Due: We, Dec 2, 2020

Math 202 / Problem Set 10 (two pages)

Convergence in \( \mathbb{R}, \mathbb{C} \)

1) Let \((a_n) \in \mathcal{S}(\mathbb{C})\) and \(r \in \mathbb{R}_{\geq 0}\) be given such that \((a_nr^n)\) is bounded in \(\mathbb{C}\). Check all the details of the proof that \(\sum_n a_nr^n\) is absolutely convergent for all \(z \in \mathbb{C}\) with \(|z| < r\).

2) Recall the formal power series \(\exp(t), \sin(t), \cos(t) \in \mathbb{Q}[[t]]\), i.e., Problems 5), 6) from HW #8. Show that \(\exp(z), \sin(z), \cos(z)\) are absolutely convergent for \(t = z, z \in \mathbb{C}\), and satisfy:
   - a) \(\exp(z_1+z_2) = \exp(z_1) \exp(z_2), \exp(zi) = \cos(z) + \sin(z)i,\) and the addition formulas:
     \(\cos(z_1+z_2) = \cos(z_1) \cos(z_2) - \sin(z_1) \sin(z_2), \sin(z_1+z_2) = \sin(z_1) \cos(z_2) + \sin(z_2) \cos(z_1).\)
   - b) If \(x \in \mathbb{R}\), then \(\exp(x), \cos(x), \sin(x) \in \mathbb{R}\), and \(\cos^2(x) + \sin^2(x) = 1.\)
   - c) Euler Formula: For \(z = a + bi\) one has: \(\exp(a+bi) = \exp(a)(\cos(b) + \sin(b)i)\), hence:
     \(|\exp(z)| = |\exp(a)| = |\exp(\Re(z))|, \exp(z)/|\exp(z)| = \exp(bi) = \cos(b) + \sin(b)i.\)

3) Which of the following series are convergent, and if so, what is the sign of the represented number:
   - a) The harmonic series: \(\sum_{n>0} \frac{1}{n};\) Leibniz alternating series: \(\sum_{n>0} (-1)^n\frac{1}{n}\)
   - b) \(\zeta(2) = \sum_{n>0} \frac{1}{n^2};\)
   - c) \(\sin(1) = \sum_n (-1)^n\frac{1}{(2n+1)!}; \cos(1) = \sum_n (-1)^n\frac{1}{(2n)!}.\)
   - d) \((\frac{1}{2})^n = \sum_n \frac{a_n}{2^n}.\) More general, \((1+z)^\alpha = \sum_n \frac{\alpha(n)}{n!} z^n\) for \(z \in \mathbb{C}.\)
     [Here, by definitions, \(\binom{\alpha}{0} \overset{\text{def}}{=} 1, \text{and } \binom{\alpha}{n} \overset{\text{def}}{=} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \text{ for } n > 0.\)]

Continuity of \(f_\alpha\) and \(\exp_\alpha, \log_\alpha\)

4) Prove in all detail that the power-\(\alpha\) maps \(f_\alpha : (0, \infty) \rightarrow (0, \infty)\) and the exponential functions \(\exp_\alpha : \mathbb{R} \rightarrow (0, \infty)\) are continuous.

Conclude: \(f_\alpha, \alpha \neq 0, \text{and } \exp_\alpha, a > 0, a \neq 1\) are continuous bijective. Moreover, \(f_\alpha^{-1} = f_{\frac{1}{\alpha}}.\)

[Hint: Use HW #9, Problem 7), etc. ...]

Terminology: The inverse map of \(\exp_\alpha\) is \(\log_\alpha : (0, \infty) \rightarrow \mathbb{R},\) the \textit{logarithm in basis }\(a.\)

5) Prove that \(\log_\alpha : (0, \infty) \rightarrow \mathbb{R}\) is continuous, surjective and satisfies:
   - a) \(\log_\alpha(x_1x_2) = \log_\alpha(x_1) + \log_\alpha(x_2)\) and \(\log_\alpha(x^\alpha) = \alpha \log_\alpha(x) \ \forall \ x, x_1, x_2 \in \mathbb{R}, \alpha \in \mathbb{R}.\)
   - b) \(\log_\alpha\) is strictly monotone. For which bases \(a\) is \(\log_\alpha\) increasing/decreasing?

Continuity of analytic functions: Let \(F = \mathbb{R}, \mathbb{C}\)

Recall that \(f(t) := \sum_n a_n t^n \in \mathbb{F}[[t]]\) having \(D_f := \{z \in F \mid |z| < \rho\} = \{|z| < \rho\}\) as domain of absolute convergence, gives rise to the \textit{analytic function} \(f : D_f \rightarrow F, \ z \mapsto f(z) := \sum_n a_n z^n.\)

6) Check all the details of proofs of the assertions from the class:
   - a) If \(z \in D_f, \) then \(\sum_{n>0} na_n z^{n-1}\) is absolutely convergent.
   - b) \(f : D_f \rightarrow F\) is continuous on \(D_f.\)
   - c) If \(z, z+h \in D_f, \) then \(\lim_{h \rightarrow 0} \frac{f(z+h)-f(z)}{h} := L\) exists, and \(L = \sum_{n>0} na_n z^{n-1}.\)
Recall: $\exp(z) := \sum_n \frac{z^n}{n!}$ is absolutely convergent for $z \in F$.

**Notation:** $e := \sum_n \frac{1}{n!} \in \mathbb{R}$ is Euler’s number. **Note:** $e = 2.71828182845904523536028747135263602874713526624977572$.

7) Prove that $e^x = \exp_e(x) := \sum_n \frac{x^n}{n!} =: \exp(x)$ for $x \in \mathbb{R}$ along the following lines:

a) $\lim_{m_n \to \infty} \left(1 + \frac{z}{m_n}\right)^{m_n} = \lim_{n \to \infty} \left(1 + \frac{z}{n}\right)^n = \sum_n \frac{z^n}{n!}$ for all $z \in \mathbb{C}$. **Hence** $e = \lim_{m_n \to \infty} (1 + \frac{1}{m_n})^{m_n}$.

b) Let $a_n = \frac{l_n}{k_n} \to x$, $x \geq 0$, where $l_n, k_n \in \mathbb{N}_{>0}$. For $m_n := mk_n$ with $m \to \infty$ one has $\exp_e(l_n/k_n) := \lim_{m_n \to \infty} \left(1 + \frac{l_n}{m_n}\right)^{m_n} = \lim_{n \to \infty} \left(1 + \frac{l_n}{m_n}\right)^{m_n} = \sum_m \frac{(l_n/m)^m}{m!} = \exp(l_n/k_n)$.

c) If $a_n \to x$, then $\exp_e(a_n) = \exp(a_n)$, and $\exp_e(a_n) \to \exp_e(x)$, $\exp(a_n) \to \exp(x)$ **[WHY]**. **Conclude** that $e^x = \exp_e(x) = \exp(x)$ **[WHY]**.

**Topology, limits, continuity**

Recall: For a metric space $X, d$ we denote by $B_r(x) := \{x' \in X \mid d(x, x') < r, \}$, respectively $\overline{B}_r(x) := \{x' \in X \mid d(x, x') \leq r \}$ the open/closed, balls of center $x \in X$ and radius $r \geq 0$.

8) Prove in all detail the assertions from the class:

a) $B_r(x)$ is an open set, and $\overline{B}_r(x)$ is a closed set in $X$.

b) The set of all the closed subsets of a metric space $X$ satisfy:

(i) $\emptyset, X$ are closed subsets of $X$

(ii) If $(A_i)_{i \in I}$, $I$ arbitrary, are closed subsets, then $\cap_i A_i$ is a closed subset.

(iii) If $(A_i)_{i \in I}$, $I$ finite, are closed subsets, then $\cup_i B_i$ is a closed subset.

c) A subset $A \subset X$ is closed iff for all $(x_n)_n \in S(A)$ one has: If $x_n \to x \in X$ then $x \in A$.

9) Prove in all detail the **Heine-Borel Thm** for metric spaces, asserting:

A subspace $O \neq Y \subset X$ is compact iff every sequence $(x_n)_n \in S(Y)$ has a convergent subsequence $(x_{n_k})_k$ with $x_{n_k} \to x \in Y$. In particular, $Y \subset X$ is bounded and complete, i.e., $\exists y \in Y, \epsilon > 0$ s.t. $Y \subset B_r(y)$, and every $(x_n)_n \in S_C(Y)$ is convergent in $Y$.

10) Let $X, Y$ be metric spaces, $I \subset \mathbb{R}$ be an interval. Check all the details of the proofs of:

a) Let $f : I \to \mathbb{R}$ be continuous. Then $f$ is injective iff $f$ is strictly monotone.

b) Let $X$ be compact, $f : X \to Y$ be continuous bijective. Then $f^{-1} : Y \to X$ is continuous.

**Optional**

I) Recall that, see HW # 9, Problem 5), the power functions $f_\alpha$ satisfy:

(i) $f_\alpha \cdot f_\beta = f_{\alpha + \beta}$ and $f_\alpha \circ f_\beta = f_{\alpha \beta} = f_\beta \circ f_\alpha$ for all $\alpha, \beta \in \mathbb{R}$.

(ii) $f_\gamma \circ (f_\alpha \cdot f_\beta) = (f_\gamma \circ f_\alpha) \cdot (f_\gamma \circ f_\beta)$ for all $\alpha, \beta, \gamma \in \mathbb{R}$.

What algebraic structure is $\mathcal{F} := \{f_\alpha \mid \alpha \in \mathbb{R}\}$ endowed with the composition laws $\cdot, \circ$?

II) Recall, see HW # 9, Problem 6), that the exponential functions $\exp_\alpha$ satisfy:

(i) $\exp_a \cdot \exp_b = \exp_{ab}$ for all $a, b > 0$.

What algebraic structure is $\mathcal{F} := \{\exp_a \mid a > 0, a \neq 1\}$ endowed with the composition law $\cdot$?