Thm (Cont & Compactness)
Let \( f : X \to Y \) be a cont. map of metric spaces.
1) If \( X_0 \subset X \) is compact, then \( f(X_0) \subset Y \) is compact.
2) If \( X \) is compact, and \( f \) is bijective, then the inverse map \( f^{-1} : Y \to X \) is continuous.

Proof
By Heine-Borel Thm, enough to show:
Every \((y_n)_n \in f(f(X_0))\) has a conv. subseq. \( y_{n_k} \to y \in f(X_0)\).
Now, for \((y_n)_n\) with \( y_n \in f(X_0)\), consider \((x_n)_n \in f(X)\)
so \( y_n = f(x_n) \forall n \in \mathbb{N}\). Since \( X_0 \) compact,
by the Heine-Borel Thm, \( \exists x_{n_k} \to x \in X_0 \).
Hence \( f \) cont \( \Rightarrow f(x_{n_k}) \to f(x) \) in \( f(X_0) \) (why?)
Thus setting \( y := f(x) \), get \( y \in X_0 \) & \( y_{n_k} \to y \).

To 2): By def, one has \( y = f(x) \) iff \( x = f^{-1}(y) \).
By contrast, suppose \( \exists y_n \to y \) s.t. \( x_n := f^{-1}(y_n) \to x := f^{-1}(y) \).
Hence \( \exists \varepsilon > 0 \) and a subseq. \((x_{n_k})_k\) s.t. \( d(x_{n_k}, x) > \varepsilon \ \forall k \) (why?).
Since \( X \) compact \( \Rightarrow \exists (x_{n_{k_l}})_l \) s.t. \( x_{n_{k_l}} \to x' \in X \).
Then \( f \) cont \( \Rightarrow y_{n_{k_l}} = f(x_{n_{k_l}}) \to f(x') =: y' \), hence \( y = y' \) (why?)
Thus \( x' = x \) (why?), hence \( d(x_{n_{k_l}}, x) < \varepsilon \) for \( l \gg 0 \)
Corollary. Let \( f : X \to \mathbb{R} \) be cont, \( X \) compact. Then \( f \) has points of absolute minimum \( x_m \), and absolute maximum \( x_M \), i.e. satisfying:
\[
 f(x_m) \leq f(x) \leq x_M \quad \forall x \in X.
\]

**Proof.** \( f(X) \subset \mathbb{R} \) is compact, hence bonded and closed \((\text{why?})\), hence one has \((\text{why?})\):
\[
y_m = \min(f(X)), \quad y_M = \max(f(X)) \text{ exist.}
\]
Hence setting \( y_m = f(x_m), \ y_M = f(x_M), \) etc ....

---

**The ring of cont. functions**

**Situation:**
- \( X \) metric space
- \( F = \mathbb{R} \) or \( F = \mathbb{C} \) as metric spaces w.r.t. \( d \).
- For \( f, g : X \to F \) and \( x = +1 \), consider \( f + g : X \to F \).
- For \( a \in F \), \( f : X \to F \), consider \( af : X \to F \).
- If \( g(x) \neq 0 \) for \( x \in X_0 \), consider \( f/g : X_0 \to F \).
**Prop**(E(x,F)) In the above notation, TF has:

1) Suppose that \( \lim_{x \to x_0} f(x) = y, \lim_{x \to x_0} g(x) = y' \).
   Then \( \lim_{x \to x_0} (f \times g)(x) = y'x'y' \), \( \lim_{x \to x_0} \alpha f(x) = ay' \).
   Further, if \( y' \neq 0 \), \( \exists \delta > 0 \) s.t. \( g(x) \neq 0 \ \forall x \in B_{\delta}(x_0) \), and \( \lim_{x \to x_0} (f/g)(x) = y'/y'' \).

2) If \( f, g \) are cont at \( x_0 \), then \( f \times g \) and \( \alpha f \) are cont. at \( x_0 \). Further, if \( g(x_0) \neq 0 \), then \( f/g \) is cont at \( x_0 \).

**Proof ex** (Use Prop (Limits), and the corresp. properties of convergent sequences)

**Corollary** In the above notation, let us denote \( C(x,F) = \{ f : X \to F | f \text{ cont} \} \subseteq \text{Maps}(X,F) \).
Then \( C(x,F) \subseteq \text{Maps}(X,F) \) is a subring and an F-algebra.
Moreover, if \( f \in C(x,F), f(x) \neq 0 \ \forall x \in X \), then \( 1/f \in C(x,F) \).

**Proof ex** (Everything follows from Prop(C(x,F)) above).
**Continuity of elementary Functions**

**Thm** The following functions are continuous:

1) The polynomial function $f_p : \mathbb{F} \to \mathbb{F}, x \mapsto p(x)$ with $p(t) \in \mathbb{F}[t]$ a fixed polynomial.

2) The power-$\alpha$ function $f_\alpha : (0, \infty) \to (0, \infty), x \mapsto x^\alpha$ with $\alpha \in \mathbb{R}$ fixed real number.

3) The exponential function $\exp_a : \mathbb{R} \to (0, \infty)$ for $a \in \mathbb{R}_{>0}, a \neq 1$. (What if $a=1$?)

4) The logarithmic function $\log_a : (0, \infty) \to \mathbb{R}$.

**Proof.** To 1): If $x_n \to x$, then $f(x_n) = p(x_n) \to p(x)$ (why?), etc.

To 2), 3): See HW 9, Problem 7, etc.

To 4): Let $X := [a, b] \subset \mathbb{R}, a < b$. Then $X$ compact (why?) and $\exp_a : X \to Y$ with $Y = [a^a, a^b]$ for $a > 1$, etc., and its inverse map $\log_a : Y \to X$ is continuous (why?). Hence $\log_a : (0, \infty) \to \mathbb{R}$ is continuous (why?).
Continuity of analytic functions

Situation: \( F = \mathbb{R}, \mathbb{C}; \ B_{R} := B_{R}(0) \) for \( R \in [0, \infty] \)

- \( \mathcal{F}[t] = \{ \sum_{n} a_{n} t^{n} \mid a_{n} \in F \} \) ring of formal power series

Lemma: If \( (a_{n})_n \in \mathcal{F}_{b}(F) \), then \( \sum_{n} a_{n} x^{n} \) abs. conv. for \( |x| < r \).

Proof: Let \( \varepsilon_{0} > 0 \) satisfy \( |a_{n} x^{n}| < \varepsilon_{0} \ \forall n \).

Then \( |a_{n} x^{n}| = |a_{n} r^{n}| \frac{x^{n}}{r^{n}} < \varepsilon_{0} \frac{x^{n}}{r^{n}} \ \forall n \) (why?)

And \( r_{0} := \frac{|x|}{r} < 1 \). Therefore one has:

\[
\sigma_{n} := \sum_{i \leq n} |a_{i} x^{i}| < \varepsilon_{0} \sum_{i = 0}^{n} |\frac{x}{r}|^{i} \leq \varepsilon_{0} \frac{1 - r_{0}^{n+1}}{1 - r_{0}} < \frac{\varepsilon_{0}}{1 - r_{0}}
\]

thus \( (\sigma_{n})_{n} \) is convergent (why?), hence by definition, \( \sum_{n} a_{n} x^{n} \) is abs. conv.

Def: \( \rho := \sup \{ r \mid (a_{n} r^{n})_n \text{ bounded} \} \in \mathbb{R}_{\geq 0} \cup \{ \infty \} \)

is the radius of convergence of \( f(t) = \sum_{n} a_{n} t^{n} \).

The map \( f: D_{\rho} \to F, \ x \mapsto f(x) := \sum_{n} a_{n} x^{n}, \ D_{\rho} = B_{\rho}(0) \), is the analytic function defined by \( f(t) \).

Rem: Let \( f(t) = \sum_{n} a_{n} t^{n}, \ g(t) = \sum_{n} b_{n} t^{n}, \ h(t) = \sum_{n} a_{n} b_{n} t^{n} \in \mathcal{F}[t] \).

Then the radii of convergence satisfy: \( \rho_{h} \geq \rho_{f} \cdot \rho_{g} \).

In part, \( \sum_{n} a_{n} t^{n}, \sum_{n} a_{n} x^{n} \) have equal radii of conv.

Proof: \( \exists \chi \) (Hint: \( g(t) = \sum_{n} a_{n} x^{n} \) has \( \rho_{g} = 1 \), etc).

Problem: What is that?
**Ex** \( \exp(t), \sin(t), \cos(t) \) are absolutely convergent on the whole \( \mathbb{C} \) (why?)

**Thm** Let \( f(t) = \sum_n a_n t^n \) have \( D_f \neq \emptyset \). Then the analytic function \( f: D_f \to \mathbb{F}, x \mapsto \sum_n a_n x^n \) is cont. In particular, \( \exp, \sin, \cos \) are continuous on \( \mathbb{C} \).

**proof**: Let \( x_0 \in D_f \) be given, hence \( |x_0| < \rho \).

Choose any \( \delta > 0 \) s.t. \( |x_0| + \delta < \rho \), hence \( B_{\delta}(x_0) \subseteq D_f \) (why?)

Then \( x \in B_{\delta}(x_0) \Rightarrow |x| + |x_0| < \rho \) (why?), and consider any \( r > 0 \) s.t. \( |x| + |x_0| < r < \rho \). Then:

\[
 f(x) - f(x_0) = \sum_n a_n (x^n - x_0^n) = (x-x_0) \sum_n a_n \sum_{i+j=n} x^i z_0^j,
\]

and since \( |x|, |x_0| < r \), get \( |x^i z_0^j| < r^{n-1} \) for \( i+j=n-1 \)

Therefore, \( |f(x) - f(x_0)| \leq |x-x_0| \sum_n |a_n| \cdot n \cdot r^{n-1} \) (why?)

Finally, since \( r < \rho \), one has \( (a_n \cdot r^n) \) bounded, hence \( (|a_n| \cdot n \cdot r^{n-1}) \) bounded (why? see Rem above)

Thus \( \sum_n |a_n| \cdot r^n \) is conv, say \( \sum_n |a_n| \cdot n \cdot r^n = \varepsilon_0 \).

Then one has \( |f(x) - f(x_0)| < |x-x_0| \cdot \varepsilon_0 \), implying that \( f(x) \) cont at \( x = x_0 \) (why?).