$\mathbb{Z}/p$ metabelian birational p-adic section conjecture for varieties

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Abstract

We generalize the $\mathbb{Z}/p$ *metabelian* birational p-adic section conjecture for curves, as introduced and proved in Pop [P2], to all complete smooth varieties, provided $p > 2$. [The condition $p > 2$ seems to be of technical nature only, and it might be removable.]

1. Introduction

The (birational) [p-adic] section conjecture (SC) originates from Grothendieck [G1], [G1], see rather [GGA], and weaker/conditional forms of the SC are a part of the *local theory* in anabelian geometry, see e.g. Faltings [Fa] and Szamuely [Sz]. In spite of serious efforts to tackle the SC, only the *full Galois birational p-adic* SC is completely resolved, see Koenigsmann [Ko2] for the case of curves, and Stix [St] for higher dimensional varieties. On the other hand, a much stronger form of the birational p-adic SC for curves, to be precise, the $\mathbb{Z}/p$ *metabelian* birational p-adic SC for curves was proved in Pop [P2]. The aim of this note is to prove a similarly strong result for the *higher dimensional varieties,* at least in the case $p > 2$.

For reader's sake and to make the presentation self contained (to some extent), we begin by recalling a few notations and well known facts, see e.g. the Introduction in [P2]. First, for an arbitrary (perfect) base field k, and complete smooth geometrically integral k-varieties X, let $K = k(X)$ be the function field of X. Let $\tilde{K}|K$ be some Galois extension, $\tilde{k} \subseteq \tilde{K}$ be the relative algebraic closure of k in $\tilde{K}$, and consider the resulting canonical exact sequence of Galois groups:

$$1 \to \mathrm{Gal}(\tilde{K}|K\tilde{k}) \longrightarrow \mathrm{Gal}(\tilde{K}|K) \xrightarrow{\tilde{p}_K} \mathrm{Gal}(\tilde{k}|k) \to 1\,.$$

Let $\tilde{X} \to X$ be the normalization of X in the field extension $K \hookrightarrow \tilde{K}$. For $x \in X$ and $\tilde{x} \in \tilde{X}$ above x, let $T_x \subseteq Z_x$ be the inertia/decomposition, groups of $\tilde{x}|x$, and $G_x := \mathrm{Aut}\big(\kappa(\tilde{x})|\kappa(x)\big)$ be the residual automorphism group. By decomposition theory, one has a canonical exact sequence

$$(*) \qquad 1 \to T_x \to Z_x \to G_x \to 1\,.$$

Next suppose that x is k-rational, i.e., $\kappa(x) = k$. Since $\tilde{k} \subset \kappa(\tilde{x})$, the projection $Z_x \xrightarrow{\tilde{p}_K} \mathrm{Gal}(\tilde{k}|k)$ gives rise to a canonical surjective homomorphism $G_x \to \mathrm{Gal}(\tilde{k}|k)$, which in general is not injective. On the other hand, if $\tilde{k} \hookrightarrow \kappa(\tilde{x})$ is purely inseparable, then $G_x \to \mathrm{Gal}(\tilde{k}|k)$ is an isomorphism. Hence if the exact sequence $(*)$ splits, then $\tilde{p}_K$ has sections $\tilde{s}_x : \mathrm{Gal}(\tilde{k}|k) \to Z_x \subset \mathrm{Gal}(\tilde{K}|K)$,

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which we call sections above x. And notice that the conjugacy classes of sections $\tilde{s}_x$ above x build a "bouquet", which is in a canonical bijection with the (non-commutative) continuous cohomology pointed set $\mathrm{H}^1_{\mathrm{cont}}\big(\mathrm{Gal}(\tilde{k}|k), T_x\big)$ defined via the split exact sequence $(*)$.

Parallel to the case of points $x \in X$, one has a similar situation for k-valuations v of K as follows. For any prolongation $\tilde{v}$ of v to $\tilde{K}$, we denote by $T_v \subseteq Z_v$ the inertia/decomposition groups of $\tilde{v}|v$, and by $G_v = Z_v/T_v$ the residual automorphism group. As above, if $\kappa(v) = k$ and $\tilde{k} \hookrightarrow \kappa(\tilde{v})$ is purely inseparable, one has: First, the canonical homomorphism $G_v \to \mathrm{Gal}(\tilde{k}|k)$ is an isomorphism. Second, if the exact sequence $1 \to T_v \to Z_v \to G_v \to 1$ splits, then the projection $\tilde{p}_K : \mathrm{Gal}(\tilde{K}|K) \to \mathrm{Gal}(\tilde{k}|k)$ has a section $\tilde{s}_v : \mathrm{Gal}(\tilde{k}|k) \to Z_v \subseteq \mathrm{Gal}(\tilde{K}|K)$, which we call a section above v. And the conjugacy classes of sections $\tilde{s}_v$ above v build a "bouquet" which is in a canonical bijection with the (non-commutative) continuous cohomology pointed set $\mathrm{H}^1_{\mathrm{cont}}\big(\mathrm{Gal}(\tilde{k}|k), T_v\big)$ defined via the canonical split exact sequence above.

Finally, if $\tilde{K}|K$ contains a separable closure $K^{\mathrm{s}}|K$ of K, hence $\tilde{k}$ contains a corresponding separable closure k^{s} of k, then $\kappa(s)^{\mathrm{s}} \subseteq \kappa(\tilde{x}), \kappa(\tilde{v})$ and G_x and G_v are the absolute Galois groups of $\kappa(x)$, respectively $\kappa(v)$. Further, in this situation, $1 \to T_v \to Z_v \to G_v \to 1$ is split, see e.g., [KPR]. Thus if $\kappa(v) = k$, sections above v exist. In particular, if $x \in X(k)$ is a k-rational point, then choosing v such that $\kappa(x) = \kappa(v)$, it follows that sections above x exist as well, because every section above v is a section above x as well. We mention though that in general the bouquet of sections above x is much richer than the one of sections above v. Namely, by general decomposition theory, one has $T_v \subset T_x$, and $\mathrm{H}^1_{\mathrm{cont}}\big(\mathrm{Gal}(\tilde{k}|k), T_v\big) \to \mathrm{H}^1_{\mathrm{cont}}\big(\mathrm{Gal}(\tilde{k}|k), T_x\big)$ is a strict inclusion in general.

• Next let p be a fixed prime number. We denote by $K'|K$ the (maximal) $\mathbb{Z}/p$ elementary abelian extension of K, and by K'' the maximal $\mathbb{Z}/p$ elementary abelian extension of K' (in some fixed algebraic closure of K). Then $K''|K$ is a Galois extension, which we call the $\mathbb{Z}/p$ metabelian extension of K, and its Galois group $\mathrm{Gal}(K''|K)$ is called the metabelian Galois group of K. Note that $k' := \overline{k} \cap K'$, and $k'' := \overline{k} \cap K''$ are the $\mathbb{Z}/p$ elementary abelian extension, respectively the $\mathbb{Z}/p$ metabelian extension, of k. Finally, consider the canonical surjective projections:

$$pr'_K : \mathrm{Gal}(K'|K) \to \mathrm{Gal}(k'|k), \quad pr''_K : \mathrm{Gal}(K''|K) \to \mathrm{Gal}(k''|k).$$

We will say that a group theoretical (continuous) section $s' : \mathrm{Gal}(k'|k) \to \mathrm{Gal}(K'|K)$ of pr'_K is liftable, if there exists a section $s'' : \mathrm{Gal}(k''|k) \to \mathrm{Gal}(K''|K)$ of pr''_K which lifts s' to $\mathrm{Gal}(k''|k)$.

Note that if $p \neq \mathrm{char}$, and the p^{th} roots of unity μ_p are contained in k, hence in K, then by Kummer Theory we have: $K' = K[\sqrt[p]{K}\,]$, and $K'' = K'[\sqrt[p]{K'}\,]$, and similarly for k.

Theorem A. *In the above notations, let $k|\mathbb{Q}_p$ be finite with $\mu_p \subset k$. Then the following hold:*

1) *Every k-rational point $x \in X$ gives rise to a bouquet of conjugacy classes of liftable sections $s'_x : \mathrm{Gal}(k'|k) \to \mathrm{Gal}(K'|K)$ above x, which is in bijection with $\mathrm{H}^1\big(\mathrm{Gal}(k'|k), T_x\big)$.*
2) *Let $p > 2$, and $s' : \mathrm{Gal}(k'|k) \to \mathrm{Gal}(K'|K)$ be a liftable section. Then there exists a unique k-rational point $x \in X$ such that s' equals one of the sections s'_x as defined above.*

Actually one can reformulate the question addressed by Theorem A in terms of p-adic valuations, and get the following stronger result, see Section 2, C), for notations, definitions, and a few facts on (formally) p-adic valuations v, e.g., the p-adic rank d_v of v, and p-adically closed fields, respectively Ax–Kochen [A–K] and Prestel–Roquette [P–R], for proofs:

Theorem B. *Let k be a p-adically closed field with p-adic valuation v of p-adic rank d_v, and suppose that $\mu_p \subset k$. Let $K|k$ be an arbitrary field extension. Then the following hold:*

1) *Let w be a p-adic valuation of K of p-adic rank $d_w = d_v$. Then w prolongs v to K, and gives rise to a bouquet of conjugacy classes of liftable sections $s'_w : \mathrm{Gal}(k'|k) \to \mathrm{Gal}(K'|K)$ above w.*
2) *Let $p > 2$, and $s' : \mathrm{Gal}(k'|k) \to \mathrm{Gal}(K'|K)$ be a liftable section. Then there exists a unique p-adic valuation w of K of p-adic rank $d_w = d_v$ such that $s' = s'_w$ for some s'_w as above.*

Remark/Definition. As mentioned in Pop [P2], the condition $\mu_p \subset k$ is a necessary condition in the above theorems. Nevertheless, as mentioned in loc.cit., if μ_p is not contained in the base field, assertions similar to Theorems A and B above hold in the following form: Let $l|\mathbb{Q}_p$ be some finite extension, $Y \to l$ a complete geometrically integral smooth variety with function field $L = \kappa(Y)$. Let $k|l$ be a finite Galois extension with $\mu_p \subset k$. Setting $K := Lk$, consider the field extensions $K'|K \hookrightarrow K''|K$ and $k'|k \hookrightarrow k''|k$ as above. Then $k' = K' \cap \bar{l}$ and $k'' = K'' \cap \bar{l}$, and $K'|L$ and $K''|L$, as well as $k'|l$ and $k''|l$ are Galois extensions too, and one gets surjective canonical projections

$$pr'_L : \mathrm{Gal}(K'|L) \to \mathrm{Gal}(k'|l), \quad pr''_L : \mathrm{Gal}(K''|L) \to \mathrm{Gal}(k''|l).$$

In these notations and context we will say that a section $s'_L : \mathrm{Gal}(k'|l) \to \mathrm{Gal}(K'|L)$ of pr'_L is liftable, if there exists a section $s''_L : \mathrm{Gal}(k''|l) \to \mathrm{Gal}(K''|L)$ of pr''_L which lifts s'_L.

This being said, one has the following extensions of Theorem A and Theorem B:

Theorem A^0. *In the above notations and hypothesis, the following hold:*

1) *Every l-rational point $y \in Y$ gives rise to a bouquet of conjugacy classes of liftable sections $s'_y : \mathrm{Gal}(k'|l) \to \mathrm{Gal}(K'|L)$ above y, which is in bijection with $\mathrm{H}^1\big(\mathrm{Gal}(k'|l), T_y\big)$.*
2) *Let $p > 2$ and $s'_L : \mathrm{Gal}(k'|l) \to \mathrm{Gal}(K'|L)$ be a liftable section. Then there exists a unique l-rational point $y \in Y$ such that s'_L equals one of the sections s'_y as defined above.*

Theorem B^0. *Let l be a p-adically closed field with p-adic valuation v, and let $L|l$ be an arbitrary field extension. Then in the above notations the following hold:*

1) *Let w be a p-adic valuation of L with $d_w = d_v$. Then w prolongs v to L, and gives rise to a bouquet of conjugacy classes of liftable sections $s'_w : \mathrm{Gal}(k'|l) \to \mathrm{Gal}(K'|L)$ above w.*
2) *Let $p > 2$ and $s'_L : \mathrm{Gal}(k'|l) \to \mathrm{Gal}(K'|L)$ be a liftable section. Then there exists a unique p-adic valuation w of L such that $d_w = d_v$, and s'_L equals one of the sections s'_w as above.*

Remark. As mentioned in Pop [P2], the $\mathbb{Z}/p$ metabelian form of the birational p-adic SC for curves implies the corresponding full Galois SC, which was proved in Koenigsmann [Ko2]. The same hold correspondingly for higher dimensional varieties, provided $p > 2$, thus implying Stix [St] result in this case. Since the proof of the implication under discussion in the case of general varieties is word-by-word the same as that from [P2] loc.cit., we will not reproduce it here.

An interesting application of the results and techniques developed here is the following fact concerning the p-adic Section Conjecture for varieties: Let $k|\mathbb{Q}_p$ be a finite extension, and X

a complete smooth k-variety. Then there exists a finite effectively computable family of finite geometrically $\mathbb{Z}/p$ elementary abelian (ramified) covers $\varphi_i : X_i \to X$, $i \in I$, satisfying:

i) $\cup_i \varphi_i\big(X_i(k)\big) = X(k)$, i.e., every $x \in X(k)$ "survives" in at least one of the covers $X_i \to X$.

ii) A section $s : G_k \to \pi_1(X, *)$ can be lifted to a section $s_i : G_k \to \pi_1(X_i, *)$ for some $i \in I$ if and only if s arises from a k-rational point $x \in X(k)$ in the way described above.

The main technical tools for the proof of the above theorems are:

- The techniques developed in Pop [P2] (which refine facts/methods initiated in [P1]).
- The theory of rigid elements, as developed by several people: Ware [Wa], Arason–Jacob–Ware [AJW], Koenigsmann [Ko1], Efrat [Ef], etc. See Topaz [To] as the definitive reference.

As a final remark, we notice that the condition $p > 2$ in the results above originates from the weaker results about recovering valuations from rigid elements in the case $p = 2$. This technical condition might be removable, but some new ideas/techniques might be necessary to do so, see the comment at the beginning of the proof of assertion 2) of Theorem B in section 3.

2. Reviewing a few known facts

For reader's sake, in this section we review a few known facts about valuation theory, decomposition theory, and (formally) p-adic fields, but do not reproduce proofs.

A) *Generalities about valuations and their Hilbert decomposition theory*

For an arbitrary field K and an arbitrary valuation v of K, we denote usually by $\mathcal{O}_v, \mathfrak{m}_v$ the valuation ring/ideal of v, by $vK = K^\times/\mathcal{O}^\times$ the value group of v, and by $Kv =: \mathcal{O}_v/\mathfrak{m}_v := \kappa(v)$ the residue field of v. Further, $U_v^1 := 1 + \mathfrak{m}_v \subset U_v$ denote the groups of principal v-units, respectively v-units. One has the following canonical exact sequences:

$$1 \to \mathfrak{m}_v \to \mathcal{O}_v \to Kv \to 0 \quad \text{and} \quad 1 \to U_v^1 \to \mathcal{O}_v^\times \to (Kv)^\times \to 1.$$

The set of ideals of $\mathcal{O}_v$ is totally ordered with respect to inclusion. The subrings $\mathcal{O}_1 \subseteq K$ with $\mathcal{O}_v \subseteq \mathcal{O}_1$ are precisely the localizations $\mathcal{O}_1 := (\mathcal{O}_v)_{\mathfrak{m}_1}$ with $\mathfrak{m}_1 \in \operatorname{Spec}(\mathcal{O}_v)$, and moreover, $\mathfrak{m}_1 \subset \mathcal{O}_v$, and $(\mathcal{O}_v)_{\mathfrak{m}_1}$ is a valuation ring with valuation ideal $\mathfrak{m}_1$. Further, if v_1 is the corresponding valuation of K, then $\mathcal{O}_0 := \mathcal{O}_v/\mathfrak{m}_1$ is a valuation ring of Kv_1 with valuation ideal $\mathfrak{m}_0 := \mathfrak{m}_v/\mathfrak{m}_1$, say of a valuation v_0 of Kv_1. We say that v_1 is a coarsening of v, and denote $v_1 \leqslant v$ and $v_0 := v/v_1$.

Conversely, if v_1 is a valuation of K and v_0 is a valuation on the residue field Kv_1, then the preimage of the valuation ring $\mathcal{O}_{v_0} \subseteq Kv_1$ under $\mathcal{O}_{v_1} \to Kv_1$ is a valuation ring $\mathcal{O} \subseteq \mathcal{O}_{v_1}$ having as valuation ideal the preimage $\mathfrak{m} \subset \mathcal{O}$ of $\mathfrak{m}_{v_0}$. Hence if v is the valuation defined by $\mathcal{O}$ on K, then $Kv = (Kv_1)v_0$ and one has a canonical exact sequence of totally ordered groups:

$$0 \to v_0(Kv_1) \to vK \to v_1K \to 0.$$

The relation between coarsening and decomposition theory is as follows. Let $\tilde{K}|K$ be a Galois extension, and $\tilde{v}|v$ be a prolongation of v to $\tilde{K}$. Then the coarsenings $\tilde{v}_1$ of $\tilde{v}$ are in a canonical bijection with the coarsenings v_1 of v via $\mathcal{O}_{\tilde{v}_1} \mapsto \mathcal{O}_{v_1} := \mathcal{O}_{\tilde{v}_1} \cap K$, thus $\mathcal{O}_{\tilde{v}_1} = \mathcal{O}_{\tilde{v}} \cdot \mathcal{O}_{v_1}$. Let $\tilde{v}_1|v_1$ be given coarsenings of $\tilde{v}|v$ and $\tilde{K}\tilde{v}_1|Kv_1$ be the corresponding residue field extension. Then $\tilde{v}_0 := \tilde{v}/\tilde{v}_1$ is canonically a prolongation of $v_0 := v/v_1$.

FACT 1. *Let $T_{\tilde v} \subseteq Z_{\tilde v}$ and $T_{\tilde v_1} \subseteq Z_{\tilde v_1}$ be the corresponding inertia/decomposition groups, and set $G_{\tilde v_1} = \mathrm{Aut}(\tilde K\tilde v_1 \,|\, Kv_1)$. Then one has a canonical exact sequence $1 \to T_{\tilde v_1} \to Z_{\tilde v_1} \to G_{\tilde v_1} \to 1$, and the inertia/decomposition groups satisfy:*

a) *$Z_{\tilde v} \subseteq Z_{\tilde v_1}$ and $T_{\tilde v} \supseteq T_{\tilde v_1}$. Further, $T_{\tilde v_1}$ is a normal subgroup of $Z_{\tilde v}$.*

b) *Via $1 \to T_{\tilde v_1} \to Z_{\tilde v_1} \to G_{\tilde v_1} = Z_{\tilde v_1}/T_{\tilde v_1} \to 1$, one has that $T_{\tilde v_0} = T_{\tilde v}/T_{\tilde v_1}$ and $Z_{\tilde v_0} = Z_{\tilde v}/T_{\tilde v_1}$.*

B) *Hilbert decomposition in elementary abelian extensions*

Let K be a field of characteristic prime to p containing μ_n, where $n = p^e$ is a power of the prime number p, and let $\tilde K = K[\sqrt[n]{K}]$ be the maximal $\mathbb{Z}/n$ elementary abelian extension of K. Let v be a valuation of K, and $\tilde v$ some prolongation of v to $\tilde K$, and $V_{\tilde v} \subseteq T_{\tilde v} \subseteq Z_{\tilde v}$ be the ramification, the inertia, and the decomposition, groups of $\tilde v|v$, respectively. We remark that because $\mathrm{Gal}(\tilde K|K)$ is commutative, the groups $V_{\tilde v}$, $T_{\tilde v}$, and $Z_{\tilde v}$ depend on v only. Therefore we will simply denote them by V_v, T_v, and Z_v. Finally, we denote by $K^Z \subseteq K^T \subseteq K^V$ the corresponding fixed fields in $\tilde K$. One has the following, see e.g. POP [P2], Section 2 (where the case $n = p$ is dealt with; but the proof is similar for general $n = p^e$ and we will not reproduce the details here):

FACT 2. *In the above notations, the following hold:*

1) *Let $U^v := 1 + p^{2e}\mathfrak{m}_v$. Then $\sqrt[n]{U^v} \subset K^Z$, and $K^Z = K[\sqrt[n]{1+\mathfrak{m}_v}\,]$, provided $\mathrm{char}(Kv) \neq p$. In particular, if w_1 and w_2 are independent valuations of K, then $Z_{w_1} \cap Z_{w_2} = 1$.*

2) *If $p \neq \mathrm{char}(Kv)$, then $V_v = 1$ and $\tilde K\tilde v = \widetilde{Kv}$, hence $G_v := Z_v/T_v = \mathrm{Gal}(\widetilde{Kv}|Kv)$ in this case. And if $p = \mathrm{char}(Kv)$, then $V_v = T_v$, and the residue field $\tilde K\tilde v$ contains $(Kv)^{\frac{1}{n}}$ and a maximal $\mathbb{Z}/n$ elementary abelian extension of Kv.*

3) *Let $L := K^{\mathrm{h}}_v$ be the Henselization of K with respect to v. Then $\tilde L = L\tilde K$ is a maximal $\mathbb{Z}/n$ elementary extension of L. Therefore we have $\mathrm{Gal}(\tilde L|L) \cong Z_{\tilde v}$ canonically.*

C) *Formally p-adic fields and p-adic valuations*

We recall a few basic facts about p-adic valuations and (formally) p-adically closed fields, see AX–KOCHEN [A–K] and PRESTEL–ROQUETTE [P–R] for more details.

1) A valuation v of a field k is called (formally) p-adic, if its residue field kv is a finite field, say $\mathbb{F}_q$ with $q = p^{f_v}$ elements, and the value group vk has a minimal positive element 1_v such that $v(p) = e_v \cdot 1_v$ for some natural number $e_v > 0$. The number $d_v := e_v f_v$ is called the p-adic rank (or degree) of the p-adic valuation v. Note that a field k carrying a p-adic valuation v must necessarily have $\mathrm{char}(k) = 0$, as $v(p) \neq \infty$, and $\mathrm{char}(kv) = p$.

2) Let v be a p-adic valuation of k with valuation ring $\mathcal{O}_v$. Then $\mathcal{O}_1 := \mathcal{O}_v[1/p]$ is the valuation ring of the unique maximal proper coarsening v_1 of v, which is called the canonical coarsening of v. Note that setting $k_0 := kv_1$, and $v_0 := v/v_1$ the corresponding valuation on k_0 we have: v_0 is a p-adic valuation of k_0 with $e_{v_0} = e_v$ and $f_{v_0} = f_v$, hence $d_{v_0} = d_v$, and moreover, v_0 is a discrete valuation of k_0. In particular, the following hold:
   a) v has rank one iff v_1 is the trivial valuation iff $v = v_0$.
   b) Giving a p-adic valuation v of a field k of p-adic rank $d_v = e_v f_v$ is equivalent to giving a place $\mathfrak{p}$ of k with values in a finite extension $\boldsymbol{k}_0$ of $\mathbb{Q}_p$ such that the residues field $k_0 := k\mathfrak{p}$ of $\mathfrak{p}$ is dense in $\boldsymbol{k}_0$, and $\boldsymbol{k}_0|\mathbb{Q}_p$ has ramification index e_v and residual degree f_v.
   c) If $v_i < v$ is a strict coarsening of v, then $v_i \leqslant v_1$, and the quotient valuation v/v_i on the residue field kv_i is a p-adic valuation with $e_{v/v_i} = e_v$ and $f_{v/v_i} = f_v$, thus $d_{v/v_i} = d_v$. (Actually, $\kappa(v_i/v_1) \cong kv_1$ and $\kappa(v_i/v) \cong kv$ canonically.)

3) Let v be a p-adic valuation of k, and $l|k$ a finite field extension, and $w|v$ denote the prolongations of v to l. Then the following hold:

   a) All prolongations $w|v$ are p-adic valuations. Further, the *fundamental equality* holds for the finite extension $l|k$, i.e, $[l:k] = \sum_{w|v} e(w|v)f(w|v)$, where $e(w|v)$ and $f(w|v)$ are the ramification index, respectively the residual degree of $w|v$.

   b) For each $w|v$, let w_1 be the canonical coarsening of w, and $w_0 = w/w_1$ be the canonical quotient on the residue field lw_1. Then by general decomposition theory of valuations one has: $e(w|v) = e(w_1|v_1)e(w_0|v_0)$ and $f(w|v) = f(w_0|v_0)$. Further, $e_w = e(w_0|v_0)\,e_v$, and $f_w = f(w|v)f_v$, thus $d_w = e(w_0|v_0)f(w|v)\,d_v$.

   c) In particular, if $l|k$ is Galois, and w^Z is the restriction of w to the decomposition field l^Z of w, then $e(w|w^Z) = e(w|v)$ and $f(w|w^Z) = f(w|v)$, thus w^Z is a p-adic valuation having p-adic rank equal to the one of v. Further, the same is true for infinite Galois extensions $l|k$.

4) A field k is called (formally) p-adically closed, if k carries a p-adic valuation v such that for every finite extension $\tilde{k}|k$ one has: If v has a prolongation $\tilde{v}$ to $\tilde{k}$ with $d_{\tilde{v}} = d_v$, then $\tilde{k} = k$. One has the following characterization of the p-adically closed fields: For a field k endowed with a p-adic valuation v, and its canonical coarsening v_1, the following are equivalent:

   i) k is p-adically closed with respect to v.

   ii) v is Henselian, and $v_1 k$ is divisible (maybe trivial).

   iii) v_1 is Henselian, $v_1 k$ is divisible (maybe trivial), and the residue field $k_0 := kv_1$ is relatively algebraically closed in its $v_0 = v/v_1$ completion $\boldsymbol{k}_0$ (itself is a finite extension of $\mathbb{Q}_p$).

   Further, the p-adic valuation of a p-adically closed field is definable and unique.

5) Finally, for every field k endowed with a p-adic valuation v, there exist p-adic closures $\widehat{k}, \widehat{v}$ such that $d_{\widehat{v}} = d_v$. Moreover, the space of the k-isomorphy classes of p-adic closures of k, v has a concrete description as follows: Let v_1 be the canonical coarsening of v, and $\boldsymbol{k}_0|\mathbb{Q}_p$ the completion of the residue field of $k_0 = kv_1$ with respect to the discrete valuation $v_0 = v/v_1$. Recalling the canonical exact sequence $1 \to I_{v_1} \longrightarrow D_v \xrightarrow{pr} G_{\boldsymbol{k}_0} \to 1$, one has that the space of the isomorphy classes of p-adic closures of k, v is in bijection with the space of conjugacy classes of sections of pr, thus with $\mathrm{H}^1_{\mathrm{cont}}(G_{\boldsymbol{k}_0}, I_{v_1})$.

6) In the above notations, the following hold:

   a) Let k, v be a p-adically closed field. Then $k_0 = kv_1$ is p-adically closed (with respect to v_0), and k^{abs} is actually the relative algebraic closure of $\mathbb{Q}$ in k_0. Further, $\overline{k} = k\overline{\mathbb{Q}}$.

   b) The elementary equivalence class of a p-adically closed field k is determined by both: The *absolute subfield* $k^{\mathrm{abs}} := k \cap \overline{\mathbb{Q}} = k_0 \cap \overline{\mathbb{Q}}$ of k, and the *completion* $\boldsymbol{k}_0$ of $k_0 = kv_1$ with respect to v_0 (which equals the completion of k^{abs} with respect to v_0 as well).

   c) If N is p-adically closed with respect to the p-adic valuation w, and $k \subseteq N$ is a subfield which is relatively closed in N, then k is p-adically closed with respect to $v := w|_k$, and v and w have equal p-adic ranks, and N and k are elementary equivalent.

   d) If $N|k$ is an extension of p-adically closed fields of the same rank, the following hold:

      - $\tilde{k}|k \mapsto N\tilde{k}$ defines a bijection fom the algebraic extensions $\tilde{k}|k$ of k onto those of N.
      - The canonical projection $G_N \to G_k$ is an isomorphism.

   e) In particular, if $L|l$ is an extension of p-adically closed fields of the same rank, in the notations from the Introduction, the following canonical projections are isomorphisms:

$$(\dagger) \qquad pr'_L : \mathrm{Gal}(K'|L) \to \mathrm{Gal}(k'|l), \quad pr''_L : \mathrm{Gal}(K''|L) \to \mathrm{Gal}(k''|l).$$

D) *Valuations and rigid elements*

We recall the result Arason–Elman–Jacob [AEJ], Thm 2.16, see also Koenigsmann [Ko1], Ware [Wa], Efrat [Ef], and especially Topaz [To] for much more about this. The point here is that one can recover valuations of an arbitrary field K from particular subgroups $T \subset K^\times$ as follows: Let $T \subset K^\times$ be a subgroup with $-1 \in T$. We say that $x \in K^\times \backslash T$ is T-rigid, if $1 + x \in T \cup xT$; and by abuse of language, we say that K is T-rigid, if all $x \in K^\times \backslash T$ are T-rigid.

Theorem 3 (Arason-Elman-Jacob). *In the above notations, let $T \subset K^\times$ be a subgroup with $-1 \in T$ such that K is T-rigid. Then there exist a valuation v of K whose valuation ideal $\mathfrak{m}_v$ satisfies $1 + \mathfrak{m}_v \subseteq T$, and whose valuation ring $\mathcal{O}_v$ has the property that $\left|\mathcal{O}_v^\times/(T \cap \mathcal{O}_v^\times)\right| \leqslant 2$.*

3. Proof of Theorem B

To 1): Let $\widehat{K}, \widehat{w}$ be a p-adic closure of K, w. Then $\widehat{w}$ prolongs w and has p-adic rank $d_{\widehat{w}} = d_w$, thus equal to d_v by the fact that $d_w = d_v$. Therefore, since k is p-adically closed, k must be relatively algebraically closed in $\widehat{K}$. Conclude by applying relation (†) from Section 2, C), 6), e), with $l := k$ and $L := \widehat{K}$, and taking into account that the isomorphisms $\mathrm{Gal}(\widehat{K}''|\widehat{K}) \to \mathrm{Gal}(k''|k)$ factors through $\mathrm{Gal}(K''|K) \to \mathrm{Gal}(k''|k)$, thus gives rise to a liftable section of $\mathrm{Gal}(K'|K) \to \mathrm{Gal}(k'|k)$.

To 2): The proof of assertion 2) is divided in three main steps, whereas the hypothesis $p > 2$ is used only in Step 2). This might be relevant when trying to address the case $p = 2$.

Step 1. By Kummer theory, $pr'_K : \mathrm{Gal}(K'|K) \to \mathrm{Gal}(k'|k)$ is Pontrjagin dual to the canonical embedding $k^\times/p \to K^\times/p$. Second, given a liftable section $s' : \mathrm{Gal}(k'|k) \to \mathrm{Gal}(K'|K)$ of pr'_K, it follows by Kummer theory that the Pontrjagin dual of $s' : \mathrm{Gal}(k'|k) \to \mathrm{Gal}(K'|K)$ is a surjective projection $K^\times/p \to k^\times/p$, whose kernel $\Sigma/p \subset K^\times/p$ is a complement of $k^\times/p \subset K^\times/p$. That means, s' gives rise canonically to a presentation of $K^\times/p$ as a direct sum

$$(\dagger) \qquad K^\times/p = \Sigma/p \cdot k^\times/p\,.$$

For every k-subfield $K_\alpha \subset K$ which is relatively algebraically closed in K, one has a commutative diagram of surjective projections

$$\begin{array}{ccccc} \mathrm{Gal}(K''|K) & \longrightarrow & \mathrm{Gal}(K''_\alpha|K_\alpha) & \longrightarrow & \mathrm{Gal}(k''|k) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Gal}(K'|K) & \longrightarrow & \mathrm{Gal}(K'_\alpha|K_\alpha) & \longrightarrow & \mathrm{Gal}(k'|k) \end{array}$$

and s' gives rise canonically to a liftable section s'_α of $pr'_\alpha : \mathrm{Gal}(K'_\alpha|K_\alpha) \to \mathrm{Gal}(k'|k)$, etc. In particular, one has corresponding canonical presentations as direct sums

$$(\dagger)_\alpha \qquad K_\alpha^\times/p = \Sigma_\alpha/p \cdot k^\times/p$$

defined by the sections $s'_\alpha : \mathrm{Gal}(k'|k) \to \mathrm{Gal}(K'_\alpha|K_\alpha)$ induced by $s' : \mathrm{Gal}(k'|k) \to \mathrm{Gal}(K'|K)$.

Claim. *In the above notions one has: $\Sigma_\alpha/p = \Sigma/p \cap K_\alpha^\times/p$, thus Σ/p determines Σ_α/p.*

Indeed, let $D_\alpha := \mathrm{im}(s'_\alpha) \subset \mathrm{Gal}(K'_\alpha|K_\alpha)$. Then by the definition of s'_α, it follows that D_α is the image of $D = \mathrm{im}(s')$ under the canonical projection $\mathrm{Gal}(K'|K) \to \mathrm{Gal}(K'_\alpha|K_\alpha)$. In other words, by Pontrjagin duality, the projection $K_\alpha^\times/p \to k^\times/p$ factors through the inclusion $K_\alpha^\times/p \hookrightarrow K^\times/p$. Hence Σ_α/p is mapped into Σ/p under $K_\alpha^\times/p \hookrightarrow K^\times/p$, which proves th Claim.

Now let $T/p := \Sigma/p \cdot \mathcal{O}_v^\times/p$ and $T \subset K^\times$ be the corresponding subgroup (thus containing the p^{th} powers in $K^\times$). Then for every k-subfield $K_\alpha \subset K$ which is relatively algebraically closed in K, by the remarks above one has that $T_\alpha := T \cap K_\alpha^\times \subset K_\alpha^\times$ satisfies $T_\alpha/p = \Sigma_\alpha/p \cdot \mathcal{O}_v^\times/p$.

Finally, let $(K_\alpha)_\alpha$ be the family of all the k-subfields $K_\alpha \subset K$ which are relatively algebraically closed in K and satisfy $\text{tr.deg}(K_\alpha|k) = 1$. Then by Theorem B of Pop [P2], for every subfield K_α, there exits a unique p-adic valuation w_α of K_α prolonging the p-adic valuation v of k to K_α and having the same p-adic rank as v. Our final aim is to show that there exists a (unique) p-adic valuation w of K such that w_α is the restriction of w to K_α for each K_α.

Lemma 4. *In the above notations, K_α is T_α-rigid. Further, $T = \cup_\alpha T_\alpha$, and K is T-rigid.*

Proof. We first show that $\mathcal{O}_{w_\alpha} \subset T_\alpha$. Indeed, let v be the p-adic valuation of k, and further consider: First, the canonical coarsening v_1 of v, and the canonical p-adic valuation $v_0 := v/v_1$ on the residue field $k_0 := kv_1$ of v_1. Second, consider the p-adic valuation w_α of K_α, and let $w_{\alpha 1}$ and $w_{\alpha 0} := w_\alpha/w_{\alpha 1}$, and $K_{\alpha 0}$ be correspondingly defined. Notice that $w_\alpha|_k = v$ implies that $w_{\alpha 1}|_k = v_1$ and $w_{\alpha 0}|_{k_0} = v_0$. The following hold:

a) First, by Fact 2, it follows that $\sqrt[p]{1 + p^2\mathfrak{m}_{w_\alpha}}$ is contained in the decomposition field of w_α over K, which is actually the fixed field of Z_{w_α} in K'_α. Second, the fixed field of $\text{im}(s')$ in K'_α is, by the mere definitions, generated as a field extension of K by $\sqrt[p]{\Sigma_\alpha}$. Thus since $\text{im}(s') \subset Z_{w_\alpha}$, it follows by Kummer theory that $1 + p^2\mathfrak{m}_{w_\alpha} \subset \Sigma_\alpha$.

b) Since, by the mere definition, one has $\mathfrak{m}_{w_{\alpha 1}} \subset \mathfrak{m}_{w_\alpha}$ and p is invertible in $\mathcal{O}_{w_{\alpha 1}}$, it follows that $1 + \mathfrak{m}_{w_{\alpha 1}} \subset 1 + p^2\mathfrak{m}_{w_\alpha}$. Thus, one has finally $1 + \mathfrak{m}_{w_{\alpha 1}} \subset \Sigma_\alpha$ as well.

c) Since w_α and v have the same p-adic rank, it follows by the discussion in section 2, C, 5), that $w_{\alpha 0}$ and v_0 are discrete p-adic valuations of the same p-adic rank, hence k_0 is dense in $K_{\alpha 0}$. Therefore, since $w_{\alpha 0}|v_0$ are discrete valuations, and k_0 is dense in $K_{\alpha 0}$ under $k_0 \hookrightarrow K_{\alpha 0}$, one has that $\mathcal{O}_{w_{\alpha 0}}^\times = \mathcal{O}_{v_0}^\times \cdot (1 + p^2\mathfrak{m}_{w_{\alpha 0}})$ and $K_{\alpha 0}^\times = k_0^\times \cdot (1 + p^2\mathfrak{m}_{w_{\alpha 0}})$ as well.

d) Since $K_{\alpha 0}^\times = \mathcal{O}_{w_{\alpha 1}}^\times/(1+\mathfrak{m}_{w_{\alpha 1}})$, $k_0^\times = \mathcal{O}_{v_1}^\times/(1+\mathfrak{m}_{v_1})$, and $1+p^2\mathfrak{m}_{w_{\alpha 0}} = (1+p^2\mathfrak{m}_{w_\alpha})/(1+\mathfrak{m}_{w_{\alpha 1}})$, from the equality $K_{\alpha 0}^\times = k_0^\times \cdot (1 + p^2\mathfrak{m}_{w_{\alpha 0}})$ above, it follows that $\mathcal{O}_{w_{\alpha 1}}^\times = \mathcal{O}_{v_1}^\times \cdot (1 + p^2\mathfrak{m}_{w_\alpha})$.

e) Similarly, the equalities $\mathcal{O}_{w_{\alpha 0}}^\times = \mathcal{O}_{w_\alpha}^\times/(1 + \mathfrak{m}_{w_{\alpha 1}})$ and $\mathcal{O}_{w_{\alpha 0}}^\times = \mathcal{O}_{v_0}^\times \cdot (1 + p^2\mathfrak{m}_{w_{\alpha 0}})$, imply that $\mathcal{O}_{w_\alpha}^\times = \mathcal{O}_v^\times \cdot (1 + p^2\mathfrak{m}_{w_\alpha})$.

Hence since $\mathcal{O}_v^\times, 1 + p^2\mathfrak{m}_{w_\alpha} \subset T_\alpha$, one finally has $\mathcal{O}_{w_\alpha}^\times = \mathcal{O}_v^\times \cdot (1 + p^2\mathfrak{m}_{w_\alpha}) \subset T_\alpha$, as claimed.

We next show that K_α is T_α-rigid. To do so, we first notice that by the discussion above, for any fixed element $\pi \in \mathcal{O}_v$ of minimal positive value $1_v \in vk$, the following holds: Let $x \in \mathcal{O}_{w_{\alpha 1}}^\times$ be an arbitrary $w_{\alpha 1}$-unit. Then there exist $m \in \mathbb{Z}$, $\epsilon \in \mathcal{O}_v^\times$, $x_1 \in 1 + p^2\mathfrak{m}_{w_\alpha}$ such that

$$(\sharp) \qquad x = \pi^m \epsilon\, x_1.$$

Now let $x \in K_\alpha^\times \backslash T_\alpha$ be given. Then one has the following possibilities:

1) $w_{\alpha 1}(x) > 0$. Then $1 + x$ is a principal $w_{\alpha 1}$-unit, and therefore, $1 + x \in \Sigma_\alpha$ by assertion b) above. Since $\Sigma_\alpha \subset T_\alpha$, we conclude that $1 + x \in T_\alpha$.

2) $w_{\alpha 1}(x) < 0$. Then $1 + x = x(1 + x^{-1})$. Since $w_{\alpha 1}(x^{-1}) > 0$, by the discussion above, it follows that $1 + x^{-1} \in T_\alpha$. Therefore, one finally has that $1 + x \in xT_\alpha$.

3) $w_{\alpha 1}(x) = 0$, or equivalently, $x \in \mathcal{O}_{w_{\alpha 1}}^\times$. Let $x = \pi^m \epsilon\, x_1$ be as given at $(\sharp)$ above. One has:

α) If $m > 0$, then $x \in \pi^m \cdot \mathcal{O}_{w_\alpha}^\times$, thus $1 + x \in \mathcal{O}_{w_\alpha}^\times$ as well. Hence by the relation ($\sharp$) above, $1 + x = \eta_1 \cdot \eta_0$ for some $\eta_1 \in 1 + p^2\mathfrak{m}_{w_\alpha} \subset \Sigma_\alpha$, $\eta_0 \in \mathcal{O}_v^\times$. Thus finally, $1 + x \in T_\alpha$.

β) If $m < 0$, then $1 + x = x(1 + x^{-1})$, and x^{-1} has value $-m > 0$. But then, by the first case above, $1 + x^{-1} \in T_\alpha$. Hence $1 + x = x(1 + x^{-1}) \in xT_\alpha$, thus $1 + x \in x\,T_\alpha$.

γ) If $m = 0$, then $x \in \mathcal{O}_{w_\alpha}^\times \subset T_\alpha$, thus $x \notin K_\alpha^\times \backslash T_\alpha$.

For the T-rigidity of K, let $x \in K\backslash T$ be given. If $x \in k$, then $x \in k\backslash\mathcal{O}_v^\times$ (by the definition of T). An easy case by case analysis, namely $v(x) > 0$ or $v(x) < 0$, shows that $1 + x \in \mathcal{O}_v^\times \cup x\,\mathcal{O}_v^\times$, etc. Finally, if $x \notin k$, then letting $K_\alpha \subset K$ be the relative algebraic closure of $k(x)$ in K, one has: Since $x \in K\backslash T$, one must have $x \in K_\alpha\backslash T_\alpha$. Thus by the discussion above, it follows that $1 + x \in T_\alpha \cup x\,T_\alpha$, and therefore, $1 + x \in T \cup x\,T$, etc.

This concludes the proof of Lemma 4. □

Step 2. Using Lemma 4 above and applying the Arason–Elman–Jacob Theorem 3, we get: There exists a valuation w on K such that $|\mathcal{O}_w^\times/(\mathcal{O}_w \cap T)| \leqslant 2$ and $1 + \mathfrak{m}_w \subset T$. Hence letting $\mathcal{O}_w^\times T \subset K^\times$ be the subgroup generated by T and $\mathcal{O}_w$, one has $\mathcal{O}_w/(\mathcal{O}_w \cap T) = (\mathcal{O}_w^\times T)/T$, thus $|(\mathcal{O}_w^\times T)/T| \leqslant 2$. We claim that one actually has $\mathcal{O}_w^\times \subset T$. Indeed, first, one has $k^\times = \mathcal{O}_v^\times \cdot \pi^{\mathbb{Z}}$ as direct sum, hence $(k^\times/p)/(\mathcal{O}_v^\times/p) = \pi^{\mathbb{Z}/p}$. Second, by definitions one has that $K^\times/p = \Sigma/p \cdot k^\times/p$ and $T/p = \Sigma/p \cdot \mathcal{O}_v^\times/p$, both of which being direct sums. Thus finally one gets that

$$K^\times/p = \Sigma/p \cdot k^\times/p = \Sigma/p \cdot \mathcal{O}_v^\times/p \cdot \pi^{\mathbb{Z}/p} = T/p \cdot \pi^{\mathbb{Z}/p}$$

where the dot denotes direct sums, in particular, one has $|K^\times/T| = |(K^\times/p)/(T/p)| = p$. Hence considering the canonical inclusions of groups $T \subseteq \mathcal{O}_w^\times T \subseteq K^\times$, we get

$$p = |K^\times/T| = |K^\times/(\mathcal{O}_w^\times T)| \cdot |(\mathcal{O}_w^\times T)/T|\,.$$

Since $|(\mathcal{O}_w^\times T)/T| \leqslant 2$ and $2 < p$, it follows that $|(\mathcal{O}_w^\times T)/T| = 1$ is the only possibility, hence $T = \mathcal{O}_w^\times T$, and finally, $\mathcal{O}_w^\times \subseteq T$. Hence we conclude that $|K^\times/\mathcal{O}_w^\times| \geqslant p$, and therefore:

- The valuation w is a *non-trivial* valuation of K.

Step 3. Recalling that $\mathcal{O}_w^\times \subset T$, one has that the canonical projection $K^\times/\mathcal{O}_w^\times \to K^\times/T$ is surjective. Therefore, if $b \in K$ is a generator of $K^\times/T$, e.g., $b = \pi \in k_0$ has $v_0(\pi) = 1$, then b is not a w-unit and $w(b)$ is not divisible by p in $wK = K^\times/\mathcal{O}_w^\times$, hence wK is not divisible by p.

For every subfield $K_\alpha \subset K$ as in the proof of Lemma 4, let $v_\alpha := w|_{K_\alpha}$ be the restriction of w to K_α. Then $\mathcal{O}_{v_\alpha} = \mathcal{O}_w \cap K_\alpha$, and therefore, $\mathcal{O}_{v_\alpha}^\times$ is contained in $T_\alpha = T \cap K_\alpha$.

Lemma 5. *The restriction $v_\alpha := w|_{K_\alpha}$ of w to K_α equals the p-adic valuation w_α.*

Proof. By the first part of the proof of Lemma 4, we have that $\mathcal{O}_{w_\alpha}^\times \subset T_\alpha$. Since $\mathcal{O}_{v_\alpha}^\times \subseteq T_\alpha$ as well, it follows that the element wise product $\mathcal{O}_{w_\alpha}^\times\mathcal{O}_{v_\alpha}^\times$ is contained in T_α. Since T_α is a proper subgroup of $K_\alpha^\times$ it follows that that $\mathcal{O}_{w_\alpha}^\times\mathcal{O}_{v_\alpha}^\times \neq K^\times$ as well. The following is well known valuation theoretical non-sense: Let $\mathfrak{n}$ be the largest common ideal of $\mathcal{O}_{v_\alpha}$ and $\mathcal{O}_{w_\alpha}$. Then $\mathcal{O} := \mathcal{O}_{v_\alpha}\mathcal{O}_{w_\alpha}$ equals both the localization of $\mathcal{O}_{v_\alpha}$ at $\mathfrak{n}$ and the localization $\mathcal{O}_{w_\alpha}$ at $\mathfrak{n}$. Further, $\mathcal{O}$ is the smallest valuation ring of K which contains both $\mathcal{O}_{v_\alpha}$ and $\mathcal{O}_{w_\alpha}$; or equivalently, $\mathcal{O}$ is the valuation ring of the finest common coarsening of v_α and w_α. We now claim that one has:

$$\mathcal{O}^\times = \mathcal{O}_{v_\alpha}^\times\,\mathcal{O}_{w_\alpha}^\times.$$

Indeed, let v_α^1 and w_α^1 be the valuations of $\kappa(\mathfrak{n}) := \mathcal{O}/\mathfrak{n}$ defined by $\mathcal{O}_{w_\alpha}/\mathfrak{n}$, respectively $\mathcal{O}_{v_\alpha}/\mathfrak{n}$.

Then v_α^1 and w_α^1 are independent, and one has exact sequences

$$1 \to (1+\mathfrak{n}) \to \mathcal{O}_{v_\alpha}^\times \to \mathcal{O}_{v_\alpha^1}^\times \to 1 \quad \text{and} \quad 1 \to (1+\mathfrak{n}) \to \mathcal{O}_{w_\alpha}^\times \to \mathcal{O}_{w_\alpha^1}^\times \to 1.$$

Since v_α^1 and w_α^1 are independent valuations of $\kappa(\mathfrak{n})$, one has that $\mathcal{O}_{v_\alpha^1}^\times \mathcal{O}_{w_\alpha^1}^\times = \kappa(\mathfrak{n})^\times$, and therefore

$$\big(\mathcal{O}_{v_\alpha}^\times \mathcal{O}_{w_\alpha}^\times\big)/(1+\mathfrak{n}) = \kappa(\mathfrak{n})^\times.$$

On the other hand, one also has $\mathcal{O}^\times/(1+\mathfrak{n}) = \kappa(\mathfrak{n})^\times$. Further, $1+\mathfrak{n}$ is contained in both $\mathcal{O}_{v_\alpha}^\times$ and $\mathcal{O}_{w_\alpha}^\times$, hence we conclude that $\mathcal{O}_{v_\alpha}^\times \mathcal{O}_{w_\alpha}^\times = \mathcal{O}^\times$, as claimed.

By contradiction, suppose that $\mathcal{O}_{v_\alpha} \neq \mathcal{O}_{w_\alpha}$. Recall that the valuation ring $\mathcal{O}_{w_\alpha}$ has finite residue field, hence $\mathcal{O}_{w_\alpha}$ is minimal among the valuation rings of K_α, and in particular, $\mathcal{O}_{v_\alpha}$ cannot be contained in $\mathcal{O}_{w_\alpha}$. Therefore, in the above notations, one has that $\mathcal{O}_{w_\alpha} \subset \mathcal{O}$ strictly, or equivalently, $\mathfrak{n} \subset \mathfrak{m}_{w_\alpha}$ is a strict inclusion. On the other hand, if $b \in k$ is any element of minimal positive value 1_v, then $\mathfrak{m}_{w_\alpha} = b\,\mathcal{O}_{w_\alpha}$, and therefore, $b \notin \mathfrak{n}$. Thus we have

$$b \in \mathcal{O}^\times = \mathcal{O}_{v_\alpha}^\times \mathcal{O}_{w_\alpha}^\times \subseteq T_\alpha,$$

contradicting the fact that $w(b)$ generates $wK/w(T) \cong \mathbb{Z}/p$. Thus we conclude that one must have $\mathcal{O}_{w_\alpha} = \mathcal{O}_{v_\alpha}$, and Lemma 5 is proved. □

We next claim that w is a p-adic valuation of K having p-adic rank $d_w = d_v$. Indeed, for $t \in \mathcal{O}_w$, let $K_\alpha \subset K$ be the relative algebraic closure of $k(t)$ in K. Then $K_\alpha|k$ has transcendence degree $\leqslant 1$, and therefore, $w|_{K_\alpha} = w_\alpha$ is the p-adic valuation w_α by Lemma 5. In particular, if $b \in k$ is such that $v(b) = 1_v$ is the minimal positive element of $v(k^\times)$, it follows that $w_\alpha(b)$ is the minimal positive element of $w_\alpha K_\alpha$ under $vk \hookrightarrow w_\alpha K_\alpha$, and further, $kv = K_\alpha w_\alpha$ is the finite field of cardinality $f_v = f_{w_\alpha}$. One has the following:

a) $w(b)$ is the minimal positive element of $w(K^\times)$. Indeed, for $t \in \mathfrak{m}_w$, in the above notations one has: $w(t) = w_\alpha(t) \geqslant w_\alpha(b) = w(b)$.
b) $kv = Kw$ thus $f_v = f_{w_\alpha}$. Indeed, if $t \in \mathcal{O}_w$, then in the above notations, the residue $\bar{t} \in Kw$ satisfies: $\bar{t} \in K_\alpha w_\alpha = kv$.

Therefore, w is a p-adic valuation of rank $d_w = d_v$, which is unique, by the uniqueness of $w_\alpha = w|_{K_\alpha}$ for every subfield K_α. This concludes the proof of Theorem B.

4. Proof of the other announced results

A) *Proof of Theorem A*

The following stronger assertion holds (from which Theorem A immediately follows):

Theorem 6. *Let $k|\mathbb{Q}_p$ be a finite extension containing the p^{th} roots of unity, and let $k_0 \subseteq k$ be a subfield which is relatively algebraically closed in k. Let X_0 be a complete smooth k_0-variety, and $K_0 = k_0(X)$ be the function field of X_0. The following hold:*

1) *Every k-rational point $x \in X_0$ gives rise to a bouquet of conjugacy classes of liftable sections s'_x of $\mathrm{Gal}(K'_0|K_0) \to \mathrm{Gal}(k'_0|k_0)$ above x.*
2) *Suppose that $p > 2$, and let s' be a liftable section of $\mathrm{Gal}(K'_0|K_0) \to \mathrm{Gal}(k'_0|k_0)$. Then there exists a unique k-rational point $x \in X_0$ such that s' equals one of the sections s'_x above.*

Proof. The proof is very similar to the proof of Theorem A of Pop [P2]. We repeat here the arguments briefly for reader's sake.

To 1): Let v be the valuation of k. We notice that by Section 2, C), b), there exists a bijection from the set of (equivalence classes of) p-adic valuations w of $K_0 = \kappa(X_0)$ with $d_w = d_v$ onto the set of bouques of liftable sections above k-rational points x of X_0, which sends each w to the corresponding bouquet of liftable sections above the center x of the canonical coarsening w_1 on $X = X_0 \times_{k_0} k$. Conclude by applying assertion 1) of Theorem B.

To 2): Since $k_0 \subseteq k$ is relatively algebraically closed, it follows that k_0 is p-adically closed. Let v be the valuation of k and of all subfields of k. Since k_0 is p-adically closed, we can apply Theorem B and get: For every liftable section s' of $\mathrm{Gal}(K_0'|K_0) \to \mathrm{Gal}(k_0'|k_0)$, there exists a unique p-adic valuation w of K_0 which prolongs v to K_0 and has p-adic rank equal to the p-adic rank of v, such that s' is a section above w. Let w_1 be the canonical coarsening of w. Then we have:

Case 1. The valuation w_1 is trivial.

Then w is a discrete p-adic valuation of K prolonging v to K, having the same residue field and the same value group as v. Equivalently, the completions of k_0 and K_0 are equal, hence equal to k. Therefore, w is uniquely determined by the embedding $\imath_w : (K_0, w) \hookrightarrow (k, v)$. In geometric terms, $\imath_w$ defines a k-rational point x of X_0, etc.

Case 2. The valuation w_1 is not trivial.

Then w_1 is a k_0-rational place of K_0, hence defines a k_0-rational point x_0 of X_0; hence by base change, a k-rational point x of X_0 as well, etc. □

B) *Proof of Theorem B^0*

The proof is almost identical with the one of Theorem B^0 from Pop [P2]. The proof of assertion 1) is identical with the proof of assertion 1) of Theorem B, thus we omit it. Concerning assertion 2), let $s'_L : \mathrm{Gal}(k'|l) \to \mathrm{Gal}(K'|L)$ be a liftable section of $pr'_L : \mathrm{Gal}(K'|L) \to \mathrm{Gal}(k'|l)$. Then considering the restriction

$$s' := pr'_L|_{\mathrm{Gal}(k'|k)} : \mathrm{Gal}(k'|k) \to \mathrm{Gal}(K'|K),$$

it follows by mere definitions that s' is a liftable section of $pr'_K : \mathrm{Gal}(K'|K) \to \mathrm{Gal}(k'|k)$. Hence by Theorem B, there exists a unique p-adic valuation w^1 of K which prolongs the p-adic valuation v_k of k to K and has $d_{w^1} = d_{v_k}$, and $s' = s_{w^1}$ in the usual way.

Let $w = w^1|_L$ be the restriction of w^1 to L. Then w prolongs the valuation v of l to L. We claim that w^1 is the unique prolongation of w to K. Indeed, let $w^2 := w^1 \circ \sigma_0$ with $\sigma_0 \in \mathrm{Gal}(k|l)$, be a further prolongation of w to K. Then if $(w^i)'$ is a prolongation of w^i to K', $i = 1, 2$, and $\sigma \in \mathrm{im}(s'_L)$ is a preimage of σ_0, then $(w^2)' := (w^1)' \circ \sigma$ is a prolongation of w^2 to K'. Therefore, if $Z_{w^1} \subset \mathrm{Gal}(K'|K)$ is the decomposition group above w^1, then $Z_{w^2} := \sigma Z_{w^1} \sigma^{-1}$ is the decomposition group above w^2. On the other hand, $\mathrm{im}(s') \subseteq Z_{w^1}$ by Theorem B. Since $\mathrm{Gal}(k'|k)$ is a normal subgroup of $\mathrm{Gal}(k'|l)$, it follows that $\mathrm{im}(s')$ is normal in $\mathrm{im}(s'_L)$. Hence if $\sigma \in \mathrm{im}(s'_L)$, it follows that $\sigma\big(\mathrm{im}(s')\big)\sigma^{-1} = \mathrm{im}(s')$, and therefore, one has:

$$Z_{w^1} \supseteq \mathrm{im}(s') = \sigma\big(\mathrm{im}(s')\big)\sigma^{-1} \subseteq \sigma Z_{w^1} \sigma^{-1} = Z_{w^2}$$

Hence $\mathrm{im}(s') \subset Z_{w^1}, Z_{w^2}$, thus by the uniqueness assertion of Theorem B, we must have $w^1 = w^2$. Equivalently, if $\sigma \in \mathrm{im}(s'_L)$, then $\sigma Z_{w^1} \sigma^{-1} = Z_{w^1}$, and therefore $\sigma \in Z_{w^1}$. Finally conclude that $d_w = d_v$, as claimed, and this concludes the proof of Theorem B^0.

C) *Proof of Theorem A^0*

The following stronger assertion holds (from which Theorem A^0 follows immediately):

Theorem 7. *Let $l|\mathbb{Q}_p$ be a finite extension. Let $l_0 \subset l$ be a relatively algebraically closed subfield, and $k_0|l_0$ a finite Galois extension with $\mu_p \subset k_0$. Let Y_0 be a complete smooth geometrically integral variety over l_0. Let $L_0 = \kappa(Y_0)$ the function field of Y_0, and $K_0 = L_0k_0$.*

1) *Every l-rational point $y \in Y_0$ gives rise to a bouquet of conjugacy classes of liftable sections s'_y of $\mathrm{Gal}(K'_0|L_0) \to \mathrm{Gal}(k'_0|l_0)$ above y.*
2) *Let $p > 2$, and $s' : \mathrm{Gal}(k'_0|l_0) \to \mathrm{Gal}(K'_0|L_0)$ be a liftable section of $\mathrm{Gal}(K'_0|L_0) \to \mathrm{Gal}(k'_0|l_0)$. Then there exists a unique l-rational point $y \in Y_0(l)$ such that s' equals one of the sections s'_y introduced at point 1) above.*

Proof. The proof is identical with the proof of Theorem 4.1 above, with the only difference that one uses Theorem B^0, instead of Theorem B. □

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