is the inverse of $\exp(x)$ w.r.t. multiplication in $R[[x]]$.

[**Hint** to a): Recall that two power series $f(x,y), g(x,y) \in R[[x,y]]$ are equal $\overset{\text{def}}{\longleftrightarrow}$ $f(x,y)$, $g(x,y)$ have the same coefficients. Compare the coefficients of $x^i y^j$ in $\exp(x+y)$ and $\exp(x) \cdot \exp(y)$, etc.…]

7) Prove that in $R[[x, y]]$ one has the identities of power series:

a) $-\sin(x) = \sin(-x)$, $\cos(-x) = \cos(x)$, and $\sin(0) = 0$, $\cos(0) = 1$ in $\mathbb{Q}$ (WHY).

b) $\cos(x+y) = \cos(x) \cdot \cos(y) - \sin(x) \cdot \sin(y)$, $\sin(x+y) = \sin(x) \cdot \cos(y) + \cos(x) \cdot \sin(y)$

d) $\cos(x)^2 + \cos(x)^2 = 1_{R[[x]]}$

[**Hint:** W.l.o.g., suppose that there is $i \in R$ s.t. $i^2 = -1$, e.g. the field of *rational complex numbers* $\mathbb{Q}(i) \subset R$ (What is that?). Then $\exp(ix) = \cos(x) + i\sin(x)$ (WHY), and $\exp(ix + iy) = \exp(ix) \cdot \exp(iy)$ (WHY), etc.…]

8) Show that $\exp(z)$, $\sin(z)$, $\cos(z)$ are absolutely convergent for all $z \in \mathbb{C}$, and satisfy:

a) $\exp(z_1 + z_1) = \exp(z_1) \cdot \exp(z_2)$, $\exp(z\boldsymbol{i}) = \cos(z) + \sin(z)\boldsymbol{i}$, and the addition formulas: $\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2)$, $\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \sin(z_2)\cos(z_1)$.

b) If $x \in \mathbb{R}$, then $\exp(x), \cos(x), \sin(x) \in \mathbb{R}$, and $\cos^2(x) + \sin^2(x) = 1$.

c) **Euler Formula**: For $z = a + b\,\boldsymbol{i}$ one has: $\exp(a + b\,\boldsymbol{i}) = \exp(a)\Big(\cos(b) + \sin(b)\boldsymbol{i}\Big)$, hence:

$$|\exp(z)| = \exp(a) = \exp\Big(\Re(z)\Big), \quad \exp(z)/|\exp(z)| = \exp(b\,\boldsymbol{i}) = \cos(b) + \sin(b)\boldsymbol{i}\,.$$