Recall: \( e = \sum \frac{1}{n!} \in \mathbb{R} ; \ (g \circ f)(x) = g(f(x)) \cdot f'(x) \)

**Differentiability (cont) / Week 13(c)**

- Recall that an analytic function
  \[ f: \mathbb{D}_f \rightarrow \mathbb{F}, \quad f(x) = \sum_{n=0}^{\infty} a_n x^n \]
  is differentiable on \( \mathbb{D}_f \), and \( f'(x) = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1} \).
- Recall that \( \exp_e: \mathbb{R} \rightarrow (0,\infty), \ \exp_e(x) = e^x \)
  \text{AND} \ \exp: \mathbb{C} \rightarrow \mathbb{C}, \ \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \]
  satisfy \( e^x = \exp(x) \ \forall x \in \mathbb{R} \), see HW #10, Prob. 1.
- In particular \( \exp_e' = \exp_e \) on \( \mathbb{R} \) (why?).

**Thm (Elem. Functions)**

The elementary functions \( \exp_a, \log_a, \) for one are differentiable on their domains, and one has:

\[ \exp_a'(z) = \exp_a(x) \cdot \log_a, \quad \log_a'(x) = \frac{1}{\log_a x}, \quad f(x) = x^p \cdot x^q. \]

**Proof:**
Recall: \( f: X \rightarrow Y \) bij. Then \( f^{-1}: Y \rightarrow X \) by \( f'(y) = x \) iff \( f(x) = y \).
Further: \( \log_a = \exp_{a^{-1}} \) and \( \exp_{e} = \exp_{e} = \exp \).

Hence \( \log_a (\exp_a(x)) = x \Rightarrow \log_a'(\exp_a(x)) \cdot \exp_a'(x) = 1. \)

Since \( \exp_a'(x) = \exp_a(x) = y \), get: \( \log_a'(y) = \frac{1}{x} \forall y \in (0,\infty) \).
Thus one has:

- \( \exp_a(x) = a^x = e^{\log(a^x)} = e^{x \log(a)} = \exp(x \cdot \log(a)) \), hence: \( \exp_a'(x) = \exp'(x \cdot \log(a)) \cdot (x \cdot \log(a))' \) (why?)

\[ = \exp(x \cdot \log(a)) \cdot \log(a) = a^x \cdot \log(a) \] (why?)

- \( \log_a'(x) = \frac{1}{\log_a \cdot x} \) (why?)

- \( f_a(x) = x^a = \exp(\log(x^a)) = \exp(a \log(x)) \), hence \( f_a'(x) = \exp'(a \log(x)) \cdot (a \cdot \log(x))' \)

\[ = x^a \cdot a \cdot \frac{1}{x} = ax^{a-1} = \frac{d}{dx} x^a \] (why?)

---

1) Graphs of \( \exp_a, \log_a \)

\( \exp_a : \mathbb{R} \rightarrow (0, \infty) \); \( \log_a : (0, \infty) \rightarrow \mathbb{R} \)
2) Graph of \( f_\alpha \)

\[
\begin{align*}
y & = x^\alpha \\
y & = x \\
\end{align*}
\]

Ex: Study/Explain why the graphs look like this.

More about \( \sin \) & \( \cos \)

Recall:

\[
\begin{align*}
\sin : \mathbb{C} & \to \mathbb{C}, \quad \sin(x) = \sum_n (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
\cos : \mathbb{C} & \to \mathbb{C}, \quad \cos(x) = \sum_n (-1)^n \frac{x^{2n}}{(2n)!} \\
\end{align*}
\]

Addition Formulas:

\[
\begin{align*}
\sin(x+y) & = \sin(x) \cos(y) + \sin(y) \cos(x) \\
\cos(x+y) & = \cos(x) \cos(y) - \sin(x) \sin(y) \\
\end{align*}
\]

Parity:

\[
\begin{align*}
\cos(-x) & = \cos(x), \text{ i.e., } \cos \text{ is an even function.} \\
\sin(-x) & = -\sin(x), \text{ i.e., } \sin \text{ is an odd function.} \\
\end{align*}
\]

The Identity:

\[
1 = \cos^2(x) + \sin^2(x) \quad \forall x \in \mathbb{C}
\]
There exists $\theta > 0$ s.t. the following hold:

1) $\cos(x) > 0$ for $x \in (0, \theta)$ and $\cos(\theta) = 0$.
2) $\sin, \cos$ are periodic with period $4\theta$, i.e., $\sin(x+4\theta) = \sin(x)$, $\cos(x+4\theta) = \cos(x)$, and satisfy:

<table>
<thead>
<tr>
<th>$\sin$</th>
<th>$0$</th>
<th>$1$</th>
<th>$0$</th>
<th>$-1$</th>
<th>$0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cos$</td>
<td>$1$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$0$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

**Graphs:**

**proof:** One has: $\cos(0) = 1$, $\cos(2) < 0$ (why?)

By the IVT: $\exists \theta \in [0, 2]$ s.t. $\cos(\theta) = 0$.

And let $\theta := \inf \{ x \in \mathbb{R}^+ \mid \cos(x) = 0 \}$. Then one has:

- $\cos(\theta) = 0$ (why?), hence $\theta > 0$ (why?)
- $\cos(x) > 0 \ \forall x \in (0, \theta)$ (why?)
• Since \( \sin'(x) = \cos(x) \), get: \( \sin \) strictly increasing \([0, \theta]\), and \( \sin(0) = 0, \sin(\theta) = 1 \). In particular, \( \sin(x) > 0 \) for \( x \in (0, \theta) \) (why?).

• Therefore, \( \cos'(x) = -\sin(x) < 0 \) for \( (0, \theta) \), hence \( \cos(x) \) strictly decreasing on \([0, \theta]\).

• For \( x \in [\theta, 2\theta] \) one has: let \( x = \theta + x' \), \( x' \in [0, \theta] \). Then \( \sin(x) = \sin(\theta + x') = \cos(x') \) (why? addition formulas).

\[
\begin{align*}
\cos(x) &= \cos(\theta + x') = -\sin(x') \\
\end{align*}
\]

• Analyze similarly \( x \in [2\theta, 3\theta], x \in [3\theta, 4\theta] \) AND conclude that \( \sin(x + 4\theta) = \sin(x) \) \( \cos(x + 4\theta) = \cos(x) \).

**NOTATION/REMARK** \( J = 2\theta \) is the famous number...

---

**The Story of** \( J \)

**Def.** \( J = \lambda(C(0,r))/2\pi \)

(Why "Def"? What is \( \lambda(C(0,r)) \))
• How to compute $\pi$?

• Archimedes (~250 BC)

"Approximate the circle by inscribed regular polygons, e.g. square, octagon, 16-gon,... and get better and better approx. for $\ell(C(0, r))$.

• The usual approx. for $\pi$ used by Greeks was

$$\pi \approx \frac{22}{7} = 3.142...$$

\[(3.14159...)

• Several Chinese Mathematicians (100-500 AD)

(Huy) $3.1415926 < \pi < 3.1415927$

e.g. $\pi \approx \frac{355}{113}$

• Squaring the circle (Greeks ~ 5th century BC?)

Using ruler & compass, given the unit length, construct a square and a circle of the same area.

FAMOUS FORMULAS/NUMBERS: $\sqrt{2}; e^{\pi i} = -1; E = mc^2$
Gauss (1802) All the numbers \( \alpha \in \mathbb{R} \) which can be constructed with ruler \& compass are roots of irreducible polynomials \( p(t) \in \mathbb{Z}[t] \) of degree \( 2^n \), \( n \geq 0 \). Moreover, the roots can be computed by solving successively quadratic equations (involving previously computed numbers).

Lindemann-Weierstrass (1882) \( e \) is a transcendental number, i.e., it is not a root of any non-zero polynomial \( p(t) \in \mathbb{Q}[t] \). In particular, squaring the circle with ruler and compass is impossible!

Computing \( e \) using (power) series

Recall: \( \sin: [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1] \), \( \cos: [0, \pi] \to [-1, 1] \)

\( \tan: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R} \), \( \cotan: (0, \pi) \to \mathbb{R} \) are bij.
The corresponding inverse functions are:

\[
\begin{align*}
\arcsin &= \sin^{-1}: [-1,1] \rightarrow \left[ \frac{-\pi}{2}, \frac{\pi}{2} \right] \\
\arccos &= \cos^{-1}: [-1,1] \rightarrow [0,\pi] \\
\arctan &= \tan^{-1}: \mathbb{R} \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2}) \\
\arccotan &= \cotan^{-1}: \mathbb{R} \rightarrow (0, \pi)
\end{align*}
\]

The derivatives are (why?):

\[
\begin{align*}
\arcsin'(x) &= \frac{1}{\sqrt{1-x^2}} ; \\
\arccos'(x) &= -\frac{1}{\sqrt{1-x^2}} ; \\
\arctan'(x) &= \frac{1}{1+x^2} ; \\
\arccotan'(x) &= -\frac{1}{1+x^2}
\end{align*}
\]

Using \( \arctan \):

\[
\begin{align*}
\arctan \left( \frac{1}{\sqrt{3}} \right) &= \frac{\pi}{6} ; \\
\arctan(1) &= \frac{\pi}{4} ; \\
\arctan'(x) &= \frac{1}{1+x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n+1} , \quad |x| < 1
\end{align*}
\]

Hence \( \arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n+1} \) (why?).

\[
\begin{align*}
\frac{\pi}{6} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{3}^{2n+1}} ; \\
\frac{\pi}{4} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \\
\frac{\pi}{3} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+1}} (\text{Leibniz})
\end{align*}
\]