Due: Thr Sept 16, 2021

Math 202 (Proofs – Analysis) / HW # 1 (two pages)

Recall that in order to disprove a particular assertion, one must give an example in which the hypothesis is satisfied, but the conclusion is false.

Example:

• Prove/disprove: Every natural number is a sum of two squares of natural numbers.

Solution. The assertion is false, because 3 is not a sum of two squares of natural numbers.

Basics of logical deduction

Recall the notations from the class: Given assertions p, q, one defines their (logical) disjunction $p \vee q$, the (logical) conjunction $p \,\&\, q$, and the (logical) negation $\neg\, p$.

1) Using the table of truth, prove the following assertions from class:

a) $\neg(p \vee q)$ is the same as $\neg p \,\&\, \neg q$.
b) $\neg(p \,\&\, q)$ is the same as $\neg p \vee \neg q$.

2) Show that using parentheses is essential for building desired unambiguous assertions. Namely, the assertion $p \vee q \,\&\, r$ is ambiguous. Indeed, the possible interpretations are

$$\text{(i)}\quad (p \vee q) \,\&\, r \qquad\qquad \text{(ii)}\quad p \vee (q \,\&\, r)$$

and prove that the assertions (i) and (ii) are not equivalent.

3) Using the table of truth, prove the *associativity, commutativity,* and *distributivity* properties of $\vee$ and &, as mentioned in class:

a) $(p \vee q) \vee r$ is the same as $p \vee (q \vee r)$, and $(p \,\&\, q) \,\&\, r$ is the same as $p \,\&\, (q \,\&\, r)$.
b) $p \vee q$ is the same as $q \vee p$, and $p \,\&\, q$ is the same as $q \,\&\, p$.
c) $(p \vee q) \,\&\, r$ is the same as $(p \,\&\, r) \vee (q \,\&\, r)$, and $(p \,\&\, q) \vee r$ is the same as $(p \vee r) \,\&\, (q \vee r)$.

Recall the quantifiers $\forall$ and $\exists$, their usage, and in particular the their negations: If $p(x)$ is an assertion depending on x, then one has:

i) $\neg\, \big(\forall\, x\ p(x)\big)$ is the same as $\exists\, x\, \big(\neg\, p(x)\big)$.
ii) $\neg\, \big(\exists\, x\ p(x)\big)$ is the same as $\forall\, x\, \big(\neg\, p(x)\big)$.

4) Recall that 0 is, by definition, a natural number. Consider the assertion in plain English:

($*$) *Every natural number less than 101 is a sum of three squares of natural numbers.*

a) Write the above assertion using quantifiers.
b) What is the negation of the above assertion in plain English?
c) Write the negation of the above assertion using quantifiers.
d) Is the above assertion true?

5) Recall that a real number x is a square of a real number iff $x \geqslant 0$. Consider the implication:

$$x, y \in \mathbb{R} \Rightarrow \exists\, z \in \mathbb{R} \quad \text{s.t.} \quad x^2 + y^2 = z^2$$

a) Formulate the above implication as an assertion in plain English.

b) Write the negation of the above implication, both with qualifiers, and in plain English.

c) Prove that the implication above is true, both directly, and arguing by contradiction.

Working with sets

Recall that we always work in the Zermelo-Fraenkel System of Axioms ZF (check Notes!) and spend some time trying to get used to the axiomatic way of thinking and working with sets!!!

6) Let $A, B, X, Y, \dots$ denote sets. Prove the assertions:

a) X is the only element of $\{X\}$, and $\{X\}$ is not a subset of X.

b) $\{\{A\}, \{A, B\}\} = \{\{X\}, \{X, Y\}\}$ iff $A = X$, $B = Y$.

Terminology: One denotes $(X, Y) := \{\{X\}, \{X, Y\}\}$, and called it the (ordered) pair with entries (or coordinates) A, B.

c) Prove that $(X, Y) = (Y, X)$ iff $X = Y$. Hence ordered pairs are **not** commutative.

7) Let A, B be an arbitrary set, $\mathcal{P}(A \cup B)$ be the power set of $A \cup B$, and $\mathcal{P}\big(\mathcal{P}(A \cup B)\big)$ be the power set of $\mathcal{P}(A \cup B)$. Prove the following:

a) For all $X \in A, Y \in B$ one has that $\{X\}, \{X, Y\} \in \mathcal{P}(A \cup B)$.
What is the converse assertion of a), and is the converse assertion true?

b) For all $X \in A, Y \in B$ one has that $(X, Y) \in \mathcal{P}\big(\mathcal{P}(A \cup B)\big)$.

8) Let A, B, C, D be given sets, and x be elements, e.g., real numbers. Answer the following:

a) Using $\cup$, $\cap$, $\setminus$ and A, B, C, D write down the sets of all x which satisfy:
i) $\big(x \in A$ or $x \in B\big)$ & $x \in C$ & $x \notin D$; ii) $x \in A$ or $\big(x \in B$ & $x \in C\big)$ & $x \notin D$.

b) Write as a union of disjoint intervals the sets of the real numbers $x \in \mathbb{R}$ satisfying:
i) $\big(x < 20$ & $x^2 < 100\big)$ or $x \notin (-\infty, -1]$; ii) $x < 20$ & $\big(x^2 < 100$ or $x \notin (-\infty, -1]\big)$.

$(*)$ Does the place of the parentheses matter?

9) Using the definitions of $\cup$, $\cap$, $\setminus$, $\times$, prove the following:

a) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ and $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
More general: $(\cap_{i \in I} A_i) \cup C = \cap_{i \in I}(A_i \cup C)$ and $(\cup_{i \in I} A_i) \cap C = \cup_{i \in I}(A_i \cap C)$.

b) *de Morgan* laws: $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$, $C \setminus (A \cap B) = C \setminus A \cup C \setminus B$.
More general: $C \setminus (\cup_{i \in I} A_i) = \cap_{i \in I}(C \setminus A_i)$ and $C \setminus (\cap_{i \in I} A_i) = \cup_{i \in I}(C \setminus A_i)$.

c) $(A \cup B) \times C = (A \times C) \cup (B \times C)$ and $(A \cap B) \times C = (A \times C) \cap (B \times C)$.
More general: $(\cup_{i \in I} A_i) \times C = \cup_{i \in I}(A_i \times C)$ and $(\cap_{i \in I} A_i) \times C = \cap_{i \in I}(A_i \times C)$.