PRO-ℓ GALOIS THEORY OF ZARISKI PRIME DIVISORS

by

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Abstract. — In this paper we show how to recover a special class of valuations (which generalize in a natural way the Zariski prime divisors) of function fields from the Galois theory of the functions fields in discussion. These valuations play a central role in the birational anabelian geometry and related questions.

***Résumé* (Théorie de Galois pro-ℓ des diviseurs premiers de Zariski)**
Dans cet article nous montrons comment retrouver une classe spéciale de valuations de corps de fonctions (qui généralisent naturellement les diviseurs premiers de Zariski) à partir de la théorie de Galois des corps de fonctions en question. Ces valuations jouent un rôle central en géométrie anabélienne birationnelle et pour d'autres questions connexes.

1. Introduction

The aim of this paper is to give a first insight into the way the pro-ℓ Galois theory of function fields over algebraically closed base fields of characteristic $\neq \ell$ encodes the Zariski prime divisors of the function fields in discussion. We consider the following context:

• ℓ is a fixed rational prime number.

• $K|k$ are unction fields with k algebraically closed of characteristic $\neq \ell$.

• $K(\ell)|K$ is the maximal pro-ℓ Galois extension of K in some separable closure of K, and $G_K(\ell)$ denotes its Galois group.

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It is a Program initiated by Bogomolov [B] at the beginning of the 1990's which has as ultimate goal to recover (the isomorphy type of) the field K from the Galois group $G_K(\ell)$. Actually, Bogomolov expects to recover the field K even from the Galois information encoded in PGal^c_K, which is the quotient of $G_K(\ell)$ by the second factor in its central series. Unfortunately, at the moment we have only a rough idea (maybe a hope) about how to recover the field K from $G_K(\ell)$, and not a definitive answer to the problem. Nevertheless, this program is settled and has a *positive answer,* in the case k is *an algebraic closure of a finite field,* Pop [P4]; see also Bogomolov–Tschinkel [B–T2] for the case of function fields of smooth surfaces with trivial fundamental group and $\ell, \mathrm{char}(k) \neq 2$.

It is important to remark that ideas of this type were first initiated by Neukirch, who asked whether the isomorphism type of a number field F is encoded in its absolute Galois group. The final result in this direction is the celebrated result by Neukirch, Iwasawa, Uchida (with previous partial results by Ikeda, Komatsu, etc.) which roughly speaking asserts that the *isomorphy types of global fields are functorially encoded in their absolute Galois groups.* Nevertheless, it turns out that the result above concerning global fields is just a first piece in a very broad picture, namely that of *Grothendieck's anabelian geometry,* see Grothendieck [G1], [G2]. Grothendieck predicts in particular, that the finitely generated infinite fields are functorially encoded in their absolute Galois groups. This was finally proved by the author Pop [P2], [P3]; see also Spiess [Sp].

The strategy to prove the above fact is to first develop a "Local theory", which amounts of recovering local type information about a finitely generated field from its absolute Galois group. And then "globalize" the local information in order to finally get the field structure. The local type information consists of recovering the Zariski prime divisors of the finitely generated field. These are the discrete valuations which are defined by the Weil prime divisors of the several normal models of the finitely generated field in discussion.

In this manuscript, we will mimic the Local theory from the case of finitely generated infinite fields, and will develop a geometric pro-ℓ Local theory, whose final aim is to recover the so called quasi-divisorial valuations of a function field $K|k$ form $G_K(\ell)$ —notations as at the beginning of the Introduction. We remark that this kind of results played a key role in Pop [P4], where only the case $k = \overline{\mathbb{F}}_p$ was considered.

We mention here briefly the notions introduced later and the main results proved later in the paper —notations as above.

Let v be some valuation of $K(\ell)$, and for subfields Λ of $K(\ell)$ denote by $v\Lambda$ and Λv the value group, respectively residue field, of the restriction of v to Λ. And let $T_v \subseteq Z_v$ be the inertia, respectively decomposition, group of v in $G_K(\ell) = \mathrm{Gal}\big(K(\ell)|K\big)$.

First recall, see Section 3, A), that a Zariski prime divisor v of $K(\ell)$ is any valuation of $K(\ell)$ whose restriction $v|_K$ to K "comes from geometry", i.e., the valuation ring of $v|_K$ equals the local ring $\mathcal{O}_{X,x_v}$ of the generic point x_v of some Weil prime divisor of some normal model $X \to k$ of $K|k$. Thus $vK \cong \mathbb{Z}$ and $Kv|k$ is a function field satisfying $\mathrm{td}(Kv\,|\,k) = \mathrm{td}(K|k) - 1$. Now it turns out that Z_v has a "nice" structure as follows:

$$T_v \cong \mathbb{Z}_\ell \quad \text{and} \quad Z_v \cong T_v \times G_{Kv}(\ell) \cong \mathbb{Z}_\ell \times G_{Kv}(\ell).$$

We will call the decomposition groups Z_v of Zariski prime divisors v of $K(\ell)|k$ divisorial subgroups of $G_K(\ell)$ or of K.

Now in the case k is an algebraic closure of a finite field, it turns out that a *maximal subgroup of $G_K(\ell)$ which is isomorphic to a divisorial subgroup is actually indeed a divisorial subgroup of $G_K(\ell)$,* see [P4]; this follows nevertheless from Proposition 4.1 of this manuscript, as k has no no-trivial valuations in this case.

On the other hand, if k has positive Kronecker dimension (i.e., it is not algebraic over a finite field), then the situation becomes more intricate, as the non-trivial valuations of k play into the game. Let us say that a valuation v of $K(\ell)$ is a quasi-divisorial valuation, if it is minimal among the valuations of $K(\ell)$ having the properties: $\mathrm{td}(Kv\,|\,kv) = \mathrm{td}(K|k) - 1$ and $vK/vk \cong \mathbb{Z}$, see Definition 3.4, and Fact 5.5, 3). Note that the Zariski prime divisors of $K(\ell)$ are quasi-divisorial valuations of $K(\ell)$.

On the Galois theoretic side we make definitions as follows: Let Z be a closed subgroup of $G_K(\ell)$.

i) We say that Z a divisorial like subgroup of $G_K(\ell)$ or of K, if Z is isomorphic to a divisorial subgroup of some function field $L|l$ such that $\mathrm{td}(L|l) = \mathrm{td}(K|k)$, and l algebraically closed of characteristic $\neq \ell$.

ii) We will say that Z is quasi-divisorial, if Z is divisorial like and maximal among the divisorial like subgroups of $G_K(\ell)$.

Finally, for $t \in K$ a non-constant function, let K_t be the relative algebraic closure of $k(t)$ in K. Thus $K_t|k$ is a function field in one variable, and one has a canonical projection $p_t : G_K(\ell) \to G_{K_t}(\ell)$.

In these notations, the main results of the present manuscript can be summarized as follows, see Proposition 4.1, Key Lemma 4.2, and Proposition 4.6.

Theorem 1.1. — *Let $K|k$ be a function field with $\mathrm{td}(K|k) > 1$, where k is algebraically closed of characteristic $\neq \ell$. Then one has:*

(1) *A closed subgroup $Z \subset G_K(\ell)$ is quasi-divisorial $\iff$ Z is maximal among the subgroups Z' of $G_K(\ell)$ which have the properties:*

i) *Z' contains closed subgroups isomorphic to $\mathbb{Z}_\ell^d$, where $d = \mathrm{td}(K|k)$.*

ii) *Z' has a non-trivial pro-cyclic normal subgroup T' such that Z'/T' has no non-trivial Abelian normal subgroups.*

(2) *The quasi-divisorial subgroups of* $G_K(\ell)$ *are exactly the decomposition groups of the quasi-divisorial valuations of* $K(\ell)$.

(3) *A quasi-divisorial subgroup* Z *of* $G_K(\ell)$ *is a divisorial subgroup of* $G_K(\ell)$ $\iff$ $p_t(Z)$ *is open in* $G_{K_t}(\ell)$ *for some non-constant* $t \in K$.

Among other things, one uses in the proof some ideas by Ware and Arason–Elman–Jacob, see e.g. Engler–Nogueira [E–N] for $\ell = 2$, Engler–Koenigsmann [E–K] in the case $\ell \neq 2$, and/or Efrat [Ef] in general. And naturally, one could use here Bogomolov [B], Bogomolov–Tschinkel [B–T1]. We would also like to remark that this kind of assertions —and even stronger but more technical ones— might be obtained by employing the local theory developed by Bogomolov [B], and Bogomolov–Tschinkel [B–T1].

Concerning applications: Proposition 4.1 plays an essential role in tackling Bogomolov's Program in the case the base field k is an algebraic closure of a global field (and hopefully, in general); and Proposition 4.6 is used in a proof of the so called Ihara/Oda–Matsumoto Conjecture. (These facts will be published soon).

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2. Basic facts from valuation theory

A) *On the decomposition group* (See e.g. [En], [BOU], [Z–S].)

Consider the following context: $\tilde{K}|K$ is some Galois field extension, and v is a valuation on $\tilde{K}$. For every subfield Λ of $\tilde{K}$ denote by $v\Lambda$ and Λv the valued group, respectively the residue field of Λ with respect to (the restriction of) v on Λ. We denote by $p = \mathrm{char}(\tilde{K}v)$ the residue characteristic. Further let Z_v, T_v, and V_v be respectively the decomposition group, the inertia group, and the ramification group of v in $\mathrm{Gal}(\tilde{K}|K)$, and K^V, K^T, and K^Z the corresponding fixed fields in $\tilde{K}$.

Fact 2.1. — The following are well known facts from Hilbert decomposition, and/or ramification theory for general valuations:

1) $\tilde{K}v\,|\,Kv$ is a normal field extension. We set $G_v := \mathrm{Aut}(\tilde{K}v\,|\,Kv)$. Further, $V_v \subset T_v$ are normal subgroups of Z_v, and one has a canonical exact sequence

$$1 \to T_v \to Z_v \to G_v \to 1\,.$$

One has $v(K^T) = v(K^Z) = vK$, and $Kv = K^Z v$. Further, $K^T v\,|\,Kv$ is the separable part of the normal extension $\tilde{K}v\,|\,Kv$, thus it is the maximal Galois sub-extension of $\tilde{K}v\,|\,Kv$. Further, $K^V\,|\,K^T$ is totally tamely ramified.

2) Let $\mu_{\tilde{K}v}$ denote the group of roots of unity in $\tilde{K}v$. There exists a canonical pairing as follows: $\Psi_{\tilde{K}} : T_v \times v\tilde{K}/vK \to \mu_{\tilde{K}v}$, $(g,\, vx) \mapsto (gx/x)v$, and the following hold: The left kernel of $\Psi_{\tilde{K}}$ is exactly V_v. The right kernel of $\Psi_{\tilde{K}}$ is trivial if $p = 0$, respectively equals the Sylow p-group of $v\tilde{K}/vK$ if $p > 0$. In particular, T_v/V_v is Abelian, V_v is trivial if $\mathrm{char}(K) = 0$, respectively equals the unique Sylow p-group of T_v if $\mathrm{char}(K) = p > 0$. Further, $\Psi_{\tilde{K}}$ is compatible with the action of G_v.

3) Suppose that $v' \leq v$ is a coarsening of v, i.e., $\mathcal{O}_v \subseteq \mathcal{O}_{v'}$. Then denoting $v_0 = v/v'$ the valuation induced by v on $\tilde{K}v'$, and by Z_{v_0} its decomposition group in $G_{v'} = \mathrm{Aut}(\tilde{K}v'|Kv)$, one has: $T_v \subseteq Z_v$ are the preimages of $T_{v_0} \subseteq Z_{v_0}$ in $Z_{v'}$ via the canonical projection $Z_{v'} \to G_{v'}$. In particular, $T_{v'} \subseteq T_v$ and $Z_v \subseteq Z_{v'}$.

Fact 2.2. — Let ℓ be a rational prime number. In the notations and the context from Fact 2.1 above, suppose that K contains the ℓ^∞ roots of unity, and fix once for all an identification of the Tate ℓ-module of $\mathbb{G}_{m,K}$ with $\mathbb{Z}_\ell(1)$, say

$$\imath : \mathbb{T}_\ell \to \mathbb{Z}_\ell(1).$$

And let the Galois extensions $\tilde{K}|K$ considered at Fact 2.1 satisfy $K^{\ell,\mathrm{ab}} \subseteq \tilde{K} \subseteq K(\ell)$, where $K^{\ell,\mathrm{ab}}$ is the maximal Abelian extension of K inside $K(\ell)$. Finally, we consider valuations v on $\tilde{K}$ such that Kv has characteristic $\neq \ell$. Then by the discussion above we have: $V_v = \{1\}$, and further: $v\tilde{K}$ is the ℓ-divisible hull of vK; and the residue field extension $\tilde{K}v\,|\,Kv$ is separable and also satisfies the properties above we asked for $\tilde{K}|K$ to satisfy.

1) For $n = \ell^e$, there exists a unique sub-extension $K_n|K^T$ of $\tilde{K}|K^T$ such that $K_n|K^Z$ is a Galois sub-extension of $\tilde{K}|K^Z$, and $vK_n = \frac{1}{n}\,vK^T = \frac{1}{n}\,vK$. On the other hand, the multiplication by n induces a canonical isomorphism $\frac{1}{n}\,vK \,/\, vK \cong vK/n$. Therefore, the pairing $\Psi_{\tilde{K}}$ gives rise to a non-degenerate pairing

$$\Psi_n : T_v/n \times vK/n \to \mu_n \xrightarrow{\imath} \mathbb{Z}/n(1),$$

hence to isomorphisms $\theta^{v,n} : vK/n \to \mathrm{Hom}(T_v, \mu_n)$, $\theta_{v,n} : T_v/n \to \mathrm{Hom}(vK, \mu_n)$. In particular, taking limits over all $n = \ell^e$, one obtains a canonical isomorphism of G_v-modules

$$\theta_v : T_v \to \mathrm{Hom}\big(vK, \mathbb{Z}_\ell(1)\big).$$

2) Next let $\mathcal{B} = (vx_i)_i$ be an $\mathbb{F}_\ell$-basis of the vector space $vK\,/\,\ell$. For every x_i, choose a system of roots $(\alpha_{i,n})_n$ in $\tilde{K}$ such that $\alpha_{i,n}^\ell = \alpha_{i,n-1}$ (all $n > 0$), where $\alpha_{i,0} = x_i$. Then setting $K^0 = K[(\alpha_{i,n})_{i,n}] \subset \tilde{K}$, it follows that v is totally ramified in $K^0|K$, and vK^0 is ℓ-divisible. Therefore, $K^0v = Kv$, and the inertia group of v in $\tilde{K}|K^0$ is trivial. In particular, T_v has complements in Z_v, and $T_v \cong \mathbb{Z}_\ell^{\mathcal{B}}(1)$ as G_v-modules.

3) Since by hypothesis $\mu_{\ell^\infty} \subseteq K$, the action of G_v on $\mathbb{Z}_\ell^{\mathcal{B}}(1) \cong T_v$ is trivial. In particular, setting $\delta_v := |\mathcal{B}| = \dim_{\mathbb{F}_\ell}(vK/\ell)$ we finally have:

$$Z_v \cong T_v \times G_v \;\cong\; \mathbb{Z}_\ell^{\delta_v} \times G_v.$$

B) *Two results of F. K. Schmidt*

In this subsection we will recall the pro-ℓ form of two important results of F. K. Schmidt and generalizations of these like the ones in Pop [P1], The local theory, A. See also Endler–Engler [E–E].

Let ℓ be a fixed rational prime number. We consider fields K of characteristic $\neq \ell$ containing the ℓ^∞ roots of unity. For such a field K we denote by $K(\ell)$ a maximal pro-ℓ Galois extension of K. Thus the Galois group of $K(\ell)|K$ is the maximal pro-ℓ quotient of the absolute Galois group G_K of K. We will say that K is ℓ-closed, if $K(\ell) = K$, or equivalently, if every element $x \in K$ is an ℓ^{th} power in K.

We next define the pro-ℓ core of a valuation on $K(\ell)$. This is the pro-ℓ correspondent of the core of a valuation on the separable closure of K, as defined in [P1], The local theory, A. The construction is as follows: In the above context, let v be a valuation of $K(\ell)$ which is pro-ℓ Henselian on some subfield Λ of $K(\ell)$. Consider the set $\mathcal{V}'$ of all the coarsenings v' of v such that $\Lambda v'$ is pro-ℓ closed, and set $\mathcal{V} = \mathcal{V}' \cup \{v\}$. Note that $\mathcal{V}'$ might be empty. By general valuation theory, $\mathcal{V}$ has an infimum whose valuation ring is the union of all the valuation rings $\mathcal{O}_{v'}$ $(v' \in \mathcal{V})$. We denote this valuation by

$$v_{\text{pro-}\ell,\Lambda} = \inf \mathcal{V}$$

and call it the pro-ℓ Λ-core of v. Finally, for a given valuation v on $K(\ell)$ we denote by $v_{\text{pro-}\ell,K}$ the pro-ℓ K^Z-core of v, where K^Z is the decomposition field of v in $K(\ell)$. With this definition of a "core", the Proposition 1.2 and Proposition 1.3 from [P1] remain true in the following form:

Proposition 2.3. — *Let v be a non-trivial valuation of $K(\ell)$, and suppose that K^Z is not pro-ℓ closed. Then the pro-ℓ K^Z-core $v_{\text{pro-}\ell,K}$ of v is non-trivial and lies in $\mathcal{V}$. Consequently:*

(1) *Kv is pro-ℓ closed $\iff$ $K^Z v_{\text{pro-}\ell,K}$ is pro-ℓ closed $\iff$ $Z_v = T_v$.*

(2) *If v has rank one or Kv is not pro-ℓ closed, then v equals its pro-ℓ K^Z-core.*

Proof. — The proof is word by word identical with the one from loc.cit. □

The following result is the announced pro-ℓ form of the results of F. K. Schmidt.

Proposition 2.4. — *Suppose that K is not pro-ℓ closed. Let w^1, w^2 be two valuations on $K(\ell)$ such that they are pro-ℓ Henselian on some sub-extension $\Lambda|K$ of $K(\ell)|K$. Then their pro-ℓ Λ-cores are comparable.*

Consequently, let $K \subseteq L \subseteq \Lambda \subseteq K(\ell)$ be sub-extensions of $K(\ell)|K$, and let v be a valuation on $K(\ell)$ that is pro-ℓ Henselian on Λ and equals its pro-ℓ Λ-core. Then the following hold:

(1) *If $\Lambda|L$ is normal, then v is pro-ℓ Henselian on L and equals its pro-ℓ L-core. In particular, Z_v is self-normalizing in $G_K(\ell)$.*

(2) *If* $\Lambda|L$ *is finite, then* v *is pro-*ℓ *Henselian on* L *and* $v = v_{pro\text{-}\ell,L}$. *In particular,* Z_v *is not a proper open subgroup of a closed subgroup of* $G_K(\ell)$.

Proof. — The proofs of these assertions are identical to the ones in [P1], loc.cit., thus we will omit them here. For the fact that Z_v is self-normalizing, let N be its normalizer in $G_K(\ell)$, and set $\Lambda = K^Z$ and $L = K(\ell)^N$. Then $\Lambda|L$ is normal, so v is pro-ℓ Henselian on L by the first part of (1) above. Thus finally $\Lambda = K^Z = L$, i.e., $N = Z_v$. □

3. Zariski prime divisors and quasi-divisorial valuations

In the sequel we consider function fields $K|k$ over algebraically closed base fields k of characteristic $\neq \ell$.

A) *Zariski prime divisors*

Remark/Definition 3.1. — Recall that a valuation v of K is called a k-valuation, if v is trivial on k. For a k-valuation v of K the following conditions are equivalent:

i) v is discrete, and its residue field Kv is a function field in $\mathrm{td}(K|k) - 1$ variables.

ii) $\mathrm{td}(Kv\,|\,k) = \mathrm{td}(K|k) - 1$.

A k-valuation v on K with the above equivalent properties is called a Zariski prime divisor of $K|k$. See Appendix, A), where a geometric description of the Zariski prime divisors of $K|k$ is given.

The aim of this subsection is to give a first insight in the pro-ℓ Galois theory of the Zariski prime divisors of $K|k$. By abuse of language, we say that a k-valuation v of $K(\ell)$ is a Zariski prime divisor, if v is the prolongation of some Zariski prime divisor of $K|k$ to $K(\ell)$. Since $K(\ell)\,|\,K$ is algebraic, hence $K(\ell)v\,|\,Kv$ is algebraic too, it follows that a k-valuation of $K(\ell)$ is a Zariski prime divisor if and only if $\mathrm{td}(K(\ell)v\,|\,k) = \mathrm{td}(K(\ell)\,|\,k) - 1$.

With this convention, for a Zariski prime divisor v of $K(\ell)|k$ we denote the decomposition group of v in $G_K(\ell)$ by Z_v. By general decomposition theory we have: Two Zariski prime divisors of $K(\ell)|k$ have the same restriction to K if and only if they are conjugated under $G_K(\ell)$.

Definition 3.2. — In the notations from above, we will say that Z_v is a divisorial subgroup of $G_K(\ell)$ or of the function field $K|k$ (at the Zariski prime divisor v, if this is relevant for the context).

More generally, a closed subgroup $Z \subset G_K(\ell)$ which is isomorphic to a divisorial subgroup Z_w of a function field $L|l$ with $\mathrm{td}(L|l) = \mathrm{td}(K|k)$ and l algebraically closed of characteristic $\neq \ell$ is called a divisorial like subgroup of $G_K(\ell)$ or of $K|k$.

By the remark above, the divisorial subgroups as well as the divisorial like subgroups of $G_K(\ell)$ form full conjugacy classes of closed subgroups of $G_K(\ell)$. Some of the significant properties of the divisorial subgroups are summarized in the following:

Proposition 3.3. — *Let v be a Zariski prime divisor of $K(\ell)|k$. Let $Z_v \subset G_K(\ell)$ be the divisorial subgroup at v, T_v the inertia group of Z_v, and $G_v = Z_v/T_v$ the Galois group of the corresponding Galois residue field extension $K(\ell)v\,|\,Kv$. Then the following hold:*

(1) $K(\ell)v = (Kv)(\ell)$, hence $G_v = G_{Kv}(\ell)$. Further, $T_v \cong \mathbb{Z}_\ell$ and $Z_v \cong T_v \times G_{Kv}(\ell)$ as pro-finite groups.

In particular, a divisorial like subgroup of $G_K(\ell)$ is any closed subgroup Z which is isomorphic to $\cong \mathbb{Z}_\ell \times G_{L_1}(\ell)$ for some function field $L_1|l$ with l algebraically closed of characteristic $\neq \ell$ and $\mathrm{td}(L_1|l) = \mathrm{td}(K|k) - 1$.

(2) T_v is the unique maximal Abelian normal subgroup of Z_v.

(3) Z_v is self-normalizing in $G_K(\ell)$, and it is maximal among the subgroups of $G_K(\ell)$ which have a non-trivial pro-cyclic normal subgroup.

Proof. — Assertion (1) follows immediately from Fact 2.2, 2), 3).

Concerning (2), let $T \subseteq Z_v$ be an Abelian normal closed subgroup. Then the image $\overline{T}$ of T in $G_{Kv}(\ell)$ is an Abelian normal subgroup of $G_{Kv}(\ell)$. Thus the assertion follows from the following:

Claim. If $L|k$ is a function field, then $G_L(\ell)$ has no non-trivial Abelian normal subgroups.

In order to prove the Claim, recall that if $L \neq k$, then L is a Hilbertian field; see e.g. [F–J], Ch.16 for basic facts concerning Hilbertian fields. Let $L_1|L$ be a proper Galois sub-extension of $L(\ell)|L$. Then by Kummer theory, $L(\ell)|L_1$ is not finite. Choose any proper finite sub-extension $L_2|L_1$ of $L(\ell)|L_1$. Then by Weissauer's Theorem, L_2 is a Hilbertian field. Since every finite split embedding problem with Abelian kernel over L_2 is properly solvable, it follows that L_2 has "many" finite Galois ℓ-extensions which are not Abelian. Therefore, $L(\ell)|L_1$ cannot be an Abelian extension.

To (3): First, since a Zariski prime divisor has rank one, it follows by Proposition 2.3 that it equals its absolute pro-ℓ core. Thus by Proposition 2.4, (1), it follows that its decomposition group is self-normalizing. Concerning the maximality: Let Z' be a subgroup of $G_K(\ell)$ having a non-trivial pro-cyclic normal subgroup T' and satisfying $Z_v \subseteq Z'$. We show that $Z' = Z_v$, and that $T' = T_v$ provided T' is a maximal pro-cyclic normal subgroup.

Indeed, since $Z_v \subseteq Z'$, and T' is normal in Z', we have: T' is normal in $G := T_vT'$; and one has an exact sequence $1 \to T' \to G \to \overline{T}_v \to 1$, where $\overline{T}_v = T_v/(T_v \cap T')$ is a quotient of $T_v \cong \mathbb{Z}_\ell$. Let $\Lambda = K(\ell)^G$ be the fixed field of G, thus $\Lambda(\ell) = K(\ell)$. Since Λ contains the algebraically closed field k, it contains the ℓ^∞ roots of unity. Thus by Kummer Theory, the following two finite ℓ-elementary Abelian groups $G^{\rm ab}/\ell$ and $\Lambda^\times/\ell$ are (Pontrjagin) dual to each other. On the other hand, by the definition of G —see the above exact sequence, we have: Either $G^{\rm ab}/\ell \cong \mathbb{Z}/\ell$, or $G^{\rm ab}/\ell \cong (\mathbb{Z}/\ell)^2$. Applying again Kummer Theory in its pro-ℓ setting, we get:

a) If $G^{\rm ab}/\ell \cong \mathbb{Z}/\ell$, then $\Lambda^\times/\ell$ is cyclic. Thus $\Lambda(\ell)|\Lambda$ is pro-cyclic. And in turn, $G \cong \mathbb{Z}_\ell$.

b) If $G^{\rm ab}/\ell \cong (\mathbb{Z}/\ell)^2$, then $\Lambda^\times/\ell$ is generated by exactly two elements. Thus $G = G_\Lambda(\ell)$ has $\mathbb{Z}_\ell \times \mathbb{Z}_\ell$ as a quotient. Hence taking into account the exact sequence above $1 \to T' \to G \to \overline{T}_v \to 1$, it follows that $G \cong \mathbb{Z}_\ell \times \mathbb{Z}_\ell$.

From this we conclude that G is abelian in both cases a) and b) above; hence T_v is a normal subgroup of $G = T_vT'$. Since v is pro-ℓ Henselian on the fixed field K^T of T_v and equals its pro-ℓ K^T-core, it follows by the above cited result of F. K. Schmidt, Proposition 2.4, (1), that Λ is pro-ℓ Henselian with respect to v. Thus denoting by $K^{T'}$ the fixed field of T' in $K(\ell)$, we have $\Lambda \subseteq K^{T'}$. Hence v is pro-ℓ Henselian on $K^{T'}$. Finally, as T' is a normal subgroup of Z', it follows by loc.cit. again that v is pro-ℓ Henselian on the fixed field $K^{Z'}$ of Z'. Since Z_v is the decomposition group of v in $G_K(\ell)$, it finally follows that $Z' \subseteq Z_v$. Thus $Z' = Z_v$, and $T' = T_v$ provided T' is a maximal pro-cyclic normal subgroup of Z'. □

B) *Quasi-divisorial valuations*

The ultimate goal of the Galois theory of the Zariski prime divisors of $K|k$ is to identify these divisors as corresponding to the conjugacy classes of particular divisorial like subgroups. Nevertheless, the only kind of extra information one might use in such a characterization should be of group theoretic nature, originating in the pro-ℓ Galois theory of function fields. Obviously, the best one can expect is that "morally" the converse of Proposition 3.3 above should also be true; this means that if $Z \subset G_K(\ell)$ is a divisorial like subgroup, then it should originate from a Zariski prime divisor of $K(\ell)|k$, which should be unique. *Unfortunately,* this cannot be true, as indicated below.

- First, every open subgroup $Z' \subseteq Z_v$ of a divisorial subgroup is a divisorial like subgroup. Indeed, such an open subgroup is a divisorial subgroup for some properly chosen finite sub-extension $K'|K$ of $K(\ell)|K$. Thus in general, a divisorial like subgroup is not a divisorial subgroup.

An obvious way to remedy this failure for the converse of Proposition 3.3 is by restricting ourselves to considering *maximal* divisorial like subgroup Z of $G_K(\ell)$ only, and then ask whether every such a maximal subgroup Z is divisorial.

- Unfortunately, there is a more subtle source of divisorial like subgroups of $G_K(\ell)$ coming from so called defectless valuations v on $K(\ell)$ of relative rational rank 1, which generalize in a natural way the Zariski prime divisors of $K|k$.

In order to explain these phenomena, we introduce notations/notions as follows: Let v be an arbitrary valuation on $K(\ell)$. Since k is algebraically closed, $v\,k$ is a totally ordered $\mathbb{Q}$-vector space (which is trivial, if the restriction of v to k is trivial). We will denote by r_v the rational rank of the torsion free group vK/vk, and by abuse of language call it the *rational rank of* v. Next remark that the residue field kv is algebraically closed too, and $Kv\,|kv$ is some field extension (not necessarily a function field!). We will denote $\mathrm{td}_v = \mathrm{td}(Kv\,|\,kv)$ and call it the residual transcendence degree. By general valuation theory, see e.g. [BOU], Ch.6, §10, 3, one has the following:

$$r_v + \mathrm{td}_v \leq \mathrm{td}(K|k).$$

We will say that v has no (transcendence) defect, or that v is defectless, if the above inequality is an equality, i.e., $r_v + \mathrm{td}_v = \mathrm{td}(K|k)$. See Appendix, B), for basic facts concerning defectless valuations, in particular for a "recipe" which produces all the defectless valuations v of $K(\ell)$.

Remark/Definition 3.4. — For a valuation v of $K(\ell)$ the following conditions are equivalent:

i) The valuation v is minimal among the valuations w of $K(\ell)$ satisfying $r_w = 1$ and $\mathrm{td}_w = \mathrm{td}(K|k) - 1$.

ii) v has no relative defect and satisfies: First, $r_v = 1$, and second, $r_{v'} = 0$ for any proper coarsening v' of v.

A valuation of K with the equivalent properties i), ii), above is called quasi-divisorial.

In particular, by Appendix, Fact 5.5, 2), b), it follows that if v is quasi-divisorial, then $Kv|kv$ is a function field with $\mathrm{td}(Kv\,|\,kv) = \mathrm{td}(K|k) - 1$, and second, $vK/vk \cong \mathbb{Z}$.

Further, every Zariski prime divisor of $K|k$ is a quasi-divisorial valuation. And a quasi-divisorial valuation v of $K(\ell)$ is a Zariski prime divisor if and only if v is trivial on k.

The aim of this subsection is to give a first insight in the pro-ℓ Galois theory of the quasi-divisorial valuation of $K|k$.

Proposition 3.5. — *Let v be a valuation on $K(\ell)$ having no relative defect such that $r_v = 1$ and $\mathrm{char}(Kv) \neq \ell$. Let $T_v \subset Z_v$ and $G_v = Z_v/T_v$ be defined as usual. Then the following hold:*

(1) *$K(\ell)v = (Kv)(\ell)$, hence $G_v = G_{Kv}(\ell)$. Further, $T_v \cong \mathbb{Z}_\ell$ and $Z_v \cong T_v \times G_{Kv}(\ell)$ as pro-finite groups. In particular, Z_v is a divisorial like subgroup of $G_K(\ell)$.*

(2) *T_v is the unique maximal Abelian normal subgroup of Z_v.*

(3) *Suppose that $\mathrm{td}(K|k) > 1$. Then Z_v is maximal among the divisorial like subgroups of $G_K(\ell)$ if and only if v is a quasi-divisorial valuation on $K(\ell)$.*

Proof. — For assertion (1), recall that $Kv\,|\,kv$ is a function field which satisfies $\mathrm{td}(Kv\,|\,kv) = \mathrm{td}(K|k) - 1$. One concludes by applying Fact 2.2, 2), 3) and taking into account that $vK/\ell \cong \mathbb{Z}/\ell$.

The proof of assertion (2) is identical with the proof of the corresponding assertion in the case of divisorial subgroups of $G_K(\ell)$ in Proposition 3.3.

To (3): First, suppose that Z_v is maximal among the divisorial like subgroups of $G_K(\ell)$. Let v' be the unique coarsening of v which is a quasi-divisorial valuation of $K(\ell)$, as described in Appendix, Fact 5.5, 3). We claim that $v = v'$. Indeed, let $v_0 = v/v'$ be the valuation induced by v on the residue field $K(\ell)v'$. Then by general decomposition theory for valuations, see e.g. Fact 2.1, 3), it follows via the canonical exact sequence

$$1 \to T_{v'} \to Z_{v'} \to G_{Kv'}(\ell) \to 1,$$

that Z_v is exactly the preimage of Z_{v_0} in $Z_{v'}$. Since v' has $r_{v'} = 1$ and is defectless, by assertion (1) of the Proposition, it follows that $Z_{v'}$ is divisorial like. On the other hand, $Z_v \subseteq Z_{v'}$ is maximal among divisorial like subgroups of $G_K(\ell)$, hence $Z_v = Z_{v'}$. But then $Z_{v_0} = G_{Kv'}(\ell)$, and so v_0 is pro-ℓ Henselian on the function field $Kv'|kv'$. Since $Kv'|kv'$ has transcendence degree equal to $\mathrm{td}(K|k) - 1 > 0$, the valuation v_0 must be the trivial valuation. Thus finally $v = v'$, and v is quasi-divisorial.

For the converse, suppose that v is quasi-divisorial. We show that Z_v is maximal among the subgroups Z' of $G_K(\ell)$ having a non-trivial pro-cyclic normal subgroup T'. The proof is more or less identical with the proof of assertion (3) of Proposition 3.3. Indeed, in the above notations (which are identical with the ones from loc.cit.) and reasoning like there, we have: The fixed field Λ of $G := T_vT'$ is pro-ℓ Henselian with respect to v. Next recall that K^T is the fixed field of T_v in $K(\ell)$. Hence $\Lambda \subset K^T$.

Claim. The pro-ℓ, K^T-core $v_{\text{pro-}\ell,K^T}$ equals v.

To simplify notations, let $L = K^T$ be the fixed field of T_v in $K(\ell)$, and $w = v_{\text{pro-}\ell,L} = v_{\text{pro-}\ell,K^T}$. Then $L(\ell) = K(\ell)$, and $G_L(\ell) = T_v$. And further, $G_L(\ell) \subset Z_w$, as w is pro-ℓ Henselian on L. By contradiction, suppose that $w < v$ is a proper coarsening of v. Since v is quasi-divisorial, it follows that wK is divisible, thus wL is divisible too. Therefore, $T_w = 1$ by Fact 2.2; and the canonical projection below is an isomorphism

$$\pi_w : Z_w \to G_{Kw}(\ell).$$

On the other hand, since $K^Tv = Lv$ is pro-ℓ closed, and $w = v_{\text{pro-}\ell,L}$, it follows by Proposition 2.3, (1), that Lw is pro-ℓ closed too. Thus $\pi_w(G_L(\ell)) = G_{Lw}(\ell) = 1$, hence $G_L(\ell)$ is trivial. Contradiction! The Claim is proved.

We return to the proof of assertion (3): Taking into account that $w = v_{\text{pro-}\ell,K^T}$ is pro-ℓ Henselian on Λ, and that $\Lambda \subseteq K^T = L$, by using the Claim above and the properties of pro-ℓ core, we get:

$$v = v_{\text{pro-}\ell,K^T} \leq v_{\text{pro-}\ell,\Lambda} \leq v.$$

Hence finally $v_{\text{pro-}\ell,\Lambda} = v$. Therefore, as in loc.cit., it follows that v is pro-ℓ Henselian on the fixed field $K(\ell)^{Z'}$. Equivalently, $Z' \subseteq Z_v$. □

4. Characterization of the (quasi-)divisorial subgroups

A) *Arithmetical nature of quasi-divisorial subgroups*

Recall the notations from the Introduction: A closed subgroup $Z \subset G_K(\ell)$ is called quasi-divisorial, if it is divisorial like and maximal among the divisorial like subgroups of $G_K(\ell)$. The aim of this section is to show that the quasi-divisorial subgroups are of arithmetical/geometrical nature. More precisely, we will prove the following:

Proposition 4.1. — *Let $K|k$ be a function field with $d = \mathrm{td}(K|k) > 1$. Then the following hold:*

(1) *Every divisorial like subgroup of $G_K(\ell)$ is contained in a unique quasi-divisorial subgroup of $G_K(\ell)$.*

(2) *Suppose that Z is a quasi-divisorial subgroup of $G_K(\ell)$. Then there exists a unique quasi-divisorial valuation v on $K(\ell)$ such that $Z_v = Z$. Moreover,* $\mathrm{char}(Kv) \neq \ell$.

Proof. — The main step in the proof is the following Key Lemma, which in some sense plays the same role as the q-Lemma in the Local theory from [P1]. Nevertheless, its proof requires other techniques, see below.

Key Lemma 4.2. — *In the context of the Theorem, let $Z \subset G_K(\ell)$ be a subgroup with the following properties:*

i) *Z contains closed subgroups Z_0 isomorphic to $\mathbb{Z}_\ell^d$ with $d = \mathrm{td}(K|k)$.*

ii) *Z has non-trivial Abelian normal subgroups.*

Then there exists a non-trivial valuation v of $K(\ell)$ which is pro-ℓ Henselian on the fixed field K^Z of Z in $K(\ell)$ and satisfies the following: v has no relative defect, and $r_v > 0$, and $\mathrm{char}(Kv) \neq \ell$, *and $T_v \cap Z \cong \mathbb{Z}_\ell^{r_v}$.*

Moreover, if Z has a pro-cyclic normal subgroup T such that Z/T has no non-trivial Abelian normal subgroups, then there exists a unique quasi-divisorial valuation v such that $Z \subseteq Z_v$ and $T = T_v \cap Z$.

Proof. — The main ingredient in the proof of Key Lemma is the following result, see e.g. Engler–Nogueira [E–N] for $\ell = 2$, and Engler–Koenigsmann [E–K] and/or Efrat [Ef] in general. And naturally, one could use here Bogomolov [B], Bogomolov–Tschinkel [B–T1].

Fact 4.3. — Let Λ be a field such that $G_\Lambda(\ell) \cong \mathbb{Z}_\ell \rtimes \mathbb{Z}_\ell$. Then there exists a non-trivial valuation w on $K(\ell)$ which is ℓ-Henselian on Λ such that $w\Lambda$ is not ℓ-divisible, and the residual characteristic $\mathrm{char}(\Lambda w) \neq \ell$.

As a consequence of Fact 4.3 we have the following:

Fact 4.4. — Let $K|k$ be as above, and $d = \mathrm{td}(K|k)$. Then by induction on d it follows: Every Abelian subgroup $G \subset G_K(\ell)$ is of the form $G \cong \mathbb{Z}_\ell^s$ for some $s = s_G \leq d$.

Using these facts and the techniques developed in Pop [P1], Local Theory, one can easily prove the above Key Lemma as follows:

Let K^Z be the fixed field of Z in $K(\ell)$. Let $T' \cong \mathbb{Z}_\ell^r$ be a maximal non-trivial Abelian normal subgroup of Z. We remark that Z contains a subgroup $G \subseteq Z$ as the one in the Fact 4.3 above such that $G \cap T'$ is non-trivial. Indeed, if $r > 1$, then T' contains a subgroup $G \cong \mathbb{Z}_\ell \times \mathbb{Z}_\ell$. If on the other side $r = 1$, then we choose any $g \in Z$ not in T'. Then denoting by G the closed subgroup generated by T' and g, we get a subgroup $G \subseteq Z$ with the desired properties.

Next let Λ be the fixed field of G in $K(\ell)$. Then by Fact 4.3 above, there is a valuation w on $K(\ell)$ which is pro-ℓ Henselian on Λ and satisfies: $w\Lambda$ is not ℓ-divisible, and Λw has characteristic $\neq \ell$.

In particular, w is also pro-ℓ Henselian on the fixed field $\Lambda^{T' \cap G}$ of $T' \cap G$ in $K(\ell)$. Let v be the pro-ℓ $\Lambda^{T' \cap G}$-core of w. We claim that v has the properties we asked for in the Key Lemma.

Claim 1. v is pro-ℓ Henselian on K^Z.

Indeed, taking into account that $T' \cap G$ is a normal subgroup of T', it follows by Proposition 2.4, (1), that v is pro-ℓ Henselian on the fixed field $K^{T'}$ of T' in $K(\ell)$. On the other hand, since $K^{T'} \subseteq \Lambda^{T' \cap G}$, and v equals its pro-ℓ $\Lambda^{T' \cap G}$-core, it follows that v equals its pro-ℓ $K^{T'}$-core. Thus reasoning as above, since T' is a normal subgroup in Z, it follows that v is pro-ℓ Henselian on the fixed field K^Z of Z in $K(\ell)$. And moreover, v equals its pro-ℓ K^Z-core.

Claim 2. v has no relative defect.

Indeed, let Λ_0 be the fixed field of a subgroup $Z_0 \cong \mathbb{Z}_\ell^d$ of $Z \subset G_K(\ell)$. Taking into account that v is pro-ℓ Henselian on Λ_0, applying Fact 2.2, 3), we have:

$$\mathbb{Z}_\ell^d \cong Z_0 = G_{\Lambda_0}(\ell) \cong \mathbb{Z}_\ell^{\delta_0} \times G_{\Lambda_0 v}(\ell),$$

where δ_0 equals the dimension of the $\mathbb{F}_\ell$ vector space $v\Lambda_0/\ell$. And since Z_0 is Abelian, $G_{\Lambda_0 v}(\ell)$ is Abelian too, say $G_{\Lambda_0 v}(\ell) \cong \mathbb{Z}_\ell^s$. Hence $d = \delta_0 + s$. On the other hand, we have the inequalities as follows:

a) $\delta_0 \leq \delta_v = r_v$, deduced from the exact sequence $0 \to vk \to vK \to \mathbb{Z}^{r_v} \to 0$. Here vk is a divisible group, and δ_v is the dimension of the $\mathbb{F}_\ell$ vector space vK/ℓ.

b) $s \leq \mathrm{td}(\Lambda_0 v\,|\,kv) = \mathrm{td}(Kv\,|\,kv) = \mathrm{td}_v$, by Fact 4.4.

Therefore we have: $d = \delta_0 + s \leq r_v + \mathrm{td}_v \leq d$, and hence the inequalities at a), b) above are equalities, and $r_v + \mathrm{td}_v = d$. Thus v has no relative defect.

In order to show that $T_v \cap Z \cong \mathbb{Z}_\ell^{r_v}$, we use the conclusion of the discussion above: We have namely proved that via $\pi_v : Z_v \to G_{Kv}(\ell)$ one has: $Z_0 \cap \ker(\pi_v) \cong \mathbb{Z}_\ell^{\delta_0} = \mathbb{Z}_\ell^{r_v}$. Therefore, since $\ker(\pi_v) = T_v$, we have:

$$\mathbb{Z}^{r_v} \cong Z_0 \cap T_v \subseteq Z \cap T_v \subseteq T_v \cong \mathbb{Z}^{r_v}.$$

Thus finally we get $T_v \cap Z \cong \mathbb{Z}_\ell^{r_v}$.

Finally, we address the last assertion of the Key Lemma, and consider the case when Z has a pro-cyclic normal subgroup T such that Z/T has no non-trivial Abelian subgroups. Let v be the valuation just constructed above. Since T_v is the center of Z_v, it follows that $Z \cap T_v \cong \mathbb{Z}_\ell^{r_v}$ is contained in the center of Z. Hence by the hypothesis on Z and T it follows that $Z \cap T_v$ is contained in $T \cong \mathbb{Z}_\ell$, thus in particular $r_v = 1$. Furthermore, since $\mathbb{Z}_\ell \cong T_v \cap Z \subseteq T \cong \mathbb{Z}_\ell$, it follows that $T_v \cap Z$ is an open subgroup of T. Thus

$$\pi_v(T) \cong T/(T \cap T_v) = T/(Z \cap T_v)$$

is a finite cyclic subgroup of $G_{Kv}(\ell)$. Since $G_{Kv}(\ell)$ is torsion free, it follows that $\pi_v(T) = 1$, thus $T = T_v \cap Z$.

In order to conclude, replace v by its minimal coarsening v' which is a quasi-divisorial valuation, as in Appendix, Fact 5.5, 3). Then by Fact 2.1, 3), one has $Z_v \subseteq Z_{v'}$. And by Proposition 3.5, (3), v' is the only quasi-divisorial valuation of $K(\ell)$ such that $Z_v \subseteq Z_{v'}$.

This concludes the proof of the Key Lemma. □

We now come to the proof of the Proposition 4.1. Assertion (1) is an immediate consequence of the Key Lemma and the fact that (by Proposition 3.3) the divisorial like subgroups satisfy the conditions i), ii), of the Key Lemma. For Assertion (2), the uniqueness of v follows from Proposition 2.4 by taking into account that a quasi-divisorial valuation equals its absolute pro-ℓ core (as its residue field is not pro-ℓ closed if $d > 1$). □

B) *Characterization of the divisorial subgroups*

We now show that using the information encoded in "sufficiently many" 1-dimensional projections one can characterize the divisorial subgroups among all the quasi-divisorial subgroups. First some preparations:

Fact 4.5. — Let $K|k$ be a function field in $d = \mathrm{td}(K|k) > 1$ variables, and $L|k$ a function subfield of $K|k$ with $\mathrm{td}(L|k) > 0$. In particular, we can and will view $L(\ell)$ as a subfield of $K(\ell)$. Finally let v be a valuation of $K(\ell)$, and denote by K^Z and L^Z its decomposition groups in $G_K(\ell)$, respectively $G_L(\ell)$. Then the following hold:

1) If v is defectless on $K(\ell)$, then v is defectless on $L(\ell)$. Moreover, if v is a quasi-divisorial valuation of $K(\ell)$, then v is either trivial or a quasi-divisorial valuation on $L(\ell)$.

2) One has $L^Z \subseteq K^Z$. Further, denoting by L' the relative algebraic closure of L^Z in K^Z, it follows that $L'|L^Z$ is finite.

Proof. — The first assertion follows by the additivity of r_v and td_v in towers of fields in the case of defectless valuations. For the second assertion see the proof of Corollary 1.18 from [P1]. $\square$

Next we define 1-dimensional projections of $G_K(\ell)$ as follows: For every $t \in K\backslash k$, let K_t be the relative algebraic closure of $k(t)$ in K. Then $K_t|k$ is a function field in one variable. Moreover, if t is "general", then $K_t = k(t)$ is a rational function field over k. Turning our attention to Galois theory, the inclusion $\imath_t : K_t \to K$ gives rise to a surjective restriction homomorphism

$$p_t : G_K(\ell) \to G_{K_t}(\ell).$$

We now are ready to announce the recipe for detecting the divisorial valuations of $K|k$ using the projections p_t.

Proposition 4.6. — *Let $K|k$ be a function field as usual, and suppose that with $\mathrm{td}(K|k) > 1$. Then for a given quasi-divisorial subgroup $Z \subset G_K(\ell)$, the following assertions are equivalent:*

i) *Z is a divisorial subgroup of $G_K(\ell)$.*

ii) *$\exists\ t \in K\backslash k$ such that $p_t(Z) \subseteq G_{K_t}(\ell)$ is an open subgroup.*

Proof. — First, suppose that $Z = Z_v$ is the divisorial subgroup defined by some divisorial valuation v on $K(\ell)$. Then choosing t such that t is a v-unit and tv is transcendental over k in Kv, we have: v is trivial on $L := K_t$. And in particular, v is pro-ℓ Henselian on L, i.e., $L^Z = L$. Therefore, in the notations from the Fact 4.5 above, it follows by loc.cit. that the relative algebraic closure L' of $L^Z = L$ in K^Z is a finite extension of $L = K_t$. Thus $p_t(Z) = G_{L'}(\ell) \leq G_{K_t}(\ell)$ is an open subgroup.

Conversely, suppose that $p_t(Z)$ is an open subgroup of $G_{K_t}(\ell)$ for some non-constant $t \in K$. Equivalently, the relative algebraic closure L' of the field $L := K_t$ in K^Z is a finite extension of L. Now denoting by L^Z the decomposition field of v in $L(\ell)$, it follows by Fact 4.5 that $L \subseteq L^Z \subseteq L'$. Therefore, $L^Z|L$ is finite, thus $L^Z|k$ is a function field in one variable over k. And since v is a pro-ℓ Henselian valuation of this function field, it follows that v must be trivial on L^Z. In particular, v is trivial on k. Hence v is a Zariski prime divisor of K. $\square$

5. Appendix

A) *Geometric interpretation of the Zariski prime divisors*

Let $K|k$ be a function field. A model $X \to k$ of $K|k$ is any integral k-variety $X \to k$ whose function field $k(X)|k$ is identified with $K|k$. In particular, we then identify the structure sheaf $\mathcal{O}_X$ with a sheaf of k-subalgebras of $K|k$. In particular, the restriction morphisms of $\mathcal{O}_X$, say $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ for $V \subset U$ are simply the inclusions. Therefore, we have the following:

Fact 5.1. — On the set of all the models $X_i \to k$ of $K|k$ there exists a naturally defined domination relation as follows: $X_j \geq X_i$ if and only if there exists a surjective morphism $\varphi_{ji} : X_j \to X_i$ which at the structure sheaf level is defined by inclusions. Let $\mathcal{P}_K$ be the set of all projective normal models of K. Some basic results in algebraic geometry guarantee the following, see e.g., Zariski–Samuel [Z–S], Ch.VI, especially §17, Mumford [M], Ch.I, etc.:

1. Every model $X_i \to k$ is contained as an open subvariety in a complete model $\tilde{X}_i \to k$ of $K|k$ (Nagata's Theorem).
2. Every complete model $X \to k$ is dominated by some $X_i \in \mathcal{P}_K$ (Chow's Lemma).
3. The set $\mathcal{P}_K$ is increasingly filtered with respect to $\geq$, hence it is a surjective projective system.
4. Given any function $f \neq 0$ in K, there exist projective models $X_i \to k$ such that for each $x_i \in X_i$ one has: Either f of $1/f$ is defined at x_i.
5. Let v be a k-valuation of $K|k$ with valuation ring $\mathcal{O}_v$. Then for every model $X \to k$ of $K|k$, there exists at most one point $x \in X$ such that $\mathcal{O}_v$ dominates the local ring $\mathcal{O}_{X,x}$ (Valuation criterion).

   If such a point exists, we will say that x is the center of v on X.
6. If in the above context, $X \to k$ is proper, then every v has a center on X (valuation criterion).

Remark/Definition 5.2. — We denote by $\mathfrak{R}_{K|k}$ the space of all the k-valuations of $K|k$, and call it the Riemann space of $K|k$. There exists a canonical identification of $\mathfrak{R}_{K|k}$ with $\varprojlim_i X_i$ as follows:

Let v be a k-valuation of $K|k$ with valuation ring $\mathcal{O}_v$. For every projective —thus proper— model $X_i \to k$ of $K|k$, let $x_i \in X_i$ be the center of v on X_i. Clearly, if $X_j \geq X_i$, say via a morphism $\phi_{ji} : X_j \to X_i$, then $\phi_{ji}(x_j) = x_i$. Therefore $(x_i)_i \in \mathfrak{R}_K = \varprojlim_i X_i$. Further, for the local rings we have: $\mathcal{O}_{X_i,x_i} \subseteq \mathcal{O}_{X_j,x_j}$. Thus $R_v := \cup_{X_i}\mathcal{O}_{X_i,x_i}$ is a k-subalgebra of $K|k$, which finally turns to be exactly the valuation ring $\mathcal{O}_v$.

Conversely, given $(x_i)_i \in \varprojlim_i X_i$, one has: $\mathcal{O}_{X_i,x_i} \subseteq \mathcal{O}_{X_j,x_j}$ if $j \geq i$. Thus $\mathcal{O} := \cup_{X_i}\mathcal{O}_{X_i,x_i}$ is a k-subalgebra of $K|k$. Using Fact 3.1, 4) above, it follows that for every $f \neq 0$ from K one has: Either f of $1/f$ lie in $\mathcal{O}_{X_i,x_i}$ for i sufficiently large; thus f of $1/f$ lie in $\mathcal{O}$. Hence $\mathcal{O}$ is a k-valuation ring of a k-valuation v of $K|k$.

Remark/Definition 5.3. — We remark that for a point $v = (x_i)_i$ in $\mathfrak{R}_{K|k}$ as above, the following conditions are equivalent, one uses e.g. [BOU], Ch.IV, §3, or [P1], The Local Theory:

i) For i sufficiently large, x_i has co-dimension 1, or equivalently, x_i is the generic point of a prime Weil divisor of X_i. Hence v is the discrete k-valuation of K with valuation ring $\mathcal{O}_{X_i,x_i}$.

ii) v is discrete, and $Kv|k$ is a function field in $\mathrm{td}(K|k)-1$ variables.

iii) $\mathrm{td}(Kv\,|\,k) = \mathrm{td}(K|k)-1$.

A valuation v on K with the above equivalent properties is called a Zariski prime divisor of $K|k$. As a corollary of the observations above we have:

The space of all Zariski prime divisors of K is the union of the spaces of Weil prime divisors of all normal models of $K|k$ (if we identify every Weil prime divisor with the discrete valuation on K it defines).

B) *Defectless valuations*

We consider function fields $K|k$ as at the beginning of the Subsection 3, B). In the notations from there, for a valuation v of K we denote by r_v and td_v the rational rank, respectively the residual transcendence degree of v. We recall the following basic facts concerning defectless valuations.

Fact 5.4. — Let v be the composition $v = v_0\,v'$ of a valuation v' on $K(\ell)$ and a valuation v_0 on $K_0(\ell)$, where $K_0 := Kv'$ is a field extension of the algebraically closed field $k_0 := kv'$.

1) One has a canonical diagram with exact rows as follows:

$$\begin{array}{ccccccccc} 0 & \to & v_0k_0 & \to & vk & \to & v'k & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & v_0K_0 & \to & vK & \to & v'K & \to & 0 \end{array}$$

The vertical maps are inclusions, therefore one has an exact sequence of torsion free groups

$$(*) \qquad 0 \to v_0K_0/v_0k_0 \to vK/vk \to v'K/v'k \to 0\,.$$

From this we deduce the following:

a) $r_{(\cdot)}$ is *additive* in the following sense: $r_v = r_{v'} + r_{v_0}$.

b) There exists a unique minimal coarsening say v' of v such that $r_{v'} = r_v$. And this coarsening is defined by some convex divisible subgroup Δ' of vK.

2) Further, one has $\mathrm{td}_v = \mathrm{td}_{v_0}$, thus $\mathrm{td}_{v'} \geq \mathrm{td}_v + r_{v_0}$.

3) Hence v is defectless $\iff$ both v' and v_0 are defectless.

Proof. — The only less obvious fact might be assertion b). In order to prove it, suppose first that v' is a coarsening of v such that $r_{v'} = r_v$. Then from the exact sequence $(*)$ above we deduce that $r_{v_0} = 0$. Thus v_0K_0/v_0k_0 is a torsion free group of rational rank equal to 0. Hence $v_0K_0 = v_0k_0$. Since $k_0 = kv'$ is algebraically closed, $v_0 k_0$ is divisible; hence the convex subgroup $\Delta_{v'} := v_0 K_0$ of vK defining v' is divisible. One gets Δ' by taking the union of all the convex divisible subgroups of the form $\Delta_{v'}$ for v' having $r_{v'} = r_v$.

Note that $\Delta' = vK$ if and only if $r_v = 0$. □

Fact 5.5. — Let v be a defectless valuation on $K(\ell)$.

1) The *fundamental equality* holds for every finite extension $L|K$ of K. This means, that if $v_1, \dots, v_r$ are the finitely many prolongations of $v_0 := v|_K$ to L, then

$$[L : K] = \textstyle\sum_i e(v_i|v_0)\, f(v_i|v_0).$$

This is a non-trivial fact, see e.g., Kuhlmann [Ku]. It is a generalization of results by several people starting with Deuring, Grauert–Remmert's Stability Theorem from the rigid algebraic geometry, etc.

2) In particular, using the description of the defectless valuations given by Fact 5.6 below, one gets the following:

a) $vK \,/\, vk \cong \mathbb{Z}^{r_v}$.

b) $Kv\,|\,kv$ is a function field in td_v variables.

3) The unique minimal coarsening v' of v such that $r_v = r_{v'}$ —defined in Fact 5.4, 1), b), is defectless and is defined by the *unique maximal divisible convex subgroup* Δ' of vK.

In particular, for every defectless valuation v of $K(\ell)$ with $r_v = 1$, there exists a unique coarsening v' of v which is a quasi-divisorial valuation of $K|k$.

Fact 5.6. — A recipe which produces all possible valuations without relative defect on $K|k$ with given invariants $r = r_v$ and $\mathrm{td} = \mathrm{td}_v$ is as follows:

First let v be a defectless valuation on $K(\ell)$, say with invariants $r = r_v$ and $\mathrm{td} = \mathrm{td}_v$. Since $vK(\ell)/vK$ is a torsion group, it follows that vK/vk has rational rank equal to r. We do the following:

- Choose a system $(t_1, \dots, t_r)$ of elements of $K^\times$ such that setting $\gamma_i = v\,t_i$, the resulting system $(\gamma_1, \dots, \gamma_r)$ of elements of vK is linearly independent in vK/vk.

- Further choose a system of v-units $\mathcal{T}_0 = (t_{r+1}, \dots, t_d)$ in K such that $(t_{r+1}v, \dots, t_d v)$ is a transcendence basis of $Kv\,|\,kv$.

Using e.g. [BOU], Ch.6, §10, 3, it follows that $\mathcal{T} = (t_1, \dots, t_d)$ is a system of elements of K which is algebraically independent over k. Since v is defectless by hypothesis, it follows that $d = \mathrm{td}(K|k)$, i.e., $\mathcal{T}$ is a transcendence basis of $K|k$. Thus denoting by $w_{\mathcal{T}_0}$ and $w_{\mathcal{T}}$ the restrictions of v to the rational function fields $k(\mathcal{T}_0) \subset k(\mathcal{T})$ respectively, the following hold:

i) $w_{\mathcal{T}_0}$ is the so called $\mathsf{generalized\ Gauß}$ valuation defined by $v_k := v|_k$ and $\mathcal{T}_0$ on $k(\mathcal{T}_0)$, i.e., for every polynomial $p(\mathbf{t})$ in the system of variables $\mathbf{t_0} = (t_{r+1}, \dots, t_d)$, say $p(\mathbf{t}_0) = \sum_{\mathbf{i}} a_{\mathbf{i}} \mathbf{t}_0^{\mathbf{i}} \in k[\mathcal{T}_0]$ one has:

$$w_{\mathcal{T}_0}\big(p(\mathbf{t}_0)\big) = \min_{\mathbf{i}} v_k(a_{\mathbf{i}}) = v\big(p(\mathbf{t}_0)\big).$$

ii) $w_{\mathcal{T}}$ is the unique prolongation of $w_{\mathcal{T}_0}$ to $k(\mathcal{T})$ such that for all t_i $(1 \le i \le r)$ one has:

$$w_{\mathcal{T}}(t_i) = \gamma_i = v(t_i).$$

And finally, $v|_K$ is one of the finitely many prolongations of $v_{\mathcal{T}}$ to the finite extension $K|k(\mathcal{T})$.

In particular, $\Gamma := vk(\mathcal{T}) = w_{\mathcal{T}}k(\mathcal{T}) = vk + \gamma_1\mathbb{Z} + \cdots + \gamma_r\mathbb{Z}$ is a totally ordered group such that Γ/vk is a free Abelian group of rank r. And the residue field $k(\mathcal{T})v_{\mathcal{T}}$ is the rational function field $kv(t_{r+1}v, \dots, t_d v)$.

The observation above can be "reversed" in order to produce defectless valuations on $K(\ell)$ as follows. Consider:

- Valuations v_k on k together with totally ordered groups of the form $\Gamma = v_k k + \gamma_1\mathbb{Z} + \cdots + \gamma_r\mathbb{Z}$ with $\Gamma/v_k k$ a free Abelian group of rank r.

- Transcendence bases $\mathcal{T} = (t_1, \dots, t_d)$ of $K|k$.

We set $\mathcal{T}_0 = (t_{r+1}, \dots, t_d)$. On the rational function fields $k(\mathcal{T}_0)$ we consider the generalized Gauß valuation $w_{\mathcal{T}_0}$ defined by v_k and $\mathcal{T}_0$; and we denote by $w_{\mathcal{T}}$ the unique prolongation of $w_{\mathcal{T}_0}$ to $k(\mathcal{T})$ satisfying $w_{\mathcal{T}}(t_i) = \gamma_i$ if $i \le r$.

Then finally we have:

For every $w_{\mathcal{T}}$ obtained as indicated above, all its prolongations v to $K(\ell)$ are defectless valuations on $K(\ell)$.

Conversely, every valuation without relative defect on $K|k$ is obtained by the above recipe.

In particular, a valuation v without relative defect on $K(\ell)$ is a Zariski prime divisor $\iff$ v is trivial on k and has $r_v = 1$

Remark 5.7. — It would be very desirable to have a geometric description of the space of all the quasi-divisorial valuations of $K(\ell)$, thus generalizing the construction of all the Zariski prime divisors of $K|k$ given in the subsection A) above. Unfortunately, at the moment we are not able to do this in a satisfactory way.

One could do this along the same lines as in subsection A) above for a special class of quasi-divisorial valuations, which are the so called constant reductions à la Deuring, see Roquette [R], followed by Zariski prime divisors of the residue function fields of such constant reductions. This situation arises in an *arithmetical way* as follows, see loc.cit.:

Let R be the valuation ring of a valuation v_k of the base field k. Let $\mathcal{X}_0 = \operatorname{Proj} R[X_0, \dots, X_d]$ be the d-dimensional projective space over R, where $d = \operatorname{td}(K|k)$. Let $\mathcal{T}$ be a transcendence basis of $K|k$, and identify $k(\mathcal{T})$ with the function field of $\mathcal{X}_0$

via $t_i = X_i/X_0$. Then the local ring of the generic point η of the special fiber of $\mathcal{X}_0$ is exactly the valuation ring of the Gauß $\mathcal{T}, v_k$-valuation $v_{\mathcal{T}}$ on $k(\mathcal{T})$. Finally, if $\mathcal{X} \to \mathcal{X}_0$ is the normalization of $\mathcal{X}_0$ in the function field extension $k(\mathcal{T}) \hookrightarrow K$, then denoting by η_i the generic points of the special fiber of $\mathcal{X}$, it follows that the corresponding local rings $\mathcal{O}_{\mathcal{X},\eta_i}$ are exactly the valuation rings of the prolongations v_i of $v_{\mathcal{T}}$ to K. Obviously, if v_i is a constant reduction of $K|k$, then $r_{v_i} = 0$ and $\mathrm{td}_{v_i} = \mathrm{td}(K|k)$. By abuse of language, we will say that a valuation v_0 of $K(\ell)$ is a constant reduction, if $\mathrm{td}(Kv_0|\,kv_0) = \mathrm{td}(K|k)$. Thus the constant reductions of $K(\ell)$ are exactly the prolongations to $K(\ell)$ of the "usual" constant reductions of $K|k$ defined above.

Finally, if v_0 is a given constant reduction of $K(\ell)$, and v_1 is a Zariski prime divisor of the residue field $K(\ell)v_0 = (Kv_0)(\ell)$ of v_0, then the composition of the two valuations $v = v_1 v_0$ is a quasi-divisorial valuation on $K(\ell)$ which we call a c.r.-divisorial valuation.

References

[B] F. A. Bogomolov, *On two conjectures in birational algebraic geometry*, in Algebraic Geometry and Analytic Geometry, ICM-90 Satellite Conference Proceedings, ed A. Fujiki et all, Springer Verlag Tokyo 1991. ↑

[B–T1] F. A. and Tschinkel Bogomolov Y., *Commuting elements in Galois groups of function fields* (2002). ↑

[B–T2] ———, *Reconstruction of function fields*, 2003. ↑

[BOU] N. Bourbaki, *Algèbre commutative*, Hermann Paris 1964. ↑

[D] P. Deligne, *Le groupe fondamental de la droite projective moins trois points*, Galois groups over **Q**, Math. Sci. Res. Inst. Publ. **16**, 79–297, Springer 1989. ↑

[Ef] I. Efrat, *Construction of valuations from K-theory*, Mathematical Research Letters **6** ((1999)), 335–344. ↑

[En] O. Endler, *Valuation theory*, Universitext, Springer-Verlag, New York-Heidelberg, 1972. ↑

[E–E] O. and Engler Endler A. J., *Fields with Henselian Valuation Rings*, Math. Z. **152** (1977), 191–193. ↑

[E–K] A. J. and Koenigsmann Engler J., *Abelian subgroups of pro-p Galois groups*, Trans. AMS **350** (1998), 2473–2485. ↑

[E–N] A. J. and Nogueira Engler J. B., *Maximal Abelian normal subgroups of Galois pro-2-groups*, J. Algebra **166** (994), 481ndash505. ↑

[F–J] M. and Jarden Fried M., *Field Arithmetic*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge A Series of Modern Surveys in Mathematics, Vol. 11, Springer Verlag 2004. ↑

[GGA] Geometric Galois Actions I, LMS LNS Vol **242**, eds L. Schneps – P. Lochak, Cambridge Univ. Press 1998. ↑

[G1] A. Grothendieck, *Letter to Faltings, June 1983*, See [GGA]. ↑

[G2] ———, *Esquisse d'un programme, 1984*, See [GGA]. ↑

[Ik] M. Ikeda, *Completeness of the absolute Galois group of the rational number field*, J. reine angew. Math. **291** (1977), 1–22. ↑

[Ko] H. Koch, *Die Galoissche Theorie der p-Erweiterungen*, Math. Monogr. **10**, Berlin 1970. ↑

[K] J. Koenigsmann, *From p-rigid elements to valuations (with a Galois characterization of p-adic fields)*, J. reine angew. Math. **465** (1995), 165–182. ↑

[Km] K. Komatsu, *A remark to a Neukirch's conjecture*, Proc. Japan Acad. **50** (1974), 253–255. ↑

[Ku] F.-V. Kuhlmann, *Book on Valuation Theory*, See: `http://math.usask.ca/~fvk/Fvkbook.htm` (in preparation). ↑

[K–P–R] F.-V. Kuhlmann Pank, *Immediate and purely wild extensions of valued fields*, Manuscripta math. **55** (1986), 39–67. ↑

[M] D. Mumford, *The ref book of varieties and schemes*, LNM ???, Springer Verlag 1966. ↑

[N1] J. Neukirch, *Über eine algebraische Kennzeichnung der Henselkörper*, J. reine angew. Math. **231** (1968), 75–81. ↑

[N2] ———, *Kennzeichnung der p-adischen und endlichen algebraischen Zahlkörper*, Inventiones math. **6** (1969), 269–314. ↑

[N3] ———, *Kennzeichnung der endlich-algebraischen Zahlkörper durch die Galoisgruppe der maximal auflösbaren Erweiterungen*, J. reine angew. Math. **238** (1969), 135–147. ↑

[Pa] A. N. Parshin, *Finiteness Theorems and Hyperbolic Manifolds*, in: The Grothendieck Festschrift III, ed P. Cartier et all, PM Series Vol 88, Birkhäuser Boston Basel Berlin 1990. ↑

[P1] F. Pop, *On Grothendieck's conjecture of birational anabelian geometry*, Ann. of Math. **138** (1994), 145–182. ↑

[P2] ———, *On Grothendieck's conjecture of birational anabelian geometry II*, Heidelberg–Mannheim Preprint series Arithmetik II, N^0 **16**, Heidelberg 1995. ↑

[P3] ———, *Alterations and birational anabelian geometry*, 519–532, in: Resolution of singularities, Birkhäuser PM Series, Vol. **181**; eds Herwig Hauser et al, Birkhäuser Verlag, Basel 2000. ↑

[P4] ———, *Pro-ℓ birational anabelian geometry over alg. closed fields I*, See: `http://arxiv.org/pdf/math.AG/0307076`. ↑

[R] P. Roquette, *Zur Theorie der Konstantenreduktion algebraischer Mannigfaltigkeiten*, J. reine angew. Math. **200** (1958), 1–44. ↑

[S] J.-P. Serre, *Cohomologie Galoisienne*, LNM 5, Springer 1965. ↑

[Sp] M. Spiess, *An arithmetic proof of Pop's Theorem concerning Galois groups of function fields over number fields*, J. reine angew. Math. **478** (1996), 107–126. ↑

[U1] K. Uchida, *Isomorphisms of Galois groups of algebraic function fields*, Ann. of Math. **106** (1977), 589–598. ↑

[U2] ———, *Isomorphisms of Galois groups of solvably closed Galois extensions*, Tôhoku Math. J. **31** (1979), 359–362. ↑

[U3] ———, *Homomorphisms of Galois groups of solvably closed Galois extensions*, J. Math. Soc. Japan **33** (1981), ???–???. ↑

[W] R. Ware, *Valuation Rings and rigid Elements in Fields*, Can. J. Math. **33** (1981), 1338–1355. ↑

[Z–S] O. and Samuel Zariski P., *Commutative Algebra*, Vol II, Springer-Verlag, New York, 1975. ↑

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