Arithmetic in the fundamental group of a p-adic curve

— On the p-adic section conjecture for curves —

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Abstract — We establish a valuative version of Grothendieck's section conjecture for curves over p-adic local fields. The image of every section is contained in the decomposition subgroup of a valuation which prolongs the p-adic valuation to the function field of the curve.

1. Introduction

This note addresses the arithmetic of rational points on curves over p-adic fields with ramification theory of general valuations and the étale fundamental group as the principal tools.

1.1. **The fundamental group.** Let $\bar{k}$ be a fixed separable closure of an arbitrary field k, and let $\mathrm{Gal}_k := \mathrm{Gal}(\bar{k}|k)$ be the absolute Galois group of k.

Let X/k be a geometrically connected variety, and let $\overline{X} := X \times_k \bar{k}$ be the base change of X to $\bar{k}$. The étale fundamental group $\pi_1(X, \bar{x})$ with base point $\bar{x}$ is an extension

$$1 \to \pi_1(\overline{X}, \bar{x}) \to \pi_1(X, \bar{x}) \to \mathrm{Gal}_k \to 1, \tag{1.1}$$

where $\pi_1(\overline{X}, \bar{x})$ is the geometric fundamental group of X with base point $\bar{x}$. In the sequel, we denote the extension (1.1) by $\pi_1(X/k)$ and ignore the base points $\bar{x}$, because they will be irrelevant for our discussion.

1.2. **The conjecture.** To a rational point $a \in X(k)$ the functoriality of π_1 gives rise to a section $s_a : \mathrm{Gal}_k \to \pi_1(X)$ of $\pi_1(X/k)$. The functor π_1 depends a priori on a pointed space but yields a well defined $\pi_1(\overline{X})$-conjugacy class $[s_a]$ of sections. The section conjecture of Grothendieck gives a conjectural description of the set of all the sections in an arithmetic situation as follows.

Conjecture 1 (see Grothendieck [Gr83])**.** *Let X be a smooth, projective and geometrically connected curve of genus ≥ 2 over a number field k. Then the map $a \mapsto [s_a]$ is a bijection from the set of rational points $X(k)$ onto the set of $\pi_1(\overline{X})$-conjugacy classes of sections of $\pi_1(X/k)$.*

Actually, Grothendieck originally made a more general conjecture allowing k to be a finitely generated extension of $\mathbb{Q}$. Moreover, Grothendieck noticed that $a \mapsto [s_a]$ is injective if k is a number field as a consequence of the Mordell-Weil Theorem, see [Sx08] Appendix B for details.

The focus of the present paper is the following local version of Grothendieck's section conjecture.

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Conjecture 2 (p-adic version of the section conjecture)**.** *Let $k/\mathbb{Q}_p$ be a finite extension, and let X/k be a smooth, projective and geometrically connected curve of genus ≥ 2. Then the map $a \mapsto [s_a]$ is a bijection from the set of rational points $X(k)$ onto the set of $\pi_1(\overline{X})$-conjugacy classes of sections of $\pi_1(X/k)$.*

To prove the section conjecture for curves over a field k as above, it suffices to show that for all finite étale geometrically connected covers $X' \to X$, if $\pi_1(X'/k)$ has a section, then $X'(k)$ is non-empty. This follows by a well known limit argument used already in the work of Neukirch, and was introduced to anabelian geometry by Nakamura, while Tamagawa [Ta97] Prop 2.8 emphasized its significance to the section conjecture, see [Ko05] Lem 1.7 or [Sx10] Appendix C.

1.3. **Evidence for the section conjecture.** The first known examples of curves over number fields that satisfy the section conjecture were probably given in [Sx10] and later [HS09], and also [Sx09]. More recently, Hain [Ha10] succeeds to verify the section conjecture for the generic curve of genus $g \geq 5$. These are nevertheless *no sections examples* in the sense that there are no sections of $\pi_1(X/k)$ and hence no rational points. But as we mentioned above, the ostensibly dull case of curves with neither sections nor points is exactly the crucial class of examples.

The p-adic version of the section conjecture has recently moved into the focus of several investigations, as pieces of evidence for the p-adic section conjecture emerged in recent years. The most convincing piece consists perhaps in Koenigsmann's [Ko05] proof of a birational analogue of the p-adic section conjecture for curves, see also Pop [P10] for a $\mathbb{Z}/p\mathbb{Z}$-metabelian form of the birational p-adic section conjecture.

On the other hand, the birational world seems to be quite different from the world of curves, because Hoshi [Ho10] showed that a geometrically pro-p version of the section conjecture over p-adic fields or number fields does not hold.

1.4. **The valuative section conjecture.** Before announcing the main result, let us give a valuation theoretic perspective of the section conjectures above.

First, recall that for every complete normal curve X/k, its closed points $a \in X$ are in bijection with the set $\mathrm{Val}_k(K)$ of equivalence classes of non-trivial k-valuations w_a of the function field $K = k(X)$ of X in such a way that the residue field $\kappa(w_a)$ of the valuation w_a associated to $a \in X$ equals $\kappa(a)$. Precisely, the local ring $\mathcal{O}_{X,a}$ at a closed point $a \in X$ is a discrete k-valuation ring of K with valuation $w_a \in \mathrm{Val}_k(K)$. Conversely, if w is a non-trivial k-valuation of K, then by the valuative criterion of properness, the valuation ring R_w of w dominates the local ring $\mathcal{O}_{X,a}$ of a unique point $a \in X$. Since $R_w \neq K$, we have $\mathcal{O}_{X,a} \neq K$ and $a \in X$ is a closed point. Hence $R_w = \mathcal{O}_{X,a}$, and R_w is a discrete k-valuation ring of K.

Let $\tilde{K} = k(\tilde{X})$ be the function field of the univeral pro-étale cover $\tilde{X}$ of X. The extension $\tilde{K}/K$ is Galois with Galois group identified with $\pi_1(X)$. For every k-valuation w on K and every prolongation $\tilde{w}$ to $\tilde{K}$ we denote by $\mathrm{D}_{\tilde{w}}$ the decomposition subgroup of $\tilde{w}$ in $\pi_1(X)$. If $w = w_a$ with $a \in X(k)$ then the projection $\mathrm{D}_{\tilde{w}_a} \to \mathrm{Gal}_k$ is an isomorphism. Hence its inverse gives rise to a section of $\pi_1(X/k)$, the $\pi_1(\overline{X})$-conjugacy class of which agrees with $[s_a]$.

Conjecture 3 ($\mathrm{Val}_k(K)$ section conjecture)**.** *Let k be a number field or a finite extension of $\mathbb{Q}_p$. Then in the above notations, for every section $s : \mathrm{Gal}_k \to \pi_1(X)$ of $\pi_1(X/k)$ there exists a valuation $w \in \mathrm{Val}_k(K)$ with $\kappa(w) = k$ and a prolongation $\tilde{w}$ of w to $\tilde{K}$ with $s(\mathrm{Gal}_k) \subseteq \mathrm{D}_{\tilde{w}}$.*

1.5. **The main result.** Let k be a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathfrak{o} \subset k$ and p-adic valuation v. The following richer birational geometric picture unfolds, see Section 2 for more details and precise references.

• *Geometry*: For a smooth, projective, geometrically connected curve X/k we consider the set of all its proper flat normal models $\mathscr{X}_i \to \mathrm{Spec}(\mathfrak{o})$. The set $\{\mathscr{X}_i\}_i$ is partially ordered with respect to the domination relation (inducing the identity in X/k). We consider $\varprojlim \mathscr{X}_i$ as an abstract set. There is a canonical identification

$$\mathrm{Val}_{\mathfrak{o}}(K) = \varprojlim \mathscr{X}_i$$

where $\mathrm{Val}_{\mathfrak{o}}(K)$ is the subspace of the Riemann–Zariski space of the field K consisting of the $\mathfrak{o}$-valuations, i.e., valuations w whose valuation ring R_w satisfies $\mathfrak{o} \subseteq R_w$. Indeed, for $(x_i) \in \varprojlim \mathscr{X}_i$ the ring $R = \varinjlim \mathcal{O}_{\mathscr{X}_i, x_i}$ is a valuation ring of K that contains $\mathfrak{o}$, because the x_i lie on models over $\mathrm{Spec}(\mathfrak{o})$. Conversely, for $w \in \mathrm{Val}_{\mathfrak{o}}(K)$ with valuation ring R_w the valuative criterion of properness yields for every proper model $\mathscr{X}_i$ a unique point x_i such that R_w dominates $\mathcal{O}_{\mathscr{X}_i, x_i}$. The points x_i form a compatible system $(x_i) \in \varprojlim \mathscr{X}_i$ and $R_w = \varinjlim \mathcal{O}_{\mathscr{X}_i, x_i}$ holds.

- *Valuations*: The set $\mathrm{Val}_{\mathfrak{o}}(K)$ of $\mathfrak{o}$-valuations of K is a disjoint union

$$\mathrm{Val}_{\mathfrak{o}}(K) = \mathrm{Val}_k(K) \amalg \mathrm{Val}_v(K),$$

where $\mathrm{Val}_v(K)$ is the set of valuations w of K which prolong v from k to K. We notice that there is a canonical embedding

$$\mathrm{Val}_k(K) \hookrightarrow \mathrm{Val}_v(K)$$

as follows. Let $w_a \in \mathrm{Val}_k(K)$ be the k-valuation, corresponding to a closed point $a \in X$. Then the residue field $\kappa(w_a) = \kappa(a)$ is a finite extension of k, hence v has a unique prolongation $v_{\kappa(a)}$ to $\kappa(w_a)$. The valuation theoretic composition $w := v_{\kappa(a)} \circ w_a$ yields a valuation of K which prolongs v, thus w lies in $\mathrm{Val}_v(K)$. Conversely, if $w \in \mathrm{Val}_v(K)$ is a valuation with valuation ring R, then $R[1/p]$ is a valuation ring of K that contains $k = \mathfrak{o}[1/p]$. In particular, if $R[1/p] \neq K$, then $R[1/p] = R_{w_a}$ for some $w_a \in \mathrm{Val}_k(K)$. For $v_{\kappa(a)}$ as above, the valuation theoretic composition $v_{\kappa(a)} \circ w_a$ is exactly the valuation w we started with.

We say that $w \in \mathrm{Val}_{\mathfrak{o}}(K)$ **originates from a k-rational point**, if there exists $a \in X(k)$ such that either $w = w_a$ or w is the image of w_a under the canonical embedding $\mathrm{Val}_k(K) \hookrightarrow \mathrm{Val}_v(K)$.

The main result of the present paper is the following positive answer to the $\mathrm{Val}_{\mathfrak{o}}(K)$ variant of the section conjecture instead of the $\mathrm{Val}_k(K)$ section conjecture above.

Main Theorem. *Let k be a finite extension of $\mathbb{Q}_p$. Let X/k be a smooth, projective, geometrically connected curve of genus ≥ 2. Then for every section $s : \mathrm{Gal}_k \to \pi_1(X)$ of $\pi_1(X/k)$ there exists an $\mathfrak{o}$-valuation $w \in \mathrm{Val}_{\mathfrak{o}}(K)$ of the function field $K = k(X)$ and a prolongation $\tilde{w}$ to the function field $\tilde{K} = k(\tilde{X})$ of the universal pro-étale cover $\tilde{X}$ such that $s(\mathrm{Gal}_k)$ is contained in the decomposition group $\mathrm{D}_{\tilde{w}}$ of $\tilde{w}$ in $\pi_1(X)$.*

In Theorem 37 we will prove actually a more general assertion concerning hyperbolic curves X/k that are not necessarily projective. A smooth, geometrically connected curve X/k is called **hyperbolic,** if X has negative ℓ-adic Euler-characteristic $\chi(\overline{X}) = 2 - 2g - r$. Here r is the number of geometric points needed to smoothly compactify $\overline{X}$ over $\overline{k}$ and g is the genus of the smooth compactification. Recall that in characteristic zero, X being hyperbolic is equivalent to $\pi_1(\overline{X})$ being non-abelian.

The section conjecture for hyperbolic curves asserts that every conjugacy class of sections of $\pi_1(X/k)$ is defined as indicated above by a k-rational point of the smooth compactification of X, or equivalently by a k-valuation v of K with residue field $\kappa(v) = k$.

In some sense the Main Theorem and Theorem 37 give an optimal local version of the section conjecture, were Conjecture 2 to fail. Indeed, if Conjecture 2 fails, then it fails for a good reason, namely that the projection map $\mathrm{D}_{\tilde{w}} \to \mathrm{Gal}_k$ from a decomposition subgroup $\mathrm{D}_{\tilde{w}} \subset \pi_1(X)$ of some valuation $w \in \mathrm{Val}_{\mathfrak{o}}(K)$ admits a section, although the valuation w does not originate from a k-rational point. In this respect the Main Theorem above reduces the p-adic section conjecture to a completely local problem, namely to confirm that $\mathrm{D}_{\tilde{w}} \to \mathrm{Gal}_k$ does not split if w does not originate from a k-rational point. The proof of the birational version of the p-adic section conjecture as in [P10] follows the above strategy with $\pi_1(X)$ replaced by Gal_K.

Finally, it was pointed out by Kedlaya that in yet another interpretation of the Main Theorem above a section of $\pi_1(X/k)$ gives — if not a k-rational point as predicted by Conjecture 2 — at least a k-*Berkovich* point which is responsible for the section. In light of the above explanations, it remains to be studied, which k-Berkovich points might contribute sections of $\pi_1(X/k)$.

1.6. **Outline of the paper.** Since $\mathrm{D}_{\tilde{w}}$ is the stabilizer of $\tilde{w}$ under the action of $\pi_1(X)$, the property $s(\mathrm{Gal}_k) \subset \mathrm{D}_{\tilde{w}}$ for a section s translates into the existence of a fixed point under the Galois action by Gal_k via s, see Section 8. The starting point of our search for a fixed point comes from the Brauer group method, see Section 7, which relies surprisingly only on the ℓ-part of $\pi_1(\overline{X})$. The results on the ℓ-part of $\pi_1(\overline{X})$ provided in Sections 4–6 make use in a subtle way of the p-part in $\pi_1(\overline{X})$, in particular Tamagawa's non-resolution [Ta04], in order to move apart the ℓ-parts of inertia groups corresponding to different prime divisors. We ultimately conclude the existence of a fixed point in Section 8 from a well known combinatorial lemma on group actions on trees.

In Section 2 we provide the reader with a complete geometric description of the valuation theory for p-adic curves, which is otherwise not well documented in the literature. Section 3 adds a geometric description of the valuation theoretic *Hilbert Zerlegungstheorie*.

Contents

2. The zoo of valuations for 2-dimensional semilocal fields

Let k be a complete discrete valued field with valuation ring $\mathfrak{o}$ and perfect residue field κ, e.g., k is a finite extension of $\mathbb{Q}_p$. Let v denote the canonical valuation on k.

Let K be the function field of a smooth projective geometrically connected curve X over k. In this section we discuss the space $\mathrm{Val}_{\mathfrak{o}}(K) = \mathrm{Val}_k(K) \cup \mathrm{Val}_v(K)$ of equivalence classes of valuations w on K whose valuation ring R_w contains $\mathfrak{o}$, or equivalently, the restriction of w to k is either the trivial valuation or equals v.

2.1. The Riemann–Zariski space of $\mathfrak{o}$-valuations.

2.1.1. *Models.* In this paper a **model** or more precisely a **regular model with strict normal crossing** of X over $\mathfrak{o}$ is a regular scheme $\mathscr{X}$ which is flat and proper over $\mathrm{Spec}(\mathfrak{o})$ together with a k-isomorphism of X with the generic fibre $\mathscr{X}_k$ such that the reduced special fibre $\mathscr{X}_{\kappa,\mathrm{red}}$ is a divisor with strict normal crossings on $\mathscr{X}$. In particular, unfortunately a stable model in general is not a model in the sense of this paper. By a result of Lichtenbaum, [Li68] Thm 2.8, models are automatically projective over $\mathrm{Spec}(\mathfrak{o})$. We denote the underlying topological space of $\mathscr{X}$ by $\mathscr{X}^{\mathrm{top}}$, whereas $\mathscr{X}_{\mathrm{cons}}$ denotes $\mathscr{X}^{\mathrm{top}}$ when given the finer constructible topology.

2.1.2. *The center.* For $w \in \mathrm{Val}_{\mathfrak{o}}(K)$ the valuative criterion of properness implies a canonical map $\mathrm{Spec}(R_w) \to \mathscr{X}$ which maps the closed point of $\mathrm{Spec}(R_w)$ to the **center** $x_w \in \mathscr{X}_{\mathrm{cons}}$ of the valuation w on the model $\mathscr{X}$. Maps between different models, which are the identity on X, respect the center of a valuation. The resulting map

$$\mathrm{center} : \mathrm{Val}_{\mathfrak{o}}(K) \to \varprojlim \mathscr{X}_{\mathrm{cons}}, \tag{2.1}$$

where the projective limit ranges over all models of K, identifies $\varprojlim \mathscr{X}_{\rm cons}$ with $\mathrm{Val}_{\mathfrak{o}}(K)$ which is a subspace of the **Riemann–Zariski space** of K/k.

2.1.3. *The valuation ring.* The centers of a valuation $w \in \mathrm{Val}_{\mathfrak{o}}(K)$ determine the valuation ring $R_w = \varinjlim \mathcal{O}_{\mathscr{X},x_w}$ where the direct limit ranges over all models. The inverse map to (2.1) is described as follows. To a compatible system of points $a_{\mathscr{X}} \in \mathscr{X}_{\rm cons}$ on all models we associate first the ring $R_a = \varinjlim \mathcal{O}_{\mathscr{X},a_{\mathscr{X}}}$. The ring R_a is the valuation ring of a valuation w of K because for every $f \in K^*$ at least one of f and f^{-1} belongs to R_a, see [Bo98] VI §1.2 . Indeed, the indeterminacy of f, that is the set of points where neither f nor f^{-1} is defined, disappears on a fine enough model.

2.1.4. *The patch topology.* The **patch topology** on $\mathrm{Val}_{\mathfrak{o}}(K)$ is defined as the topology induced from the pro-finite product topology by the injective map

$$\mathrm{sign}\ :\ \mathrm{Val}_{\mathfrak{o}}(K) \hookrightarrow \prod_{f\in K^*}\{-,0,+\}$$

that assigns to a valuation w the collection of **signs** of the value $w(f)$ for each $f \in K^*$, where the sign of f is $+$ if $w(f) > 0$, it is $-$ if $w(f) < 0$ and the sign is 0 if $w(f) = 0$. The condition on a collection of signs to belong to a valuation ring, namely that the subset in K of 0 and the nonnegative elements forms a ring which contains at least one of f, f^{-1} for each $f \in K^*$, is a closed condition. Hence $\mathrm{Val}_{\mathfrak{o}}(K)$ is a pro-finite space, in particular it is compact and hausdorff.

The map center : $\mathrm{Val}_{\mathfrak{o}}(K) \to \varprojlim \mathscr{X}_{\rm cons}$ defined in (2.1) is a homeomorphism from $\mathrm{Val}_{\mathfrak{o}}(K)$ endowed with the patch topology to $\varprojlim \mathscr{X}_{\rm cons}$ with respect to the $\varprojlim$-topology. The subset

$$\mathrm{Val}_v(K) = \{w \in \mathrm{Val}_{\mathfrak{o}}(K)\ ;\ w|_k = v\} \subset \mathrm{Val}_{\mathfrak{o}}(K)$$

is a closed subset in the patch topology described by the condition that $w(\pi) > 0$ for a uniformizer π of $\mathfrak{o}$. The set $\mathrm{Val}_v(K)$ corresponds to the subset $\varprojlim \mathscr{X}_{\kappa,\rm cons} \subset \varprojlim \mathscr{X}_{\rm cons}$ where $\mathscr{X}_\kappa \subset \mathscr{X}$ is the special fibre.

2.2. Types of valuations.

2.2.1. *Height.* We define the **height** of a valuation $w \in \mathrm{Val}_{\mathfrak{o}}(K)$ as the well defined number $\mathrm{ht}(w) \in \{0,1,2\}$ such that $\mathrm{ht}(x_w) = \mathrm{ht}(w)$ for all sufficiently fine models with respect to the system of all models. The unique valuation of height 0 is the trivial valuation.

2.2.2. *Height 1.* Valuations of height 1 are the discrete valuations associated to prime divisors on an arbitrary model fine enough such that the respective divisor appears. The corresponding prime divisor is either **vertical**, i.e., it maps to the closed point of $\mathrm{Spec}(\mathfrak{o})$, or **horizontal**, i.e., it maps finitely to $\mathrm{Spec}(\mathfrak{o})$. The first are called of **type 1v** whereas the latter valuations are called of **type 1h**.

Notation 4. The usual notation for a valuation of K of type 1v will be α. The corresponding prime divisor of a fine enough model will be denoted by Y_α and by abuse of notation has a generic point denoted by α again. The precise meaning of α will always be clear from the context.

2.2.3. *Height 2.* All the remaining valuations are of height 2 and thus have all their centers at closed points of the special fibre.

Let w be a valuation of height 2. For each valuation α of height 1 we define the distance of w to α on the model $\mathscr{X}$ as the infimum of the number of irreducible components in a 1-dimensional connected subscheme $Z \subset \mathscr{X}$ which contains the center of α and of w. If w keeps finite distance to any valuation of height 1 as we vary over the system of all models, then there is a unique valuation α of height 1 with a closed point y on the associated divisor such that w is the composition of α with the valuation v_y on the residue field of α associated to y. So $w = v_y \circ \alpha$ is called of **type 2v** (resp. **type 2h**) if α is **vertical** (resp. **horizontal**.)

The remaining valuations have centers which move away from any valuation of height 1 and are called of **(coarse) type 2u (unbounded)**.

2.3. **Rigid analytic viewpoint.** Valuations of height 2 can be understood in terms of the associated rigid analytic space X^{rig}. For a model $\mathscr{X}$ we get a specialisation map

$$\mathrm{sp}_{\mathscr{X}} : X^{\mathrm{rig}} \to \mathscr{X}_{\kappa,\mathrm{red}}$$

from the rigid space to the set of closed points of the special fibre. The preimage of a smooth closed point of $\mathscr{X}_{\kappa,\mathrm{red}}$ is an open disc, the preimage of a node of $\mathscr{X}_{\kappa,\mathrm{red}}$ is an annulus.

To a valuation w of height 2 we associate the system $C_{\mathscr{X}} = \mathrm{sp}_{\mathscr{X}}^{-1}(x_w)$ of preimages of the centers indexed by the system of all models. The system of subsets $C_{\mathscr{X}}$ is monotone decreasing with respect to inclusion when the model becomes finer. The valuation is uniquely determined by the system of the $C_{\mathscr{X}}$ as

$$R_w = \bigcup_{\mathscr{X}} \mathcal{O}_{\mathscr{X}}(C_{\mathscr{X}}) = \bigcup_{\mathscr{X}} \big\{ f \in K;\ f \text{ defined on } C_{\mathscr{X}}, \|f\|_{C_{\mathscr{X}},\infty} \leq 1 \big\},$$

where $\|f\|_{C_{\mathscr{X}},\infty}$ is the sup-norm of f on $C_{\mathscr{X}}$. The various types belong to distinctive geometric pictures of the system of the $C_{\mathscr{X}}$ as follows.

2.3.1. *Type 2h.* For fine enough models, $C_{\mathscr{X}}$ is an open disc with fixed center $x \in X^{\mathrm{rig}}$ and radius converging to 0 with finer and finer models.

2.3.2. *Type 2v.* For fine enough models, $C_{\mathscr{X}}$ is an annulus, such that the correspponding annuli for finer and finer models share one common boundary.

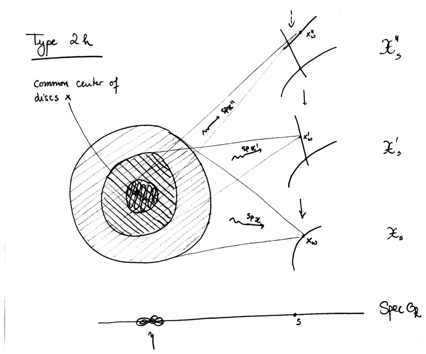

FIGURE 1. type 2h

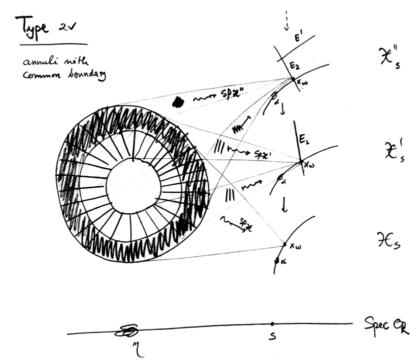

FIGURE 2. type 2v

2.4. **Height 2 but unbounded distance.** The valuations of type 2u can be described and arranged into types in more detail as follows.

Every closed point y in the reduced special fibre carries invariants $(e_{y,\alpha}), f_y$ equal to the tuple $(e_{y,\alpha})$ of the multiplicities of the components on which y lies in the special fibre $\mathscr{X}_\kappa$ and the residue field degree f_y of y over κ. Any closed point x in the generic fibre $X = \mathscr{X}_K$ that specialises to y has to have residue field $\kappa(x)$ with $e_{\kappa(x)/k} = \sum_\alpha m_\alpha e_{y,\alpha}$ with $m_i \in \mathbb{N}_{\geq 1}$ and $f_y | f_{\kappa(x)/k}$. On the other hand, there is always an x with the minimal possible values of e, f.

2.4.1. *Type $2u_{\mathrm{smooth}}$.* For a valuation w of type 2u the value $\sum_\alpha e_{x_w,\alpha}$ remains bounded if and only if for fine enough models ultimately all centers x_w belong to the smooth locus of the reduced special fibre. Such a valuation is called of **type $\mathbf{2u}_{\mathrm{smooth}}$** or **$\mathbf{2u}_{\mathrm{sm}}$ (ultimately smooth)**.

2.4.2. *Type $2u_{\mathrm{node}}$.* We call a valuation w of **type $\mathbf{2u}_{\mathrm{node}}$** or **$\mathbf{2u}_{\mathrm{n}}$ (ultimately node)** if for all fine enough models the center lies in a node of the reduced special fibre.

2.4.3. *Type $2u_{\mathrm{alt}}$.* For a valuation of type 2u, if neither type $2u_{\mathrm{node}}$ nor type $2u_{\mathrm{smooth}}$ applies, then the center x_w in the pro-system of models alternate between the smooth locus of the reduced special fibre and its nodes, and hence these are called of **type $2u_{\mathrm{alternating}}$** or **$2u_{\mathrm{alt}}$(unbounded alternating)**.

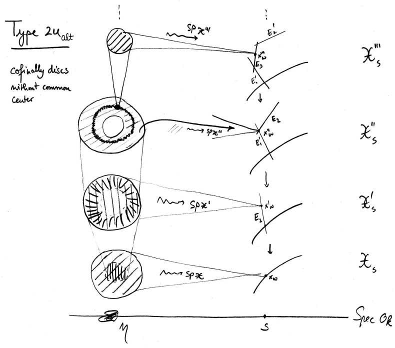

FIGURE 3. type $2u_{\mathrm{alternating}}$

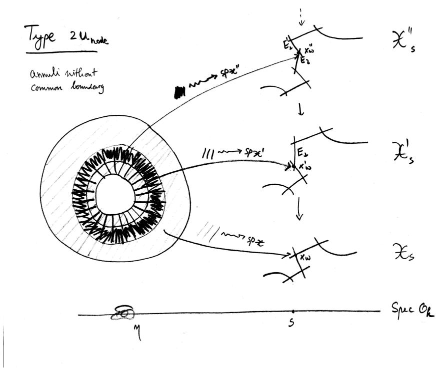

FIGURE 4. type $2u_{\mathrm{node}}$

2.4.4. *Rigid analytic description of type $2u_{\mathrm{smooth}}$.* For a cofinal set of models, $C_{\mathscr{X}}$ is an open disc without common center in X^{rig}. The radius of the discs converges to 0 with finer and finer models. There is a unique limit point in $X(\widehat{k^{\mathrm{alg}}}) \setminus X(k^{\mathrm{alg}})$, where $\widehat{k^{\mathrm{alg}}}$ is the completion of k^{alg}.

2.4.5. *Rigid analytic description of type $2u_{\mathrm{node}}$.* For fine enough models, $C_{\mathscr{X}}$ is a p-adic annulus, such that the corresponding annuli for finer and finer models share no common boundaries.

2.5. **Algebraic structure.** The information on the algebraic structure associated to a valuation w according to its type is summarized in the following table. The **rational rank** of w or better its value group Γ_w is defined as $\dim_{\mathbb{Q}}(\Gamma_w \otimes \mathbb{Q})$, see [Bo98] VI §10.2. And the **rank** of w, *hauteur* in [Bo98] VI §4.4, is the Krull dimension $\dim \mathrm{Spec}(R_w)$ of its valuation ring R_w.

type	height	value group	$\mathbb{Q}$-rank	rank	on k	residue field
0	0	1	0	0	trivial	K
1h	1	$\mathbb{Z}$	1	1	trivial	finite over k
1v	1	$\mathbb{Z}$	1	1	v	function field over κ of transcendence degree 1
2h	2	$\mathbb{Z} \oplus \mathbb{Z}$ lex.	2	2	v	finite over κ
2v	2	$\mathbb{Z} \oplus \mathbb{Z}$ lex.	2	2	v	finite over κ
$2u_{\mathrm{n}}$	2	$\mathbb{Z} \oplus \mathbb{Z}\gamma \subset \mathbb{R}$	2	1	v	finite over κ
$2u_{\mathrm{sm}}$	2	$\mathbb{Z}$	1	1	v	infinite, algebraic over κ
$2u_{\mathrm{alt}}$	2	$\bigcup_n \frac{1}{e_n}\mathbb{Z}$ with $\lim e_n = \infty$	1	1	v	algebraic over κ

The residue field for a valuation w of type $2u_{\mathrm{smooth}}$ has to be algebraic over κ of infinite degree. Indeed, otherwise the extension $\mathfrak{o} \prec R_w$ had finite residue degree $f = [\kappa(w) : \kappa]$ and finite index of value groups $e = (w(K) : v(k))$, which implies that K as a k vector space has $\dim_k K = ef$, a contradiction. In particular, if we ultimately pick smooth centers x_w and the residue field degree $[\kappa(x_w) : \kappa]$ remains finite, then we actually deal with a valuation of type 2h.

2.6. **Valuations of the universal cover.** From now on we fix a geometric generic point $\bar{\eta} : \mathrm{Spec}(\Omega) \to X$ of X as base point. Let $\tilde{K}$ be the function field of the associated pointed universal pro-étale cover $\tilde{X}$ of X, i.e, $\tilde{K} \subset \Omega$ is the maximal algebraic extension of K which is unramified over X. We conclude that $\pi_1(X, \bar{\eta})$ equals $\mathrm{Gal}(\tilde{K}/K)$.

2.6.1. *The Riemann–Zariski space of the universal cover.* The prolongation $\mathrm{Val}_{\mathfrak{o}}(\tilde{K})$ of $\mathrm{Val}_{\mathfrak{o}}(K)$ to $\tilde{K}$ endowed with the patch topology is a projective limit

$$\mathrm{Val}_{\mathfrak{o}}(\tilde{K}) \xrightarrow{\sim} \varprojlim_{K'} \mathrm{Val}_{\mathfrak{o}}(K')$$

of the spaces $\mathrm{Val}_{\mathfrak{o}}(K')$ equipped with the patch topology, where K' ranges over all finite intermediate extensions K'/K in $\tilde{K}/K$. As above, one has a homeomorphism

$$\mathrm{center} : \mathrm{Val}_{\mathfrak{o}}(\tilde{K}) \xrightarrow{\sim} \varprojlim_{X' \subset \mathscr{X}'} \mathscr{X}'_{\mathrm{cons}}$$

and the subset

$$\mathrm{Val}_v(\tilde{K}) = \{\tilde{w} \in \mathrm{Val}_{\mathfrak{o}}(\tilde{K}) \ ; \ \tilde{w}|_k = v\} \subset \mathrm{Val}(\tilde{K})$$

is a closed subset in the patch topology described by the condition that $\tilde{w}(\pi) > 0$ for a uniformizer π of $\mathfrak{o}$. Thus $\mathrm{Val}_v(\tilde{K})$ is a compact, hausdorff, pro-finite space which furthermore is canonically a pro-finite limit

$$\mathrm{center} : \mathrm{Val}_v(\tilde{K}) \xrightarrow{\sim} \varprojlim_{K', \mathscr{X}'} \mathscr{X}'_{\kappa,\mathrm{cons}} \tag{2.2}$$

of the pro-finite spaces $\mathscr{X}'_{\kappa,\mathrm{cons}}$, where $\mathscr{X}'_{\kappa,\mathrm{cons}}$ is the reduced special fibre of $\mathscr{X}'$ endowed with the constructible topology.

2.6.2. *Types and the universal cover.* The canonical restriction map $\mathrm{Val}_{\mathfrak{o}}(K') \to \mathrm{Val}_{\mathfrak{o}}(K)$ is surjective, and for $w' \mapsto w$, by the fundamental inequality, the residue field extension $\kappa(w')/\kappa(w)$ is finite and the inclusion of value groups $w(K) \subset w'(K')$ has finite index, see [Bo98] VI §8. Hence the type of a valuation is preserved under the restriction map $\mathrm{Val}_{\mathfrak{o}}(K') \to \mathrm{Val}_{\mathfrak{o}}(K)$, and the classification into types also applies to valuations in $\mathrm{Val}_{\mathfrak{o}}(\tilde{K})$.

2.6.3. *Notational convenience.* The map $\mathrm{Val}_{\mathfrak{o}}(\tilde{K}) \to \mathrm{Val}_{\mathfrak{o}}(K)$ will be denoted by $\tilde{w} \mapsto w = \tilde{w}|_K$ which implicitly could also imply a choice of a preimage $\tilde{w}$ of the valuation w if the latter happens to appear first.

2.7. **Generalizations.** The content of Section 2 generalizes without modification, except for the rigid analytic point of view in Section 2.3, to the 2-dimensional setting fibred over a base of dimension ≤ 1. Namely, let $X \to B$ be a proper map of connected normal excellent schemes with $\dim(X) = 2$ and $\dim(B) \leq 1$. Then the preceeding discussion leads to a classification and description of the space of valuations of the function field of X which dominate a valuation of the function field of B associated to a point of B.

Hypothesis. From now on, if not explicitly stated otherwise, we will work under the hypothesis that k is a finite extension of $\mathbb{Q}_p$, hence in particular the residue field $\kappa = \mathbb{F}$ is a finite field. Further, all finite extensions of k are locally compact fields.

3. Unramified *Hilbert Zerlegungstheorie*

3.1. **Nearby points.** For a geometric point y on a model $\mathscr{X}$ we set $\mathscr{X}_y^{\mathrm{h}} = \mathrm{Spec}(\mathcal{O}^{\mathrm{h}}_{\mathscr{X},y})$ for the **scheme of nearby points** and $\mathscr{X}_y^{\mathrm{sh}} = \mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{\mathscr{X},y})$ for the **scheme of strictly nearby points**. The intersection with the generic fibre we denote by

$$\mathscr{U}_y^{\mathrm{h}} = \mathrm{Spec}(\mathcal{O}^{\mathrm{h}}_{\mathscr{X},y} \otimes_{\mathfrak{o}} k) \subset \mathscr{X}_y^{\mathrm{h}} \qquad \text{and} \qquad \mathscr{U}_y^{\mathrm{sh}} = \mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{\mathscr{X},y} \otimes_{\mathfrak{o}} k) \subset \mathscr{X}_y^{\mathrm{sh}}.$$

For y equal to the center x_w of a valuation $w \in \mathrm{Val}_{\mathfrak{o}}(K)$, more precisely, for a choice of geometric point above the closed point of the valuation ring which induces a geometric point $\overline{x}_w$ above each center, we abbreviate

$$\mathscr{U}_w^{\mathrm{h}} := \mathscr{U}_{\bar{x}_w}^{\mathrm{h}} \subseteq \mathscr{X}_w^{\mathrm{h}} := \mathscr{X}_{\bar{x}_w}^{\mathrm{h}} \qquad \text{and} \qquad \mathscr{U}_w^{\mathrm{sh}} := \mathscr{U}_{\bar{x}_w}^{\mathrm{sh}} \subseteq \mathscr{X}_w^{\mathrm{sh}} := \mathscr{X}_{\bar{x}_w}^{\mathrm{sh}}.$$

In the limit over all models $\mathscr{X}$ of K we get

$$U_w^{\mathrm{h}} = \varprojlim_{\mathscr{X}} \mathscr{U}_w^{\mathrm{h}} \subseteq X_w^{\mathrm{h}} = \varprojlim_{\mathscr{X}} \mathscr{X}_w^{\mathrm{h}} \qquad \text{and} \qquad U_w^{\mathrm{sh}} = \varprojlim_{\mathscr{X}} \mathscr{U}_w^{\mathrm{sh}} \subseteq X_w^{\mathrm{sh}} = \varprojlim_{\mathscr{X}} \mathscr{X}_w^{\mathrm{sh}}.$$

We note that U_w^{h} (resp. U_w^{sh}) is a limit of affine p-adic curves over k (resp. k^{nr}). In particular, the cohomological dimension of U_w^{sh} for étale constructible sheaves is at most 2.

3.2. **Hilbert decomposition and inertia group.** Let us fix a choice of a geometric generic point $\bar{\xi}_y$ of $\mathscr{U}_y^{\mathrm{sh}}$ such that $(\mathscr{U}_y^{\mathrm{sh}}, \bar{\xi}_y) \to (X, \bar{\eta})$ becomes a pointed map.

The **decomposition group** resp. **inertia group** in the sense of Hilbert at y is given by the image D_y, resp. I_y, of the natural map $\pi_1(\mathscr{U}_y^{\mathrm{h}}, \bar{\xi}_y) \to \pi_1(X, \bar{\eta})$, resp. $\pi_1(\mathscr{U}_y^{\mathrm{sh}}, \bar{\xi}_y) \to \pi_1(X, \bar{\eta})$, induced by the inclusions. We suppress the choice of base points in the notation for decomposition and inertia groups.

3.3. **Decomposition and interia group of a valuation.** For $w \in \mathrm{Val}_{\mathfrak{o}}(K)$ let $\bar{\xi}_w$ be a geometric generic point of U_w^{sh} such that $(U_w^{\mathrm{sh}}, \bar{\xi}_w) \to (X, \bar{\eta})$ becomes a pointed map. The compatibility of $\bar{\xi}_w$ with $\bar{\eta}$ describes a unique prolongation $\tilde{w}$ of w to $\tilde{K}$ by the property $K_{\tilde{w}}^{\mathrm{sh}} = \tilde{K} \cdot K_w^{\mathrm{sh}}$ and similarly $K_{\tilde{w}}^{\mathrm{h}} = \tilde{K} \cdot K_w^{\mathrm{h}}$ in Ω. Here K_w^{h} (resp. K_w^{sh}) is a (strict) henselisation of K in w, and similarly for $\tilde{w}$. We easily observe the following lemma.

Lemma 5. *For a valuation $w \in \mathrm{Val}_{\mathfrak{o}}(K)$ of height 2 but not of type 2h we have $\mathrm{Spec}(K_w^{\mathrm{h}}) = U_w^{\mathrm{h}}$, whereas for w type 2h refining α of type 1h the nearby points U_w^{h} equals the spectrum of the valuation ring the exention of α to K_w^{h} and moreover equals $U_\alpha^{\mathrm{h}} = X_\alpha^{\mathrm{h}}$.* □

The **decomposition group** (resp. **inertia group**) in the sense of valuation theory of w, or more precisely the prolongation $\tilde{w}|w$ to a valuation of $\tilde{K}$, is given by the image $\mathrm{D}_{\tilde{w}|w}$ of $\pi_1(U_w^{\mathrm{h}}, \bar{\xi}_w) \to \pi_1(X, \bar{\eta})$, resp. the image $\mathrm{I}_{\tilde{w}|w}$ of $\pi_1(U_w^{\mathrm{sh}}, \bar{\xi}_w) \to \pi_1(X, \bar{\eta})$.

The dependence on $\tilde{w}$ is through the choice of a path connecting the base points $\bar{\xi}_w$ and $\bar{\eta}$ to the effect of conjugating $\mathrm{D}_{\tilde{w}|w}$ and $\mathrm{I}_{\tilde{w}|w}$ within $\pi_1(X, \bar{\eta})$. If no confusion arises, we will simplify the notation to $\mathrm{D}_w = \mathrm{D}_{\tilde{w}|w}$ (resp. $\mathrm{I}_w = \mathrm{I}_{\tilde{w}|w}$).

3.4. **Reconciliation of valuation theory and arithmetic geometry.** The relation between the two viewpoints of inertia and decomposition groups comes from the compliance of the functor π_1 with affine projective limits. We may assume that $\bar{\xi}_w$ induces $\bar{\xi}_{\bar{x}_w} \in \mathscr{U}_w^{\mathrm{sh}}$ for every model of X, and then find

$$\mathrm{D}_{\tilde{w}|w} = \varprojlim_{\mathscr{X}} \mathrm{D}_{x_w} \qquad \text{and} \qquad \mathrm{I}_{\tilde{w}|w} = \varprojlim_{\mathscr{X}} \mathrm{I}_{x_w}, \tag{3.1}$$

where the limits are in fact simply intersections of closed subgroups in $\pi_1(X, \bar{\eta})$.

Moreover, let α be a valuation of height 1 and y a geometric point localised in a closed point of the divisor Y_α associated to α on a suitable model $\mathscr{X}$. Then we have the following diagram when the corresponding geometric points are compatibly chosen.

$$\begin{tikzcd}
 & & \mathrm{Spec}(\Omega) \arrow[dll, "\bar{\xi}_\alpha"'] \arrow[dl, "\bar{\xi}_y"] \arrow[ddd, "\bar{\eta}"] \\
U_\alpha^{\mathrm{sh}} \arrow[r, dotted] \arrow[d] & \mathscr{U}_y^{\mathrm{sh}} \arrow[d] & \\
U_\alpha^{\mathrm{h}} \arrow[dr, dotted] & \mathscr{U}_y^{\mathrm{h}} \arrow[d] & \\
 & \mathscr{U}_y^{\mathrm{Nis}_\alpha} \arrow[r] & X
\end{tikzcd}$$

The scheme $\mathscr{U}_y^{\mathrm{Nis}_\alpha}$ is the generic fibre of $\mathscr{X}_y^{\mathrm{Nis}_\alpha}$ which is the maximal strict étale neighbourhood in between $\mathscr{X}_y^{\mathrm{h}} \to \mathscr{X}$ which is Nisnevich at α, i.e., such that the point α splits in the image of $\bar{\xi}_\alpha$ after an appropriate choice is fixed. With $\mathrm{D}_{y,\alpha} = \mathrm{im}\Big(\pi_1(\mathscr{U}_y^{\mathrm{Nis}_\alpha}, \bar{\xi}_y) \to \pi_1(X, \bar{\eta})\Big)$, we find

$$\begin{array}{ccccccc}
\mathrm{I}_\alpha & \subseteq & \mathrm{I}_y & \subseteq & \mathrm{D}_y & & \\
\cap & & & & \cap & & \\
\mathrm{D}_\alpha & & \subseteq & & \mathrm{D}_{y,\alpha} & \subseteq & \pi_1(X, \bar{\eta})
\end{array} \tag{3.2}$$

Let $w \in \mathrm{Val}_{\mathfrak{o}}(K)$ be a valuation of rational rank 2, i.e. of type 2h or 2v, and a refinement of the valuation α, then in the limit over all models we deduce from (3.2) and (3.1) that

$$\mathrm{I}_\alpha \subseteq \mathrm{I}_w \subseteq \mathrm{D}_w \subseteq \mathrm{D}_\alpha$$

because $\mathrm{D}_\alpha = \varprojlim_{\mathscr{X}} \mathrm{D}_{x_w,\alpha}$.

4. Detecting inertia of type 1v in the kernel of specialisation

4.1. **The kernel of** sp**.** Let $\mathscr{X}/\mathfrak{o}$ be a model of the smooth, projective, geometrically connected curve X/k as defined in 2.1.1. The reduced special fibre $Y = \mathscr{X}_{\mathbb{F},\mathrm{red}}$ is by assumption a strict normal crossing divisor on $\mathscr{X}$. Let $Y = \bigcup_\alpha Y_\alpha$ be the decomposition into irreducible components Y_α which are smooth, projective curves with field of constants $\mathbb{F}_\alpha$. The specialisation map of fundamental groups is a surjection

$$\mathrm{sp} : \pi_1(X) \twoheadrightarrow \pi_1(\mathscr{X}) = \pi_1(Y)$$

the kernel of which we denote by $\mathscr{N}_{X|\mathscr{X}}$. The inertia group I_w of a valuation $w \in \mathrm{Val}_{\mathfrak{o}}(K)$ lies in $\mathscr{N}_{X|\mathscr{X}}$. Those for valuations of type 1v generate $\mathscr{N}_{X|\mathscr{X}}$ as a pro-finite group by Zariski-Nagata purity of the branch locus.

4.2. **Cohomology on the model.** Let $n \in \mathbb{N}$ be invertible on $\mathscr{X}$. Let $i : Y \hookrightarrow \mathscr{X}$ be the closed immersion of the reduced special fibre. Standard computations in étale cohomology show $\mathrm{R}^q\, i^! \mu_n = 0$ for $q = 0, 1$, and yield the local cycle class map

$$\bigoplus_\alpha \underline{\mathbb{Z}/n\mathbb{Z}}_{Y_\alpha} \xrightarrow{\sim} \mathrm{R}^2\, i^! \mu_n \tag{4.1}$$

The sheaves $\mathrm{R}^q\, i^! \mu_n$ for $q \geq 3$ are also known by work of Gabber and Fujiwara [Fu02] §8 Consequence 3, see also [Il02] Thm 7.2, but are not part of our computation. It follows that $\mathrm{H}^1_Y(\mathscr{X}, \mu_n)$ vanishes and

$$\mathrm{H}^2_Y(\mathscr{X}, \mu_n) = \bigoplus_\alpha \mathrm{H}^0(Y_\alpha, \mathbb{Z}/n\mathbb{Z}).$$

By proper base change we have $\mathrm{H}^2\left(\mathscr{X}, \mu_n\right) = \mathrm{H}^2\left(Y, \mu_n\right)$, and the relevant part of the localisation sequence reads

$$0 \to \mathrm{H}^1\left(\mathscr{X}, \mu_n\right) \to \mathrm{H}^1\left(X, \mu_n\right) \xrightarrow{\mathrm{res}} \bigoplus_{\alpha} \mathrm{H}^0\left(Y_\alpha, \mathbb{Z}/n\mathbb{Z}\right) \xrightarrow{\rho_n} \mathrm{H}^2\left(Y, \mu_n\right).$$

Unraveling the definitions for the map ρ_n yields the composite

$$\bigoplus_{\alpha} \mathbb{Z}/n\mathbb{Z} \to \mathrm{Pic}(\mathscr{X}) \otimes \mathbb{Z}/n\mathbb{Z} \xrightarrow{\mathrm{c}_1} \mathrm{H}^2\left(\mathscr{X}, \mu_n\right) \xrightarrow{i^*} \mathrm{H}^2\left(Y, \mu_n\right)$$

which maps (n_α) to $i^* \mathrm{c}_1\left(\mathcal{O}_{\mathscr{X}}(\sum n_\alpha Y_\alpha)\right)$. The map res_α, the α component of res, can be computed by excision and functoriality of the localisation sequence as follows. By abuse of notation, we denote by α also the valuation of type 1v in $\mathrm{Val}_{\mathfrak{o}}(K)$ corresponding to Y_α. Let $\overline{\alpha}$ be a geometric point localised in the generic point of Y_α. Then, using the notation of Section 3.1 and the tame character we get a commutative diagram with isomorphisms as indicated.

$$\begin{tikzcd}
\mathrm{H}^1(\pi_1(X), \mu_n) \arrow[r, "\sim"] \arrow[d] & \mathrm{H}^1\left(X, \mu_n\right) \arrow[r, "\mathrm{res}"] \arrow[d] & \mathrm{H}^2_Y\left(\mathscr{X}, \mu_n\right) \arrow[r, equal] \arrow[d] & \bigoplus_\alpha \mathrm{H}^0\left(Y_\alpha, \mathbb{Z}/n\mathbb{Z}\right) \arrow[d, "\mathrm{pr}_\alpha"] \\
\mathrm{H}^1\left(\mathrm{I}_\alpha, \mu_n\right) \arrow[r, "\sim", "\mathrm{inf}"'] & \mathrm{H}^1\left(\mathscr{U}^{\mathrm{sh}}_\alpha, \mu_n\right) \arrow[r, "\sim"] & \mathrm{H}^2_{\overline{\alpha}}\left(\mathscr{X}^{\mathrm{sh}}_\alpha, \mu_n\right) \arrow[r, equal] & \mathbb{Z}/n\mathbb{Z}
\end{tikzcd} \tag{4.2}$$

The inflation map inf in the diagram is an isomorphism because the map $\pi_1(\mathscr{U}^{\mathrm{sh}}_\alpha) \twoheadrightarrow \mathrm{I}_\alpha$ is an isomorphism on the prime to p part due to enough tame ramification along each Y_α, see Proposition 10 (3) below. Consequently, the map res_α is essentially the map induced by restriction from $\pi_1(X)$ to I_α.

4.3. **Cohomology of the special fibre.** Let $\overline{Y} = Y \times_{\mathbb{F}} \mathbb{F}^{\mathrm{alg}}$ be the geometric reduced special fibre. Let $\mathcal{I}_{\alpha,n}$ be the permutation module $\mathbb{Z}/n\mathbb{Z}[\mathrm{Hom}_{\mathbb{F}}(\mathbb{F}_\alpha, \mathbb{F}^{\mathrm{alg}})]$ as a $\mathrm{Gal}_{\mathbb{F}} = \mathrm{Gal}(\mathbb{F}^{\mathrm{alg}}/\mathbb{F})$-module. The degree map of the components describe a $\mathrm{Gal}_{\mathbb{F}}$-equivariant isomorphism

$$\mathrm{H}^2\left(\overline{Y}, \mu_n\right) = \bigoplus_{\alpha} \mathcal{I}_{\alpha,n}.$$

The relevant cohomology of Y computes via the Leray spectral sequence as

$$0 \to \mathrm{H}^1\left(\mathbb{F}, \mathrm{H}^1(\overline{Y}, \mu_n)\right) \to \mathrm{H}^2\left(Y, \mu_n\right) \xrightarrow{(\deg_\alpha)} \bigoplus_{\alpha} \mathrm{H}^0\left(\mathbb{F}, \mathcal{I}_{\alpha,n}\right) \to 0,$$

where $\deg_\alpha$ is the degree map on the component Y_α. If we fix an $\mathbb{F}$-embedding $\mathbb{F}_\alpha \subset \mathbb{F}^{\mathrm{alg}}$, then

$$\mathcal{I}_{\alpha,n} = \mathrm{Ind}^{\mathbb{F}}_{\mathbb{F}_\alpha}\left(\mathbb{Z}/n\mathbb{Z}\right)$$

becomes canonically isomorphic to the induced module with respect to $\mathrm{Gal}_{\mathbb{F}_\alpha} \subset \mathrm{Gal}_{\mathbb{F}}$.

4.4. **Unramified extensions of the base.** Now we perform the limit of the above computations over unramified extensions k'/k viewing the result as $\mathrm{Gal}_{\mathbb{F}} = \pi_1(\mathfrak{o})$-modules. In other words, we take the stalk at $\mathrm{Spec}(\mathbb{F}^{\mathrm{alg}}) \to \mathrm{Spec}(\mathfrak{o})$ of the higher direct images for $\mathscr{X}/\mathfrak{o}$. The unramified base changes do not destroy the good properties that $\mathscr{X}$ has by assumption as a model, and no modification by blow-ups is necessary. We get an exact sequence of $\mathrm{Gal}_{\mathbb{F}}$-modules as follows.

$$0 \to \mathrm{H}^1\left(\overline{Y}, \mu_n\right) \to \mathrm{H}^1\left(X \times_k k^{\mathrm{nr}}, \mu_n\right) \to \bigoplus_{\alpha} \mathrm{Ind}^{\mathbb{F}}_{\mathbb{F}_\alpha}(\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\overline{\rho}_n} \bigoplus_{\alpha} \mathrm{Ind}^{\mathbb{F}}_{\mathbb{F}_\alpha}(\mathbb{Z}/n\mathbb{Z}). \tag{4.3}$$

The map $\overline{\rho}_n = (\deg_\alpha) \circ \rho_n$ is a matrix with entries from $\mathrm{End}(\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$ with rows and columns indexed by the $\mathrm{Gal}_{\mathbb{F}}$-set of irreducible components of $\overline{Y}$. This is nothing but the intersection matrix for the reduced geometric special fibre modulo n.

4.5. **ℓ-adic coefficients.** The local cycle class (4.1) is compatible with change of coefficients $\mu_n \subset \mu_{nd}$ with $p \nmid d$ via the commutative diagram

$$\begin{tikzcd} \bigoplus_\alpha \underline{\mathbb{Z}/n\mathbb{Z}}_{Y_\alpha} \arrow[r, "\cong"] \arrow[d, "\cdot d"] & \mathrm{R}^2\, i^! \mu_n \arrow[d] \\ \bigoplus_\alpha \underline{\mathbb{Z}/nd\mathbb{Z}}_{Y_\alpha} \arrow[r, "\cong"] & \mathrm{R}^2\, i^! \mu_{nd}. \end{tikzcd}$$

Taking the direct limit of (4.3) for $n = \ell^r$, $r \geq 0$ we obtain an exact sequence of $\mathrm{Gal}_{\mathbb{F}}$-modules

$$(4.4)\quad 0 \to \mathrm{H}^1\left(\overline{Y}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)\right) \to \mathrm{H}^1\left(X_{k^{\mathrm{nr}}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)\right) \to \bigoplus_\alpha \mathrm{Ind}_{\mathbb{F}_\alpha}^{\mathbb{F}}\left(\mathbb{Q}_\ell/\mathbb{Z}_\ell\right) \xrightarrow{\overline{\rho}} \bigoplus_\alpha \mathrm{Ind}_{\mathbb{F}_\alpha}^{\mathbb{F}}\left(\mathbb{Q}_\ell/\mathbb{Z}_\ell\right)$$

Here $\overline{\rho}$ is a matrix with entries from $\mathrm{End}(\mathbb{Q}_\ell/\mathbb{Z}_\ell) = \mathbb{Z}_\ell$ with rows and columns indexed by the $\mathrm{Gal}_{\mathbb{F}}$-set of irreducible components of $\overline{Y}$, which is the intersection matrix for the reduced geometric special fibre, and moreover takes values in $\mathbb{Z} \subseteq \mathbb{Z}_\ell$. As an integral matrix $\overline{\rho}$ is symmetric, negative semi-definite with radical given by the rational mutiples of the divisor of the special fibre with its multiplicities, see Mumford [Mu61] §1.

4.6. **Unramified extensions of the model.** We compute the limit of (4.4) over all finite étale covers $\mathscr{X}'$ of $\mathscr{X}$. The comments on the preservation of the good properties of the model still hold true, so we can use (4.4) for all covers. With X' the generic fibre and Y' the special fibre of $\mathscr{X}'$ as above we have

$$\varinjlim_{\mathscr{X}'} \mathrm{H}^1\left(\overline{Y'}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)\right) = 0 \quad \text{and} \quad \varinjlim_{\mathscr{X}'} \mathrm{H}^1\left(X' \times_k k^{\mathrm{nr}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)\right) = \mathrm{H}^1\left(\mathscr{N}_{X|\mathscr{X}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)\right)$$

by compatibility of cohomology of pro-finite groups and discrete coefficients with limits. If $\mathscr{X}'$ corresponds to an open subgroup $H \subset \pi_1(\mathscr{X})$, then the part of $\mathrm{H}^2_{Y'}\left(\mathscr{X}', \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)\right)$ due to components of Y' above Y_α is given by

$$\mathrm{Maps}_{\pi_1(Y_\alpha)}\left(\pi_1(Y)/H, \mathbb{Q}_\ell/\mathbb{Z}_\ell\right) = \left[\mathrm{Maps}_{\pi_1(Y_\alpha)}\left(\pi_1(Y), \mathbb{Q}_\ell/\mathbb{Z}_\ell\right)\right]^H.$$

In the limit over all $\mathscr{X}' \to \mathscr{X}$ we obtain the smooth induction

$$\mathrm{Ind}_{\pi_1(Y_\alpha)}^{\pi_1(Y)}\left(\mathbb{Q}_\ell/\mathbb{Z}_\ell\right) = \bigcup_H \left[\mathrm{Maps}_{\pi_1(Y_\alpha)}\left(\pi_1(Y), \mathbb{Q}_\ell/\mathbb{Z}_\ell\right)\right]^H.$$

The transfer maps in the limit $\varinjlim \mathrm{H}^2\left(\overline{Y}', \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)\right)$ multiply by the respective degrees. All components of positive genus are dominated by components with degree an arbitrary high power of ℓ, even abelian covers, as $\pi_1^{\mathrm{ab}}(\overline{Y}_\alpha) \otimes \mathbb{Z}_\ell \hookrightarrow \pi_1^{\mathrm{ab}}(\overline{Y} \otimes \mathbb{Z}_\ell)$ shows. Therefore in the limit only the components Y_β of genus $g_\beta = 0$ survive. Alltogether, we get the sequence

$$(4.5)\qquad 0 \to \mathrm{H}^1\left(\mathscr{N}_{X|\mathscr{X}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)\right) \to \bigoplus_\alpha \mathrm{Ind}_{\pi_1(Y_\alpha)}^{\pi_1(Y)}\left(\mathbb{Q}_\ell/\mathbb{Z}_\ell\right) \xrightarrow{\mathscr{R}} \bigoplus_{\beta,\ g_\beta = 0} \mathrm{Ind}_{\pi_1(Y_\beta)}^{\pi_1(Y)}\left(\mathbb{Q}_\ell/\mathbb{Z}_\ell\right).$$

Let $\mathrm{I}_\alpha^{\mathrm{ab},\ell} = \mathrm{I}_\alpha^{\mathrm{ab}} \otimes \mathbb{Z}_\ell$ be the ℓ-Sylow group of $\mathrm{I}_\alpha^{\mathrm{ab}}$. Taking Pontrjagin duality with Tate-twist, i.e, $\mathrm{Hom}\left(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)\right)$, and using (4.2) we get the exact sequence of pro-finite $\pi_1(Y)$-modules

$$(4.6)\qquad \bigoplus_{\beta,\ g_\beta=0} \mathrm{I}_\beta^{\mathrm{ab},\ell}[[\pi_1(Y)/\pi_1(Y_\beta)]] \xrightarrow{\mathscr{R}^\vee(1)} \bigoplus_\alpha \mathrm{I}_\alpha^{\mathrm{ab},\ell}[[\pi_1(Y)/\pi_1(Y_\alpha)]] \to \mathscr{N}_{X|\mathscr{X}}^{\mathrm{ab}} \otimes \mathbb{Z}_\ell \to 0.$$

Here we have used the notation $M[[G/G_0]]$ for a finitely generated $\mathbb{Z}_\ell$-module M and a closed subgroup G_0 of a profinite group G to denote

$$M[[G/G_0]] = \varprojlim_H M \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[H \backslash G/G_0],$$

where H ranges over the open normal subgroups of G and $\mathbb{Z}_\ell[H\backslash G/G_0]$ is the permutation module on the set $H\backslash G/G_0$ with coefficients in $\mathbb{Z}_\ell$. The dual of the induced module $\mathrm{Ind}_{G_0}^G(\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ equals $\mathbb{Z}_\ell[[G/G_0]]$ due to the identification

$$(\mathbb{Z}/\ell^n\mathbb{Z})[H\backslash G/G_0] = \mathrm{Hom}\left(\mathrm{Maps}_{G_0}(G/H, \frac{1}{\ell^n}\mathbb{Z}/\mathbb{Z}), \mathbb{Q}_\ell/\mathbb{Z}_\ell\right)$$

mapping $HgG_0 \in H\backslash G/G_0$ to the evaluation $f \mapsto f(g^{-1})$ for $f \in \mathrm{Maps}_{G_0}(G/H, \frac{1}{\ell^n}\mathbb{Z}/\mathbb{Z})$.

The composition of $\mathscr{R}^\vee(1)$ in (4.6) with the projection to the part of components of genus zero yields a map, which is a projective limit indexed over finite étale covers $Y' \to Y$ of maps as follows

$$\bigoplus_{\beta,\ g_\beta=0} \mathrm{I}_\beta^{\mathrm{ab},\ell}[[\pi_1(Y')\backslash\pi_1(Y)/\pi_1(Y_\beta)]] \to \bigoplus_{\beta,\ g_\beta=0} \mathrm{I}_\beta^{\mathrm{ab},\ell}[[\pi_1(Y')\backslash\pi_1(Y)/\pi_1(Y_\beta)]].$$

For each $Y' \to Y$ the map is given by a matrix with the intersection pairing of $Y' \subset \mathscr{X}'$ restricted to the genus zero components. If in Y at least one component has genus at least 1, then this matrix is negative definite, see Mumford [Mu61] §1, and hence the map is injective and remains so in the projective limit over all Y'.

Proposition 6. *Let $\alpha_1, \ldots, \alpha_r \in \mathrm{Val}_{\mathfrak{o}}(K)$ be valuations of type 1v which belong to distinct components of Y with positive genus. Then the natural map*

$$\bigoplus_{i=1}^r \mathrm{I}_{\alpha_i}^{\mathrm{ab},\ell} \hookrightarrow \mathscr{N}_{X|\mathscr{X}}^{\mathrm{ab}} \otimes \mathbb{Z}_\ell$$

is injective and the ℓ-Sylow subgroups of any two distinct I_{α_i} intersect trivially in $\pi_1(X)$.

Proof: The computation above shows that the images of the natural map

$$\bigoplus_{i=1}^r \mathrm{I}_{\alpha_i}^{\mathrm{ab},\ell} \to \bigoplus_{\alpha} \mathrm{I}_\alpha^{\mathrm{ab},\ell}[[\pi_1(Y)/\pi_1(Y_\alpha)]]$$

and of $\mathscr{R}^\vee(1)$ meet only trivially. $\square$

5. The logarithmic point of view towards inertia

5.1. **The tame character.** Let Γ_w be the value group of $w \in \mathrm{Val}_v(K)$. Let $\hat{\mathbb{Z}}'(1)$ be the prime to p Tate module of roots of unity in the separable closure $\kappa(w)^{\mathrm{sep}}$ of $\kappa(w)$. The **tame character** at w is the surjective homomorphism

$$\chi : \mathrm{Gal}_{K_w^{\mathrm{sh}}} \twoheadrightarrow \mathrm{Hom}\left(\Gamma_w, \hat{\mathbb{Z}}'(1)\right)$$

that maps $\sigma \in \mathrm{Gal}_{K_w^{\mathrm{sh}}}$ to the homomorphism

$$\chi_\sigma : \gamma \mapsto \left(\sigma(\sqrt[n]{t_\gamma})/\sqrt[n]{t_\gamma}\right)_n$$

with t_γ being an arbitrary element of value $w(t_\gamma) = \gamma$. The kernel of the tame character χ is the p-Sylow group of $\mathrm{I}_w = \mathrm{Gal}_{K_w^{\mathrm{sh}}}$.

5.2. **Enters the logarithmic fundamental group.** A model $\mathscr{X}$ can be naturally equipped with a log-regular fs-log structure by the divisor $\mathscr{X}_{\mathbb{F},\mathrm{red}}$. We obtain a quotient

$$\pi_1(X, \bar{\eta}) \twoheadrightarrow \pi_1^{\log}(\mathscr{X})$$

which has a tractable group structure by a logarithmic van Kampen theorem applied to the logarithmic special fibre.

Proposition 7. *(1) For a map $f : \mathscr{X}' \to \mathscr{X}$ between models of X the induced map*

$$\pi_1^{\log}(f) \ : \ \pi_1^{\log}(\mathscr{X}') \to \pi_1^{\log}(\mathscr{X})$$

is an isomorphism.

(2) Let X admit a stable model $\mathscr{X}_{\mathrm{stable}}$. Then $\mathscr{X}_{\mathrm{stable}}$ admits an fs log structure which is log regular, and for any model $\mathscr{X}$ of X the natural map $f : \mathscr{X} \to \mathscr{X}_{\mathrm{stable}}$ the induced map

$$\pi_1^{\log}(f) \ : \ \pi_1^{\log}(\mathscr{X}) \to \pi_1^{\log}(\mathscr{X}_{\mathrm{stable}})$$

is an isomorphism.

Proof: In both cases f is a composition of blow-ups which can be enriched to logarithmic blow-up maps. A logarithmic blow-up map yields an isomorphism of log fundamental groups by [FK95] 2.4, see also [Il02] Thm 6.10 or [Sx02] Cor 3.3.11. $\square$

5.3. **Logarithmic inertia groups.** We denote by $\mathrm{I}_w^{\log}$ (resp. $\mathrm{I}_y^{\log}$) the image of I_w (resp. I_y) in $\pi_1^{\log}(\mathscr{X})$, which is a pro-finite group of order prime to p. The log structure on $\mathscr{X}$ induces a log structure on $\mathscr{X}_y^{\mathrm{sh}}$ for every y. The image of

$$\pi_1^{\log}(\mathscr{X}_y^{\mathrm{sh}}, \bar{\xi}_y) \to \pi_1^{\log}(\mathscr{X}, \bar{\eta})$$

is nothing but $\mathrm{I}_y^{\log}$.

Lemma 8. *Let y be a geometric point of $\mathscr{X}$ which lies above a point of the special fibre.*

(1) *The natural map*

$$\pi_1^{\log}(\mathscr{X}_y^{\mathrm{sh}}, \bar{\xi}_y) \to \mathrm{Hom}\left(\mathscr{O}^*(\mathscr{U}_y^{\mathrm{sh}})/\mathscr{O}^*(\mathscr{X}_y^{\mathrm{sh}}), \hat{\mathbb{Z}}'(1)\right)$$

induced by the tame character is an isomorphism.

(2) *Let y lie over the generic point of the component Y_α of the special fibre associated to a valuation α of type 1v. Then we have canonically*

$$\hat{\mathbb{Z}}'(1) = \mathrm{Hom}\left(\mathscr{O}^*(\mathscr{U}_\alpha^{\mathrm{sh}})/\mathscr{O}^*(\mathscr{X}_\alpha^{\mathrm{sh}}), \hat{\mathbb{Z}}'(1)\right) = \pi_1^{\log}(\mathscr{X}_\alpha^{\mathrm{sh}}, \bar{\xi}_\alpha)$$

(3) *Let y lie over a closed point of the special fibre. Then the canonical map*

$$\bigoplus_{\alpha\, ;\ y \in Y_\alpha} \hat{\mathbb{Z}}'(1) \to \pi_1^{\log}(\mathscr{X}_y^{\mathrm{sh}}, \bar{\xi}_y),$$

given essentially by (2) above and restriction of units is an isomorphism. Here α ranges over the valuations associated to components Y_α of the special fibre with $y \in Y_\alpha$.

Proof: This standard result in log geometry follows from Abhyankar's Lemma and Zariski–Nagata purity of the branch locus, see [Il02] Example 4.7 or [Sx02] Cor 3.1.11. $\square$

The following lemma describes the behaviour of logarithmic inertia groups under changes of the model.

Lemma 9. *Let $f : \mathscr{X}' \to \mathscr{X}$ be a blow-up of a closed point of the model $\mathscr{X}$, above which we have the geometric point y. Let α denote the valuation associated to a component Y_α of the special fibre of $\mathscr{X}$ which contains y, and let ε denote the valuation associated to the exceptional divisor E of the blow-up on $\mathscr{X}'$. Let z (resp. x) be geometric points of $\mathscr{X}'$ which lift y and lie on E, such that z also lies on the strict transform of Y_α (resp. such that x lies in the smooth locus of the reduced special fibre of $\mathscr{X}'$). The map f yields maps*

$$f_{z,y} : \pi_1^{\log}(\mathscr{X}_z'^{,\mathrm{sh}}, \bar{\xi}_z) \to \pi_1^{\log}(\mathscr{X}_y^{\mathrm{sh}}, \bar{\xi}_y) \quad \text{and} \quad f_{x,y} : \pi_1^{\log}(\mathscr{X}_x'^{,\mathrm{sh}}, \bar{\xi}_x) \to \pi_1^{\log}(\mathscr{X}_y^{\mathrm{sh}}, \bar{\xi}_y)$$

with the following description in terms of the canonical coordinates provided by Lemma 8 (2).

(1) *Let y be a node of the reduced special fibre of $\mathscr{X}$ with the other component through y besides Y_α being the component Y_β associated to the valuation β. Then $f_{z,y}$ is the isomorphism*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \hat{\mathbb{Z}}'(1) \oplus \hat{\mathbb{Z}}'(1) \xrightarrow{\sim} \hat{\mathbb{Z}}'(1) \oplus \hat{\mathbb{Z}}'(1)$$

with respect to the ordering (α, ε) and (α, β). And $f_{x,z}$ is the diagonal injection

$$\hat{\mathbb{Z}}'(1) \hookrightarrow \hat{\mathbb{Z}}'(1) \oplus \hat{\mathbb{Z}}'(1).$$

(2) *Let y be a smooth point of the reduced special fibre of $\mathscr{X}$. Then $f_{z,y}$ is the surjection given by the sum*

$$\hat{\mathbb{Z}}'(1) \oplus \hat{\mathbb{Z}}'(1) \twoheadrightarrow \hat{\mathbb{Z}}'(1),$$

and $f_{x,y}$ is the identity isomorphism

$$\hat{\mathbb{Z}}'(1) \xrightarrow{\sim} \hat{\mathbb{Z}}'(1).$$

Proof: It all comes down to compute the valuations of the pull back to $\mathscr{X}'$ of local parameters at y. Let the center of the blow up be the ideal (u, v) with $u = 0$ describing Y_α and, if present $v = 0$ describing Y_β. Then near z we find that $v = 0$ describes E while $u/v = 0$ describes the stric transform of Y_α. Hence $\varepsilon(u) = \varepsilon(v) = 1$ which leads to the matrix in (1). The remaining calculations are of the same kind but simpler. □

Let I_k (resp. $\mathrm{I}_k^{\mathrm{tame}} = \hat{\mathbb{Z}}'(1)$) be the inertia (resp. tame inertia) group of Gal_k (resp. its tame quotient $\mathrm{Gal}_k^{\mathrm{tame}}$). The projection $\pi_1(X, \bar{\eta}) \to \mathrm{Gal}_k$ maps the inertia (resp. the log inertia) groups associated to points or valuations to I_k (resp. $\mathrm{I}_k^{\mathrm{tame}}$).

Proposition 10. *Let $w \in \mathrm{Val}_{\mathfrak{o}}(K)$ be a valuation. The structure of $\mathrm{I}_w^{\log}$ is as follows.*

(1) *If w is of type 1h, then $\mathrm{I}_w^{\log} = 1$.*
(2) *If w is of type 2h with $w = v_y \circ \alpha$, then $\mathrm{I}_w^{\log} = \hat{\mathbb{Z}}'(1)$ with the natural map $\mathrm{I}_w^{\log} \to \mathrm{I}_k^{\mathrm{tame}}$ being multiplication by the ramification index $e(v_y/v)$.*
(3) *If w is of type 1v, then $\mathrm{I}_w^{\log} = \hat{\mathbb{Z}}'(1)$ with the natural map $\mathrm{I}_w^{\log} \to \mathrm{I}_k^{\mathrm{tame}}$ being multiplication by the ramification index $e(w/v)$.*
(4) *If w is a valuation of height 2 and on some model $\mathscr{X}$ the center x_w lies only on one component of the special fibre associated to the valuation α, then $\mathrm{I}_w^{\log} \subseteq \mathrm{I}_\alpha^{\log} = \hat{\mathbb{Z}}'(1)$. This applies in particular to valuations of type 2h, $2u_{\mathrm{sm}}$ and $2u_{\mathrm{alt}}$.*
(5) *If the center of a valuation w of height 2 is a node of the reduced special fibre on all models, then*

$$\hat{\mathbb{Z}}'(1) \oplus \hat{\mathbb{Z}}'(1) \twoheadrightarrow \mathrm{I}_w^{\log} = \langle \mathrm{I}_\alpha^{\log}, \mathrm{I}_\beta^{\log} \rangle$$

where α, β are the valuations of type 1v which correspond to the components through the center of w on a given model $\mathscr{X}$ of X/k, and approriate base points have been chosen.

Proof: All assertions follow from Proposition 7 and Lemma 9. □

We would like to stress, that if in (5) in fact we have equality $\hat{\mathbb{Z}}'(1) \oplus \hat{\mathbb{Z}}'(1) = \mathrm{I}_w^{\log}$, then the decomposition as a direct sum depends on the choice of model according to Lemma 9 (1).

Proposition 11. *Let X/k have stable reduction $\mathscr{X}_{\mathrm{stable}}/\mathfrak{o}$. Let $f : \mathscr{X} \to \mathscr{X}_{\mathrm{stable}}$ be the minimal regular resolution of the stable model. Let $y \in \mathscr{X}_{\mathrm{stable}}$ be a node with singularity of type A_n in the intersection of the two distinct components $Y_{\alpha_1}, Y_{\alpha_2}$, such that $f^{-1}(y)$ equals a chain of irreducible divisors $E_1, \ldots, E_{n-1}$, that links the strict transforms E_0, E_n of $Y_{\alpha_1}, Y_{\alpha_2}$, i.e., such that*

(i) *E_i meets E_{i-1} and E_{i+1} each in a single node for $i = 1, \ldots, n-1$,*
(ii) *and $E_i \cong \mathbb{P}^1$ for $i = 1, \ldots, n-1$.*

Let ε_i be the valuation of type 1v associated to E_i for $i = 1, \ldots, n-1$. Then for $i = 1, \ldots, n-1$ the natural map

$$\mathbb{Z}_\ell(1) = \mathrm{I}^{\log}_{\varepsilon_i} \otimes \mathbb{Z}_\ell \to \mathrm{I}^{\log}_y \otimes \mathbb{Z}_\ell \subset \left(\mathrm{I}^{\log}_{\alpha_1} \oplus \mathrm{I}^{\log}_{\alpha_2}\right) \otimes \mathbb{Q}_\ell = \mathbb{Q}_\ell(1) \oplus \mathbb{Q}_\ell(1)$$

is given by multiplication with $(\frac{i}{n}, \frac{n-i}{n})$ and thus remains injective after projection to each component.

Proof: Étale locally around y the situation is as follows. The local ring is $R = \mathfrak{o}[u,v]/(uv - \pi^n)$ with u and v parameters along Y_{α_1} and Y_{α_2} and n is the thickness of the double point singularity, that equals the length of the chain of E_i's connecting the strict transforms E_0, E_n in the minimal resolution $f : \mathscr{X} \to \mathscr{X}_{\mathrm{stable}}$.

The map f enhances to a log blow up and thus has a combinatorial description within the fs monoid $Q = \overline{\mathrm{M}}_y = R[1/\pi]^*/R^*$ which is spanned by u, v and π. We give a description of the dual monoids because in the end $\mathrm{Hom}(Q, \hat{\mathbb{Z}}'(1))$ equals the tame inertia $\mathrm{I}^{\log}_y$ at y. In coordinates dual to u, v we find

$$Q^\vee = \{(a,b) \in \frac{1}{n}(\mathbb{N}_0)^2 \ ; \ a + b \in \mathbb{Z}\}.$$

The log blow up corresponds to a subdivision of $Q^\vee$ as follows. The component E_i comes from the dual of the submonoid $P_i^\vee \subset Q^\vee$ generated by $(\frac{i}{n}, \frac{n-i}{n})$ and hence the node $E_i \cap E_{i+1}$ is given by the dual of

$$\langle (\frac{i}{n}, \frac{n-i}{n}), (\frac{i+1}{n}, \frac{n-i-1}{n}) \rangle \subset Q^\vee,$$

for $i = 0, \ldots, n-1$. From the fact, that this monoid is isomorphic to $\mathbb{N}^2$ we see again that $\mathscr{X}$ is indeed regular in the nodes $E_i \cap E_{i+1}$. Moreover, using the special values $i = 0$ and $i = n-1$ it follows that indeed

$$\mathrm{I}^{\log}_y \otimes \mathbb{Z}_\ell = Q^\vee \otimes \mathbb{Z}_\ell(1) \subset \left(\mathrm{I}^{\log}_{\alpha_1} \oplus \mathrm{I}^{\log}_{\alpha_2}\right) \otimes \mathbb{Q}_\ell = \mathbb{Q}_\ell(1) \oplus \mathbb{Q}_\ell(1)$$

with respect to the identity map on coordinates $(a,b) \in Q^\vee \otimes \mathbb{Z}_\ell(1) \mapsto (a,b) \in \mathbb{Q}_\ell(1) \oplus \mathbb{Q}_\ell(1)$. The proposition now follows from the identification

$$\mathrm{I}^{\log}_{\varepsilon_i} \otimes \mathbb{Z}_\ell = P_i^\vee \otimes \mathbb{Z}_\ell(1) = (\frac{i}{n}, \frac{n-i}{n}) \cdot \mathbb{Z}_\ell(1) \subset Q^\vee \otimes \mathbb{Z}_\ell(1) = \mathrm{I}^{\log}_y \otimes \mathbb{Z}_\ell.$$

□

Corollary 12. *Let X admit a stable model $\mathscr{X}_{\mathrm{stable}}$, and let $\mathscr{X}$ be a model of X with natural map $f : \mathscr{X} \to \mathscr{X}_{\mathrm{stable}}$. Let $\tilde{w}$ be a prolongation of the valuation $w \in \mathrm{Val}_{\mathfrak{o}}(K)$ of height 2, and let $y \in \mathscr{X}_{\mathrm{stable}}$ be the image $f(x_w)$ of the center of w under f. The logarithmic inertia $\mathrm{I}^{\log}_{\tilde{w}|w}$ is a subgroup of $\mathrm{I}^{\log}_y$ and the intersection*

$$\mathrm{I}^{\log}_{\tilde{w}|w} \cap \bigoplus_\alpha \mathrm{I}^{\log}_{\tilde{\alpha}|\alpha}$$

is of finite index in $\mathrm{I}^{\log}_{\tilde{w}|w}$, where α ranges over the valuations of type 1v associated to irreducible components of the special fibre of $\mathscr{X}_{\mathrm{stable}}$ that contain y, and the $\tilde{\alpha}$ are prolongations to $\tilde{K}$ such that $\mathrm{I}^{\log}_{\tilde{\alpha}|\alpha} \subset \mathrm{I}^{\log}_y$. More precisely, on every log étale cover of $\mathscr{X}_{\mathrm{stable}}$ the center of $\tilde{\alpha}$ is determined as the generic point of a component of the special fibre passing through the point above y which determines the embedding $\mathrm{I}^{\log}_y \subset \pi_1^{\log}(\mathscr{X})$.

Proof: This all follows from Proposition 7 and Proposition 11 except for the claim that $\bigoplus_\alpha \mathrm{I}^{\log}_{\tilde{\alpha}|\alpha}$ with $\tilde{\alpha}|\alpha$ as in the statement is indeed a subgroup in $\mathrm{I}^{\log}_y$. The latter is a consequence of Proposition 6 and Lemma 18 below. □

5.4. **The kernel of (log-)specialisation prime-to-p.**

Proposition 13. *Let $j : X \subset \mathscr{X}$ be the inclusion of a smooth, projective curve X/k as an open subset a model $\mathscr{X}$. The natural map*

$$\mathscr{N}_{X|\mathscr{X}} \to \mathscr{N}^{\log}_{X|\mathscr{X}}$$

induced by $\pi_1(j) : \pi_1(X) \to \pi_1^{\log}(\mathscr{X})$ from the kernel $\mathscr{N}_{X|\mathscr{X}} = \ker\big(\mathrm{sp} : \pi_1(X) \twoheadrightarrow \pi_1(\mathscr{X})\big)$ to the kernel $\mathscr{N}^{\log}_{X|\mathscr{X}}$ of the 'forget log map' $\pi_1(\varepsilon) : \pi_1^{\log}(\mathscr{X}) \twoheadrightarrow \pi_1(\mathscr{X})$ induces an isomorphism on the maximal prime-to-p quotients.

Proof: The log scheme $\mathscr{X}$ is log regular and has X as its locus of trivial log structure. The map $\pi_1(j) : \pi_1(X) \to \pi_1^{\log}(\mathscr{X})$ induces an isomorphism on maximal prime-to-p quotients by Fujiwara–Kato's purity for the log fundamental group [FK95] Thm 3.1, see also [Il02] Thm 7.6. The proposition follows in the limit from this reasoning applied to arbitrary finite étale covers $\mathscr{X}' \to \mathscr{X}$ and the corresponding generic fibre $X' \subset \mathscr{X}'$. □

6. Attractive components and the dark forest

6.1. **Preliminaries.**

6.1.1. *Disentangling the dual graph.* For a proper model $\mathscr{X}/\mathfrak{o}$ the reduced special fibre $Y = \mathscr{X}_{\mathbb{F},\mathrm{red}}$ has a dual graph $\Gamma = \Gamma_Y$ which describes the combinatorics of the components Y_α of Y with their mutual intersection. The completely split fundamental group $\pi_1^{\mathrm{cs}}(Y)$ is the quotient of $\pi_1(Y)$ which describes mock covers of Y, i.e., those finite étale covers which are geometrically completely split over every generic point of Y.

Lemma 14. *Two different components of the cover of Y corresponding to the maximal geometrically abelian exponent* 2 *quotient of $\pi_1^{\mathrm{cs}}(Y)$ intersect at most once.*

Proof: This is simply topological covering theory of finite graphs. □

The lemma says that any two components of the reduction disentangle after taking a finite étale cover, even with good completely split reduction.

6.1.2. *Sturdy reduction.* We recall the following result from [Mz96] Lemma 2.9.

Lemma 15 (Mochizuki)**.** *Every model $\mathscr{X}$ admits a finite log étale cover $\mathscr{X}' \to \mathscr{X}$ such that every strict transform of a component of the stable model in $\mathscr{X}'$ has genus at least* 2 *(such degenerations are called **sturdy** in* [Mz96]*).* □

Remark 16. The $\mathscr{X}'$ im Lemma 15 is usually not regular, but can be turned into a model by a minimal desingularisation of rational A_n-singularities in the nodes.

6.2. **Visible and attractive valuations of type 1v.**

Definition 17. A valuation of type 1v is **visible** if the associated component of the special fibre is covered by a component of positive genus in the reduction of a model of a suitable finite étale cover of the generic fibre. In this case we also call the associated component visible. A valuation of type 1v is **invisible** if it is not visible.

Let α be a valuation of type 1v and let Y_α be its associated component of the reduced special fibre $\mathscr{X}_{\mathbb{F},\mathrm{red}}$ of a model $\mathscr{X}$ of X/k. When we endow $\mathscr{X}$ with the vertical log structure coming from the special fibre $Y \hookrightarrow \mathscr{X}$ and moreover Y_α by the induced log structure, then we know from Lemma 14 and [Mz96] Prop 4.2 or [Sx02] Prop 6.2.11, that

$$\pi_1^{\log}(Y_\alpha) \hookrightarrow \pi_1^{\log}(\mathscr{X}, \bar{\eta})$$

is injective, whenever Y_α is a **stable** component, i.e, the strict transform of a component from the stable model of X. In particular every stable component acquires positive genus in a finite Kummer étale logarithmic cover of $\mathscr{X}$. The lemma below is immediate and we omit its proof.

Lemma 18. *For a valuation α of type 1v the following are equivalent.*

(a) *α is visible.*
(b) *Y_α is dominated by a component of the stable model of a finite étale cover of the generic fibre.*
(c) *The genus of a component from a model of a finite étale cover of the generic fibre which dominate Y_α tends to infinity in a cofinal tower of all finite étale covers of the generic fibre.*

Remark 19. For a given finite étale cover of the generic fibre there are only finitely many components of the special fibre which have positive genus. As $\pi_1(\overline{X}, \bar{\eta})$ is topologically finitely generated, the category of all finite étale covers has up to scalar extension only countably many isomorphism classes. Hence, there are at most countably many visible components on each model. As in our case the residue field of k is finite, every X/k admits only countably many components of the special fibres of its models up to taking strict transforms, so that we have no cardinality reason to argue that not all components are visible.

Definition 20. A valuation $w \in \mathrm{Val}_{\mathfrak{o}}(K)$ of type 1v is called **attractive** if for a prolongation $\tilde{w} \in \mathrm{Val}(\tilde{K})$ and any $g \in \pi_1(X, \bar{\eta})$, such that the union $\mathrm{Syl}_\ell(\mathrm{I}_{\tilde{w}|w})$ of the ℓ-Sylow groups of $\mathrm{I}_{\tilde{w}|w}$ meets the union of the ℓ-Sylow groups of $\mathrm{I}_{g(\tilde{w})|w}$ nontrivially, lies already in $\mathrm{D}_{\tilde{w}|w}$.

The notion of w being attractive does a priori depend on a prolongation $\tilde{w} \in \mathrm{Val}(\tilde{K})$, but in fact clearly is independent of such a choice.

Proposition 21. *A visible component is attractive.*

Proof: The group $\pi_1(X)$ is torsion free because it has finite cohomological dimension. Hence any nontrivial intersection of ℓ-Sylows of inertia at $\tilde{w}|w$ or $g(\tilde{w})|w$ must be infinite. In particular, the intersection remains nontrivial after replacing X by a finite étale cover such that $\tilde{w}$ and $g(\tilde{w})$ correspond to distinct components of the reduced special fibre of a suitable model. Now the proposition follows at once from Proposition 6. □

6.3. **Bridge elements and twigs.** We discuss the combinatorial structure of the union of all invisible components. Because of Lemma 18 the invisible components are of genus 0 over some field extension $\mathbb{F}'$ of $\mathbb{F}$ and meet the rest of the special fibre in at most a divisor of degree 2 over $\mathbb{F}'$. An invisible component is furthermore of one of the following two kinds.

Definition 22. A **bridge element** is an invisible irreducible component of the special fibre of a model $\mathscr{X}$, which is contained in a **bridge**, i.e, a chain of components $E_0 = Y_\alpha, E_1, \ldots, E_{e-1}$, $E_e = Y_{\alpha'}$ in the reduced special fibre $\mathscr{X}_{\mathbb{F},\mathrm{red}}$ where

(i) E_i meets E_{i-1} and E_{i+1} in a double point,
(ii) E_i is invisible for $i = 1, \ldots, e-1$,
(iii) Y_α and $Y_{\alpha'}$ are visible and not necessarily distinct components, the **bridge heads** of the bridge.

A valuation of type 1v is called a **bridge element** if the associated irreducible component on a fine enough model is a bridge element. Valuations α, α' of type 1v which give rise to the bridge heads $Y_\alpha, Y_{\alpha'}$ are also called **bridge heads**.

Remark 23. Due to Lemma 18, a bridge element can only be dominated by bridge elements in refinements of models or in models of finite, generically étale covers. Hence a valuation of type 1v belongs to a bridge on every model on which its associated divisor appears.

The complement of the set of visible components and bridge elements is a disjoint union of trees of invisible components.

Definition 24. A **twig** is an invisible component of the special fibre of a model $\mathscr{X}$ that is not a bridge element. A connected component of the union of all twigs is called an **invisible tree** on $\mathscr{X}$ and the union of all invisible trees forms the **dark forest** on $\mathscr{X}$. An invisible tree meets the union of bridge elements and visible components in exactly one closed point y of a visible component or bridge element α. The pair (α, y), or by abuse just α, is called the **root** of the invisible tree. The **root of a twig** is the root of the tree to which the given twig belongs.

Question 25. An important question that resists all our efforts to resolve it asks whether there are non-empty dark forests or even whether there are invisible components at all. The important work of Tamagawa [Ta04] towards this question only guarantees that any model has non-stable rational components which are visible.

6.4. **Resolution of non-singularities.** For the sake of reference we extract the following lemma from Tamagawa's work on resolution of non-singularities [Ta04].

Lemma 26. *Let y_1, y_2 be distinct closed points on a visible component Y_α of the reduced special fibre of a model $\mathscr{X}$ of X/k. Then there is a finite étale cover $X' \to X$ and a model $\mathscr{X}'$ of X' which allows an extension of the cover $f : \mathscr{X}' \to \mathscr{X}$, such that the following holds.*

(i) *We have distinct visible components $Y_{\alpha_1}, Y_{\alpha'}, Y_{\alpha_2}$ in the reduced special fibre of $\mathscr{X}'$.*
(ii) *$Y_{\alpha'}$ dominates Y_α under the map f.*
(iii) *$f(Y_{\alpha_i}) = y_i$ for $i = 1, 2$.*
(iv) *$Y_{\alpha'}$ and Y_{α_i} for $i = 1, 2$ intersect above y_i or are connected by a bridge, the bridge elements of which map to y_i under f.*

Proof: That we can find a cover $X' \to X$ with a model $\mathscr{X}'$ that satisfies (i)-(iii) follows directly from [Ta04] Thm 0.2 (v). Note that [Ta04] works with components of the stable model. By Lemma 18, an auxiliary cover allows first to replace Y_α by a component of the stable model.

Then, as our models are assumed to be regular, we have to resolve the singularities of the stable model, that is rational A_n-singularities, which only contributes additional chains of $\mathbb{P}^1$'s. By choosing Y_{α_i} visible and at minimal distance from $Y_{\alpha'}$ along such a chain yields the desired components. □

6.5. **The invisible can be attractive.** The sole purpose of this section is to prove the following proposition.

Proposition 27. *Bridge elements are attractive.*

Proof: Let $\beta \in \mathrm{Val}_{\mathfrak{o}}(K)$ be a valuation of type 1v that is a bridge element, and let $\tilde{\beta}$ be a prolongation to $\tilde{K}$, with respect to which we define inertia group $\mathrm{I}_{\tilde{\beta}|\beta}$ and decomposition group $\mathrm{D}_{\tilde{\beta}|\beta}$ of β. Let $g \in \pi_1(X, \bar{\eta})$ be an element such that

$$\mathrm{Syl}_\ell(\mathrm{I}_{\tilde{\beta}|\beta}) \cap \mathrm{Syl}_\ell(\mathrm{I}_{g(\tilde{\beta})|\beta}) \neq \{1\} \tag{6.1}$$

We have to show, that $g \in \mathrm{D}_{\tilde{\beta}|\beta}$, or equivalently, that g as a covering transformation of any intermediate finite Galois cover $X' \to X$ of the universal pro-étale cover $\tilde{X}/X$ fixes the component $Y_{\beta'}$ corresponding to the restriction of $\tilde{\beta}$ on an arbitrary fine enough Galois equivariant model $\mathscr{X}'$ of X' with function field K'.

We impose several extra conditions on X' and its model $\mathscr{X}'$ which are satisfied for a cofinal set of covers and models. It suffices to consider those for the proof of the proposition.

(1) Let $Y_{\alpha_1}, Y_{\alpha_2}$ be the bridge heads of the bridge of which β is a bridge element. We may assume that the components of $\mathscr{X}'$ above $Y_{\alpha_1}, Y_{\alpha_2}$ are of positive genus.
(2) The stable reduction theorem for curves, see [Ab00], shows that after a suitable finite Galois extension k'/k the base change $X' = X \times_k k'$ admits a stable model $\mathscr{X}'_{\text{stable}}$ over the ring of integers $\mathfrak{o}'$ of k'.

(3) By Lemma 15 we may furthermore assume, that every irreducible component of $\mathscr{X}'_{\text{stable}}$ has positive genus,
(4) and by Lemma 14 that there are no selfintersections for components of $\mathscr{X}'_{\text{stable}}$.

By the theory of minimal models there is a canonical map $f : \mathscr{X}' \to \mathscr{X}'_{\text{stable}}$. The fibres of f are connected trees of $\mathbb{P}^1$'s which meet the stable components of $\mathscr{X}'$ in one or two closed points. The bridge, of which β' is a bridge element, except for possibly its bridge heads, must be contained in a fibre $f^{-1}(y)$. But note that the bridge not necessarily equals the fibre.

Because of (1), the image $y = f(Y_{\beta'})$ must be a node of the stable model $\mathscr{X}'_{\text{stable}}$. Let $Y_{\sigma_1}, Y_{\sigma_2}$ be the, by (4) distinct, irreducible components of $\mathscr{X}'_{\text{stable}}$ which meet in $y = f(Y_{\beta'})$, and let σ_1, σ_2 denote the associated valuations of $\mathrm{Val}(K')$ of type 1v.

The map $f : \mathscr{X}' \to \mathscr{X}'_{\text{stable}}$ induces and isomorphism

$$\pi_1^{\log}(f) \;:\; \pi_1^{\log}(\mathscr{X}') \xrightarrow{\sim} \pi_1^{\log}(\mathscr{X}'_{\text{stable}}),$$

by Proposition 7. It is in this group, that the following reflections on log inertia groups take place. We deduce from Proposition 11 and Lemma 9, that the natural map

$$\mathbb{Q}_\ell(1) = \mathrm{I}^{\log}_{\beta'} \otimes \mathbb{Q}_\ell \to \mathrm{I}^{\log}_y \otimes \mathbb{Q}_\ell = \left(\mathrm{I}^{\log}_{\sigma_1} \oplus \mathrm{I}^{\log}_{\sigma_2}\right) \otimes \mathbb{Q}_\ell = \mathbb{Q}_\ell(1) \oplus \mathbb{Q}_\ell(1)$$

is injective in each component.

From the property (6.1) that is satisfied by g by assumption and the fact that $\pi_1(X)$ is torsion free, we deduce that

$$\mathrm{Syl}_\ell(\mathrm{I}_{\tilde{\beta}|\beta'}) \cap \mathrm{Syl}_\ell(\mathrm{I}_{g(\tilde{\beta})|\beta'}) \neq \{1\}$$

and so $\left(\mathrm{I}^{\log}_{\sigma_1} \oplus \mathrm{I}^{\log}_{\sigma_2}\right) \otimes \mathbb{Z}_\ell$ and $\left(\mathrm{I}^{\log}_{g(\sigma_1)} \oplus \mathrm{I}^{\log}_{g(\sigma_2)}\right) \otimes \mathbb{Z}_\ell$ meet nontrivially, namely in the image of the intersection of an ℓ-Sylow from $\mathrm{I}_{\beta'}$ and an ℓ-Sylow from $\mathrm{I}_{g(\beta')}$. Moreover, the intersection is nontrivial in such a way, that by Proposition 6 the set

$$\{Y_{\sigma_1}, Y_{\sigma_2}, g(Y_{\sigma_1}), g(Y_{\sigma_2})\}$$

must have cardinality 2, and therefore $g(\{Y_{\sigma_1}, Y_{\sigma_2}\}) = \{Y_{\sigma_1}, Y_{\sigma_2}\}$. We conclude that the node $y \in \mathscr{X}'_{\text{stable}}$ is mapped to a node $g(y) \in Y_{\sigma_1} \cap Y_{\sigma_2}$. If y is distinct from $g(y)$, then by applying Lemma 26 we may replace X' and $\mathscr{X}'$ further to the effect that for the new data

$$\{Y_{\sigma_1}, Y_{\sigma_2}, g(Y_{\sigma_1}), g(Y_{\sigma_2})\}$$

has cardinality 4, a contradiction. Hence we may assume that $g(y) = y$, which implies, that $Y_{\beta'}$ and $g(Y_{\beta'}) = Y_{g(\beta')}$ lies in the same fibre of $f : \mathscr{X}' \to \mathscr{X}'_{\text{stable}}$. Moreover, property (6.1) implies more precisely, that

$$\mathrm{I}^{\log}_{\beta'} \otimes \mathbb{Q}_\ell = \mathrm{I}^{\log}_{g(\beta')} \otimes \mathbb{Q}_\ell$$

as subgroups of $\mathrm{I}^{\log}_y \otimes \mathbb{Q}_\ell = \left(\mathrm{I}^{\log}_{\sigma_1} \oplus \mathrm{I}^{\log}_{\sigma_2}\right) \otimes \mathbb{Q}_\ell$. We deduce $g(\beta') = \beta'$ from Lemma 28 below, as desired to complete the proof of the proposition. □

Lemma 28. *Let $f : \mathscr{X}' \to \mathscr{X}'_{\text{stable}}$ be the canonical map from a model to the stable model. Let $Y_{\sigma_1}, Y_{\sigma_2}$ be two distinct irrducible compoents of positive genus of the stable model, that meet in a node y. Let $E_0, \ldots, E_n$ be the chain of components of $\mathscr{X}'$, i.e., E_i and E_{i+1} intersect in a single node of $\mathscr{X}'$ for $i = 0, \ldots, n-1$, given by $f^{-1}(y)$, that connects the strict transforms E_0, E_n of $Y_{\sigma_1}, Y_{\sigma_2}$. Let β'_i be the valuation of type 1v associated to E_i for $i = 0, \ldots, n$.*

Then the image under the canonical map

$$\mathbb{Q}_\ell(1) = \mathrm{I}^{\log}_{\beta'_i} \otimes \mathbb{Q}_\ell \to \mathrm{I}^{\log}_y \otimes \mathbb{Q}_\ell = \left(\mathrm{I}^{\log}_{\sigma_1} \oplus \mathrm{I}^{\log}_{\sigma_2}\right) \otimes \mathbb{Q}_\ell = \mathbb{Q}_\ell(1) \oplus \mathbb{Q}_\ell(1)$$

is spanned by $(t_i, 1 - t_i)$ for a strictly monotone decreasing sequence

$$t_0 = 1 > t_1 > \ldots > t_{n-1} > t_n = 0$$

of rational values $t_i \in \mathbb{Q}$. In particular, no two of these images coincide.

Proof: The chain $f^{-1}(y)$ can be constructed in two steps. There is an intermediate model, the minimal regular model $\mathscr{X}' \to \mathscr{X}'_{\min} \xrightarrow{f_{\min}} \mathscr{X}'_{\text{stable}}$, which requires first to resolve the rational singularity of $\mathscr{X}'_{\text{stable}}$ in the node. This has been studied in detail in Proposition 11. The result for the strict transforms of the irreducible components of $f_{\min}^{-1}(y)$ is in accordance with the claim, as we find the sequence $t_i = (n-i)/n$.

As a second step we may refine the chain by blowing up nodes on regular models, which is covered in the analysis of Lemma 9. Here the answer is that the new exceptional component in between components with parameter s and t will have image of ℓ-adic log inertia described by the parameter $(s+t)/2$, hence the arithmetic mean, which is always strictly contained in between the two different values s, t. We conclude by induction.

Two finish the proof of the lemma we check that $(t, 1-t)\mathbb{Q}_\ell = (s, 1-s)\mathbb{Q}_\ell$ implies that there is a $\lambda \in \mathbb{Q}_\ell^*$ with $\lambda(t, 1-t) = (s, 1-s)$. Summing up the coordinates yields $\lambda = 1$ and thus $t = s$ as claimed. $\square$

7. Sections and the Brauer group method

7.1. Local–global principles for the Brauer group. Lichtenbaum constructs for a smooth projective curve X over the p-adic field k a perfect duality pairing [Li69] §5 Thm 4

$$\mathrm{Br}(X) \times \mathrm{Pic}(X) \to \mathrm{Br}(k) = \mathbb{Q}/\mathbb{Z}. \tag{7.1}$$

The vanishing of the left kernel of (7.1) translates into the injectivity of the map

$$\mathrm{Br}(X) \hookrightarrow \prod_{a \in X_0} \mathrm{Br}(\kappa(a)), \tag{7.2}$$

which evaluates a Brauer class on X in every closed point $a \in X$.

Let $\mathscr{X}$ be a model of X/k. A closed point $a \in X$ has the henselian local scheme $Z_a \subset \mathscr{X}$ as its Zariski-closure in $\mathscr{X}$. The unique closed point of Z_a is the topological intersection $y_a = Z_a \cap \mathscr{X}_{\mathbb{F}}$ with the special fibre of the model. Because Z_a is henselian, its inclusion to $\mathscr{X}$ lifts to the scheme of nearby points $\mathscr{X}^{\mathrm{h}}_{y_a}$, and the lift induces a map $\mathrm{Spec}(\kappa(a)) \to \mathscr{U}^{\mathrm{h}}_{y_a}$ that lifts the point $a \in X$. It follows immediately from (7.2) that we also have a local global principle

$$\mathrm{Br}(X) \hookrightarrow \prod_{y \in \mathscr{X}_0} \mathrm{Br}(\mathscr{U}^{\mathrm{h}}_y) \tag{7.3}$$

with respect to the nearby points $\mathscr{U}^{\mathrm{h}}_y$ associated to the closed points $y \in \mathscr{X}_0$ of a model. In the direct limit over all models of X/k we find the composition

$$\mathrm{Br}(X) \hookrightarrow \varinjlim_{\mathscr{X}} \prod_{y \in \mathscr{X}_0} \mathrm{Br}(\mathscr{U}^{\mathrm{h}}_y) \to \prod_{w \in \mathrm{Val}_v(K)} \mathrm{Br}(K^{\mathrm{h}}_w). \tag{7.4}$$

The last map follows from the restriction map

$$\varinjlim_{\mathscr{X}} \mathrm{Br}(\mathscr{U}^{\mathrm{h}}_{x_w}) \xrightarrow{\sim} \mathrm{Br}(U^{\mathrm{h}}_w) \hookrightarrow \mathrm{Br}(K^{\mathrm{h}}_w), \tag{7.5}$$

and is injective by purity for the Brauer group. Moreover, if w is not of type 2h, the map (7.5) is an isomorphism by the compatibility of henselisation and the Brauer group with direct limits.

Proposition 29. *Let $\mathscr{X}$ be a model of X/k. Let $A \in \mathrm{Br}(X)$ be a Brauer class. Then the set*

$$\{y \in \mathscr{X}_{\mathbb{F},\mathrm{red}} \ ; \ A \text{ is nontrivial in } \mathrm{Br}(\mathscr{U}^{\mathrm{h}}_y)\}$$

is closed in the constructible topology $\mathscr{X}_{\mathrm{cons}}$.

Proof: We only have to argue that if A vanishes in $\mathrm{Br}(\mathscr{U}^{\mathrm{h}}_\alpha)$ for some generic point α of $\mathscr{X}_{\mathbb{F},\mathrm{red}}$, then A vanishes in $\mathrm{Br}(\mathscr{U}^{\mathrm{h}}_y)$ for all but finitely many closed points y in the closure of α. But if A vanishes in $\mathrm{Br}(\mathscr{U}^{\mathrm{h}}_\alpha)$, then this occurs already on $V = \mathscr{V} \times_{\mathscr{X}} X$ for some strict étale neighbourhood $\mathscr{V} \to \mathscr{X}$ of α. For allmost all y in question the natural map $\mathscr{U}_y^{\rm h} \to X$ factors over V and therefore A also vanishes in those $\mathrm{Br}(\mathscr{U}_y^{\rm h})$. □

Corollary 30. *Let $A \in \mathrm{Br}(X)$ be a Brauer class. Then the set*

$$\{w \in \mathrm{Val}_{\mathfrak{o}}(K)\ ;\ A \text{ is nontrivial in } \mathrm{Br}(K_w^{\rm h})\}$$

is closed in the patch topology on $\mathrm{Val}_{\mathfrak{o}}(K)$.

Proof: Corollary 30 follows at once from Proposition 29 because $\mathrm{Val}_{\mathfrak{o}}(K)$ with the patch topology is homeomorphic with $\varprojlim_{\mathscr{X}} \mathscr{X}_{\rm cons}$. □

Proposition 29 and the fact that a projective limit of nonempty compact spaces is nonempty shows that the composite map in (7.4) is also injective, see originally [P88] Thm 4.5 and moreover [Wi95] for an alternative proof without model theory. More precisely, by taking limits and exploiting the compactness of the patch topology we find the following generalization of (7.4):

Theorem 31 ([P88] Thm 4.5)**.** *Let M/k be a function field of transcendence degree 1 over k. Then the following restriction map is injective*

$$\mathrm{Br}(M) \hookrightarrow \prod_{w|v} \mathrm{Br}(M_w^{\rm h}), \tag{7.6}$$

where the product ranges over all valuations w of M extending the p-adic valuation v on k.

7.2. **Computation of the Brauer group of a henselisation — case of height 2.** We are interested in controlling the kernel of $\mathrm{Br}(k) \to \mathrm{Br}(K_w^{\rm h})$ for a valuation $w \in \mathrm{Val}_v(K)$. We will compute for each model $\mathscr{X}$ a relevant subgroup of $\mathrm{Br}(\mathscr{U}_y^{\rm h})$ and take the limit over all models as in (7.5).

Let $y \in \mathscr{X}$ be a closed point of a model. The cohomology sheaves with support of $\mathbb{G}_m$ for $i : Y_y^{\rm h} := \mathscr{X}_y^{\rm h} \setminus \mathscr{U}_y^{\rm h} \hookrightarrow \mathscr{X}_y^{\rm h}$ are $\mathrm{R}^q\, i^! \mathbb{G}_m = 0$ for $q = 0, 2$ and

$$\mathrm{R}^1\, i^! \mathbb{G}_m = j_* \mathbb{G}_m / \mathbb{G}_m \cong \bigoplus i_{Y_\alpha,*}\mathbb{Z}$$

with the isomorphism induced by the valuation w_α on the function field K of $\mathscr{X}$ defined by the components $i_\alpha : Y_\alpha \hookrightarrow \mathscr{X}_y^{\rm h}$ of $Y_y^{\rm h}$. Moreover, $i_* \mathrm{R}^3\, i^! \mathbb{G}_m = \mathrm{R}^2\, j_* \mathbb{G}_m$ with open immersion $j : \mathscr{U}_y^{\rm h} \subset \mathscr{X}_y^{\rm h}$ has stalk $(\mathrm{R}^3\, i^! \mathbb{G}_m)_{\bar{y}} = \mathrm{Br}(\mathscr{U}_y^{\rm sh})$ and therefore the map

$$\mathrm{Br}(\mathscr{U}_y^{\rm h}) \to \mathrm{H}^3_{Y_y^{\rm h}}\left(\mathscr{X}_y^{\rm h}, \mathbb{G}_m\right) \to \mathrm{H}^0\left(Y_y^{\rm h}, \mathrm{R}^3\, i^! \mathbb{G}_m\right) = \mathrm{H}^0\left(y, (\mathrm{R}^3\, i^! \mathbb{G}_m)_{\bar{y}}\right)$$

has kernel the relative Brauer group $\mathrm{Br}(\mathscr{U}_y^{\rm sh}/\mathscr{U}_y^{\rm h})$ of classes in $\mathrm{Br}(\mathscr{U}_y^{\rm h})$ which die when restricted to $\mathscr{U}_y^{\rm sh}$. In the limit over all models we get $\mathrm{Br}(U_w^{\rm sh}/U_w^{\rm h}) \subset \mathrm{Br}(K_w^{\rm h})$. By the computation of $\mathrm{Br}(k)$ along $i : \mathrm{Spec}(\mathbb{F}) \hookrightarrow \mathrm{Spec}(\mathfrak{o})$ as

$$\mathrm{Br}(k) = \mathrm{H}^3_{\mathrm{Spec}(\mathbb{F})}\left(\mathrm{Spec}(\mathfrak{o}), \mathbb{G}_m\right) = \mathrm{H}^2\left(\mathbb{F}, \mathrm{R}^1\, i^! \mathbb{G}_m\right) = \mathrm{Hom}\left(\mathrm{Gal}_{\mathbb{F}}, v(k) \otimes \mathbb{Q}/\mathbb{Z}\right) \cong \mathbb{Q}/\mathbb{Z}$$

the subgroup $\mathrm{Br}(\mathscr{U}_y^{\rm sh}/\mathscr{U}_y^{\rm h})$ receives the image of the restriction map $\mathrm{Br}(k) \to \mathrm{Br}(\mathscr{U}_y^{\rm h})$. Let Gal_y be the absolute Galois group of the residue field $\kappa(y)$ at y. Using $\mathrm{Br}(\mathscr{X}_y^{\rm h}) = \mathrm{Br}(\kappa(y)) = 0$ and $\mathrm{H}^3\left(\mathscr{X}_y^{h}, \mathbb{G}_m\right) = \mathrm{H}^3\left(\kappa(y), \mathbb{G}_m\right) = 0$, the relative cohomology sequence for $(\mathscr{X}_y^{\rm h}, \mathscr{U}_y^{\rm h})$ yields an isomorphism of $\mathrm{Br}(\mathscr{U}_y^{\rm sh}/\mathscr{U}_y^{\rm h})$ with

$$\mathrm{H}^2\left(Y_y^{\rm h}, \mathrm{R}^1\, i^! \mathbb{G}_m\right) = \bigoplus_\alpha \mathrm{Hom}\left(\mathrm{Gal}_y, w_\alpha(K) \otimes \mathbb{Q}/\mathbb{Z}\right) = \mathrm{Hom}\left(\mathrm{Gal}_y, \mathbb{G}_m(\mathscr{U}_y^{\rm h})/\mathbb{G}_m(\mathscr{X}_y^{\rm h})\right).$$

In the limit over all models we obtain the following proposition:

Proposition 32. *Let $w \in \mathrm{Val}_v(K)$ be a valuation of height 2. The map $\mathrm{Br}(k) \to \mathrm{Br}(U_w^{\rm sh}/U_w^{\rm h})$ is isomorphic to the map:*

(1) *If w is not of type 2h, then*

$$\mathrm{Hom}\left(\mathrm{Gal}_{\mathbb{F}}, v(k) \otimes \mathbb{Q}/\mathbb{Z}\right) \to \mathrm{Hom}\left(\mathrm{Gal}_{\kappa(w)}, w(K) \otimes \mathbb{Q}/\mathbb{Z}\right),$$

(2) *If $w = v_{\kappa(a)} \circ w_a$ is of type 2h refining w_a of type 1h for a closed point $a \in X$ and $v_{\kappa(a)}$ the p-adic valuation of the residue field $\kappa(a)$, then*

$$\mathrm{Hom}\left(\mathrm{Gal}_{\mathbb{F}}, v(k) \otimes \mathbb{Q}/\mathbb{Z}\right) \to \mathrm{Hom}\left(\mathrm{Gal}_{\kappa(w)}, v(\kappa(a)) \otimes \mathbb{Q}/\mathbb{Z}\right),$$

where the maps are defined by the inclusion map $v(k) \hookrightarrow w(K)$, resp. $v(k) \hookrightarrow v(\kappa(a))$, of value groups and the restriction map $\mathrm{Gal}_{\kappa(w)} \to \mathrm{Gal}_{\mathbb{F}}$. □

Corollary 33. *The class of invariant $1/\ell$ survives in $\mathrm{Br}(K_w^{\mathrm{h}})$ for a valuation w of height 2 if and only if the degree of the residue field extension $\kappa(w)/\mathbb{F}$ is prime to ℓ and the value $w(\pi)$ of a uniformizer π of k is not divisible by ℓ in the value group $w(K)$.* □

7.3. **Local–semilocal principle for the Brauer group.** Let α be a valuation of type 1v, and let Y_α be the associated divisor in suitably fine models. The scheme Y_α is a smooth projective curve over a finite extension $\mathbb{F}_\alpha/\mathbb{F}$ as field of constants. We define $\mathrm{Br}'(K_\alpha^{\mathrm{h}})$ as the preimage of $\mathrm{H}^1(\pi_1(Y_\alpha), \mathbb{Q}/\mathbb{Z}) \subset \mathrm{H}^1(\kappa(\alpha), \mathbb{Q}/\mathbb{Z}) = \mathrm{H}^2(\kappa(\alpha), \mathrm{R}^1\, i^!\mathbb{G}_m)$ under the natural map

$$\mathrm{Br}(K_\alpha^{\mathrm{h}}) \to \mathrm{H}^3_\alpha\left(\mathscr{X}_\alpha^{\mathrm{h}}, \mathbb{G}_m\right),$$

from the relative cohomology sequence and the local to global spectral sequence associated to the regular embedding $i : \mathrm{Spec}(\kappa(\alpha)) \hookrightarrow \mathscr{X}_\alpha^{\mathrm{h}}$. The subgroup $\mathrm{Br}'(K_\alpha^{\mathrm{h}})$ receives the image of the restriction map $\mathrm{Br}(k) \to \mathrm{Br}(K_\alpha^{\mathrm{h}})$. By $\mathrm{H}^3\left(\mathscr{X}_\alpha^{\mathrm{h}}, \mathbb{G}_m\right) = \mathrm{H}^3\left(\kappa(\alpha), \mathbb{G}_m\right) = 0$, we can extract from the relative cohomology sequence the exact sequence

$$0 \to \mathrm{Br}(\kappa(\alpha)) \to \mathrm{Br}'(K_\alpha^{\mathrm{h}}) \to \mathrm{H}^1\left(\pi_1(Y_\alpha), \mathbb{Q}/\mathbb{Z}\right) \to 0. \tag{7.7}$$

A valuation $w_{y\alpha} = v_y \circ \alpha$ which refines α by means of a closed point $y \in Y_\alpha$ has henselisation $K_{w_{y\alpha}}^{\mathrm{h}}$ containing K_α^{h}. The restriction map $\mathrm{Br}(K_\alpha^{\mathrm{h}}) \to \mathrm{Br}(K_{w_{y\alpha}}^{\mathrm{h}})$ respects the respective subgroups $\mathrm{Br}'(K_{w_{y\alpha}}^{\mathrm{h}}) \to \mathrm{Br}(U_{w_{y\alpha}}^{\mathrm{sh}}/U_{w_{y\alpha}}^{\mathrm{h}})$. The value group of w sits in an exact sequence of torsion free groups

$$0 \to v_y(\kappa(\alpha)) \to w_{y\alpha}(K) \to w_\alpha(K) \to 0,$$

which therefore remains exact after applying $\mathrm{Hom}(\mathrm{Gal}_y, (-) \otimes \mathbb{Q}/\mathbb{Z})$. The restriction maps on Brauer groups for all such $w_{y\alpha} := v_y \circ w_\alpha$ fit into a map

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Br}(\kappa(\alpha)) \arrow[r] \arrow[d, "\lambda_1"] & \mathrm{Br}'(K_\alpha^{\mathrm{h}}) \arrow[r] \arrow[d, "\lambda_2"] & \mathrm{Hom}(\pi_1(Y_\alpha), \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, "\lambda_3"] & 0 \\
0 \arrow[r] & \prod_{y \in Y_\alpha} \mathrm{Hom}\left(\mathrm{Gal}_y, \mathbb{Q}/\mathbb{Z}\right) \arrow[r] & \prod_{y \in Y_\alpha} \mathrm{Br}(U_{w_{y\alpha}}^{\mathrm{sh}}/U_{w_{y\alpha}}^{\mathrm{h}}) \arrow[r] & \prod_{y \in Y_\alpha} \mathrm{Hom}\left(\mathrm{Gal}_y, \mathbb{Q}/\mathbb{Z}\right) \arrow[r] & 0
\end{tikzcd} \tag{7.8}$$

of exact sequences. The homomorphism λ_1 is injective by the local–global principle for the Brauer group of the function field $\kappa(\alpha)$. The homomorphism λ_3 restricts an unramified character to the decomposition group of $y \in Y_\alpha$ and is injective because the set of Frobenius elements is dense in $\pi_1(Y_\alpha)$. Hence by the 5-Lemma we deduce the following local–semilocal principle:

Proposition 34. *The restriction map*

$$\mathrm{Br}'(K_\alpha^{\mathrm{h}}) \hookrightarrow \prod_{y \in Y_\alpha} \mathrm{Br}(U_{w_{y\alpha}}^{\mathrm{sh}}/U_{w_{y\alpha}}^{\mathrm{h}}) \tag{7.9}$$

is injective. □

7.4. **The Brauer group of the decomposition pro-cover of a section.** In this section we fix a section $s : \mathrm{Gal}_k \to \pi_1(X, \bar{\eta})$ of the fundamental group extension $\pi_1(X/k)$.

7.4.1. *The decomposition pro-cover of a section.* The section s induces a right action of Gal_k on the universal pro-étale cover $\tilde{X}$ of the curve X/k. The corresponding quotient
$$\tilde{X}_s = \tilde{X}/s(\mathrm{Gal}_k)$$
is the maximal subcover X'/X of $\tilde{X}/X$ such that the section s lifts to a section of the composition $\pi_1(X') \subset \pi_1(X,\bar{\eta}) \to \mathrm{Gal}_k$. In fact, $\pi_1(\tilde{X}_s)$ is nothing but the image $s(\mathrm{Gal}_k)$ in $\pi_1(X,\bar{\eta})$.

7.4.2. *Relative Brauer groups and sections.* The **relative Brauer group** $\mathrm{Br}(X/k)$ of the p-adic curve X/k is the kernel of the pullback map $\mathrm{Br}(k) \to \mathrm{Br}(X)$. By a theorem of Roquette and Lichtenbaum, see [Li69] Thm p.120, we know that $\mathrm{Br}(X/k)$ is cyclic of order the index of X. The index of X is defined as $\gcd(\deg(D))$, where D ranges over all k-rational divisors on X.

Proposition 35. *For a section s of $\pi_1(X/k)$ and any $\ell \neq p$ the natural map*
$$\mathrm{Br}(k) \otimes \mathbb{Z}_\ell \xrightarrow{\sim} \mathrm{Br}(\tilde{X}_s) \otimes \mathbb{Z}_\ell$$
is an isomorphism.

Proof: The Leray spectral sequence yields an exact sequence
$$0 \to \mathrm{Br}(X/k) \to \mathrm{Br}(k) \to \mathrm{Br}(X) \to \mathrm{H}^1\left(k, \mathrm{Pic}_X\right). \tag{7.10}$$
For every finite subcovers $X' \to X$ of $\tilde{X}_s$ the section s lifts canonically to a section of $\pi_1(X'/k)$. The presence of a section implies that the index is a power of p, see [Sx10] Thm 15, so that $\mathrm{Br}(X'/k)$ is a cyclic p-group. In the limit over all finite subcovers X'/X of $\tilde{X}_s$ of (7.10) we therefore find the exact sequence
$$0 \to \mathrm{Br}(k) \otimes \mathbb{Z}_\ell \to \mathrm{Br}(\tilde{X}_s) \otimes \mathbb{Z}_\ell \to \varinjlim_{\tilde{X}_s/X'/X} \mathrm{H}^1(k, \mathrm{Pic}_{X'}) \otimes \mathbb{Z}_\ell.$$
The degree sequence on every X' gives an exact sequence
$$0 \to \mathbb{Z}/\operatorname{period}(X')\mathbb{Z} \to \mathrm{H}^1(k, \mathrm{Pic}^0_{X'}) \to \mathrm{H}^1(k, \mathrm{Pic}_{X'}) \to \mathrm{H}^1(k, \mathbb{Z}) = 0.$$
The period, i.e., the order of $[\mathrm{Pic}^1] \in \mathrm{H}^1(k, \mathrm{Pic}^0)$, divides the index and thus is a power of p for any subcover $X' \to X$ of $\tilde{X}_s$. We therefore find in the limit
$$\varinjlim_{\tilde{X}_s/X'/X} \mathrm{H}^1(k, \mathrm{Pic}_{X'}) \otimes \mathbb{Z}_\ell \cong \varinjlim_{\tilde{X}_s/X'/X} \mathrm{H}^1(k, \mathrm{Pic}^0_{X'}) \otimes \mathbb{Z}_\ell \cong \mathrm{H}^1(k, \varinjlim_{\tilde{X}_s/X'/X} \mathrm{Pic}^0_{X'}) \otimes \mathbb{Z}_\ell \cong 0$$
because $\varinjlim_{\tilde{X}_s/X'/X} \mathrm{Pic}^0_{X'}$ is uniquely divisible. Indeed, for every finite subcover X'/X of $\tilde{X}_s$ and every $n \in \mathbb{N}$ we have a further subcover X'_n/X' of $\tilde{X}_s$, such that $\mathrm{Pic}^0_{X'} \to \mathrm{Pic}^0_{X'_n}$ factors over the multiplication by n map of $\mathrm{Pic}^0_{X'}$. Namely, if X' corresponds to $H \cdot s(\mathrm{Gal}_k) \subset \pi_1(X)$ with $H \subset \pi_1(X \times_k k^{\mathrm{alg}})$ open, then X'_n corresponds to $\overline{[H,H]H^n} \cdot s(\mathrm{Gal}_k)$. $\square$

7.5. Detecting a valuation from a section.

Theorem 36. *Let $s : \mathrm{Gal}_k \to \pi_1(X)$ be a section. There is a valuation $w \in \mathrm{Val}_v(K)$ with prolongation $\tilde{w}$ to $\tilde{K}$ such that*

(i) *the image $s(\mathrm{Gal}_{k,\ell})$ of an ℓ-Sylow subgroup $\mathrm{Gal}_{k,\ell}$ of Gal_k is contained in $\mathrm{D}_{\tilde{w}|w}$, and*
(ii) *the image $s(\mathrm{I}_{k,\ell})$ of the ℓ-Sylow subgroup $\mathrm{I}_{k,\ell} = \mathrm{Gal}_{k,\ell} \cap \mathrm{I}_k$ is contained in $\mathrm{I}_{\tilde{w}|w}$.*

Proof: Let $M = k(\tilde{X}_s)$ be the function field of the decomposition pro-cover $\tilde{X}_s$ of the section s. By Theorem 31 and Proposition 35 there is a valuation $w \in \mathrm{Val}_v(M)$, such that the Brauer class of invariant $1/\ell$ in $\mathrm{Br}(k)$ is nontrivial in the Brauer group of the henselisation M^{h}_w. We claim, that $s(\mathrm{Gal}_{k,\ell})$ is contained in $\mathrm{D}_{\tilde{w}|w}$ for an appropriate prolongation $\tilde{w} \in \mathrm{Val}_v(\tilde{K})$ of w.

With $\Lambda = \tilde{K} \cap M^{\mathrm{h}}_w$ the claim is equivalent to the degree of Λ/M being prime to ℓ in the sense of supernatural numbers. By construction of M there is an algebraic extension λ/k such that $\Lambda = \lambda M$ and the degree of Λ/M equals the degree of λ/k. The defining property of w implies that $\mathrm{Br}(k)\otimes\mathbb{Z}_\ell\hookrightarrow\mathrm{Br}(\lambda)\otimes\mathbb{Z}_\ell$ is injective, which forces the degree of λ/k to be prime to ℓ. This proves the claim and we have found a valuation w such that (i) holds.

In order to enforce property (ii), let us first assume that the valuation w constructed above is of height 2. In the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{I}_{\tilde{w}|w} \arrow[r] \arrow[d] & \mathrm{D}_{\tilde{w}|w} \arrow[r] \arrow[d] & \mathrm{Gal}_{\kappa(w)} \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & \mathrm{I}_k \arrow[r] & \mathrm{Gal}_k \arrow[r] & \mathrm{Gal}_{\mathbb{F}} \arrow[r] & 1 \end{tikzcd}$$

the rightmost vertical map injective. Hence the ℓ-Sylow subgroup $s(\mathrm{I}_{k,\ell})=s(\mathrm{Gal}_{k,\ell})\cap\mathrm{I}_k$ of $s(\mathrm{I}_k)$ is automatically contained in $\mathrm{I}_{\tilde{w}|w}$.

It remains to discuss the case where a priori $w=\alpha$ is a valuation of M of type 1v and A, the Brauer class of invariant $1/\ell$ in $\mathrm{Br}(k)$, vanishes in $\mathrm{Br}(M^{\mathrm{h}}_{w_{y\alpha}})$ for all valuations $w_{y\alpha}=v_y\circ\alpha$ of height 2. The exact sequence (7.7) yields in the limit an exact sequence

$$0\to\mathrm{Br}(\kappa(\alpha))\to\mathrm{Br}'(M^{\mathrm{h}}_\alpha)\to\mathrm{H}^1\big(\kappa(\alpha),\alpha(M)\otimes\mathbb{Q}/\mathbb{Z}\big). \tag{7.11}$$

The local–global principle for the Brauer group of $\kappa(\alpha)$ yields an injection

$$\mathrm{Br}(\kappa(\alpha))\hookrightarrow\prod_y\mathrm{Hom}\left(\mathrm{Gal}_{\kappa(w_{y\alpha})},\mathbb{Q}/\mathbb{Z}\right)\hookrightarrow\prod_y\mathrm{Br}\left(M^{\mathrm{h}}_{w_{y\alpha}}\right)$$

where y ranges over the closed points of Y_α and so the composite valuations $w_{y\alpha}=v_y\circ\alpha$ are the refinements of type 2v of α. Hence the restriction of A in $\mathrm{Br}'(M^{\mathrm{h}}_\alpha)$ is a ramified class, i.e., it induces a nontrivial character $\chi_A\in\mathrm{H}^1\left(\kappa(\alpha),\alpha(M)\otimes\mathbb{Q}/\mathbb{Z}\right)$. In fact χ_A is the character

$$\chi_A:\mathrm{Gal}_{\kappa(\alpha)}\to\mathrm{Gal}_{\mathbb{F}}\xrightarrow{\mathrm{Frob}\mapsto v(\pi)\otimes 1/\ell}v(k)\otimes\mathbb{Q}/\mathbb{Z}\to\alpha(M)\otimes\mathbb{Q}/\mathbb{Z},$$

where π is a uniformizer of $\mathfrak{o}$. For χ_A to be nontrivial means that the image of

$$\mathrm{D}_{\tilde{\alpha}|\alpha}=\mathrm{Gal}(\tilde{K}/M^{\mathrm{h}}_\alpha)\subset\mathrm{Gal}(\tilde{K}/M)=s(\mathrm{Gal}_k)\to\mathrm{Gal}_k\to\mathrm{Gal}_{\mathbb{F}}$$

contains the ℓ-Sylow subgroup of $\mathrm{Gal}_{\mathbb{F}}$, and the ramification index $e(\alpha/v)$ is prime to ℓ.

Let L/M be a subextension of $\tilde{K}/M$ with an unramified prolongation α_L of α and such that the residue field of α_L has Galois group

$$\mathrm{Gal}(\kappa(\tilde{\alpha})/\kappa(\alpha_L))\subset\mathrm{Gal}(\kappa(\tilde{\alpha})/\kappa(\alpha))$$

that projects isomorphically to the ℓ-Sylow subgroup $\mathrm{Gal}_{\mathbb{F},\ell}$ of $\mathrm{Gal}_{\mathbb{F}}$. Consequently, the restriction of A to $\mathrm{Br}(L^{\mathrm{h}}_{\alpha_L})$ still does not vanish because the restriction of the character χ_A remains nontrivial. By construction we have a short exact sequence

$$1\to\mathrm{I}_{\tilde{\alpha}|\alpha}\to\mathrm{D}_{\tilde{\alpha}|\alpha_L}\to\mathrm{Gal}_{\mathbb{F},\ell}\to 1. \tag{7.12}$$

The argument in the first part of the proof shows that (i) holds for L and α_L, thus showing that $s(\mathrm{Gal}_{k,\ell})\subset\mathrm{D}_{\tilde{\alpha}|\alpha_L}$. By a diagram chase with (7.12) we deduce that

$$s(\mathrm{I}_{k,\ell})\subset\mathrm{I}_{\tilde{\alpha}|\alpha}$$

and so α indeed also satisfies (ii). □

8. Every section has its place

In this section we formulate and prove the main theorem.

8.1. The main theorem as a fixed point problem. The T^G be the set of fixed points for a continuous G-action on a space T.

8.1.1. *The action.* The fundamental group $\pi_1(X,\bar\eta)$ is anti-isomorphic to the group of covering transformations of the universal pro-étale cover $\tilde X/X$. Thus $\pi_1(X,\bar\eta)$ acts on $\tilde X$ from the right and on its function field $\tilde K$ from the left. The action on $\tilde X$ is continuous in the sense that the induced action on a finite intermediate Galois cover X'/X factors through a finite quotient of $\pi_1(X,\bar\eta)$. Cofinally in the set of all intermediate covers X'/X and their models $\mathscr{X}'$ we find Galois equivariant models to which the $\pi_1(X,\bar\eta)$-action uniquely extends. The set of valuations $\mathrm{Val}_v(\tilde K)$ of $\tilde K$ extending v inherits a continuous $\pi_1(X,\bar\eta)$-action from the right as the pro-finite limit of $\pi_1(X,\bar\eta)$-spaces

$$\varprojlim_{X'\subset\mathscr{X}'} \mathscr{X}'_{\mathbb{F},\mathrm{cons}},$$

where $\mathscr{X}'$ ranges over a cofinal system of Galois equivariant models.

8.1.2. *Fixed points and decomposition subgroups.* The stabilizer of a valuation $\tilde w \in \mathrm{Val}_v(\tilde K)$ is nothing but the decomposition subgroup $\mathrm{D}_{\tilde w|w} \subset \pi_1(X,\bar\eta)$. Hence for a subgroup $G \subset \pi_1(X,\bar\eta)$ we have $G \subseteq \mathrm{D}_{\tilde w|w}$ if and only if $\tilde w$ belongs to the fixed points $\mathrm{Val}_v(\tilde K)^G$ of the induced G-action.

8.1.3. *The main theorem.*

Theorem 37. *Let X be a smooth, hyperbolic, geometrically connected curve over a finite extension k of $\mathbb{Q}_p$. Then for any section s of $\pi_1(X/k)$ there exists a valuation $\tilde w \in \mathrm{Val}_v(\tilde K)$, such that the image $s(\mathrm{Gal}_k)$ is contained in the decomposition subgroup $\mathrm{D}_{\tilde w|w}$.*

Proof: Let Σ be the image of $s(\mathrm{Gal}_k)$. By the above we have to show that the set of fixed points

$$\Big(\mathrm{Val}_v(\tilde K)\Big)^\Sigma = \Big(\varprojlim_{X'\subset\mathscr{X}'} \mathscr{X}'_{\mathbb{F},\mathrm{cons}}\Big)^\Sigma = \varprojlim_{X'\subset\mathscr{X}'} \big(\mathscr{X}'_{\mathbb{F},\mathrm{cons}}\big)^\Sigma$$

is non-empty, where $X' \subset \mathscr{X}'$ ranges over Galois equivariant models of the smooth projective compactifications of finite Galois étale covers X'/X in $\tilde X$. But for each Galois equivariant model the set of fixed points $\big(\mathscr{X}'_{\mathbb{F},\mathrm{cons}}\big)^\Sigma$ is a closed subset of the pro-finite set $\mathscr{X}'_{\mathbb{F},\mathrm{cons}}$ and thus compact. The projective limit of compact sets is non-empty if and only if each member of the limit is non-empty, which reduces the proof to the following Theorem 38. $\square$

Theorem 38. *Let $\Sigma \subset \pi_1(X,\bar\eta)$ be the image of a section $s : \mathrm{Gal}_k \to \pi_1(X,\bar\eta)$, and let X'/X be a finite Galois étale cover with a Galois equivariant model $\mathscr{X}'$. The induced action of Σ on $\mathscr{X}'_{\mathbb{F},\mathrm{cons}}$ has a nonempty set of fixed points $(\mathscr{X}'_{\mathbb{F},\mathrm{cons}})^\Sigma$.*

The proof of Theorem 38 will be given in Section 8.2 below.

8.2. **The existence of fixed points: proof of Theorem 38.** For fine enough finite étale covers X'/X the smooth compactification of X' will itself be hyperbolic. It thus suffices to consider the case of smooth projective curves X/k of genus at least 2.

Let $\Sigma \subset \pi_1(X,\bar\eta)$ be the image of a section $s : \mathrm{Gal}_k \to \pi_1(X,\bar\eta)$, and let $\Theta \subset \Sigma$ be the image $s(\mathrm{I}_k)$ of the inertia subgroup under the section s. Let $w \in \mathrm{Val}_v(K)$ be a valuation as in Theorem 36 with a prolongation $\tilde w \in \mathrm{Val}_v(\tilde K)$ such that an ℓ-Sylow Σ_ℓ of Σ is contained in the decomposition group $\mathrm{D}_{\tilde w|w}$ and the ℓ-Sylow subgroup $\Theta_\ell = \Sigma_\ell \cap \Theta$ of Θ is contained in $\mathrm{I}_{\tilde w|w}$.

Let X'/X be a finite Galois étale cover with a Galois equivariant model $\mathscr{X}'$. In order to prove Theorem 38 we may pass to a characteristic cover X''/X' and a finer equivariant model $\mathscr{X}'' \to \mathscr{X}'$. Hence we may assume that X' has field of constants a finite extension k' and admits a stable model over the valuation ring $\mathfrak{o}'$ of k'. Moreover, by Lemma 14 and Lemma 15 we may assume that

(i) the stable model $\mathscr{X}'_{\mathrm{stable}}$ of X' is sturdy, i.e., that any stable component is of genus at least 2, and
(ii) any two components of the stable model intersect at most once in the stable model.

Let now w' be the restriction of $\tilde{w}$ to K', i.e., the extension of w to K' determined by $\tilde{w}$. The intersection $\Theta_\ell \cap \mathrm{I}_{\tilde{w}|w'}$ is of finite index in Θ_ℓ and thus isomorphic to $\mathbb{Z}_\ell(1)$.

Let $\mathscr{N}_{X'|\mathscr{X}'}$ be the kernel of the specialisation map $\mathrm{sp} : \pi_1(X') \to \pi_1(\mathscr{X}')$, which contains $\mathrm{I}_{\tilde{w}|w'}$. The projection $\pi_1(X') \to \mathrm{Gal}_k$ induces a map $\mathscr{N}_{X'|\mathscr{X}'} \to \mathrm{I}_k^{\mathrm{tame}} = \hat{\mathbb{Z}}'(1)$, which maps $\Theta_\ell \cap \mathrm{I}_{\tilde{w}|w'}$ to a pro-ℓ-group of finite index in the ℓ-Sylow subgroup $\mathbb{Z}_\ell(1)$ of $\mathrm{I}_k^{\mathrm{tame}}$, hence to an infinite pro-ℓ-group. Consequently, the image of $\Theta_\ell \cap \mathrm{I}_{\tilde{w}|w'}$ in $\mathscr{N}^{\mathrm{ab}}_{X'|\mathscr{X}'} \otimes \mathbb{Z}_\ell$ is nontrivial and also isomorphic to $\mathbb{Z}_\ell(1)$.

For an element $\sigma \in \Sigma$ the image of $\Theta_\ell \cap \mathrm{I}_{\tilde{w}|w'}$ in $\mathscr{N}^{\mathrm{ab}}_{X'|\mathscr{X}'} \otimes \mathbb{Z}_\ell$ meets the image of its σ-conjugate $\sigma\Theta_\ell\sigma^{-1} \cap \mathrm{I}_{\sigma(\tilde{w})|\sigma(w')}$ nontrivially because both are contained in the image of $\Theta \cap \mathscr{N}_{X'|\mathscr{X}'}$ and we have the following well known and useful lemma.

Lemma 39. *Let $H \subset \mathrm{I}_k$ be a closed subgroup of the inertia group I_k of k. Then the maximal pro-ℓ quotient H^ℓ of H for an $\ell \neq p$ is a quotient of $\mathbb{Z}_\ell(1)$.*

Proof: The wild inertia $\mathrm{P}_k \lhd \mathrm{I}_k$ is the unique normal p-Sylow subgroup of I_k. Thus H^ℓ is a quotient of $H/(H \cap \mathrm{P}_k)$ which is a subgroup of $\mathrm{I}_k \,/\, \mathrm{P}_k \cong \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$. □

We deduce that the images of $\mathrm{I}^{\mathrm{ab}}_{\tilde{w}|w'} \otimes \mathbb{Z}_\ell$ and $\mathrm{I}^{\mathrm{ab}}_{\sigma(\tilde{w})|\sigma(w')} \otimes \mathbb{Z}_\ell$ in $\mathscr{N}^{\mathrm{ab}}_{X'|\mathscr{X}'} \otimes \mathbb{Z}_\ell$ intersect nontrivially. Due to Proposition 13 we may compute in the subquotient $\mathscr{N}^{\mathrm{log,ab}}_{X'|\mathscr{X}'} \otimes \mathbb{Z}_\ell$ of $\pi_1^{\log}(\mathscr{X}')$. If w has height 2, Corollary 12 implies that the intersection

$$\Big(\bigoplus_\alpha \mathrm{I}^{\mathrm{ab}}_{\tilde{\alpha}|\alpha} \otimes \mathbb{Z}_\ell\Big) \cap \Big(\bigoplus_\alpha \mathrm{I}^{\mathrm{ab}}_{\sigma(\tilde{\alpha})|\sigma(\alpha)} \otimes \mathbb{Z}_\ell\Big)$$

in $\mathscr{N}^{\mathrm{ab}}_{X'|\mathscr{X}'} \otimes \mathbb{Z}_\ell$ is nontrivial, where α ranges over valuations of type 1v associated to irreducible components Y_α of the special fibre of $\mathscr{X}'_{\mathrm{stable}}$ that contain the image y of the center $x_{w'} \in \mathscr{X}'_{\mathbb{F}}$ of w', and the $\tilde{\alpha}$ are prolongations to $\tilde{K}$ as in Corollary 12. If w is of type 1v, the same conclusion holds with just $\alpha = w$. Now Proposition 6 applies and shows that the intersection

$$\{\alpha \ ; \ y \in Y_\alpha\} \cap \{\sigma(\alpha) \ ; \ y \in Y_\alpha\}$$

is nonempty for every $\sigma \in \Sigma$. If the set $\{\alpha \ ; \ y \in Y_\alpha\}$ has cardinality 1 then this α is a fixed point. Otherwise the cardinality is 2 and the combinatorics of the Σ-action on $\Sigma \cdot \{\alpha \ ; \ y \in Y_\alpha\}$ conforms to the following lemma.

Lemma 40. *Let G be a finite group acting on a set M. Let $x, y \in M$ be elements, such that $M = G \cdot x \cup G \cdot y$ and such that for every $g \in G$ the set $\{x, y\}$ intersects $\{gx, gy\}$ nontrivially, then we have one of the following three cases.*

1. *$M^G \neq \emptyset$, more precisely x or y is fixed under G.*
2. *$M = \{x, y\}$ has two elements and G acts transitively.*
3. *$M = \{x, y, z\}$ has three elements and G acts transitively.*

Proof: For $z \in M$ let G_z be the stabilizer of z under the action by G. If G acts with two orbits, then $G = G_x \cup G_y$ and thus not both stabilizers are of index in G bigger than 1, hence we have case (1). The same conclusion holds if x equals y.

If G acts transitively on M and $x \neq y$, then there is a $g \in G$ with $gx = y$ and we have

$$G = G_x \cup gG_xg^{-1} \cup gG_x \cup G_xg^{-1}.$$

Because $G_x \cap gG_xg^{-1}$ contains $1 \in G$, we can estimate $\#G + 1 \leq 4 \cdot \#G_x$ and thus the index $(G : G_x)$ is at most 3. This proves the lemma. □

In the situation of the proof of Theorem 38, when we let Σ act through a finite quotient on $\Sigma \cdot \{\alpha \ ; \ y \in Y_\alpha\}$, then we obviously have a fixed point in case (1), namely the generic point of the component Y_α of the special fibre of $\mathscr{X}'_{\mathrm{stable}}$. Consequently, the generic point of the strict transform of Y_α in $\mathscr{X}'$ is fixed by Σ.

Next we lead case (3) to a contradiction. In case (3) the set $\Sigma \cdot \{\alpha \ ; \ y \in Y_\alpha\}$ consists of three distinct valuations $\alpha_1, \alpha_2, \alpha_3$ such that by Proposition 6 we have

$$\mathbb{Z}_\ell(1) \oplus \mathbb{Z}_\ell(1) \oplus \mathbb{Z}_\ell(1) = \bigoplus_{i=1}^{3} \mathrm{I}^{\mathrm{ab}}_{\tilde{\alpha}_i|\alpha_i} \otimes \mathbb{Z}_\ell \subset \mathscr{N}^{\mathrm{ab}}_{X'|\mathscr{X}'} \otimes \mathbb{Z}_\ell.$$

Moreover, the nontrivial images of the conjugates of $\Theta_\ell \cap \mathrm{I}_{\tilde{w}|w'}$ in $\mathscr{N}^{\mathrm{ab}}_{X'|\mathscr{X}'} \otimes \mathbb{Z}_\ell$ all agree. But on the other hand, if $\{\alpha \ ; \ y \in Y_\alpha\} = \{\alpha_1, \alpha_2\}$ then the image of $\sigma\Theta_\ell\sigma^{-1} \cap \mathrm{I}_{\sigma(\tilde{w})|\sigma(w')}$ is contained in $\mathrm{I}^{\mathrm{ab}}_{\sigma(\tilde{\alpha}_1)|\alpha_1} \oplus \mathrm{I}^{\mathrm{ab}}_{\sigma(\tilde{\alpha}_2)|\alpha_2}$, so lies in a coordinate plane. Because Σ acts transitively, we get a contradiction.

In case (2) of the lemma we find at least a fixed point in the stable model $\mathscr{X}'_{\mathrm{stable}}$ that is the unique node y, due to condition (ii), in which the two components Y_α with $y \in Y_\alpha$ meet. The proof of Theorem 38 will thus be completed by the following lemma.

Lemma 41. *Let X/k be a smooth projective curve of genus at least 2 that admits a stable model $\mathscr{X}_{\mathrm{stable}}$. Let $\mathscr{X}$ be a model of X that allows a finite group action by G. The natural map $f : \mathscr{X} \to \mathscr{X}_{\mathrm{stable}}$ to the stable model is then G-equivariant and the map on fixed points*

$$\mathscr{X}^G \to \mathscr{X}^G_{\mathrm{stable}}$$

is surjective.

Proof: The uniqueness of the stable model induces an action of G and forces the map f to be G-equivariant.

Let $y \in \mathscr{X}_{\mathrm{stable}}$ be a fixed point under G. Then $f^{-1}(y)$ is geometrically connected and consists either of just one point, which then necessarily is fixed by G, or is a tree of projective lines. Then the dual graph Γ_y of $f^{-1}(y)$ is a tree which inherits an action by G. By Lemma 42 below, we have a fixed vertex or a fixed edge in Γ_y. That translates into a fixed component or a fixed node in $f^{-1}(y)$, so anyway the set of fixed points in $\mathscr{X}$ above y is nonempty. $\square$

Lemma 42. *Let G be a group acting on a finite nonempty graph Γ. If Γ is a tree, then the G-action on Γ has a fixed point, which can be a vertex or an edge.*

Proof: This follows at once from [Se80] Prop 10 and its corollary, which unfortunately is only stated for trees of odd diameter, when the guaranteed fixed point is a vertex. We recall the argument in order to cover the case of even diameter.

For two vertices $x, y \in \Gamma$ the **distance** $d(x,y)$ is defined as the minimum over the number of edges in a connected subgraph of Γ that contains x and y, see [Se80] §I.2.2. We set

$$d_x = \max\{d(x,y) \ ; \ \text{ all vertices } y \in \Gamma\}$$

for any vertex $x \in \Gamma$, and call $d = \max_x\{d_x\}$ the **diameter** of Γ, see [Se80] §I.2.2. The function d_x is convex along **geodesic** paths in Γ, as can be seen from an easy case by case proof of $d_x + d_z \geq 2d_y$ for adjacent vertices

$$\overset{x}{\bullet} \text{———} \overset{y}{\bullet} \text{———} \overset{z}{\bullet}$$

Let $\Gamma' \subset \Gamma$ be the minimal connected subgraph of Γ that contains all vertices $x \in \Gamma$ such that $d_x < d$. By the convexity of the function d_x along geodesics, we find that Γ' does not contain vertices x with $d_x = d$. The tree Γ' is thus a G-equivariant subtree and has smaller diameter than Γ. In fact, the diameter of Γ' is $d-2$. By induction on the diameter it suffices to treat cases, where Γ' is empty. This leaves only the case of diameter 1 and 2 which are trivial. $\square$

9. Concluding remarks

In this last section we would like to discuss some consequences of Theorem 37.

9.1. **Sections localized at type 2h valuations and the p-adic section conjecture.** Let $s : \mathrm{Gal}_k \to \pi_1(X)$ be a section with $s(\mathrm{Gal}_k)$ contained in the decomposition group $\mathrm{D}_{\tilde{w}|w}$ of a valuation $\tilde{w} \in \mathrm{Val}_v(\tilde{K})$ of type 2h. The valuation w is a refinement of a valuation w_a of type 1h corresponding to a closed point $a \in X$ of the generic fibre. It follows that

$$s(\mathrm{Gal}_k) \subseteq \mathrm{D}_{\tilde{w}|w} = \mathrm{D}_{\tilde{w}_a|w_a} = s_a(\mathrm{Gal}_{\kappa(a)})$$

which after projection to Gal_k implies $\mathrm{Gal}_k \subseteq \mathrm{Gal}_{\kappa(a)} \subseteq \mathrm{Gal}_k$, and $s = s_a$ is the section associated to the k-rational point $a \in X(k)$ as predicted by the p-adic section conjecture.

The p-adic section conjecture thus reduces to the task of eliminating valuations $w \in \mathrm{Val}_v(K)$ of type other than 2h in Theorem 37.

9.2. **The residue field.** Let $s : \mathrm{Gal}(k) \to \pi_1(X)$ be a section and let $\tilde{w} \in \mathrm{Val}_v(\tilde{K})$ such that with $s(\mathrm{Gal}_k) \subset \mathrm{D}_{\tilde{w}|w}$. The induced map

$$\mathrm{Gal}_k \to \mathrm{D}_{\tilde{w}|w} / \mathrm{I}_{\tilde{w}|w} \to \mathrm{Gal}_{\mathbb{F}}$$

is surjective. Hence the residue field $\mathbb{F}$ of k is relatively algebraically closed in $\kappa(w)$. Therefore, if $w = \alpha$ is of type 1v, we find that $\kappa(\alpha)$ is a regular function field over $\mathbb{F}$. And if w has height 2, then $\kappa(w)$ equals $\mathbb{F}$. We conclude that the valuation w given by Theorem 37 cannot be of type $2\mathrm{u}_{\mathrm{smooth}}$.

9.3. **Sections localized at valuations of height 2.** Let $s : \mathrm{Gal}(k) \to \pi_1(X)$ be a section and let $\tilde{w} \in \mathrm{Val}_v(\tilde{K})$ be a valuation of height 2 with $s(\mathrm{Gal}_k) \subset \mathrm{D}_{\tilde{w}|w}$.

9.3.1. *The ramification.* Let $\mathscr{X}/\mathfrak{o}$ be a model of X with reduced geometric special fibre $\overline{Y}$. The **ramification** of a section $s : \mathrm{Gal}_k \to \pi_1(X)$ with respect to the model $\mathscr{X}$ is defined as the homomorphism

$$\mathrm{ram}(s) = \mathrm{sp} \circ s|_{\mathrm{I}_k} : \mathrm{I}_k \to \pi_1(\overline{Y})$$

of the composite of the restriction to the inertia subgroup $\mathrm{I}_k \subset \mathrm{Gal}_k$ with the specialisation map $\mathrm{sp} : \pi_1(X) \twoheadrightarrow \pi_1(Y)$. A section s is called **unramified** with rerspect to $\mathscr{X}$ if $\mathrm{ram}(s)$ is the trivial homomorphism. A section associated to a k-rational point that extends to an $\mathfrak{o}$-rational point of the model, in particaular any such for proper X/k, is necessarily unramified.

Let X/k be a proper, smooth hyperbolic curve. The diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{I}_{\tilde{w}|w} \arrow[r] \arrow[d] & \mathrm{D}_{\tilde{w}|w} \arrow[r] \arrow[d] & \mathrm{Gal}_{\kappa(w)} \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \mathrm{I}_k \arrow[r] \arrow[u, dotted, bend right, "s"'] & \mathrm{Gal}_k \arrow[r] \arrow[u, bend right, "s"'] & \mathrm{Gal}_{\mathbb{F}} \arrow[r] & 1
\end{tikzcd} \tag{9.1}$$

shows that for any given model $\mathscr{X}$ with reduced special fibre Y the ramification $\mathrm{ram}(s)$ of the section s vanishes. Moreover, the induced section of $\pi_1(Y/\mathbb{F})$ is the section associated to the $\mathbb{F}$-rational point given by the center x_w of the valuation w on $Y \subset \mathscr{X}$. Of course, this is predicted by the p-adic section conjecture, but in general this is not known for an arbitrary section.

9.3.2. *The center of the valuation on stable models.*

Proposition 43. *Let w_1, w_2 be valuations of height 2 with $s(\mathrm{Gal}_k) \subseteq \mathrm{D}_{\tilde{w}_i|w_i}$ for $i = 1, 2$. Let $\mathscr{X}'$ be a Galois equivariant model of a finite étale Galois cover $X' \to X$. Then the centers $y_i' = x_{w_i'}$ of the $w_i' = \tilde{w}_i|_{X'}$ map to the same closed point in the stable model $\mathscr{X}'_{\mathrm{stable}}$.*

Proof: We use the notation of the proof of Theorem 38. By Lemma 14 and Lemma 15 we may assume that the stable model $\mathscr{X}'_{\mathrm{stable}}$ of X' is sturdy, i.e., that any stable component is of genus at least 2, and any two components of the stable model intersect at most once in the stable model. In order to simplify notation we assume that $X' = X$ with stable model $\mathscr{X}_{\mathrm{stable}}$.

In $\mathscr{N}^{\mathrm{log,ab}}_{X|\mathscr{X}} \otimes \mathbb{Z}_\ell$ the log inertia groups $\mathrm{I}^{\log}_{w_i} \otimes \mathbb{Z}_\ell$ meet in the image Θ_ℓ of an ℓ-Sylow of I_k under the section s. We argue as in the proof of Theorem 38 using Proposition 6 that the irreducible components of the special fibre Y of $\mathscr{X}_{\mathrm{stable}}$ which contain $y_1 = x_{w_1}$ cannot be disjoint from those which contain $y_2 = x_{w_2}$. So there is at least one component Y_α corresponding to a valuation α of type 1v which contains both x_{w_1} and x_{w_2}.

We apply the preceding paragraph to finite étale covers $X'' \to X' \to X$ which are generic fibres of finite étale covers $\mathscr{X}'' \to \mathscr{X}'$. We deduce that the section of $\pi_1(Y/\mathbb{F})$ induced by s, namely $s_{y_1} = s_{y_2}$, factors as the corresponding section of $\pi_1(Y_\alpha/\mathbb{F})$. The injectivity of the natural map

$$Y_\alpha(\mathbb{F}) \to \{\text{conjugacy classes of sections of } \pi_1(Y_\alpha/\mathbb{F})\}$$

shows thus that $y_1 = y_2$ as claimed. $\square$

9.3.3. *The non-vanishing locus of constant Brauer classes.* By the diagram (9.1) above, the section induces a splitting of the projection $\mathrm{I}_{\tilde{w}|w} \twoheadrightarrow \mathrm{I}_k$. By the computation of log inertia groups the section thus yields a splitting of the map

$$\mathrm{Hom}\big(w(K^*), \hat{\mathbb{Z}}'(1)\big) = \mathrm{I}^{\mathrm{tame}}_{\tilde{w}|w} \twoheadrightarrow \mathrm{I}^{\mathrm{tame}}_k = \mathrm{Hom}\big(v(k^*), \hat{\mathbb{Z}}'(1)\big)$$

and this means that $v(k^*) \hookrightarrow w(K^*)$ has no cotorsion prime-to-p. By Corollary 33 it follows that for all $\ell \neq p$ the map

$$\mathrm{Br}(k) \otimes \mathbb{Z}_\ell \to \mathrm{Br}(K^{\mathrm{h}}_w) \otimes \mathbb{Z}_\ell$$

is injective.

9.3.4. *Independence of $\ell \neq p$.* Although a valuation for which the constant Brauer class of invariant $1/\ell$ does not vanish is the starting point in the proof of Theorem 36, in the course of the proof of Theorem 38 no effort is taken to keep this property. It turns out that at least for valuations of height 2 that satisfy the claim of Theorem 37 the non-vanishing of the constant Brauer class of invariant $1/\ell$ is automatic. Moreover, the potential dependence of the choice of the auxillary prime ℓ different from p does not play a role in the end.

9.4. **Uniqueness of the valuation.** It is a natural question whether the valuation $w \in \mathrm{Val}_v(K)$ as well as its prolongation $\tilde{w}$ to $\tilde{K}$ predicted by Theorem 37 are unique with the requested property.

Proposition 44. *Let $s : \mathrm{Gal}_k \to \pi_1(X)$ be a section. Then there is at most one valuation $\alpha \in \mathrm{Val}_v(K)$ of height 1 corresponding to a visible component Y_α of the special fibre of some model $\mathscr{X}$ such that $s(\mathrm{Gal}_k) \subset \mathrm{D}_{\tilde{\alpha}|\alpha}$. Moreover, if such an α exists, then there is either*

(1) *a refinement $w = v_y \circ \alpha$ of type 2v associated to a closed point $y \in Y_\alpha$ such that even $s(\mathrm{Gal}_k) \subset \mathrm{D}_{\tilde{w}|w} \subset \mathrm{D}_{\tilde{\alpha}|\alpha}$, or*

(2) *the image Θ_ℓ under s of an ℓ-Sylow of the inertia group $\mathrm{I}_k \subset \mathrm{Gal}_k$ is contained in $\mathrm{I}_{\tilde{\alpha}|\alpha}$.*

Proof: It follows essentially from Tamagawa's work on resolution of non-singularities [Ta04], more concretely from the assertion of Lemma 26, that the Galois extension of the residue field $\kappa(\alpha)$ at α corresponding to $\mathrm{Gal}_{\kappa(\alpha)} \twoheadrightarrow \mathrm{Gal}_\alpha := \mathrm{D}_{\tilde{\alpha}|\alpha} / \mathrm{I}_{\tilde{\alpha}|\alpha}$ has no prime-to-p extensions. Indeed, in the system of components Y'_α corresponding to $\tilde{\alpha}$ for finer and finer étale covers X'/X the set of nodes on Y'_α will contain any given set of closed points. But as [Mz96] Prop 4.2 or [Sx02] Prop 6.2.11, show that with the natural log structures

$$\pi_1^{\log}(Y'_\alpha) \hookrightarrow \pi_1^{\log}(\mathscr{X}', \bar{\eta})$$

is injective, we see that any tamely ramified cover of Y'_α with ramification in the set of nodes will occur as residue extension of $\kappa(\alpha)$.

Let us now assume that Θ_ℓ is not contained in $\mathrm{I}_{\tilde{\alpha}|\alpha}$, so (2) fails. The proposition then claims, that for every Galois equivariant model $\mathscr{X}'$ of a finite étale cover X'/X there is a fixed point y under the action of $\Sigma = s(\mathrm{Gal}_k)$ which is a closed point in the closure Y'_α of the center of α.

Moreover, there is a compatible system of such fixed points as the model varies. By Lemma 14 and Lemma 15 we may assume that the stable model $\mathscr{X}'_{\mathrm{stable}}$ of X' is sturdy, i.e., that any stable component is of genus at least 2, and any two components of the stable model intersect at most once in the stable model. Furthermore, it suffices to find such a fixed point y on the stable model $\mathscr{X}'_{\mathrm{stable}}$.

The image $\overline{\Theta}_\ell$ of Θ_ℓ in $\mathrm{Gal}_w = \mathrm{Gal}(\kappa(\tilde{w})|\kappa(w))$ will be a cyclotomically normalized subgroup, see [Na94] §2.1, isomorphic to $\mathbb{Z}_\ell(1)$ of $\ker(\mathrm{Gal}_w \twoheadrightarrow \mathrm{Gal}_{\mathbb{F}})$. The theory of the anabelian weight filtration as pioneered by Nakamura in [Na90] §3, [Na94] §2.1, see also [Sx08] Thm 20, still works in this context, because Gal_w is sufficiently big, and shows that $\overline{\Theta}_\ell$ is contained in an inertia subgroup of a unique node y of a suitable corresponding Y'_α. By structure transport using conjugation by elements of Gal_k through the section s we see that in fact the point y is preserved under Σ.

By passing to finer and finer covers X' and models $\mathscr{X}'$ we deduce from the uniqueness of y which is detected by the partial image $\overline{\Theta}_\ell$ of the section in Gal_w that the collection of closed points so obtained forms a compatible system in

$$\varprojlim_{X' \subset \mathscr{X}'} \mathscr{X}'_{\mathrm{stable},\mathbb{F}}$$

endowed with the constructible topology. Because the valuation α corresponds to a visible component $Y'_\alpha \subset \mathscr{X}'_{\mathrm{stable}}$ that is fixed by Σ, and because the system of closed points given by the y lies on the Y'_α, we may conclude that the corresponding closed points on the strict transforms of the Y'_α in any model are also preserved by Σ. Hence there is a refinement $w = v_y \circ \alpha$ of type 2v associated to the system of closed points $y \in Y'_\alpha$ such that $s(\mathrm{Gal}_k) \subset \mathrm{D}_{\tilde{w}|w} \subset \mathrm{D}_{\tilde{\alpha}|\alpha}$, as claimed by option (1).

It remains to prove the assertion on uniqueness. We can argue by Proposition 6 as in the proof of Proposition 43. Indeed, under option (1) or (2) the image Θ_ℓ will detect the corresponding valuation of type 1v. The only problem that might occur is solved by moving apart the the two stable components Y_α, Y_β meeting in y by an application of Lemma 26. □

Proposition 45. *Let $\mathscr{X}'$ be a Galois equivariant model of a finite étale Galois cover $X' \to X$. Let w_1, w_2 be valuations with $s(\mathrm{Gal}_k) \subseteq \mathrm{D}_{\tilde{w}_i|w_i}$ for $i = 1, 2$, such that the centers $x_{w'_i}$ of the $w'_i = \tilde{w}_i|_{X'}$ map to closed points y'_i in the stable model $\mathscr{X}'_{\mathrm{stable}}$. Then y'_1 coincides with y'_2.*

Proof: By assumption we have $\mathrm{D}_{\tilde{w}_i|w'_i} \subset \mathrm{D}_{y'_i}$ whose image in $\pi_1(Y')$ under the specialisation map coincides with the image of the sections associated to the closed points y'_i in the reduced special fibre Y' of the stable model $\mathscr{X}'_{\mathrm{stable}}$. Replacing the $\mathrm{I}^{\log}_{w'_i}$ by the $\mathrm{I}^{\log}_{y'_i}$ now the proof of Proposition 43 applies mutatis mutandis. □

Theorem 46. *Let Σ be the image of a section $s : \mathrm{Gal}(k) \to \pi_1(X)$. Let $\mathscr{X}'$ be a Galois-equivariant model of a finite étale cover $X' \to X$ with stable model $\mathscr{X}'_{\mathrm{stable}}$. Then the image of the map*

$$\mathrm{center} : \mathrm{Val}_v(\tilde{K})^\Sigma \to \mathscr{X}'_{\mathrm{stable},\mathbb{F}}$$

consists either

(1) *of a unique closed $\mathbb{F}$-rational point, or*
(2) *of the generic point of a unique component together with a closed $\mathbb{F}$-rational point on that component, or*
(3) *of the generic point of a unique component.*

Proof: Lemma 18, Proposition 44 and Proposition 45 show that the image contains at most one closed and at most one generic point, while Theorem 37 shows that the image is nonempty. It remains to argue that if the image consists of both a closed point y and a generic point α, then y is in the closure of α. But this follows clearly from Proposition 6 and the discussion of the group Θ_ℓ in the proof of both Proposition 44 and Proposition 45. □

Corollary 47. *Let X/k be a smooth hyperbolic curve which has a cofinal system of finite étale covers $X' \to X$ such that X' has a model $\mathscr{X}'$ whose components of the special fibre are visible. Let $s : \mathrm{Gal}_k \to \pi_1(X)$ be a section. Then one of the following holds.*

(1) *There is a unique $\tilde{w} \in \mathrm{Val}_v(\tilde{K})$ with $s(\mathrm{Gal}_k) \subset \mathrm{D}_{\tilde{w}|w}$.*
(2) *There exist a unique $\tilde{\alpha}$ of type 1v with a refinement $\tilde{w} = \tilde{v}_y \circ \tilde{\alpha}$ of type 2v, such that $s(\mathrm{Gal}_k) \subset \mathrm{D}_{\tilde{w}|w} \subset \mathrm{D}_{\tilde{\alpha}|\alpha}$.*

Proof: This is an immediate corollary of Theorem 46 as the assumption of all components being visible leads to a bijection

$$\mathrm{Val}_v(\tilde{K}) \xrightarrow{\sim} \varprojlim_{X' \subset \mathscr{X}'} \mathscr{X}'_{\mathbb{F}} \xrightarrow{\sim} \varprojlim_{X' \subset \mathscr{X}'} \mathscr{X}'_{\mathrm{stable},\mathbb{F}}$$

$\square$

9.5. **Final remark.** It is conceivable that by the result of Section 6.5 one may extend the range of uniqueness of Theorem 46 to also include the locus in bridges of suitable models. But as soon as there are invisible forests of trees of invisible $\mathbb{P}^1$'s for a curve X/k, it is also conceivable that those will contribute sections of $\pi_1(X/k)$ localized in the respective p-adic disc of the associated rigid analytic space but not localized in a k-rational point, thus ultimately failing the p-adic section conjecture.

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