Math 202 (Proofs — Analysis)

Pre-HW: Try to solve... (2 pages)

• Solve as much as you can of the problems below. For each problem ***make explicit the precise hypotheses*** you are using to tackle the problem.

• **Recall** that in order to disprove a particular assertion $\mathcal{P}$, one must give an example in which the hypothesis of $\mathcal{P}$ is satisfied, but the conclusion of $\mathcal{P}$ does not hold.

Example:

• Assertion $\mathcal{P}$: Every natural number is a sum of three squares of natural numbers.

Solution. The assertion $\mathcal{P}$ is false, because 7 is not a sum of three squares of natural numbers.

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1) Let m, n be positive natural numbers. Prove/disprove:
mn is an odd number if and only if both m and n are odd numbers.
Is the same true for the product $n_1 \dots n_{100}$ of any hundred positive natural numbers?

Hint: What is the general form of an odd natural number ?

2) What is the remainder of the division of 3^{2019} by 5, respectively by 16 ?

Hint: $3^4 = 81$ has remainder 1 when divided by 5, etc.

3) Prove/disprove the following: If a natural number n equals the cube of another natural number m, then m and n are divisible by the same prime numbers.

4) Let n be a positive natural number divisible by 3, but not divisible by 9.
Prove that $\sqrt{n}$ is not a rational number.

Hint: *By contradiction,* suppose that $\sqrt{n}$ is rational, hence $\sqrt{n} = k/l$ for some relatively prime natural numbers k, l (WHY). Then $l^2 n = k^2$, hence 3 divides k (WHY). Thus $k = 3k'$, hence $l^2 n = 9k'^2$ (WHY). Therefore, l is divisible by 3 (WHY), etc.

5) Find all the real numbers x such that $x^6 + x^4 - 2x^2 + 1 = 0$.
Find all the pairs of real number (x, y) such that $x^2 - 5xy^2 + 7y^4 = 0$.

6) Let $c > 0$ be a positive real number. Prove/disprove:
There exists a natural number n such that $nc > 100$.

7) Prove/disprove that the function $\sin(x)$ cannot be written as a polynomial function $f(x)$.
The same question about the exponential function 2^x.

8) List these numbers from smallest to largest:

2^{121}, 9^{55}, 7^{88}, $N :=$ (number of seconds since the birth of our universe —Google it!)

9) Using the definitions of the set operations $\cup$, $\cap$, $\setminus$, $\times$, prove the following:

a) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ and $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
b) *De Morgan* laws: $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$, $C \setminus (A \cap B) = C \setminus A \cup C \setminus B$.
c) $(A \cup B) \times C = (A \times C) \cup (B \times C)$ and $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

Hint: Recall that to prove that two sets are equal $X = Y$ one has to prove that $x \in X \Rightarrow x \in Y$, and $y \in Y \Rightarrow y \in X$.

10) Let $f : A \to B$, $g : B \to C$ be maps, and consider $g \circ f : A \to C$. Prove/disprove:

a) If f and g are injective (reps. surjective), then $g \circ f$ is injective (reps. surjective).

b) If f and g are bijective, then $g \circ f$ is bijective.

11) Prove that there is no smallest strictly positive real number $c > 0$.
[Questions: What is a real number? What does "smallest" mean?]

12) Knowing that the sum of the angles (in radians) of a triangle is π, prove that the sum of the angles (in radinas) of a convex n-gon is $(n-2)\pi$. What about *non-convex n-gons*?
[Questions: What is a radian? What is a (non-convex) n-gon? (Google it!)]

13) Prove that $S_1(n) := 1 + \cdots + n = n(n+1)/2$ for all natural numbers $n > 0$. What is the generalization of this in terms of arithmetic progressions? (Google it!). Are there similar "closed formulas" for $S_2(n) := 1^2 + \cdots + n^2$, and $S_k(n) := 1^k + \cdots + n^k$ for every k?

14) The Fibonacci sequence (Google it!) is defined recursively (what is that?), as follows:

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2} \quad \text{for } n > 1.$$

Find real numbers A, B, α, β such that $F_n = A\alpha^n + B\beta^n$ for all $n > 0$, and prove the resulting "closed formula" by induction. Are the numbers A, B, α, β unique?

- Are there similar "closed formulas" for all sequences $(x_n)_n$ defined recursively?

Recall that the binomial coefficients $\binom{n}{m}$ are defined by $\binom{k+l}{k} = \dfrac{(k+l)!}{k!\,l!} = \binom{k+l}{l}$ for all natural numbers k, l, where $0! = 1$ by definition.

15) Prove the following, e.g. by induction (on what?):

a) $\binom{n}{m}$ equals the number of subsets with m elements of the set $\{1, \dots, n\}$.

b) The binomial formula holds: $(x+y)^n = \sum_{m=0}^{n} \binom{n}{m} x^{n-m} y^m$

[Question: What are the precise hypothesis on x, y and the operations $+$ and $\cdot$ for the above formula to hold?]

Hint to b): As a first step, prove that $\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}$ for all $m, n > 0$, and use this in the induction hypothesis, etc.

16) Show that $2^n = \sum_{m=0}^{n} \binom{n}{m}$. Are there similar formulas for 3^n, 4^n, etc.?

17) Let $a_1, \dots, a_n > 0$ be real numbers. Prove the famous mean inequalities (Google it!):

$$\frac{a_1 + \cdots + a_n}{n} \geqslant \sqrt[n]{a_1 \dots a_n} \geqslant \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}}$$

Note: There are more general inequalities, the so called *Jensen inequalities* (Google it!).

Hint: (i) By induction on k, prove the assertions for $n = 2^k$; (ii) Prove that the assertions for $n+1$ implies them for n, etc.

18) Obviously, there are natural numbers n which cannot be described using less than 1000 words (say, in English). Let n_0 be the smallest such number, in other words, one has that:

$(*)$ n_0 *is the smallest natural number which cannot be described using less than 1000 words.*

OTOH, the above is a description $(*)$ of n_0 has less than 1000 words!? What is wrong here?...

19) A collection of elements X is called *normal,* if $X \notin X$, and abnormal if $X \in X$. **Clearly,** every collection of elements is either normal, or abnormal. Let $\mathcal{X}$ be the collection of all the normal collections. Show that $\mathcal{X}$ is neither normal, nor abnormal. What is wrong here?...