**Limits and Continuity**

**Week 12 (a)**

**Def** A **metric space** is any non-empty set $X$ endowed with a map $d: X \times X \rightarrow \mathbb{R}_{\geq 0}$, called a **distance**, which has the properties:

(i) $d(x, x) = 0 \ \forall x \in X$

(ii) $d(x, y) = d(y, x) \ \forall x, y \in X$ (symmetry)

(iii) $d(x, z) \leq d(x, y) + d(y, z) \ \forall x, y, z \in X$ (triangle inequality).

A subspace of $X$ is any non-empty set $X' \subset X$ endowed with the distance $d'(x, y)' = d(x, y)$ \ \forall x', y' \in X'.

**Exmpl** (iii) $|x - z| \leq |x - y| + |y - z| \iff |u + v| \leq |u| + |v|$

a) If $\emptyset \neq X \subset \mathbb{C}$, then $d(z, y) = |z - y|$ is a distance on $X$. Further, $X$ is a subspace of $\mathbb{C}$; e.g. $X = \mathbb{R}$.

b) $\mathcal{S}_b(\mathbb{C})$ endowed with $d((x_n)_m, (y_n)_m) = \sup_n |x_n - y_n|$.

If $X \subset \mathbb{C}$, then $\mathcal{S}_b(X)$ is a subspace.

c) $\text{Maps}_b(T) = \{ f: T \rightarrow \mathbb{C} | f \text{ bounded} \}$ endowed with

$$d(f, g) = \sup_{t \in T} |f(t) - g(t)|.$$
Def. Let $X$ endowed with $d$ be a metric space.

1) For $x \in X$, $\varepsilon > 0$ define $B_\varepsilon(x) = \{x' \in X | d(x,x') < \varepsilon\}$ the open ball of center $x$ and radius $\varepsilon$.

2) Similarly, $\overline{B}_\varepsilon(x) = \{x' \in X | d(x,x') \leq \varepsilon\}$ is the closed ball of center $x$ and radius $\varepsilon$.

The canonical topology of a metric space

Def. Let $X$ endowed with $d$ be a metric space.

1) A subset $U \subseteq X$ is called open, if $\forall x \in U$ there is $\varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq U$.

2) A subset $A \subseteq X$ is called closed, if its complement $\overline{X}(A)$ is open.

Prop. Let $X$ endowed with $d$ be a metric space. TFAE:

1) The open balls are open, the closed balls are closed sets.

2) The set of open sets $T_X \subseteq \mathcal{P}(X)$ satisfy:

(i) $\emptyset, X \in T_X$

(ii) $T_X$ is closed w.r.t. arbitrary unions.

(iii) $T_X$ is closed w.r.t. finite intersections

Proof. ex (Hint: word-by-word as in the case $X = (\mathbb{R}, \mathbb{R})$)
Convergence of sequences in metric spaces

Def. Let \( X, d \) be a metric space, \((x_n)_n \in \mathcal{S}(X)\).

1) \((x_n)_n\) is called Cauchy, if \( \forall \varepsilon > 0 \) one has \(d(x_n, x_m) < \varepsilon \) for \( m, n \gg 0 \).

2) \((x_n)_n\) is convergent to \( x \), i.e. \( x_n \to x \), if \( \forall \varepsilon > 0 \) one has \( d(x_n, x) < \varepsilon \) for \( n \gg 0 \).

3) \((x_n)_n\) is called bounded, if \( \exists x \in X, \varepsilon > 0 \) s.t. \( x_n \in B_{\varepsilon}(x) \forall n \).

Prop. The following hold: \( \mathcal{G}_c(X) \subset \mathcal{C}_c(X) \subset \mathcal{L}_c(X) \).

proof. ex. (Same as in \( Y(\mathbb{R}) \), etc....)

Thm. (Heine-Borel for metric spaces)

let \( X, d \) be a metric space, \( \emptyset \not= Y \subset X \).

(i) \( Y \) has the finite covering property.
(ii) \( Y \) has the finite intersection property.
(iii) Every sequence \( (x_n)_n \in \mathcal{S}(Y) \) has a convergent subsequence \( (x_{n_k})_k \), \( x_{n_k} \to x \in Y \).

Terminology. \( Y \subset X \) satisfying (i)-(iii) is called compact.

proof. ex. (Same as in the case \( X = \mathbb{R}, \mathbb{C} \))
Limits / Continuity of Functions

Let $X, Y$ be metric spaces (each w.r.t. its distance) 
$f : X \to Y$ a map, $x_0 \in X$.

1) We say that $\lim_{x \to x_0} f(x) = y$ read: limit of $f(x)$ as $x$ tends to $x_0$ exists and equals $y$
if $\forall \varepsilon > 0 \exists \delta > 0$ s.t. $d(x, x_0) < \delta \Rightarrow d(f(x), y) < \varepsilon$.

2) We say that $f$ is cont at $x_0 \in X$, if 
$\lim_{x \to x_0} f(x) = f(x_0)$.

3) We say that $f$ is continuous on $X_0 \subseteq X$, if $f$ is cont at all $x_0 \in X_0$.

Example a) $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$ is cont on $\mathbb{R}$.

Proof: for $x_0 \in \mathbb{R}, \varepsilon > 0$ need $\eta > 0$ s.t. $|x - x_0| < \eta \Rightarrow |x^2 - x_0^2| < \varepsilon$

Explanation: $|x^2 - x_0^2| = |x - x_0||x + x_0| = |x - x_0||2x_0 + (x - x_0)|$

$\leq |x - x_0|(2|x_0| + |x - x_0|) ; \eta' = 1$, i.e. $|x - x_0| < 1$.

For $|x - x_0| < 1$, get $|x^2 - x_0^2| < |x - x_0|(2|x_0| + 1)$ thus choose $\eta > 0$
s.t. $\eta \cdot M < \varepsilon$, i.e. $\eta < \varepsilon / M$.

b) $f(x) = \begin{cases} -1 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$ \quad $\lim_{x \to 0} f(x)$ does not exist. (why?)
Continuity of maps (cont) / Week 12(b)

Prop (limits) In the above notation one has:
1) let \( f : X \to Y, \ x_0 \in X \) given. TFAE:
(i) \( \lim_{{x \to x_0}} f(x) = y_0 \)
(ii) \( \forall y \in Y, \exists \ U_y \in \mathcal{U}_{x_0} \text{ s.t. } f(U) \subseteq V \)
(iii) For all \( x_n \to x_0 \) in \( X \) one has \( f(x_n) \to y_0 \) in \( Y \).

2) let \( Z \) be a metric space, and \( g : Y \to Z \).
\[ \text{If } \lim_{{x \to x_0}} f(x) = y_0, \text{ and } \lim_{{y \to y_0}} g(y) = z_0 \text{ exist,} \]
\[ \text{then } \lim_{{x \to x_0}} g(f(x)) = \lim_{{y \to y_0}} g(y) = z_0. \]

Proof To 1): \( (i) \iff (ii) \) ex (Use the def of \( U \in \mathcal{U}_{x_0} \).

\( (i) \Rightarrow (ii) \) Given: \( \lim_{{x \to x_0}} f(x) = y_0 \). To prove: If \( x_n \to x_0 \Rightarrow f(x_n) \to y_0 \)

Now: \( f(x_n) \to y_0 \iff \forall \varepsilon > 0 \exists N \text{ s.t. } n > N \Rightarrow d(f(x_n), y_0) < \varepsilon. \)
\( \iff \text{for } \varepsilon > 0 \exists \delta > 0 \text{ s.t. } d(x, x_0) < \delta \Rightarrow d(f(x), y_0) < \varepsilon \)
\( \iff \text{for } x_n \to x_0, \exists N_{\delta} \text{ s.t. } n > N_{\delta} \Rightarrow d(x_n, x) < \delta \)
\( \iff \text{for } n > N_{\delta}, \text{ one has } d(f(x_n), y) < \varepsilon \) (Why?)
(iii) ⇒ (i): Given: \( \forall \varepsilon > 0, \exists \delta > 0 \) s.t. \( d(x_n, x_0) < \delta \Rightarrow d(f(x_n), y_0) < \varepsilon \)

To prove: \( \forall \varepsilon > 0, \exists \delta > 0 \) s.t. \( d(x_n, x_0) < \delta \Rightarrow d(f(x_n), y_0) < \varepsilon \)

By contrad: suppose \( \forall \varepsilon > 0, \exists x_n \in X \) s.t.

\[ d(x_n, x_0) < \delta \land d(f(x_n), y_0) \geq \varepsilon \]

Choose \( \delta_n := \frac{1}{n} \). Then: \( \exists x_n \in X \) s.t.

\[ d(x_n, x_0) < \frac{1}{n} \land d(f(x_n), y_0) \geq \varepsilon \] \((*)\)

OTOH: \( d(x_n, x_0) < \frac{1}{n} \Rightarrow x_n \to x_0 \) \((w+?y?)\).

Hence \( f(x_n) \to y_0 \), i.e. \( \forall \varepsilon' > 0, \exists N' \) s.t.

\[ n > N' \Rightarrow d(f(x_n), y_0) < \varepsilon' \]

Choosing \( \varepsilon' > \varepsilon \), have: \( d(f(x_n), y_0) < 3 > \varepsilon \)

To 2): Use 1): let \( x_n \to x_0 \). Then \( y_n := f(x_n) \to y_0 \) \((w+?y?)\) and \( g(y_n) \to z \) \((w+y?)\).

Thus \( g(f(x_n)) = g(y_n) \to z \), etc.

**Terminology:** Condition (iii) is called sequential existence of limits.

**Corollary** let \( f: X \to Y, g: Y \to Z \) be given.

1) \( f \) is cont at \( x_0 \in X \) iff \( f(x_n) \to f(x_0) \) \( \forall x_n \to x_0 \)

2) \( f \) cont at \( x_0 \) and \( g \) cont at \( y_0 = f(x_0) \) implies \( g \of f \) cont at \( x_0 \).
Theorem (Cont.)

Let \( f: X \to Y \), \( g: Y \to Z \) be maps of metric spaces \( X, Y, Z \). One has:

1) The following are equivalent:
   \( (i) \) \( f \) is continuous on \( X \), i.e. at all \( x_0 \in X \).
   \( (ii) \) \( \forall V \subset Y \) open, its preimage \( U := f^{-1}(V) \) is open in \( X \).
   \( (iii) \) \( \forall B \subset Y \) closed, its preimage \( A := f^{-1}(B) \)
      is closed in \( X \).
   \( (iv) \) For every convergent seq. \( x_n \to x \) in \( X \),
      its image \( \{ f(x_n) \}_n \) converges to \( y = f(x) \) in \( Y \).

2) If \( f: X \to Y \) and \( g: Y \to Z \) are continuous,
   the composition \( g \circ f: X \to Z \) is cont.

Proof

To 1): \( (i) \iff (iv) \) follows from the Prop (limits).

\( (i) \implies (ii) \): For \( x \in X, y = f(x) \in V \), by Prop (limits) one has:

Now: \( y \in V \), \( V \) open \( \implies V \in \tau_Y \). Hence \( \exists U_y \in \tau_Y \)

s.t. \( f(U_y) \subset V \). But then \( U_y \subset f^{-1}(V) \) (why?)

Hence \( f^{-1}(V) \) is neigh. for all \( x \in f^{-1}(V) \).
Conclude: \( \tilde{f}^V(V) \) is open in \( X \) (why?)

\((ii) \Rightarrow (i)\) \textcolor{blue}{\text{ex (Apply Prop(Limits))}}

\((ii) \iff (iii)\). \( B \subseteq Y \) closed iff \( V := C_Y(B) \) open.

OTTOH: \( \tilde{f}^V(C_Y(B)) = C_X \tilde{f}(B) \) (why?)

Thus \( \tilde{f}^V(V) \) open iff \( \tilde{f}(B) \) closed (why?)

To 2): \textcolor{blue}{\text{ex (apply Prop(Limits))}}

\textbf{Thm (Cont & Connectedness)}

Let \( f : X \to Y \) be a contin. map of metric spaces.

Then the image \( f(X_0) \) of any connected subset \( X_0 \subseteq X \) is connected.

In particular, the following hold:

1) If \( Y \subset \mathbb{R} \), then \( f(X_0) \subset \mathbb{R} \) is an interval.

2) **The Intermediate Value Thm**

If \( Y \subset \mathbb{R} \), and \( y_1 \leq y_2 \) in \( f(X_0) \), then \( [y_1, y_2] \subset f(X_0) \)
**proof** We first prove the assertion:

\[ X_0 \subset X \text{ connected } \Rightarrow f(X_0) \subset Y \text{ connected} \]

Let \( V_1, V_2 \subset Y \) open s.t. \( f(X_0) \subset V_1 \cup V_2, V_1 \cap V_2 = \emptyset \). Then \( U_i := \overline{f(V_i)} \subset X \) are open and satisfy:

\[
U_1 \cup U_2 = \overline{f(V_1)} \cup \overline{f(V_2)}, \quad U_1 \cap U_2 = \emptyset \quad (\text{why?})
\]

\[ = \overline{f(V_1 \cup V_2)} \supset \overline{f(X_0)} \quad (\text{why?}) \]

Then \( U_1, U_2 \subset X \) are open (why? Thm (Cont)) AND \( X_0 \subset U_1 \cup U_2, U_1 \cap U_2 = \emptyset \) (why?)

OTOH, \( X_0 \) connected \( \Rightarrow X_0 \subset U_1 \) or \( X_0 \subset U_2 \).

Therefore one has:

Either \( f(X_0) \subset f(U_1) \subset V_1 \), or \( f(X_0) \subset f(U_2) \subset V_2 \).

Hence conclude that \( f(X_0) \) is connected.

To 1), 2): Use that \( f(X_0) \subset \mathbb{R} \) is connected, and that the conn. subsets of \( \mathbb{R} \) are precisely the intervals.

**Corollary** \( f : \mathbb{R} \to \mathbb{R}, f_n(x) = x^n \) satisfies:

1) \( f_n([0, \infty)) = [0, \infty) \) for \( n \) even. 2) \( f_n(\mathbb{R}) = \mathbb{R} \) if \( n \) odd.

Then \( f_n \) strictly increasing, bijective, with inverse \( \sqrt[n]{\cdot} \).

**proof. ex.**