Due: Thursday, Nov 18, 2021

Math 202 / Problem Set 10 (two pages)

The exponential $\exp(x)$ versus e^x.

1) Consider the sequences $(x_n)_n$ with $x_n := (1+\frac{x}{n})^n$, and $(y_n)_n$ with $y_n = 1+\frac{x}{1!}+\cdots+\frac{x^n}{n!}$, for every fixed $x \in \mathbb{R}$, that is, $x_n = x_n(x)$ and $y_n = y_n(x)$. Prove the following:

a) $(x_n)_n, (y_n)_n \in \mathcal{S}_{\mathrm{C}}(\mathbb{R})$ for all $x \in \mathbb{R}$, hence $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ exist in $\mathbb{R}$.

b) $x_n - y_n \to 0$ for all $x \in \mathbb{R}$, hence $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$, and therefore:

$$\exp(x) :\overset{\text{def}}{=} \textstyle\sum_n \frac{x^n}{n!} = \lim_{n\to\infty}(1+\frac{x}{n})^n \text{ for all } x \in \mathbb{R}.$$

2) Prove that $e^x :\overset{\text{def}}{=} \exp_e(x) \overset{\text{why}}{=} \exp(x) :\overset{\text{def}}{=} \sum_n \frac{x^n}{n!}$ for $x \in \mathbb{R}$ along the following lines:

a) $\lim_{m_n\to\infty}\left(1+\frac{x}{m_n}\right)^{m_n} = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \sum_n \frac{x^n}{n!}$ for all $x \in \mathbb{R}$. **Hence** $e = \lim_{m_n\to\infty}(1+\frac{1}{m_n})^{m_n}$.

b) Let $a_n = \frac{l_n}{k_n}$ with $l_n, k_n \in \mathbb{N}_{>0}$ be given, and set $m_n := mk_n$ with $m \to \infty$. Then:

$$\exp_e(\tfrac{l_n}{k_n}) \overset{\text{why}}{=} \left(\lim_{m\to\infty}\left(1+\frac{1}{m_n}\right)^{m_n}\right)^{\frac{l_n}{k_n}} \overset{\text{why}}{=} \lim_{m\to\infty}\left(1+\frac{\frac{l_n}{k_n}}{ml_n}\right)^{ml_n} \overset{\text{why}}{=} \sum_m \frac{(\frac{l_n}{k_n})^m}{m!} = \exp(\tfrac{l_n}{k_n})$$

c) If $a_n \to x$, then $\exp_e(a_n) = \exp(a_n)$, and $\exp_e(a_n) \to \exp_e(x)$, $\exp(a_n) \to \exp(x)$ (WHY).
Conclude that $e^x = \exp_e(x) = \exp(x)$ (WHY).

Terminology/Notation: $e := \lim_{n\to\infty}(1+\frac{1}{n})^n = \sum_n \frac{1}{n!}$ is called the Euler number.

Note: $e = 2.71828182845904523536028747135235360287471352662497757247\ldots$ is *transcendental* (WHAT IS THAT?).

On the field of complex numbers $\mathbb{C}$

Recall the basic facts mentioned/proved in the class (and check all the details in the proofs):

- $\mathbb{C}, +, \cdot$ is a field, $z^{-1} = \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right)$, and $\mathbb{R} \to \mathbb{C}$, $a \mapsto (a,0)$ is a field embedding.
- We identify $a \in \mathbb{R}$ with $(a,0) \in \mathbb{C}$, hence $0 = 0_R = (0,0) = 0_{\mathbb{C}}$, $1 = 1_{\mathbb{R}} = (1,0) = 1_{\mathbb{C}}$.
- We denote $\boldsymbol{i} := (0,1)$, hence $z = (a,b) = (a,0) + (b,0)\cdot(0,1) = a + b\boldsymbol{i}$.
- The conjugation $z = a+b\boldsymbol{i} \mapsto a-b\boldsymbol{i} =: \overline{z}$ is compatible with addition and multiplication.
- $|\ | : \mathbb{C} \to \mathbb{R}_{\geqslant 0}$ satisfies $|z_1+z_2| \le |z_1|+|z_2|$, $|z_1\cdot z_2| = |z_1|\cdot|z_2|$ for all $z_1, z_2 \in \mathbb{C}$.
- If $z_1, z_2 \neq 0$ the following hold (WHY):

(i) $|z_1+z_2| = |z_1|+|z_2|$ iff $z_1/z_2 \in \mathbb{R}_{>0}$. (ii) $|z_1+z_2| = \big||z_1|-|z_2|\big|$ iff $z_1/z_2 \in \mathbb{R}_{<0}$.

3) Prove the assertion from the class, and at b), give $a, b, c \in \mathbb{R}$ in terms of z_1, z_2:

a) If $z_1, z_2, z \in \mathbb{C}$, then $|z-z_1|+|z_2-z| = |z_2-z_1|$ iff $\exists\, a \in [0,1]$ s.t. $z = az_2+(1-a)z_1$.
In particular, the segment $[z_1, z_2] \subset \mathbb{C}$ equals $[z_2, z_1] = \{\, az_2+(1-a)z_1 \mid 0 \leqslant a \leqslant 1 \,\}$.

b) The line $L_{P_1P_2}$ through two points $P_1 = z_1 \neq z_2 = P_2$ is given by:
$L_{P_1P_2} = \{\, z \in \mathbb{C} \mid \exists\, \lambda \in \mathbb{R} \text{ s.t. } z = \lambda z_2 + (1-\lambda)z_1 \,\} = \{\, P = (x,y) \mid ax+by+c = 0 \,\}$.

Convergence in $\mathbb{C}$

If $(z_n)_n \in \mathcal{S}(\mathbb{C})$ with $z_n = x_n + y_n\boldsymbol{i}$, then $(x_n)_n, (y_n)_n \in \mathcal{S}(\mathbb{R})$ s.t. $(z_n)_n = (x_n)_n + (y_n)_n\boldsymbol{i}$ (WHY).
If $\sum_n z_n \in \boldsymbol{\Sigma}(\mathbb{C})$ with $z_n = x_n + y_n\boldsymbol{i}$, then $\sum_n x_n, \sum_n y_n \in \boldsymbol{\Sigma}(\mathbb{R})$ s.t. $\sum_n z_n = \sum_n x_n + \sum_n y_n\boldsymbol{i}$ (WHY).

4) For $z \in \mathbb{C}$, $(z_n)_n \in \mathcal{S}(\mathbb{C})$, prove in all detail the following (using what was done in the class):

a) (i)] $z_n \to 0_{\mathbb{C}}$ iff $|z_n| \to 0_{\mathbb{R}}$. (ii) If $z_n \to z$ in $\mathbb{C}$, then $|z_n| \to |z|$ in $\mathbb{R}$. Conversely?

b) $(z^n)_n$ is convergent iff either $z = 1$, or $|z| < 0$ and if so, $z^n \to 0_{\mathbb{C}}$.

c) $\sum_n z_n$ is (absolutely) convergent iff $\sum_n x_n$ and $\sum_n y_n$ are (absolutely) convergent.
If so, prove that $\sum_n z_n = x + y\boldsymbol{i}$ in $\mathbb{C}$ iff $\sum_n x_n = x$ and $\sum_n y_n = y$ in $\mathbb{R}$.

Basics of Topology

Make sure that you/know/review the basic facts about topological spaces X, τ_X.

- Open subsets $U \in \tau_X$, closed subsets $A = \complement_X U$, $U \in \tau_X$.
- For $Y \subset X$, recall the definitions/notions:
  - isolated/accumulation/interior point; closure $\overline{Y}$ of Y; interior $\overset{\circ}{Y}$ of Y.
  - connectedness; finite covering/intersection property; compactness.
- Make sure to check the details of the proofs of the assertions from the class:
  - Let $\mathcal{A}_X$ be the set of closed subsets, and $A_i \in \mathcal{A}_X$, $i \in I$. The following hold:
    (i) $\emptyset, X \in \mathcal{A}_X$; (ii) $\cap_i A_i \in \mathcal{A}_X$; (iii) $\cup_i A_i \in \mathcal{A}_X$ provided I is finite.
  - The following hold: (i) $\overline{Y} = Y$ iff Y is cosed; (ii) $\overset{\circ}{Y} = Y$ iff Y is open.
  - Y has the finite covering property iff Y has the finite intersection property.

5) Prove the following properties of connectedness:

a) If $Y \subset X$ is connected, then $\overline{Y} \subset X$ is connected.

b) If $Y_1, Y_2 \subset X$ are connected and $Y_1 \cap Y_2 \neq \emptyset$, then $Y_1 \cup Y_2$ is connected.
What is the corresponding assertion for $Y_1, \ldots, Y_n \subset X$ connected subsets?

6) Prove the following properties of compactness:

a) If $Y \subset X$ is compact, then $Y = \overline{Y}$.

b) If $Y \subset X$ is compact, and $Y_1 \subset Y$ is non-empty and closed, then Y_1 is compact.

c) If $Y_1, Y_2 \subset X$ are compact, so is $Y_1 \cup Y_2$. What about $Y_1 \cup \cdots \cup Y_n$ with all Y_i compact ?

Basics of metric spaces

Make sure that you/know/review the basic facts about metric space and their topology:

- The open ball $B_r(x)$ is open in X. The closed ball $\overline{B}_r(x)$ is closed in X.
- $U \in \tau_X$ iff U is union of open balls, say $U = \cup_x B_{\epsilon_x}(x)$. Hence $A = \complement_X U = \cap_x \complement_X B_{\epsilon_x}(x)$.
- $A \subset X$ is closed iff for all $(x_n)_n \in \mathcal{S}(A)$ one has: If $x_n \to x \in X$ then $x \in A$.
- For a subring $F \subset \mathbb{C}$, e.g. $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, and $e \in \mathbb{N}_{>0}$ fixed, recall $\|\ \|_1$, $\|\ \|_2$, $\|\ \|_\infty$ and the corresponding d_1, d_2, d_∞ on F^e, and that $\|x\|_\infty \leqslant \|x\|_2 \leqslant \|x\|_1 \leqslant e\|x\|_\infty \ \forall\, x \in F^e$.

7) For $(x_n)_n \in \mathcal{S}(F^e)$ set $x_n := (a_{1,n}, \ldots, a_{e,n})$ with $a_{i,n} \in F$. Answer/prove the following:

a) For $F = \mathbb{R}$, $e = 1, 2, 3$, draw the unit balls $B_1(0_{F^e})$ for each d_∞, d_1, d_2.

b) For $(x_n) \in F^e$ being convergent/Cauchy/bounded is the same for d_1, d_2, d_∞.

c) $(x_n)_n \to x := (a_1, \ldots, a_e)$ in F^e iff $a_{i,n} \to a_i$ for all i in F. What do you conclude?

d) $(x_n)_n$ is Cauchy/bounded in F^e iff $(a_{i,n})_n$ is Cauchy/bounded in F for all $i = 1, \ldots, e$.