Option / Week 14
Antiderivatives/Arc Length/Area

Def Let $F = R, C$ and $D \subset F$ conn. open subset.
Given $f: D \rightarrow F$, an antiderivative $\tilde{f}$ of $f$
is any function $\tilde{f}: D \rightarrow F$ s.t. $\tilde{f}' = f$.

Rem/Notation If $\tilde{f}_1, \tilde{f}_2$ are antiderivatives of $f$,
then $\tilde{f}_1 - \tilde{f}_2 = cI$. Conversely $\tilde{f} + ct$ is antiderivative.
The usual notation for the antiderivatives is $\int f dx$.

Exmpl
1) If $f = 0$ on $D$, then $\tilde{f} = cI$, are all the
antiderivatives of $f$ (why).
2) $f(x) = x^n$, $n \neq -1$, then $\tilde{f}(x) = \frac{1}{n+1}x^{n+1} + ct$
are the antiderivatives (why?)
3) $\sin^2 + ct$ are the antiderivatives of $\sin 2x$

Fact The elementary functions $f$, $\exp$, $\log$
$
\sin, \cos, \tan, \cotan$ have antiderivatives.

Ex Write down the antiderivatives above.

Note $f(x) = \sqrt{1-x^2}$, $\tilde{f}$ exist, but are not elementary
Length of curve arcs (path) / Area of domains

- Recall: Given points $P_k = (a_k, b_k) = z_k \in \mathbb{R}^2, k = 1, 2$

\[ d(P_1, P_2) = |z_2 - z_1| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2} \]

is the distance from $P_1$ to $P_2$.

**AIM:** Define the length of arcs of curves and the area of domains in $\mathbb{R}^2 = \mathbb{C}$.

- This should be done in a way that is compatible with the usual physical intuition concerning length and area.

I) LENGTH

**Def**

1. $l([P_1, P_2]) := d(P_1, P_2)$ is the length of $[P_1, P_2]$.

2. For $P_1, ..., P_n \in \mathbb{C} = \mathbb{R}^2$, define $[P_1, ..., P_n] := \cup_{i=1}^{n} [P_i, P_{i+1}]$, and its length $l([P_1, ..., P_n]) := \sum_{1 \leq i < n} l([P_i, P_{i+1}]) = \sum |P_i - P_{i+1}|$.

**AIM:** Define the length $l(\gamma)$ - if it exists - for any cont. path $\gamma : [a, b] \to \mathbb{C} = \mathbb{R}^2$. 
Def/Rem: Let \( \gamma: [a, b] \to \mathbb{R}^2 = \mathbb{C} \) be a map.

1) \( \gamma \) is called smooth, if \( \gamma \) is differentiable on \((a, b)\), \( \gamma' \) is continuous and bounded, and \( \gamma'(x) \neq 0 \) for \( x \in (a, b) \).

2) \( \gamma \) is piece-wise smooth, if \( \exists a = x_1 < \ldots < x_n = b \) s.t. \( \gamma_i := \gamma|_{[x_i, x_{i+1}]} \) is smooth for \( i = 1, \ldots, n-1 \).

3) For \( \gamma \) as above, we say that \( \gamma([a, b]) \) is the path.

**Exmpl:**

0) \( \gamma_1: [0, 1] \to \mathbb{R}^2 \), \( \gamma_1(t) = [t, t^2] \) is smooth.

1) \( \gamma_2: [0, 1] \to \mathbb{R}^2 \), \( \gamma_2(t) = \{ (0, 0), t = 0 \} \) is continuous, but \( \gamma_2(t) = (t, \sin(1/t), t \neq 0) \) not smooth ( WHY? )

2) \( \gamma_3: [0, 1] \to \mathbb{R}^2 \),

\[
\gamma_3(t) = (\cos(\pi t), \sin(\pi t))
\]

is smooth.

3) \( \gamma_4: [-1, 1] \to \mathbb{R}^2 \),

\[
\gamma_4(t) = (t, |t|)
\]

is piece-wise smooth.
We define the length of a path \( \gamma: [a, b] \to \mathbb{R}^2 \) in two main steps along the following lines:

**Step 1**

For every partition \( I: a = a_1 \leq \ldots \leq a_n = b, n \in \mathbb{N}_+ \) set \( P_i : = \gamma(a_i) \), and define \( \ell_i : = \sum_{i \leq i < n} \ell([P_i, P_{i+1}]) \).

**Rem** Let \( I: a = a_1 \leq \ldots \leq a_m = b \) be partitions \( J: a = b_1 \leq \ldots \leq b_n = b \).

**Define:** \( I \leq J \iff \{ a_i \}_i \subset \{ b_j \}_j \) and say: "\( J \) is finer than \( I \)."

**Equiv:** \( J = \bigcup_{i \in I} J_i \) s.t. \( J_i : a_i = b_i \leq \ldots \leq b_{i_{\text{fin}}} = a_{i+1} \).

**Fact** If \( I \leq J \), then \( \ell_I \leq \ell_J \).

**Proof:** One has \( \ell_I \overset{\text{def}}{=} \sum_{i \leq m} |z_{i+n} - z_i| \) and

\[
\ell_J \overset{\text{def}}{=} \sum_{j \leq n} |z_{j+n} - z_j| \overset{\text{w.r.t.}}{=} \sum_{i \leq n} \left( \sum_{j_i \leq j < j_{i+1}} |z_{j+n} - z_j| \right)
\]

**Now** \( z_{j_i} = z_i, z_{j_{i+1}} = z_{i+1} \) and:

\[
z_{i+n} - z_i = z_{j_{i+1}} - z_{j_i} = z_{j_{i+1}} - z_{j_i} + z_{j_{i+2}} - z_{j_{i+1}} + \ldots + z_{j_{i+n}} - z_{j_{i+1}}
\]

hence \(|\ldots| = |\ldots| \leq |\ldots| + |\ldots| + \ldots + |\ldots|, \) i.e.

\[
|z_{i+n} - z_i| \leq \sum_{j_i \leq j < j_{i+1}} |z_{j+n} - z_j|, \text{ etc.}
\]
Conclude: Let \( D = \{ I \mid I \text{ partition of } [a,b] \} \)
endowed with the partial ordering \( \leq \). Then
\( l: D \to \mathbb{R}_{\geq 0}, I \mapsto l_I \) is compatible
with orderings, i.e. \( I \leq J \implies l_I \leq l_J \) in \( \mathbb{R} \).

Step 2: Suppose that \( \forall \varepsilon > 0 \exists I_\varepsilon \) such that:

\[ (*) \quad \forall I \text{ with } I_\varepsilon \leq I \text{ one has } l_I - l_{I_\varepsilon} < \varepsilon. \]

Prop: Suppose that condition \( (*) \) above
is satisfied. Then \( X_\varepsilon = \{ l_I \mid I \in D \} \) is
bounded above, and \( l(\varepsilon) = \sup (X_\varepsilon) \)
satisfies: \( \forall \varepsilon > 0 \exists I_\varepsilon \) s.t. \( \forall I_\varepsilon \leq I \) one has:

\[ 0 \leq l(\varepsilon) - l_I < \varepsilon. \]

Proof: ex.

Def: We say that \( \delta: [a,b] \to \mathbb{R}^2 \) has a
length, if cond \( (*) \) is satisfied. If so, we
say that \( l(\delta) = \sup (X_\varepsilon) \) is the length of \( \delta \).
**Example** The case $\mathcal{L}([P_1, \ldots, P_n])$

- $n = 2$: $[P_1, P_2] = \{P_1 + a(P_2 - P_1) | 0 \leq a \leq 1\} = \text{im}(\gamma)$
  where $\gamma : [0, 1] \to \mathbb{R}^2 = \mathcal{C}$, $\gamma(a) = P_1 + a(P_2 - P_1)$.
  **NOTE:** $\gamma$ is diff, and $\gamma'(z) = P_2 - P_1$ (why?), i.e., $P_2 - P_1 = (1 - 0) \cdot \gamma'(c)$ for some $c \in (0, 1)$.
  Therefore: $L([P_1, P_2]) = (1 - 0) |\gamma'(c)|$.

- In general, for $P_1, \ldots, P_n \in \mathbb{R}^2 = \mathcal{C}$, let $0 = a_1 \leq \ldots \leq a_n = 1$ and $\gamma : \{a_1, \ldots, a_n\} \to \mathcal{C}, a_i \mapsto P_i$.
  Define $\gamma : [0, 1] \to \mathcal{C}$ s.t. $\gamma([a_i, a_{i+1}]) = [P_i, P_{i+1}]$, precisely:
  \[
  \gamma_i(a) = P_i + \frac{a - a_i}{a_{i+1} - a_i} (P_{i+1} - P_i).
  \]

Then $\gamma_i$ is diff on $(a_i, a_{i+1})$, and $\gamma'(c) = \frac{1}{a_{i+1} - a_i} (P_{i+1} - P_i)$.
  Therefore: $L([P_i, P_{i+1}]) = (a_{i+1} - a_i) \cdot |\gamma'(c)|$, thus have:
  \[
  L([P_1, \ldots, P_n]) = \sum_{1 \leq i < n} (a_{i+1} - a_i) |\gamma'(c)|, \quad c \in (a_{i+1}, a_i).
  \]
For most of the paths $\gamma: [a, b] \to \mathbb{R}^2 = \mathbb{C}$ the length $\ell(\gamma)$ DOES NOT EXIST!!

**FAMOUS EXAMPLE:** Peano curve (google it)

- **Exmpl 1** above, $l(\gamma)$ is not defined.

**Thm** Suppose that $\gamma: [a, b]$ is a smooth path, then the following hold:
  1) $|\gamma'|$ has antiderivatives $\tilde{f}: [a, b] \to \mathbb{R}$.
  2) One has $l(\gamma) = \tilde{f}(b) - \tilde{f}(a)$.

**proof:** (lader)

**Exmpl.**
  1) $\gamma: [0, 1] \to \mathbb{R}^2$, $\gamma(t) = (\cos(\pi t), \sin(\pi t))$
     $\gamma'(t) = (-\pi \sin(\pi t), \pi \cos(\pi t)) \neq 0 \forall t$
     $|\gamma'| = \pi$, hence $\tilde{f}_\gamma(t) = \pi t (w+y)$
     Hence $l(\gamma) = \tilde{f}_\gamma(1) - \tilde{f}_\gamma(0) = \pi$
  2) $\gamma: [-1, 1] \to \mathbb{R}^2$, $\gamma_1(t) = (t, \sqrt{1-t^2})$
     $\gamma_1'(t) = (1, -\frac{t}{\sqrt{1-t^2}})$, $|\gamma_1'(t)| = \frac{1}{\sqrt{1-t^2}}$, hence $\tilde{f}_{\gamma_1}(t) = \arcsin(t)$
     $l(\gamma_1) = \tilde{f}_{\gamma_1}(1) - \tilde{f}_{\gamma_1}(-1) = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi$. 

\[ \gamma([0, 1]) \]
\[ \gamma_1([-1, 1]) \]
I) \textbf{Area} \quad A(B), \quad B \subset \mathbb{R}^2 = C.

Recall: (i) The unit for the area is the "unit square" 
\[ \square = [a,a+1] \times [b,b+1], \quad \text{i.e.} \quad A(\square) = 1, \quad a,b \in \mathbb{Z} \]
(ii) The area of segments \([P_1, P_2]\) is zero.
(iii) The area is additive in the toll sense:
If \(B_1, B_2 \subset \mathbb{R}^2\) a subset whose areas is defined, then the areas of 
\(B_1 \cap B_2\) and \(B_1 \cup B_2\) is defined, and 
\[ A(B_1 \cap B_2) + A(B_1 \cup B_2) = A(B_1) + A(B_2). \]

\textbf{Ex} \quad \text{Let } B_p(a,b) = \{(x,y) | x_0 \leq x \leq x_0 + a, y_0 \leq y \leq y_0 + b\} \subset \mathbb{R}^2, \quad p_0 = (x_0, y_0), \]
\(x_0, y_0 \in \mathbb{Q}, \quad a, b \in \mathbb{Q}_{\geq 0}.\) The there is a unique way to define the area of all \(B_p(a,b),\) namely:
\[ A(B_p(a,b)) = a \cdot b. \]

\textbf{Ex} \quad \text{There is a unique way to define the area of all } \(B_p(a,b), \quad p_0 \in \mathbb{R}^2, \quad a, b \in \mathbb{R}_{\geq 0},\) \text{ namely } A(B_p(a,b)) = a \cdot b.

\textbf{Ex} \quad \text{There is a unique way to define the area of open bounded sets.
Special Case: Let $B \subset \mathbb{R}^2$ be a bounded set s.t.

(i) $B^+_B = \bigcap_{B \subset \mathbb{R}^2} B^+, A(B^+) \text{ defined.}$

(ii) $B^-_B = \bigcup_{B \subset \mathbb{R}^2} B^-, A(B^-) \text{ defined.}$

Then $X^+_B = \{ (A(B^+)) \mid B \subset \mathbb{R}^2, A(B^+) \text{ def} \}$

$X^-_B = \{ (A(B^-)) \mid B \subset \mathbb{R}^2, A(B^-) \text{ def} \}$

are bounded sets of real numbers, and clearly, $A(B^-) \leq A(B^+) \cup B^-, B^+(\text{PPP})$

**Def.** We say that $A(B)$ is defined, if

$$\inf(X^+_B) = \sup(X^-_B), \text{ and if so, define } A(B)^{\text{def}} = \inf(X^+_B) = \sup(X^-_B).$$

**Ex.** Prove that if $B_1, B_2 \subset \mathbb{R}^2$ satisfy the conditions (i), (ii) above, then $B_1 \cap B_2, B_1 \cup B_2$ satisfy conditions (i), (ii), and one has:

$$A(B_1 \cap B_2) + A(B_1 \cup B_2) = A(B_1) + A(B_2).$$
Fundamental Theorem of Calculus

We next consider bounded subsets $B_f \subseteq \mathbb{R}^2$ defined via bounded functions $f: [a, b] \rightarrow \mathbb{R}_{\geq 0}$.

$$B_f = \{(x, y) \mid a \leq x \leq b, 0 \leq y \leq f(x)\}$$

**Thm (Fundamental Theorem of Calculus)**

Suppose that $f: [a, b] \rightarrow \mathbb{R}_{\geq 0}$ is continuous. Then:

1) The area $A(B_f)$ is defined.
2) $f$ has antiderivatives $\int f$ and for every $f$ one has $A(B_f) = \int_a^b f(x) \, dx$.

**Notation**

$A(B_f) = \int_a^b f(x) \, dx$.

**Proof (later)**

**Example**

1) $f: [0, 1] \rightarrow \mathbb{R}, f(x) = x^2$;  
$A(B_f) = \frac{1}{3}$

2) $f_i: [-1, 1] \rightarrow \mathbb{R}, f_i(x) = \sqrt{1-x^2}$;  
$A(B_{f_i}) = \frac{\pi}{2}$
Proofs (Sketch)

Main tool:

Thm (Uniform Continuity Thm)
Let \( f: X \to Y \) be a cont. map of metric spaces. If \( X \) is compact, then \( f \) is uniformly continuous, i.e.
\[
\forall \varepsilon > 0 \ \exists \delta > 0 \ \text{s.t.} \ d(x, x') < \delta \implies d(f(x), f(x')) < \varepsilon.
\]

Proof ex (see HW 12, Problem 1).

Proof of Fundam Thm of Calc:

For \( f: [a, b] \to \mathbb{R} \) bounded and
\( I: a = a_1 \leq \ldots \leq a_n = b \) partition, consider:
\[
y_i^m = \inf \left\{ f(x) \mid a_i < x < a_{i+1} \right\}, \quad B_i^m = [a_i, a_{i+1}] \times [0, y_i^m].
\]
\[
y_i^n = \sup \left\{ f(x) \mid a_i < x < a_{i+1} \right\}, \quad B_i^n = [a_i, a_{i+1}] \times [0, y_i^n].
\]
\[
B_i^m = \{(a, y) \mid a_i < y < a_{i+1}, \ 0 < y < f(x)\}.
\]

Then one has (why?):

- \( A(B_i^m) = \sum_{i<n} (a_{i+1} - a_i) y_i^m \leq \sum_{i<n} (a_{i+1} - a_i) y_i^n = A(B_i^n) \)
- \( B_i^m \subset B_i^m \subset B_i^n \), hence \( B_i^m = \bigcup_{i} B_i^m \subset B_i^m \).
\textbf{Def} let \( x_i \in [a_i, a_{i+1}] \), \( y_i := f(x_i) \) be given.

1) \( \sum_{f, I} := \sum_{i=1}^{n} (a_i - a_{i-1}) y_i \) is called the \textbf{Riemann sum} attached to \( f, I, (x_i) \).

2) \( \sum_{I}^M := \sum_{i=1}^{n} (a_i - a_{i-1}) y_i^m \) is the inf-sum.

\( \sum_{I}^M := \sum_{i=1}^{n} (a_i - a_{i-1}) y_i^m \) is the sup-sum.

\textbf{Obviously} \( A(B_{I}^{m}) = \sum_{f, I}^m \leq \sum_{f, I}^M = A(B_{I}^{m}) \) (why?)

\textbf{Back to the proof}

To\(1\): let \( \varepsilon > 0 \) be given. The \( \exists \delta > 0 \) s.t. one has:

\( |x - x'| < \delta \Rightarrow |f(x) - f(x')| < \varepsilon \). Hence if \( a_i - a_{i-1} < \delta \)

then have: \( y_i^m = \min f([a_i, a_{i+1}]) \), \( y_i^m = \max f([a_i, a_{i+1}]) \)

satisfy \( y_i^m - y_i^m < \varepsilon \), thus \( \sum_{f, I}^n - \sum_{f, I}^M < (b-a)\varepsilon \) (why?).

Therefore: Given \( \varepsilon > 0 \), let \( \varepsilon' = \varepsilon/(b-a) \). For \( \delta > 0 \) as above, let \( I_\varepsilon: a = a_1 < \ldots < a_n = b \) sat. \( a_i - a_{i-1} < \delta \). Then one has:

\( B_{I_\varepsilon}^{m} \subset B_{f} \subset B_{I_\varepsilon}^{M} \), \( A(B_{I_\varepsilon}^{m}) = \sum_{f, I_\varepsilon}^m \leq \sum_{f, I_\varepsilon}^M = A(B_{I_\varepsilon}^{m}) \leq A(B_{I_\varepsilon}^{m}) + \varepsilon' \).

Conclude that \( A(B_f) \) exist and equals

\( \sup \{ A(B_{I_\varepsilon}^{m}) | \varepsilon > 0 \} = \inf \{ A(B_{I_\varepsilon}^{m}) | \varepsilon > 0 \} \) (why?)
To 2): For \( x \in [a,b] \), consider 
\[ f_x : [a,x] \rightarrow \mathbb{R}, \quad f_x(x') = f(x'), \] 
that is, 
\[ f_x = f|_{[a,x]} \] 
is the restriction of \( f \) to \([a,x]\). Then 
\[ f_x : [a,x] \rightarrow \mathbb{R}_{\geq 0} \] 
is continuous (why?), hence the area \( A(B_{f_x}) \) is defined (why?). 

Define \( \widetilde{f} : [a,b] \rightarrow \mathbb{R}, \quad \widetilde{f}(x) = A(B_{f_x}). \)

Claim: \( \widetilde{f} \) is an antiderivative of \( f \).

Step 1: \( \widetilde{f} \) is diff on \((a,b)\) and \( \widetilde{f}'(x) = f(x) \) \( \forall x_0 \).

Indeed: Recall that \( f \) is unif. cont. on \([a,b]\), hence:
\[ \forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon. \]

Let \( x_m, x^M \in [-d+x_0, x_0] \) be abs min/max pts, i.e.
\[ y_m = f(x_m) \leq f(x) \leq f(x^M) = y^M \quad \forall x \in [d+x_0, x_0]. \]

One has:
\[ (x - x_0) y_m \leq \widetilde{f}(x_0) - f(x) \leq (x - x_0) y^M, \]

(why?)

Thus:
\[ y_m \leq \frac{f(x) - f(x_0)}{x - x_0} \leq y^M \quad \text{(why?)}. \]
Thus $y^m - f(x_0) \leq \frac{\hat{f}(x) - \hat{f}(x_0)}{x - x_0} \leq y^m - f(x_0)$.

Since $x^m, x \in [-\delta + x_0, x_0]$, one has $|y^m - f(x_0)|, |y^m f(x_0)| < \varepsilon$.

Hence $-\varepsilon < \frac{\hat{f}(x) - \hat{f}(x_0)}{x - x_0} - f(x_0) < \varepsilon$ (why?)

• Analyze similarly the case $x_0 < x$, and
• conclude: $\hat{f}(x_0) = f(x_0)$.

Step 2: $\hat{f}$ cont on $[a, b]$, and $A(Bf) = \hat{f}(b) - \hat{f}(a)$

• Continuity at $x_0 = a$:

To prove:

$\forall \varepsilon > 0 \exists \delta > 0 \ s.t. \ 0 \leq x - a < \delta \Rightarrow |\hat{f}(x) - \hat{f}(a)| < \varepsilon$

Now: $f$ cont on $[a, b] \Rightarrow f$ bounded, $f(x) < \varepsilon_0$.
Then $0 \leq \hat{f}(x) - \hat{f}(a) = A(B_{fx}) \leq \varepsilon_0 (x-a)$

Hence $\lim_{x \to a} \hat{f}(x) = 0 = \hat{f}(a)$.

• Finally, $A(Bf) = \hat{f}(b) - \hat{f}(a)$

By mere definition!
**Proof of Arc length Thm**

Let \( \gamma : [a, b] \rightarrow \mathbb{C} = \mathbb{R}^2 \), \( \gamma(t) = x(t) + y(t)i \).

Then \( \gamma \) diff iff \( x, y : [a, b] \rightarrow \mathbb{R} \) diff \((\text{why?})\),

and \(|\gamma'| = (|x'|^2 + |y'|^2)^{\frac{1}{2}} \) \((\text{why?})\).

And \( \gamma' \) cont & bounded iff \( x', y' : [a, b] \rightarrow \mathbb{R} \) cont & bounded.

Let \( a < a_0 < b_0 < b \) be given. Then \( x', y' : [a_0, b_0] \rightarrow \mathbb{R} \) are unif. cont. Hence \( \exists \varepsilon' > 0 \) \( \exists \delta' > 0 \) s.t.

\[ t, t' \in [a_0, b_0], |t-t'| < \delta' \Rightarrow |x'(t) - x'(t')|, |y'(t) - y'(t')| < \varepsilon'. \]

Hence if \( a_0 =: a_1 < \ldots < a_n = b_0 \) satisfies \( a_i+1 - a_i < \delta', \) one has:

**Step 1** let \( t_i^m, t_i^M, u_i^m, u_i^M \in [a_i, a_{i+1}] \) be pts of

\[ \text{abs. min/ max for } x_i : [a_0, b_0] \rightarrow \mathbb{R}, \text{ resp } y_i : [a_0, b_0] \rightarrow \mathbb{R}. \]

Then:

\[ x_i := x'(t_i^m)^2, x_i := x'(t_i^M)^2, x_i := x'(t_i^M)^2, x_i := x'(t_i^m)^2, \forall t_i \in [a_i, a_{i+1}]. \]

\[ y_i := y'(u_i^m)^2, y_i := y'(u_i^M)^2, y_i := y'(u_i^M)^2, y_i := y'(u_i^m)^2, \forall u_i \in [a_i, a_{i+1}]. \]

Conclude \( \forall \varepsilon'' > 0 \) \( \exists \varepsilon' > 0 \) in the notation above, setting \( \Delta_i = \sqrt{x_i + y_i} \), \( \forall t_i, u_i \in [a_i, a_{i+1}] \) one has:

\[ \Delta_i \leq \sqrt{x_i(t_i)^2 + y_i(u_i)^2} \leq \Delta_i + \varepsilon'' \] \((\text{why?})\).
Step 2  In the notation from Step 1, by Lagrange's Thm applied to $x, y: [a_i, a_{i+1}] \to \mathbb{R}$ one has:

$\exists t'_i, u'_i \in [a_i, a_{i+1}]$ satisfying:

$\delta(x_{a_{i+1}}) - \delta(x_i) = (a_{i+1} - a_i) \cdot (x'(t'_i), y'(u'_i)).$

In part, $|\delta(x_{a_{i+1}}) - \delta(x_i)| = (a_{i+1} - a_i) \sqrt{x'(t'_i)^2 + y'(u'_i)^2}$

Step 3  Conclude: In the above notation one has:

$(a_{i+1} - a_i) \Delta_i \leq (a_{i+1} - a_i) |x'(t_i)|, |\delta(x_{a_{i+1}}) - \delta(x_i)| < (a_{i+1} - a_i) \Delta_i + \varepsilon''(a_{i+1} - a_i)$

Thus: $-\varepsilon''(a_{i+1} - a_i) < |\delta(x_{a_{i+1}}) - \delta(x_i)| - (a_{i+1} - a_i) |x'(t_i)| < (a_{i+1} - a_i) \varepsilon''$

and summing up for $i = 1, \ldots, n-1$, finally get:

$-\varepsilon'' \sum_{i<n} (a_{i+1} - a_i) < \sum_{i<n} |\delta(x_{a_{i+1}}) - \delta(x_i)| - \sum_{i<n} (a_{i+1} - a_i) |x'(t_i)| < \varepsilon'' \sum_{i<n} (a_{i+1} - a_i)$

hence: $-\varepsilon'' (b_o - a_0) < \sum_{l<\infty} |\delta(x_{a_{l+1}}) - \delta(x_{a_l})| - \sum_{l<\infty} (a_{l+1} - a_l) |x'(t)| < \varepsilon'' (b_o - a_0)$

Step 4  Since $|x'|$ is (uniformly) cont on $[a_0, b_0]$, if $f_x$ is an antiderivative of $|x'|$ one has:

$\forall \varepsilon > 0 \exists I_{\varepsilon}: a_0 = a_1 < \ldots < a_n = b_0 \text{ st. for all } I \geq I_{\varepsilon}$:

$(\ast) \quad |\sum_{l<l<\infty} |x'|(a_l) - (\hat{f}_x(b_o) - \hat{f}_x(a_0))| < \varepsilon/2$. 

In particular, if \( \varepsilon > \frac{3}{2} \varepsilon / (b_0 - a_0) \), one has:

\[
| e - \sum_{l \neq I} \gamma_l | < \varepsilon (b_0 - a_0) < \frac{1}{2} \varepsilon \quad \forall I \in \mathcal{I}_\varepsilon,
\]

hence \((*)\) implies:

\[
| e - \sum_{l \neq I} \gamma_l | < | e - (\tilde{f}_\varepsilon(b_0) - \tilde{f}_\varepsilon(a_0)) + \tilde{f}_\varepsilon(b_0) - \tilde{f}_\varepsilon(a_0) - \sum_{l \neq I} \gamma_l | < \frac{1}{2} \varepsilon
\]

and therefore get:

\[
| e - (\tilde{f}_\varepsilon(b_0) - \tilde{f}_\varepsilon(a_0)) | < \varepsilon
\]

Thus by mere def, it follows that setting

\[
\gamma_0 := \gamma |_{[a_0, b_0]} : [a_0, b_0] \to \mathbb{R}^2, \quad \gamma_0(t) = \gamma(t),
\]

one has that \( e(\gamma_0) = \tilde{f}_\varepsilon(b_0) - \tilde{f}_\varepsilon(a_0) \).

**Step 5** Let \( a_n < a, \ b_n > b \) with \( a_0 < b_0 \), as above,

and \( \gamma_n := \gamma |_{[a_n, b_n]}, \tilde{f}_{\gamma_n} = \sum_{n} |\lambda'_n| \).

\[
e(\gamma_n) = \tilde{f}_{\gamma_n}(b_n) - \tilde{f}_{\gamma_n}(a_n) = \sum_{a_0} |\lambda'_n| (t) \ dt.
\]

Since \( |\lambda'| \) is bounded, get \( e(\gamma_n) \) conv, etc.