

# THE OORT CONJECTURE ON LIFTING COVERS OF CURVES

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ABSTRACT. We show that the conjecture of Oort on lifting covers of curves is true. The main ingredients in the proof are a deformation argument in characteristic  $p$  and (a special case of) a very recent result by Obus–Wewers. A kind of boundedness result is given as well.

## 1. INTRODUCTION

The aim of this note is to present a proof of the (classical) Oort Conjecture, which is a question about lifting Galois covers of curves from characteristic  $p > 0$  to characteristic zero. In one form or the other, this kind of question might well be considered math folklore, and it was also well known that in general the lifting is not possible. The problem was systematically addressed and formulated by OORT at least as early as 1987, see [16] or rather [17]. The general context of the lifting question/problem is as follows: Let  $k$  be an algebraically closed field of characteristic  $p > 0$ , and  $W(k)$  be the ring of Witt vectors over  $k$ . Let  $Y \rightarrow X$  a  $G$ -cover of complete smooth  $k$ -curves, where  $G$  is a finite group. Then the question is whether there exists a finite extension of discrete valuation rings  $W(k) \hookrightarrow R$  and a  $G$ -cover  $\mathcal{Y}_R \rightarrow \mathcal{X}_R$  of complete smooth  $R$ -curves whose special fiber is the given  $G$ -cover  $Y \rightarrow X$ . The answer to this question in general is negative, because over  $k$  there are curves of genus  $g > 1$  and huge automorphism groups, see e.g. ROQUETTE [23], whereas in characteristic zero one has the Hurwitz bound  $84(g - 1)$  for the order of the automorphism group. The Oort Conjecture is about a subtle interaction between the ramification structure and the nature of the inertia groups of generically Galois covers of curves, the idea being that if the inertia groups of a  $G$ -cover  $Y \rightarrow X$  look like in characteristic zero, i.e., they are all cyclic, then the Galois cover should be smoothly liftable to characteristic zero.

**Oort Conjecture.** *Let  $k$  and  $W(k)$  be as above. Let  $Y \rightarrow X$  be a possibly ramified  $G$ -cover of complete smooth  $k$ -curves having only cyclic groups as inertia groups. Then there exists a finite extension  $R$  of  $W(k)$  over which the  $G$ -cover  $Y \rightarrow X$  lifts smoothly, i.e., there exists a  $G$ -cover  $\mathcal{Y}_R \rightarrow \mathcal{X}_R$  of complete smooth  $R$ -curves with special fiber  $Y \rightarrow X$ .*

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There is also the **local Oort conjecture**, which asserts that every finite cyclic extension  $k[[t]] \hookrightarrow k[[z]]$  is the canonical reduction of a cyclic extension  $R[[T]] \hookrightarrow R[[Z]]$  for some finite extension  $R$  of  $W(k)$ . The local/global Oort Conjectures are related as follows, see Fact 4.9, where further equivalent forms of the Oort Conjecture are given: Let  $R$  be a finite extension of  $W(k)$ , and  $\mathcal{X}_R$  be a complete smooth curve with special fiber  $X$ . Then a given  $G$ -cover  $Y \rightarrow X$ ,  $y \mapsto x$ , with cyclic inertia groups lifts smoothly over a given  $R$  iff the local cyclic extensions  $k[[t_x]] := \widehat{\mathcal{O}}_{X,x} \hookrightarrow \widehat{\mathcal{O}}_{Y,y} =: k[[t_y]]$  lift smoothly over  $R$  for all  $x \in X$  and  $y \mapsto x$ .

**Notation:** Let  $\deg_p(\mathfrak{D})$  be the power of  $p$  in the degree of the different of  $k[[t]] \hookrightarrow k[[z]]$ , and  $\deg_p(\mathfrak{D}_x)$  be correspondingly defined for the local extension  $k[[t_x]] \hookrightarrow k[[t_y]]$  at  $y \mapsto x$ .

**Theorem 1.1.** *The Oort Conjecture holds. Moreover, for every  $\delta$  there exists an algebraic integer  $\pi_\delta$  such that for every algebraically closed field  $k$ ,  $\text{char}(k) = p$ , the following hold:*

- 1) *Let  $k[[t]] \hookrightarrow k[[z]]$  be a cyclic extension with  $\deg_p(\mathfrak{D}) \leq \delta$ .*
- 2) *Let  $Y \rightarrow X$  be a  $G$ -cover with cyclic inertia groups and  $\deg_p(\mathfrak{D}_x) \leq \delta$  for all  $x \in X$ .*

*Then  $k[[t]] \hookrightarrow k[[z]]$  and  $Y \rightarrow X$  are smoothly liftable over  $W(k)[\pi_\delta]$ .*

**Historical Note:** The first evidence for the Oort Conjecture is the fact that the conjecture holds for  $G$ -covers  $Y \rightarrow X$  which have tame ramification only, i.e.,  $G$ -covers whose inertia groups are cyclic of the form  $\mathbb{Z}/m$  with  $(p, m) = 1$ . Indeed, the lifting of such  $G$ -covers follows from the famous *Grothendieck's specialization theorem* for the tame fundamental group, see e.g., SGA I. The first result which involved typical wild ramification is OORT–SEKIGUCHI–SUWA [18] which tackled the case of  $\mathbb{Z}/p$ -covers. It was followed by a quite intensive research activity, see the survey article by OBUS [13] as well as the bibliography list at the end of this note. GARUTI [5], [6] contains a lot of foundational work and beyond that showed that every  $G$ -cover  $Y \rightarrow X$  has possibly non-smooth liftings but with a well understood geometry. This aspect of the problem was revisited recently by SAIDI [24], where among other things a systematic discussion of the (equivalent) forms of the Oort Conjecture is given. The paper [8] by GREEN–MATIGNON contains further foundational work and gives a positive answer to the Oort Conjecture in the case of inertia groups of the form  $\mathbb{Z}/mp^e$  with  $(p, m) = 1$  and  $e \leq 2$ . Their result relies on the Sekiguchi–Suwa theory, see [27] and [28]. The paper BERTIN–MÉZARD [?B–M] addresses the deformation theory for covers, whereas CHINBURG–GURALNICK–HARBATER [3], [4] initiated the study of the so called *Oort groups*, and showed that the class of Oort groups is quite restrictive. Last but not least, the very recent result by OBUS–WEWERS [15] solves the Oort Conjecture in the case the inertia groups are of the form  $\mathbb{Z}/mp^e$  for  $(p, m) = 1$  and  $e \leq 3$ , and a case *critical for the method of this note*, when the upper ramification jumps are subject to some explicit restrictions, see the explanations at Remark 4.13 in section 4 for details.

**About the proof:** Concerning technical tools, we use freely a few of the foundational results from the papers mentioned above. The main novel tools for the proof are Key Lemma 3.2 and its global form Theorem 3.7, and second, Key Lemma 4.11, which is actually a special case of the main result OBUS–WEWERS [15], see section 4 for precise details and citations. All these results will be used as “black boxes” in the proof of the Oort Conjecture, given in section 4. Concerning the idea of the proof, there is little to say: The point is to first deform a given  $G$ -cover  $Y \rightarrow X$  to a cover  $\mathcal{Y}_\mathfrak{o} \rightarrow \mathcal{X}_\mathfrak{o}$  over  $\mathfrak{o} = k[[\varpi]]$  in such a way that the ramification of the deformed cover has no **essential upper jumps** as defined/introduced at the beginning of section 3, then apply the local-global principles, etc.

Maybe it's interesting to mention that the first variant of the proof (January, 2012) was shorter, but relied heavily on model theoretical tools and was not effective [concerning the finite extensions of  $W(k)$  over which the smooth lifting can be realized].

**Acknowledgements(...)** If my recollection is correct, during an MFO Workshop in 2003 or so, someone asked what should be the “characteristic  $p$  Oort Conjecture,” but it seems that nobody ever followed up (successfully) on that idea.

## 2. REVIEWING WELL KNOWN FACTS

Throughout this section,  $k$  is an algebraically closed field with  $\text{char}(k) = p > 0$ . All the other fields will be field extensions of  $k$ , in particular will be fields of characteristic  $p$ .

### A) *Reviewing higher ramification for cyclic extensions*

Let  $K$  be a complete discrete valued field of positive characteristic  $p > 0$ , with valuation ring  $R$  and having as residue field an algebraically closed field  $k$ . Let  $L|K$  be a finite Galois extension, say with Galois group  $G = \text{Gal}(L|K)$ . Since  $K$  is complete, the valuation  $v_K$  of  $K$  as a unique prolongation to  $L$ . And since the residue field  $k$  of  $K$  is algebraically closed,  $L|K$  is totally ramified, i.e.,  $[L : K] = e(L|K)$ . Finally, let  $v_L$  be the normalized valuation of  $L$ , hence  $v_L(L^\times) = \mathbb{Z}$  and  $\frac{1}{[L|K]}v_L = v_K$  on  $K$ .

Recall that the lower ramification groups of  $L|K$  or of  $G$  are defined as follows: Let  $z \in L$  be a uniformizing parameter. For every  $j$  we set  $G_j := \{\sigma \in G \mid v_L(\sigma z - z) > j\}$  and call it the  $j^{\text{th}}$  lower ramification group of  $L|K$  or of  $G$ . Clearly,  $G = G_0$  is the inertia group of  $L|K$ , and  $G_1$  is the ramification group of  $L|K$ , thus the Sylow  $p$ -group of  $G$ , and  $G_j = 1$  for  $j$  sufficiently large. In particular,  $G = G_1$  iff  $G$  is a  $p$ -group iff  $L|K$  is totally wildly ramified.

The first important fact about the lower ramification groups  $(G_j)_j$  is *Hilbert's different formula*, which gives an estimate for the degree  $\deg(\mathfrak{D}_{L|K})$  of the different of  $L|K$  in terms of the orders of the lower ramification groups, see e.g., SERRE [26], IV, §1:

$$\deg(\mathfrak{D}_{L|K}) = \sum_{j=0}^{\infty} (|G_j| - 1).$$

We further denote by  $j_\rho$  the **lower jumps** for  $L|K$ , or of  $G$ , as being the numbers satisfying  $G_{j_\rho} \neq G_{j_\rho+1}$ . In particular, setting  $j_{-1} = -1$  and  $j_0 = 0$ , and denoting the upper jumps for  $L|K$  by  $j_0 \leq j_1 \leq \dots \leq j_r$ , one has that  $j_r = \max\{j \mid G_j \neq 1\}$ .

Now suppose that  $L|K$  is a cyclic extension with  $G = \mathbb{Z}/p^e$ . Then  $G = G_0 = G_1$  by the discussion above, and every subgroup of  $G$  is a lower ramification group for  $L|K$ . Thus one has precisely  $e$  lower positive jumps  $j_1 \leq \dots \leq j_e$ , and  $G_{j_1} \geq \dots \geq G_{j_e}$  are precisely the  $e$  non-trivial subgroups of  $G = \mathbb{Z}/p^e$ . Finally, the Hilbert's Different Formula becomes:

$$\deg(\mathfrak{D}_{L|K}) = p^e - 1 + \sum_{\rho=1}^e (j_\rho - j_{\rho-1})(|G_{j_\rho}| - 1) = p^e - 1 + \sum_{\rho=1}^e (j_\rho - j_{\rho-1})(p^{e-(\rho-1)} - 1)$$

We recall that the lower ramification subgroups behave functorially in the base field, i.e., if  $K'|K$  is some finite sub-extension of  $L|K$ , and  $G' \subseteq G$  is the Galois group of  $L|K'$ , then  $G'_j = G_j \cap G'$ . Since the lower ramification groups do not behave functorially with respect

to Galois sub-extensions, one introduced the **upper ramification groups**  $G^i$  for  $i \geq -1$  of  $L|K$ , which behave functorially under taking Galois sub-extensions, see SERRE Se, IV, §3.

At least in the case of cyclic extensions  $L|K$  with Galois group  $G \cong \mathbb{Z}/p^e$ , the formula which relates the lower ramification groups  $G_j$  to the upper ramification groups  $G^i$  is explicit via *Herbrand's formula*, see e.g. SERRE [26], IV, §3. Namely, if  $\iota_0 := 0$  and  $\iota_1 \leq \dots \leq \iota_e$  are the **upper jumps** for  $L|K$ , then one has:

$$J_\rho - J_{\rho-1} = p^{\rho-1}(\iota_\rho - \iota_{\rho-1}), \quad \rho = 1, \dots, e.$$

Thus in the case  $L|K$  is cyclic with Galois group  $G = \mathbb{Z}/p^e$ , one can express the degree of the different of  $L|K$  in terms of higher ramification groups as follows:

$$\deg(\mathfrak{D}_{L|K}) = p^e - 1 + \sum_{\rho=1}^e (\iota_\rho - \iota_{\rho-1}) p^{\rho-1} (p^{e-(\rho-1)} - 1) = \sum_{\rho=1}^e (\iota_\rho + 1) (p^\rho - p^{\rho-1}).$$

### B) *Explicit formulas via Artin–Schreier–Witt Theory*

In the above notations, let  $t$  be any uniformizing parameter of the complete discrete valued field  $K$  of characteristic  $p > 0$ . Then  $K$  is canonically isomorphic to the Laurent power series field  $K = k((t))$  in the variable  $t$  over  $k$ .

Recall that the Artin–Schreier–Witt theory gives a description of the cyclic  $p$ -power extensions of  $K$  via finite length Witt vectors as follows, see e.g., LANG [12], or SERRE [26], II. Let  $\mathcal{A}$  be an integrally closed domain which is a  $k$ -algebra, and  $W_e(\mathcal{A}) = \{(a_1, \dots, a_e) \mid a_i \in \mathcal{A}\}$  be the Witt vectors of length  $e$  over  $\mathcal{A}$ . Then the Frobenius morphism  $\text{Frob}$  of  $\mathcal{A}$  lifts to the Frobenius morphism  $\text{Frob}_e$  of  $W_e(\mathcal{A})$ , and one defines the **Artin–Schreier–Witt operator**  $\wp_e := \text{Frob}_e - \text{Id}$  of  $\mathcal{A}$ . If  $\mathcal{A} \hookrightarrow \mathcal{A}^{\text{nr}}$  is a ind-étale universal cover of  $\mathcal{A}$ , one has the Artin–Schreier–Witt exact sequence

$$0 \rightarrow W_e(\mathbb{F}_p) = \mathbb{Z}/p^e \longrightarrow W_e(\mathcal{A}^{\text{nr}}) \xrightarrow{\wp_e} W_e(\mathcal{A}^{\text{nr}}) \rightarrow 0.$$

of sheaves on  $\text{Et}(\mathcal{A})$ . In particular, if  $\text{Pic}(\mathcal{A}) = 0$ , one gets a canonical isomorphism

$$W_e(\mathcal{A})/\text{im}(\wp_e) \rightarrow \text{Hom}(\pi_1(\mathcal{A}), \mathbb{Z}/p^e),$$

which gives a canonical bijection between the cyclic subgroups  $\langle \mathbf{a} \rangle \subset W_e(\mathcal{A})/\text{im}(\wp_e)$  and the integral étale cyclic extensions  $\mathcal{A} \hookrightarrow \mathcal{A}_{\mathbf{a}}$  with Galois group a quotient of  $\mathbb{Z}/p^e$  by via

$$\langle \mathbf{a} \rangle \mapsto \mathcal{A}_{\mathbf{a}} := \mathcal{A}[\mathbf{x}]$$

where  $\mathbf{x} = (x_1, \dots, x_e)$  is any solution of the equation  $\wp_e(\mathbf{x}) = \mathbf{a}$ .

In the special case  $\mathcal{A} = K = k((t))$ , one can make things more precise as follows. First, by Hensel's Lemma, the class of every element in  $W_e(K)/\text{im}(\wp_e)$  contains a representative of the form  $\mathbf{p} = (p_1, \dots, p_e)$  with  $p_\rho = p_\rho(t^{-1}) \in k[t^{-1}]$  a polynomial in the variable  $t^{-1}$  over  $k$ . Second, using the properties of the Artin–Schreier operator  $\wp_e$ , one can inductively “reduce” the terms of each  $p_\rho(t^{-1})$  which contain powers of  $t^{-1}$  to exponent divisible by  $p$ . If all the polynomials  $p_\rho(t^{-1})$  have this property, one says that  $\mathbf{p} = (p_1, \dots, p_e)$  is in **standard form**. Following GARUTI [7], Thm. 1.1, and THOMAS [29], Prop. 4.2, see also OBUS–PRIESS [14], one can describe the upper jumps  $\iota_1 \leq \dots \leq \iota_e$  for the extension  $L = K_{\mathbf{p}}$  with  $\mathbf{p} = (p_1, \dots, p_e)$  in standard form and  $p_1(t^{-1}) \neq 0$  as follows:<sup>1</sup>

$$\iota_\rho = \max \{ p \iota_{\rho-1}, \deg(p_\rho(t^{-1})) \}, \quad \rho = 1, \dots, e.$$

<sup>1</sup>Notice that we use the conventions  $\deg(0) = -\infty$  and  $\iota_0 = 0$ .

In particular,  $v_\rho \geq p^\rho \deg(p_1(t^{-1}))$ , and the highest non-zero upper ramification index, i.e., the Artin conductor of  $K_{\mathbf{p}}|K$  is  $v_e = \max \{ p^{e-\rho} \deg(p_\rho(t^{-1})) \mid \rho = 1, \dots, e \}$ .

We make the following remark for later use: For  $\mathbf{a} = (a_1, \dots, a_e)$  an arbitrary Witt vector of length  $e$  over  $K$ , and  $K_{\mathbf{a}}|K$  as above one has:  $[K_{\mathbf{a}} : K] = p^m$ , where  $m$  is minimal such that  $(a_1, \dots, a_{e-m}) \in \text{im}(\wp_{e-m})$ . In particular, if  $m < e$ , then setting  $\mathbf{b} = (b_1, \dots, b_m)$  with  $b_\rho = a_{\rho+e-m}$ , one has:  $K_{\mathbf{a}}|K$  is actually the cyclic extension  $K_{\mathbf{b}}|K$  of degree  $p^m$  of  $K$ , and one can compute the upper ramification indices of  $K_{\mathbf{a}} = K_{\mathbf{b}}$  using the discussion above.

### C) Kato's smoothness criterion

Let  $k$  be as usual an algebraically closed field of characteristic  $p > 0$ , and  $\mathfrak{o}$  a complete discrete valuation ring with quotient field  $\hat{\kappa} = \text{Quot}(\mathfrak{o})$  and residue field  $k$ . Let  $\mathcal{A} = \mathfrak{o}[[T]]$  be the power series ring over  $\mathfrak{o}$ . Then  $\mathcal{R} := \mathcal{R} \otimes_{\mathfrak{o}} \hat{\kappa} = \hat{\kappa}\langle\langle T \rangle\rangle$  is the ring of power series in  $T$  over  $\hat{\kappa}$  having  $v_{\hat{\kappa}}$ -bounded coefficients. Thus  $\mathcal{R}$  is a Dedekind ring having  $\text{Spec}(\mathcal{R})$  in bijection with the points of the open rigid disc  $\mathfrak{X} = \text{Spf } \mathcal{R}$  of radius 1 over the complete valued field  $\hat{\kappa}$ . Further,  $\mathcal{A}$  is a two dimensional complete regular ring with maximal ideal  $(\pi, T)$  with  $\mathcal{A} \rightarrow \mathcal{A}/(\pi, T) = k$ , and  $\mathfrak{X} = \text{Spec}(\mathcal{R})$  is nothing but the complement of  $V(\pi) \subset \text{Spec}(\mathcal{A})$ . Finally,  $A := \mathcal{A}/(\pi) = k[[t]]$  is the power series ring in the variable  $t := T \pmod{(\pi)}$ , thus a complete discrete valuation ring.

Let  $\mathcal{K} := \text{Quot}(\mathcal{A})$  and  $K := \text{Quot}(A) = k((t))$  be the fraction fields of  $\mathcal{A}$ , respectively  $A$ . Let  $\mathcal{K} \hookrightarrow \mathcal{L}$  be a finite separable field extension, and  $\mathcal{B} \subset \mathcal{S}$  be the integral closures of  $\mathcal{A} \subset \mathcal{R}$  in the finite field extension  $\mathcal{K} \hookrightarrow \mathcal{L}$ . Then  $\mathcal{B}$  is finite  $\mathcal{A}$ -module, and  $\mathcal{S}$  is a finite  $\mathcal{R}$ -module, in particular a Dedekind ring.

Next let  $\mathfrak{r}_1, \dots, \mathfrak{r}_r$  be the prime ideals of  $\mathcal{B}$  above  $(\pi)$ . Then each  $\mathfrak{r}_i$  has height one, and the localizations  $\mathcal{B}_{\mathfrak{r}_i}$  are precisely the valuation rings of  $\mathcal{L}$  above the discrete valuation ring  $\mathcal{A}_{(\pi)}$  of  $\mathcal{K}$ . And since  $\mathcal{K} \hookrightarrow \mathcal{L}$  is a finite separable extension, by the Finiteness Lemma, the fundamental equality holds:

$$[\mathcal{L} : \mathcal{K}] = \sum_{i=1}^r e(\mathfrak{r}_i|\pi) \cdot f(\mathfrak{r}_i|\pi),$$

where  $e(\mathfrak{r}_i|\pi)$  and  $f(\mathfrak{r}_i|\pi) = [\kappa(\mathfrak{r}_i) : K]$  is the ramification index, respectively the residual degree of  $\mathfrak{r}_i|\pi$ . Therefore, if  $v_\pi$  is the discrete valuation of  $\mathcal{K}$  with valuation ring  $\mathcal{A}_{(\pi)}$ , one has  $K = \kappa(\pi) = \text{Quot}(\mathcal{A}/(\pi)) = \kappa(v_\pi)$ , and the following are equivalent:

- i) There exists a prolongation  $w$  of  $v_\pi$  to  $\mathcal{L}$  such that  $[\mathcal{L} : \mathcal{K}] = [L : K]$ , where  $L := \kappa(w)$ .
- ii) The ideal  $\mathfrak{r} := \pi\mathcal{B}$  is a prime ideal of  $\mathcal{B}$ , or equivalently,  $\pi$  is a prime element of  $\mathcal{B}$ .

In particular, if the above equivalent conditions i), ii), hold, then  $\mathcal{B}_{\mathfrak{r}}$  is the valuation ring of  $w$ , and one has  $\kappa(\mathfrak{r}) = \text{Quot}(\mathcal{B}/(\pi)) = \kappa(w) = L$ . Further,  $w$  is the unique prolongation of  $v_\pi$  to  $\mathcal{L}$ , and  $\mathfrak{r} = \pi\mathcal{B}$  is the unique prime ideal of  $\mathcal{B}$  above the ideal  $\pi\mathcal{A}$  of  $\mathcal{A}$ .

We conclude by mentioning the following smoothness criterion, which is a special case of the theory developed in KATO [10], §5; see also GREEN-MATIGNON [8], §3, especially 3.4.

**Fact 2.1.** *In the above notations, suppose that  $v_\pi$  has a prolongation  $w$  to  $\mathcal{L}$  such that  $[\mathcal{L} : \mathcal{K}] = [L : K]$ , where  $L := \kappa(w)$  and  $K := \kappa(v_\pi)$ . Let  $A \hookrightarrow B$  be the integral closure of  $A = k[[t]] = \mathcal{A}/(\pi)$  in the field extension  $K \hookrightarrow L$ . Let  $\mathfrak{D}_{\mathcal{S}|\mathcal{R}}$  and  $\mathfrak{D}_{B|A}$  be the differentials of the extensions of Dedekind rings  $\mathcal{R} \hookrightarrow \mathcal{S}$ , respectively  $A \hookrightarrow B$ . The following are equivalent:*

- i)  $\text{Spec } \mathcal{B}$  is smooth over  $\text{Spec } \mathfrak{o}$ .

ii) The degrees of  $\mathfrak{D}_{S|\mathcal{R}}$  and  $\mathfrak{D}_{B|A}$  are equal:  $\deg(\mathfrak{D}_{S|\mathcal{R}}) = \deg(\mathfrak{D}_{B|A})$ .

If the above equivalent conditions are satisfied, then there exists  $Z \in \mathcal{B}$  such that  $\mathcal{B} = \mathfrak{o}[[Z]]$  and  $B = \mathcal{S}/(\pi) = k[[z]]$ , where  $z = Z \pmod{(\pi)}$ .

### 3. THE CHARACTERISTIC $p$ OORT CONJECTURE

**Remark/Definition 3.1.** In the context of section 2), A), let  $L|K$  be a cyclic extension of degree  $p^e := [L : K]$  with upper ramification jumps  $\iota_1 \leq \dots \leq \iota_e$ . Recall that setting  $\iota_0 = 0$ , one has that  $p\iota_{\rho-1} \leq \iota_\rho$  for all  $\rho$  with  $0 < \rho \leq e$ , and the inequality is strict if and only if  $\iota_\rho$  is not divisible by  $p$ . The division by  $p$  gives:

$$\iota_\rho - p\iota_{\rho-1} = pq_\rho + \epsilon_\rho$$

with  $0 \leq q_\rho$  and  $0 \leq \epsilon_\rho < p$ , and notice that by the remark above one has:  $0 < \epsilon_\rho$  if and only if  $(p, \iota_\rho) = 1$  if and only if  $p\iota_{\rho-1} < \iota_\rho$ .

We call  $q_\rho$  the **essential part** of the upper jump at  $\rho$ , and if  $0 < q_\rho$  we say that  $\iota_\rho$  is an **essential upper jump** for  $L|K$ , and that  $\rho$  is an **essential upper index** for  $L|K$ .

We introduce terminology as follows: Let  $\mathcal{R} \hookrightarrow \mathcal{S}$  be any generically finite Galois extension of Dedekind rings with cyclic inertia groups, and  $\mathcal{K} := \text{Quot}(\mathcal{R}) \hookrightarrow \text{Quot}(\mathcal{S}) =: \mathcal{L}$  be the corresponding cyclic extension of their quotient fields. For a maximal ideal  $\mathfrak{q} \in \text{Spec } \mathcal{S}$  above  $\mathfrak{p} \in \text{Spec } \mathcal{R}$ , let  $\mathcal{K}_{\mathfrak{p}} \hookrightarrow \mathcal{L}_{\mathfrak{q}}$  be the corresponding extension of complete discrete valued fields. We will say that  $\mathcal{R} \hookrightarrow \mathcal{S}$  has (no) **essential ramification jumps** at  $\mathfrak{p}$ , if the  $p$ -part of the cyclic extension of discrete complete valued fields  $\mathcal{K}_{\mathfrak{p}} \hookrightarrow \mathcal{L}_{\mathfrak{q}}$  has (no) essential upper ramification jumps. And we say that  $\mathcal{R} \hookrightarrow \mathcal{S}$  has no essential ramification, if  $\mathcal{R} \hookrightarrow \mathcal{S}$  has no essential ramification jumps at any  $\mathfrak{p} \in \text{Spec } \mathcal{R}$ .

In the remaining part of this subsection, we will work in a special case of the situation presented in section 2, C), which is as follows: We consider a fixed algebraically closed field  $k$  with  $\text{char}(k) = p > 0$ , let  $\mathfrak{o} = k[[\varpi]]$  be the power series ring in the variable  $\varpi$  over  $k$ , thus  $\hat{k} = k((\varpi)) = \text{Quot}(\mathfrak{o})$  is the Laurent power series in the variable  $\varpi$  over  $k$ . Let  $\mathcal{A} = k[[\varpi, t]]$  and  $\mathcal{K} = k((\varpi, t)) = \text{Quot}(\mathcal{A})$  be its field of fractions. Then  $A = \mathcal{A}/(\varpi) = k[[t]]$  and  $K = k((t)) = \text{Quot}(A)$  is the fraction field of  $A$ . Further,  $\mathcal{R} := \mathcal{R} \otimes_{\mathfrak{o}} \hat{k} = \hat{k}\langle\langle t \rangle\rangle$  is the ring of power series in  $t$  over  $\hat{k}$  having  $v_{\hat{k}}$ -bounded coefficients. Thus  $\mathcal{R}$  is a Dedekind ring having  $\text{Spec}(\mathcal{R})$  in bijection with the points of the open rigid disc  $\mathfrak{X} = \text{Spf } \mathcal{R}$  of radius 1 over the complete valued field  $\hat{k}$ . And we notice that  $\mathfrak{X} = \text{Spec}(\mathcal{R})$  is precisely the complement of  $V(\varpi) \subset \text{Spec}(\mathcal{A})$ . Finally, for a finite separable field extension  $\mathcal{K} \hookrightarrow \mathcal{L}$ , we let  $\mathcal{B} \subset \mathcal{S}$  be the integral closures of  $\mathcal{A} \subset \mathcal{R}$  in the finite field extension  $\mathcal{K} \hookrightarrow \mathcal{L}$ . Thus  $\mathcal{B}$  is finite  $\mathcal{A}$ -module, and  $\mathcal{S}$  is a finite  $\mathcal{R}$ -module, in particular a Dedekind ring.

**Key Lemma 3.2.** (Characteristic  $p$  local Oort conjecture) *In the above notations, let  $N := 1 + q_1 + \dots + q_e$  and  $x_1, \dots, x_N \in \mathfrak{m}_{\mathfrak{o}}$  be distinct points. Let  $K \hookrightarrow L$  be a cyclic  $\mathbb{Z}/p^e$ -extension with upper ramification jumps  $\iota_1 \leq \dots \leq \iota_e$ . Then there exists a cyclic  $\mathbb{Z}/p^e$ -extension  $\mathcal{K} \hookrightarrow \mathcal{L}$  such that the integral closure  $\mathcal{A} \hookrightarrow \mathcal{B}$  of  $\mathcal{A}$  in  $\mathcal{K} \hookrightarrow \mathcal{L}$  and the corresponding extension of Dedekind rings  $\mathcal{R} \hookrightarrow \mathcal{S}$  satisfy:*

1) *The morphism  $\varphi : \text{Spec } \mathcal{B} \rightarrow \text{Spec } \mathfrak{o}$  is smooth. In particular,  $\mathcal{B} = k[[\varpi, Z]]$  and the special fiber of  $\varphi$  is  $\text{Spec } B \rightarrow k$ , where  $B = k[[z]] = \mathcal{B}/(\varpi)$  and  $z = Z \pmod{(\varpi)}$ .*

2) *The canonical morphism  $\mathcal{R} \hookrightarrow \mathcal{S}$  has no essential ramification and is ramified only at points  $y_\mu \in \text{Spec } \mathcal{S}$  above the points  $x_\mu \in \text{Spec } \mathcal{R}$ ,  $1 \leq \mu \leq N$ .*

3) Let  $(\iota_{\mu,\rho})_{1 \leq \rho \leq e_\mu}$  be the upper ramification jumps at each  $y_\mu \mapsto x_\mu$ . Then  $(e_\mu)_{1 \leq \mu \leq N}$  is decreasing, and the upper jumps are given by:

- i)  $\iota_{1,\rho} = p\iota_{1,\rho-1} + \epsilon_\rho$  for  $1 \leq \rho \leq e$ .
- ii)  $\iota_{\mu,\rho} = p\iota_{\mu,\rho-1} + p - 1$  for  $1 < \mu \leq N$  and  $1 \leq \rho \leq e_\mu$ .

4) In particular, the branch locus  $\{x_1, \dots, x_N\}$  of  $\mathcal{R} \hookrightarrow \mathcal{S}$  is independent of  $k[[t]] \hookrightarrow k[[z]]$ , and the upper ramification jumps  $\mathbf{\iota}_\mu := (\iota_{\mu,1}, \dots, \iota_{\mu,e_\mu})$  at each  $y_\mu \mapsto x_\mu$ ,  $1 \leq \mu \leq N$ , depend only on the upper jumps  $\mathbf{\iota} := (\iota_1, \dots, \iota_e)$  of  $k[[t]] \hookrightarrow k[[z]]$ .

The proof of the Key Lemma 3.2 will take almost the whole section. We begin by recalling that in the notations from section 2, B), there exists  $\mathbf{p} = (p_1(t^{-1}), \dots, p_e(t^{-1}))$ , say in standard form, such that  $L = K_{\mathbf{p}}$ . The integral closure  $A \hookrightarrow B$  of  $A = k[[t]]$  in the field extension  $K \hookrightarrow L$  is of the form  $B = k[[z]]$  for any uniformizing parameter  $z$  of  $L = \text{Quot}(B)$ . And the degree of the different  $\mathfrak{D}_{L|K} := \mathfrak{D}_{B|A}$  is  $\deg(\mathfrak{D}_{L|K}) = \sum_{\rho=1}^e (\iota_\rho + 1)(p^\rho - p^{\rho-1})$ .

#### A) Combinatorics of the upper jumps

In the above context, let  $e_0$  be the number of essential upper jumps, which could be zero. If there exist essential upper jumps, i.e.,  $0 < e_0$ , let  $r_1 \leq \dots \leq r_{e_0}$  be the essential upper indices for  $L|K$ , and notice that the sequence  $(r_i)_{1 \leq i \leq e_0}$  is strictly increasing with  $r_{e_0} \leq e$ . For technical reasons (to simplify notations) we set  $r_{e_0+1} := e + 1$ , and if we need to speak about  $r_{e_0+1}$ , we call it the **improper upper index**, which for  $e_0 = 0$  would become  $r_1 = e + 1$ .

We next construct a finite strictly increasing sequence  $(d_i)_{0 \leq i \leq e_0}$  as follows: We set  $d_0 = 1$ , and we are done if  $e_0 = 0$ . If  $e_0 > 0$ , we define inductively  $d_i := d_{i-1} + q_{r_i}$  for  $1 \leq i \leq e_0$ . In particular, we see that  $N := 1 + q_1 + \dots + q_e$  is  $N = 1$  if  $e_0 = 0$ , and  $N := d_{e_0}$  otherwise.

We define an  $N \times e$  matrix of non-negative integers  $(\theta_{\mu,\rho})_{1 \leq \mu \leq N, 1 \leq \rho \leq e}$  as follows:

- If  $e_0 = 0$ , then  $N = 1$ , and the  $1 \times e$  matrix is given by  $\theta_{1,\rho} := u_\rho$ ,  $1 \leq \rho \leq e$ .
- If  $e_0 > 0$ , thus  $N > 1$ , we define:<sup>2</sup>
  - a)  $\theta_{1,\rho} = p\theta_{1,\rho-1} + \epsilon_\rho$  for  $1 \leq \rho \leq e$ .
  - b) For  $i = 1, \dots, e_0$  and  $d_{i-1} < \mu \leq d_i$  define:
    - $\theta_{\mu,\rho} = 0$  for  $1 \leq \rho < r_i$ .
    - $\theta_{\mu,\rho} = p\theta_{\mu,\rho-1} + p - 1$  for  $r_i \leq \rho \leq e$ .

Notice that in the case  $e_0 > 0$ , one has: Let  $\rho$  with  $1 \leq \rho \leq e$  be given. Consider the unique  $1 \leq i \leq e_0$  such that  $r_i \leq \rho < r_{i+1}$ . (Recall the if  $r_i = e$ , then  $r_{i+1} := e + 1$  by the convention above!) Then for all  $\mu$  with  $1 \leq \mu \leq N$  one has:  $\theta_{\mu,\rho} \neq 0$  if and only if  $\mu \leq d_i$ . The fundamental combinatorial property of  $(\theta_{\mu,\rho})_{1 \leq \mu \leq N, 1 \leq \rho \leq e}$  is given by the following:

**Lemma 3.3.** For  $1 \leq i \leq e_0$  and  $r_i \leq \rho < r_{i+1}$  one has:  $\iota_\rho + 1 = \sum_{1 \leq \mu \leq d_i} (\theta_{\mu,\rho} + 1)$ .

*Proof.* The proof follows by induction on  $\rho = 1, \dots, e$ . Indeed, if  $e_0 = 0$ , then  $N = 1$ , and there is nothing to prove. Thus supposing that  $e_0 > 0$ , one argues as follows:

- The assertion holds for  $\rho = 1$ : First, if  $r_1 > 1$ , then  $\theta_{\mu,1} = 0$  for  $1 < \mu$ , thus there is nothing to prove. Second, if  $r_1 = 1$ , then  $q_1 > 0$  and  $d_1 = 1 + q_1$ . Further, by the definitions one has:  $\theta_{1,1} = \epsilon_1$  and  $\theta_{\mu,1} = p - 1$  for  $1 < \mu \leq d_1$ , and conclude by the fact that  $u_1 = pq_1 + \epsilon_1$ .

<sup>2</sup>Here and elsewhere, we set  $\theta_{\mu,0} := 0$  as well as  $\iota_0 = 0$  and  $\iota_{\mu,0} = 0$ .

• If the assertion of Lemma 3.3 holds for  $\rho < e$ , the assertion also holds for  $\rho + 1$ : Indeed, let  $i$  be such that  $r_i \leq \rho < r_{i+1}$ .

Case 1:  $\rho + 1 < r_{i+1}$ . Then  $r_i \leq \rho + 1 < r_{i+1}$ , and in particular,  $\rho + 1$  is not an essential jump index. Hence by definitions one has that  $\iota_{\rho+1} = p\iota_\rho + \epsilon_{\rho+1}$  with  $0 \leq \epsilon_{\rho+1} < p$ . On the other hand, by the induction hypothesis we have that  $\iota_\rho = \theta_{1,\rho} + \sum_{1 < \mu \leq d_i} (\theta_{\mu,\rho} + 1)$ . Hence taking into account the definitions of  $\theta_{\mu,\rho}$ , we conclude the proof in Case 1 as follows:

$$\begin{aligned} \iota_{\rho+1} + 1 &= p\iota_\rho + \epsilon_{\rho+1} + 1 \\ &= p\theta_{1,\rho} + \sum_{1 < \mu \leq d_i} (p\theta_{\mu,\rho} + p) + \epsilon_{\rho+1} + 1 \\ &= (p\theta_{1,\rho+1} + \epsilon_{\rho+1} + 1) + \sum_{1 < \mu \leq d_i} ((p\theta_{\mu,\rho} + p - 1) + 1) \\ &= (\iota_{1,\rho+1} + 1) + \sum_{1 < \mu \leq d_i} (\theta_{\mu,\rho+1} + 1) \\ &= \sum_{1 \leq \mu \leq d_i} (\theta_{\mu,\rho+1} + 1). \end{aligned}$$

Case 2:  $\rho + 1 = r_{i+1}$ . Then  $\rho + 1$  is an essential jump index, thus by definitions one has:  $\iota_{\rho+1} = p\iota_\rho + pq_{\rho+1} + \epsilon_{\rho+1}$  with  $0 < q_{\rho+1}$  and  $0 < \epsilon_{\rho+1} < p$ ,  $d_{i+1} = d_i + q_{\rho+1}$ ,  $r_{i+1} \leq \rho + 1 < r_{i+2}$ . On the other hand, by the induction hypothesis one has  $\iota_\rho = \theta_{1,\rho} + \sum_{1 < \mu \leq d_i} (\theta_{\mu,\rho} + 1)$ . Therefore, using the definitions of  $\theta_{\mu,\rho}$  we get:

$$\begin{aligned} \iota_{\rho+1} + 1 &= p\iota_\rho + pq_{\rho+1} + \epsilon_{\rho+1} + 1 \\ &= (p\theta_{1,\rho} + \epsilon_{\rho+1} + 1) + \sum_{1 < \mu \leq d_i} (p\theta_{\mu,\rho} + p) + pq_{\rho+1} \\ &= (\iota_{1,\rho+1} + 1) + \sum_{1 < \mu \leq d_i} ((p\theta_{\mu,\rho} + p - 1) + 1) + \sum_{d_i < \mu \leq d_{i+1}} ((p - 1) + 1) \\ &= (\iota_{1,\rho+1} + 1) + \sum_{1 < \mu \leq d_{i+1}} (\theta_{\mu,\rho+1} + 1) \\ &= \sum_{1 \leq \mu \leq d_{i+1}} (\theta_{\mu,\rho+1} + 1). \end{aligned}$$

This completes the proof of Lemma 3.3. □

## B) Generic liftings

Let  $k[[t]] \hookrightarrow k[[z]]$  be a  $\mathbb{Z}/p^e$ -cyclic extension with upper ramification jumps  $\iota_1 \leq \dots \leq \iota_e$ . Recall that setting  $K = k((t))$  and  $L = k((z))$ , in the notations introduced in section 2, B), there exists  $\mathbf{p} = (p_1(t^{-1}), \dots, p_e(t^{-1}))$  such that  $L = K_{\mathbf{p}}$ , where  $\mathbf{p}$  is in standard form, i.e., either  $p_\rho(t^{-1}) = 0$  or it contains no non-zero terms in which the exponent of  $t^{-1}$  is divisible by  $p$ . And by the discussion in section 2, B), one has that

$$\iota_\rho = \max \{ p\iota_{\rho-1}, \deg(p_\rho(t^{-1})) \}, \quad \rho = 1, \dots, e.$$

**Fact 3.4.** Let  $\mathbf{c} := (h_1(t^{-1}), \dots, h_e(t^{-1}))$  be an arbitrary Witt vector with coordinates in  $k[t^{-1}]$ . For a fixed polynomial  $h(t^{-1}) \in k[t^{-1}]$ , let  $\mathbf{c}_{h,i}$  be the Witt vector whose  $i^{\text{th}}$  coordinate is  $h(t^{-1})$ , and all the other coordinates are equal to  $0 \in k[t^{-1}]$ . Then  $\tilde{\mathbf{c}} := \mathbf{c} + \mathbf{c}_{h,i}$  has coordinates  $\tilde{\mathbf{c}} = (\tilde{h}_1(t^{-1}), \dots, \tilde{h}_e(t^{-1}))$  satisfying the following:

- a)  $\tilde{h}_j(t^{-1}) = h_j(t^{-1})$  for  $j < i$ .
- b)  $\tilde{h}_i(t^{-1}) = h_i(t^{-1}) + h(t^{-1})$ .
- c)  $\deg(\tilde{h}_j(t^{-1})) \leq \max \{ \deg(h_j(t^{-1})), p^{j-i} \deg(h(t^{-1})) \}$  for all  $i < j$ .



**Definition/Remark 3.5.**

1) In the above context, let  $\mathbf{q} := (q_1(t^{-1}), \dots, q_e(t^{-1}))$  with  $q_\rho(t^{-1}) \in k[t^{-1}]$  be some generator of  $L|K$ , i.e.,  $L = K_{\mathbf{q}}$ . We say that  $\mathbf{q}$  is **normalized** if  $\nu_\rho = \deg(q_\rho(t^{-1}))$ ,  $\rho = 1, \dots, e$ . And we say that  $\mathbf{q}$  is **separable**, if each  $q_\rho(t^{-1})$  is a separable polynomial (in  $t^{-1}$ ).

2) We notice that if  $\mathbf{q} := (q_1(t^{-1}), \dots, q_e(t^{-1}))$  is some given Witt vector and  $L := K_{\mathbf{q}}$ , then  $\mathbf{q}$  is normalized if and only if it satisfies:  $\deg(q_1(t^{-1}))$  is prime to  $p$ , and for all  $1 \leq \rho < e$  one has that  $p | \deg(q_{\rho+1}(t^{-1}))$  implies  $\deg(q_{\rho+1}(t^{-1})) = p \deg(q_\rho(t^{-1}))$ .

3) Given a generator  $\mathbf{p} = (p_1(t^{-1}), \dots, p_e(t^{-1}))$  in standard form for  $L|K$ , one can construct a separable normalized generator  $\mathbf{q} = (q_1(t^{-1}), \dots, q_e(t^{-1}))$  as follows: Consider Witt vectors of the form  $\mathbf{c} := (h_1(t^{-1}), \dots, h_e(t^{-1}))$  with  $h_\rho(t^{-1}) \in k[t^{-1}]$  and  $\deg(h_\rho(t^{-1})) = \nu_{\rho-1}$  for  $1 < \rho \leq e$ , which are “inductively generic” with those properties. Then setting

$$\mathbf{q} := \mathbf{p} + \wp_e(\mathbf{c}) =: (q_1(t^{-1}), \dots, q_e(t^{-1})),$$

it follows that  $K_{\mathbf{q}} = L = K_{\mathbf{p}}$ , thus  $\mathbf{q}$  is a representative for  $\mathbf{p}$  modulo  $\wp_e(K)$ . And applying inductively Fact 3.4 above, one gets: If  $p\nu_{\rho-1} < \nu_\rho$ , then  $\deg(h_\rho(t^{-1})) = \deg(p_\rho(t^{-1})) = \nu_\rho$ . Second, if  $p\nu_{\rho-1} = \nu_\rho$ , then  $\deg(p_\rho(t^{-1})) < \nu_\rho = p\nu_{\rho-1}$ . Hence applying Fact 3.4 inductively, since  $h_\rho(t^{-1})$  is generic of degree  $\nu_{\rho-1}$ , we get:  $\deg(q_\rho(t^{-1})) = p \deg(h_{\rho-1}(t^{-1})) = p\nu_{\rho-1} = \nu_\rho$ , and each  $q_\rho(t^{-1})$  is separable.

Coming back to our general context, let  $\mathbf{p} = (p_1(t^{-1}), \dots, p_e(t^{-1}))$  be a *normalized generator* for  $L|K$ . Let  $(\theta_{\mu,\rho})_{1 \leq \mu \leq N, 1 \leq \rho \leq e}$  be the matrix of non-negative integers produced in the previous subsection A). Since  $k$  is algebraically closed, we can write each polynomial  $p_\rho(t^{-1})$  as a product of polynomials  $p_{\mu,\rho}(t^{-1})$  as follows:

$$p_\rho(t^{-1}) = \prod_{1 \leq \mu \leq N} p_{\mu,\rho}(t^{-1}),$$

with  $\deg(p_{1,\rho}(t^{-1})) = \theta_{1,\rho}$ ,  $\deg(p_{\mu,\rho}(t^{-1})) = \theta_{\mu,\rho} + 1$  for  $\mu > 1$ ,  $\theta_{\mu,\rho} \neq 0$ ,  $p_{\mu,\rho} = 1$  if  $\theta_{\mu,\rho} = 0$ .

For the given elements  $x_\mu \in \varpi\mathfrak{o}$ ,  $\mu = 1, \dots, N$  we set  $t_\mu := t - x_\mu \in \mathfrak{o}[t]$ . And for a fixed choice  $p_{\mu,\rho}(t^{-1}) \in k[t^{-1}]$ , we let  $P_{\mu,\rho}(t_\mu^{-1}) \in \mathfrak{o}[t_\mu^{-1}]$  be “generic” preimages with  $\deg(P_{\mu,\rho}(t_\mu^{-1})) = \deg(p_{\mu,\rho}(t^{-1}))$  for  $\mu = 1, \dots, N$  and  $\rho = 1, \dots, e$ . In particular, each

$$P_\rho := \prod_{\mu} P_{\mu,\rho}(t_\mu^{-1}) \in \mathfrak{o}[t_{x_1}^{-1}, \dots, t_{x_N}^{-1}] \subset \mathcal{A}_{x_1, \dots, x_N}, \quad \rho = 1, \dots, e$$

is a linear combination of monomials in  $(t - x_1)^{-1}, \dots, (t - x_N)^{-1}$  with “general” coefficients from  $\mathfrak{o}$  such that under the specialization homomorphism  $\mathcal{A}_{x_1, \dots, x_N} \rightarrow A[t^{-1}]$  they map to  $p_\rho = p_\rho(t^{-1})$ . To indicate this, we will write for short

$$P_{\mu,\rho}(t_\mu^{-1}) \mapsto p_{\mu,\rho}(t^{-1}), \quad P_\rho \mapsto p_\rho(t^{-1}).$$

We set  $\mathbf{P} := (P_1, \dots, P_e)$  and view it as a Witt vector of length  $e$  over  $\mathcal{K}$ , and consider the corresponding cyclic field extension  $\mathcal{L} := \mathcal{K}_{\mathbf{P}}$ . Let  $\mathcal{A} \hookrightarrow \mathcal{B}$  be the normalization of  $\mathcal{A}$  in  $\mathcal{K} \hookrightarrow \mathcal{L}$ . Since  $\mathcal{A} = k[[\varpi, t]]$  is Noetherian and  $\mathcal{K} \hookrightarrow \mathcal{L}$  is separable, it follows that  $\mathcal{B}$  is a finite  $\mathcal{A}$ -algebra, thus Noetherian. And since  $\mathcal{A}$  is local and complete, so is  $\mathcal{B}$ .

We next have a closer look at the branching in the finite ring extension  $\mathcal{A} \hookrightarrow \mathcal{B}$ . For that we view  $\mathcal{A} \hookrightarrow \mathcal{B}$  as a finite morphism of  $\mathfrak{o} := k[[\varpi]]$  algebras, and introduce geometric language as follows:  $\mathcal{X} = \text{Spec } \mathcal{A}$  and  $\mathcal{Y} = \text{Spec } \mathcal{B}$ . Thus  $\mathcal{A} \hookrightarrow \mathcal{B}$  defines a finite  $\mathfrak{o}$ -morphism  $\mathcal{Y} \rightarrow \mathcal{X}$ . Further let  $\mathfrak{Y} := \text{Spec } \mathcal{S} \rightarrow \text{Spec } \mathcal{R} =: \mathfrak{X}$  and  $Y := \text{Spec } \mathcal{B}/(\varpi) \rightarrow \text{Spec } \mathcal{A}/(\varpi) =: X$

be the generic fiber, respectively the special fiber of  $\mathcal{Y} \rightarrow \mathcal{X}$ . In particular,  $X = \text{Spec } A$  and  $Y \rightarrow X$  is a finite morphism. We further mention the following general fact for later use:

**Fact 3.6.** Let  $\mathcal{K} \hookrightarrow \mathcal{L}$  be a cyclic extension of degree  $[\mathcal{L} : \mathcal{K}] = p^e$  with Galois group  $G = \mathbb{Z}/p^e$ , say defined by some Witt vector  $\mathbf{a} = (a_1, \dots, a_e) \in W_e(\mathcal{K})$ . For every  $0 \leq m \leq e$  let  $\mathcal{K} \hookrightarrow \mathcal{L}_m$  be the unique sub-extension of  $\mathcal{K} \hookrightarrow \mathcal{L}$  with  $[\mathcal{L}_m : \mathcal{K}] = p^m$ , hence  $\mathcal{L}_0 = \mathcal{K}$  and  $\mathcal{L}_e = \mathcal{L}$ . Let  $v$  be a discrete valuation of  $\mathcal{K}$ , say with valuation ring  $\mathcal{O}_v \subset \mathcal{K}$  and residue field  $\mathcal{O}_v \rightarrow \kappa_v, f \mapsto \bar{f}$ , and let  $T_v \subseteq Z_v \subseteq G$  be the inertia, respectively decomposition, subgroups of  $v$  in  $G$ . Then for all  $m$  with  $1 \leq m \leq e$  the following hold:

- 1) Let  $v(a_1), \dots, v(a_m) \geq 0$ . Then  $T_v \subseteq p^m G$ , and  $Z_v \subset p^m G$  iff  $(\bar{a}_1, \dots, \bar{a}_m) \in \text{im}(\wp_m)$ .
- 2) If  $v(a_m)$  is negative and prime to  $p$ , then  $p^{m-1} G \subseteq T_v$ .

We notice that since  $\mathcal{A} = k[[\varpi, t]]$  is a two dimensional local regular ring,  $\mathcal{X}$  is a two dimensional regular scheme. Therefore, the branch locus of  $\mathcal{Y} \rightarrow \mathcal{X}$  is of pure co-dimension one. Thus in order to describe the branching behavior of  $\mathcal{Y} \rightarrow \mathcal{X}$  one has to describe the branching at the generic point  $(\varpi)$  of the special fiber  $X \subset \mathcal{X}$  of  $\mathcal{X}$ , and at the closed points  $x$  of the generic fiber  $\mathfrak{X} \subset \mathcal{X}$  of  $\mathcal{X}$ .

- *The branching at  $(\varpi)$*

We recall that  $\mathcal{K} \hookrightarrow \mathcal{L}$  is defined as a cyclic extension by  $\mathbf{P} := (P_1, \dots, P_e)$ , where each  $P_\rho$  is of the form  $P_\rho = \prod_\mu P_{\mu,\rho}(t_\mu^{-1})$  with  $P_{\mu,\rho}(t_\mu^{-1}) \in \mathfrak{o}[t_\mu^{-1}] \subset \mathcal{A}_{x_1, \dots, x_N}$  is some generic preimage of  $p_{\mu,\rho}(t_\mu^{-1})$  with degree satisfying  $\deg(P_{\mu,\rho}(t_\mu^{-1})) = \deg(p_{\mu,\rho}(t_\mu^{-1}))$ . In particular, the elements  $P_{\mu,\rho}(t_\mu^{-1})$  are not divisible by  $\varpi$  in the factorial ring  $\mathcal{A}_{x_1, \dots, x_N}$ . Therefore, the elements  $P_{\mu,\rho}(t_\mu^{-1})$  are units  $\mathcal{A}_{(\varpi)}$ . By Fact 3.6 we conclude that  $\varpi$  is not branched in  $\mathcal{K} \hookrightarrow \mathcal{L}$ , and therefore, the special fiber  $Y \rightarrow X$  of  $\mathcal{Y} \rightarrow \mathcal{X}$  is reduced. Moreover, since  $P_1 \mapsto p_1(t^{-1})$ , and the latter satisfies  $p_1(t^{-1}) \notin \wp(K)$ , it follows by Fact 3.6, 1), that  $\text{Gal}(\mathcal{L}|\mathcal{K})$  is contained in the decomposition group of  $v_\varpi$ . In other words,  $\varpi$  is totally inert in  $\mathcal{K} \hookrightarrow \mathcal{L}$ . In particular,  $\mathcal{Y} \rightarrow \mathcal{X}$  is étale above  $\varpi$ , and moreover, the special fiber  $Y \rightarrow X$  of  $\mathcal{Y} \rightarrow \mathcal{X}$  is reduced, irreducible, and generically cyclic Galois of degree  $p^e = [L : K]$ .

- *The branching at the points of the generic fiber  $x \in \mathfrak{X}$*

Recall that  $\hat{\kappa} = k((\varpi))$  and that  $\mathfrak{X} = \text{Spec } \mathcal{R}$ , where  $\mathcal{R} = \mathcal{A} \otimes_{k[[\varpi]]} \hat{\kappa}$  is the ring of power series in  $t$  with bounded coefficients from the complete discrete valued field  $\hat{\kappa}$ . [Thus  $\mathfrak{X}$  is actually the rigid open unit disc over  $\hat{\kappa}$ .] If  $x \in \mathfrak{X}$  is a closed point different from  $x_1, \dots, x_N$ , and  $\mathcal{A}_x$  is the local ring of  $\mathfrak{X}$  at  $x$ , it follows that  $P_1, \dots, P_e \in \mathcal{A}_x$ . Hence by Fact 3.6, 1), it follows that  $x$  is not branched in  $\mathcal{K} \hookrightarrow \mathcal{L}$ . Thus it is left to analyze the branching behavior of  $\mathfrak{Y} \rightarrow \mathfrak{X}$  at the closed points  $x_1, \dots, x_N \in \mathfrak{X}$ . In this process we will also compute the total contribution of the ramification above  $x_\mu$  to the total different  $\mathfrak{D}_{S|\mathcal{R}}$  for  $\mu = 1, \dots, N$ .

Recall that every  $x_\mu$  is a  $\hat{\kappa}$  rational point of  $\mathfrak{X}$ , and  $t_\mu := t - x_\mu$  is the ‘‘canonical’’ uniformizing parameter at  $x_\mu$ . Thus  $\mathcal{K}_\mu := \hat{\kappa}((t_\mu))$  is the quotient field of the completion of the local ring at  $x_\mu$ , and we denote by  $v_\mu : \mathcal{K}_\mu^\times \rightarrow \mathbb{Z}$  the canonical valuation at  $x_\mu$ . We notice that  $t_\nu = x_\mu - x_\nu + t_\mu$ , hence by the ‘‘genericity’’ of  $P_{\mu,\rho}(t_\mu^{-1})$ , we can and will suppose that  $P_{\nu,\rho}(t_\nu^{-1})$  is a  $v_\mu$ -unit in  $\mathcal{K}_\mu$ . We conclude that there exist  $v_\mu$ -units  $\eta_1, \dots, \eta_e \in \mathcal{K}_\mu$  such that denoting by  $\mathcal{L}_\mu$  the compositum of  $\mathcal{K}_\mu$  and  $\mathcal{L}$ , the cyclic extension  $\mathcal{K}_\mu \hookrightarrow \mathcal{L}_\mu$  is defined by the Witt vector:

$$\mathbf{P}_\mu = (\eta_1 P_{\mu,1}(t_\mu^{-1}), \dots, \eta_e P_{\mu,e}(t_\mu^{-1})).$$

Case 1:  $\mu = 1$ .

First, by definitions we have  $\deg(P_{1,\rho}(t_1^{-1})) = \deg(p_{1,\rho}(t^{-1})) = \theta_{1,\rho}$  for all  $\rho$ . Second, by the definitions of  $(\theta_{1,\rho})_{1 \leq \rho \leq e}$  it follows that  $(p_{1,1}(t^{-1}), \dots, p_{1,e}(t^{-1}))$  is actually a normalized system of polynomials in  $k[t^{-1}]$ . Hence by Definition/Remark 3.5, 3), it follows that the upper ramification jumps of  $\mathcal{K}_1 \hookrightarrow \mathcal{L}_1$  are precisely  $\theta_{1,1} \leq \dots \leq \theta_{1,e}$ , and in particular, one has  $[\mathcal{L}_1 : \mathcal{K}_1] = p^e$ .

Case 2:  $1 < \mu$ , hence one has  $0 < N_0$  as well.

In the notations from the previous subsection A), let  $1 \leq i \leq N_0$  maximal be such that  $d_{i-1} < \mu$ . Then by the definition of  $(\theta_{\mu,\rho})_{1 \leq \rho \leq e}$  we have:  $\theta_{\mu,\rho} = 0$  for  $\rho < r_i$ ,  $\theta_{\mu,r_i} = p - 1$ , and  $\theta_{\mu,\rho} = p\theta_{\mu,\rho-1} + p - 1$  for  $r_i \leq \rho \leq e$ . Further, again by definitions, one has  $P_{\mu,\rho}(t_\mu^{-1}) = 1$  for  $\theta_{\mu,\rho} = 0$ , i.e., for  $\rho < r_i$ . And  $P_{\mu,\rho}(t_\mu^{-1})$  is generic of degree  $\theta_{\mu,\rho} + 1$  for  $r_i \leq \rho \leq e$ . Hence the Witt vector  $\mathbf{P}_\mu := (\eta_1 P_{\mu,1}(t_\mu^{-1}), \dots, \eta_e P_{\mu,e}(t_\mu^{-1}))$  satisfies the following conditions:

- $\eta_\rho P_{\mu,\rho}(t_\mu^{-1}) = \eta_\rho$  for  $1 \leq \rho < r_i$ .
- $\eta_\rho P_{\mu,\rho}(t_\mu^{-1})$  is a generic polynomial of degree  $\theta_{\mu,\rho} + 1$  for  $r_i \leq \rho \leq e$ .

In particular, by Fact 3.6, one has that  $v_\mu$  is unramified in the sub-extension  $\mathcal{K} \hookrightarrow \mathcal{L}_{r_i-1}$  of degree  $p^{r_i-1}$  of  $\mathcal{K} \hookrightarrow \mathcal{L}$ . We claim that  $\mathcal{L}_{r_i-1} \hookrightarrow \mathcal{L}$  is actually totally ramified, or equivalently, that  $\text{Gal}(\mathcal{L}|\mathcal{L}_{r_i-1}) \subseteq \text{Gal}(\mathcal{L}|\mathcal{K})$  is the inertia group of  $v_\mu$ , hence the ramification subgroup of  $v_\mu$ , because there is no tame ramification involved. Indeed, it is sufficient to prove that this is the case after base changing everything to the maximal unramified extension  $\mathcal{K}_\mu \hookrightarrow \mathcal{K}_\mu^{\text{nr}}$  of  $\mathcal{K}_\mu$ . Recall that for  $\rho$  with  $r_i \leq \rho \leq e$  we have by definitions that  $P_{\mu,\rho}(t_\mu^{-1}) \in \mathfrak{o}[t_\mu^{-1}]$  is a generic polynomial in  $t_\mu^{-1}$  over  $\mathfrak{o} = k[[\varpi]]$  of degree  $\theta_{\mu,\rho} + 1$ . Hence for  $\rho = r_i$  we have:  $\theta_{\mu,r_i} + 1$  is divisible by  $p$  and  $P_{\mu,r_i}(t_\mu^{-1})$  is generic. But then it follows that the standard representative  $Q_{\mu,r_i}(t_\mu^{-1}) \in \hat{\kappa}^{\text{sep}}[t_\mu^{-1}]$  of  $P_{\mu,r_i}(t_\mu^{-1})$  modulo  $\wp(\mathcal{K}_\mu^{\text{nr}})$  has degree  $\theta_{\mu,\rho}$ . Thus by Fact 3.6, 3), it follows that  $v_\mu$  is totally ramified in the field extension  $\mathcal{K}_\mu^{\text{nr}} \hookrightarrow \mathcal{L}_\mu^{\text{nr}}$  and that  $p^{e-r_i+1} = [\mathcal{L}_\mu^{\text{nr}} : \mathcal{K}_\mu^{\text{nr}}]$ . Combining this with the fact that  $v_\mu$  is unramified in  $\mathcal{K} \hookrightarrow \mathcal{L}_{r_i-1}$  and  $p^{r_i-1} = [\mathcal{L}_{r_i-1} : \mathcal{K}]$ , it follows that  $\mathcal{K} \hookrightarrow \mathcal{L}_{r_i-1}$  is the ramification field of  $v_\mu$  in  $\mathcal{K} \hookrightarrow \mathcal{L}$ . Equivalently,  $\text{Gal}(\mathcal{L}|\mathcal{L}_{r_i-1}) \subseteq \text{Gal}(\mathcal{L}|\mathcal{K})$  is the ramification subgroup of  $v_\mu$ .

We next compute the degree of the local different of  $\mathcal{R} \hookrightarrow \mathcal{S}$  above  $x_\mu$ . Recall that  $\mathcal{K} \hookrightarrow \mathcal{L}$  is defined by the Witt vector

$$\mathbf{P}_\mu = (\eta_1 P_{\mu,1}(t_\mu^{-1}), \dots, \eta_e P_{\mu,e}(t_\mu^{-1})).$$

On the other hand,  $\mathbf{P}_\mu$  is equivalent modulo  $\wp_e(\mathcal{K}^{\text{nr}})$  to its standard form

$$\mathbf{P}'_\mu = (Q_{\mu,1}(t_\mu^{-1}), \dots, Q_{\mu,e}(t_\mu^{-1}))$$

with  $Q_{\mu,\rho} = 0$  for  $1 \leq \rho < r_i$  and  $\deg(Q_{\mu,\rho}(t_\mu^{-1})) = \theta_{\mu,\rho}$  for  $r_i \leq \rho \leq e$ . Hence setting

$$\mathbf{Q}_\mu = (Q_{\mu,r_i}(t_\mu^{-1}), \dots, Q_{\mu,e}(t_\mu^{-1})),$$

it follows that  $\mathbf{Q}_\mu$  is a Witt vector of length  $e - r_i + 1$ , and  $\mathbf{Q}_\mu$  is in standard form, and the cyclic extension  $\mathcal{K}_\mu^{\text{nr}} \hookrightarrow \mathcal{L}_\mu^{\text{nr}}$  is defined by  $\mathbf{Q}_\mu$ . Hence setting  $e_\mu := e - r_i + 1$ , it follows that the cyclic field extension  $\mathcal{K}_\mu^{\text{nr}} \hookrightarrow \mathcal{L}_\mu^{\text{nr}}$  has degree  $p^{e_\mu}$  and upper ramification jumps given by

$$(*) \quad \iota_{\mu,\nu} = \deg(Q_{\mu,r_i+\nu}(t_\mu^{-1})) = \theta_{\mu,r_i+\nu-1}, \quad \nu = 1, \dots, e_\mu.$$

C) *Finishing the proof of Key Lemma 3.2*

Let  $\mathfrak{D}_\mu$  be the local part above  $x_\mu$  of the global different  $\mathfrak{D}_{\mathcal{S}|\mathcal{R}}$  of the extension of Dedekind rings  $\mathcal{R} \hookrightarrow \mathcal{S}$ . Then if  $\mathcal{K} \hookrightarrow \mathcal{L}^Z \hookrightarrow \mathcal{L}^T \hookrightarrow \mathcal{L}$  are the decomposition/inertia subfields of  $v_\mu$  in the cyclic field extension  $\mathcal{K} \hookrightarrow \mathcal{L}$ , by the functorial behavior of the different, it follows that

$$\deg(\mathfrak{D}_\mu) = [\kappa(x_\mu) : \hat{\kappa}] \cdot [\mathcal{L}^T : \mathcal{K}] \cdot \deg(\mathfrak{D}_{\mathcal{L}_\mu^{\text{nr}}|\mathcal{K}_\mu^{\text{nr}}}).$$

On the other hand, since  $x_\mu \in \mathfrak{m}_\mathfrak{o}$ , one has  $\kappa(x_\mu) = \hat{\kappa}$ . Further, by the discussion above one has that  $\mathcal{L}^T = \mathcal{L}_{r_i-1}$ , thus  $p^{r_i-1} = [\mathcal{L}^T : \mathcal{K}]$ . And  $\deg(\mathfrak{D}_{\mathcal{L}_\mu^{\text{nr}}|\mathcal{K}_\mu^{\text{nr}}})$  can be computed in terms of upper ramification jumps as indicated at the end of section 1), A):

$$\deg(\mathfrak{D}_{\mathcal{L}_\mu^{\text{nr}}|\mathcal{K}_\mu^{\text{nr}}}) = \sum_{1 \leq \nu \leq e_\mu} (\iota_{\mu,\nu} + 1)(p^\nu - p^{\nu-1}).$$

Hence taking into account the discussion above, we get:

$$\begin{aligned} \deg(\mathfrak{D}_\mu) &= [\kappa(x_\mu) : \hat{\kappa}] \cdot [\mathcal{L}^T : \mathcal{K}] \cdot \deg(\mathfrak{D}_{\mathcal{L}_\mu^{\text{nr}}|\mathcal{K}_\mu^{\text{nr}}}) \\ &= p^{r_i-1} \sum_{1 \leq \nu \leq e_\mu} (\iota_{\mu,\nu} + 1)(p^\nu - p^{\nu-1}) \\ &= \sum_{1 \leq \nu \leq e_\mu} (\theta_{\mu,r_i+\nu-1} + 1)(p^{\nu+r_i-1} - p^{(\nu+r_i-1)-1}) \\ &= \sum_{r_i \leq \rho \leq e} (\theta_{\mu,\rho} + 1)(p^\rho - p^{\rho-1}) \end{aligned}$$

Recall that  $\iota_\rho + 1 = \sum'_\mu (\theta_{\mu,\rho} + 1)$  for all  $1 \leq \rho \leq e$ , where  $\sum'_\mu$  is taken over all  $\mu$  with  $\theta_{\mu,\rho} \neq 0$ . Further,  $\deg(\mathfrak{D}_1) = \sum_{1 \leq \rho \leq e} (\theta_{1,\rho} + 1)(p^\rho - p^{\rho-1})$  and  $\deg(\mathfrak{D}_\mu) = \sum_{r_i \leq \rho \leq e} (\theta_{\mu,\rho} + 1)(p^\rho - p^{\rho-1})$  for all  $1 < \mu \leq N$ . Therefore we get the following:

$$\begin{aligned} \deg(\mathfrak{D}_{\mathcal{S}|\mathcal{R}}) &= \sum_{1 \leq \mu \leq N} \deg(\mathfrak{D}_\mu) \\ &= \sum_{1 \leq \rho \leq e} (\theta_{1,\rho} + 1)(p^\rho - p^{\rho-1}) + \sum_{1 \leq i \leq N_0} \sum_{d_{i-1} < \mu \leq d_i} \sum_{r_i \leq \rho \leq e} (\theta_{\mu,\rho} + 1)(p^\rho - p^{\rho-1}) \\ &= \sum_{1 \leq \rho \leq e} (\theta_{1,\rho} + 1)(p^\rho - p^{\rho-1}) + \sum_{1 \leq i \leq N_0} \sum_{r_i \leq \rho < r_{i+1}} \sum_{d_0 < \mu \leq d_i} (\theta_{\mu,\rho} + 1)(p^\rho - p^{\rho-1}) \\ &= \sum_{1 \leq i \leq N_0} \sum_{r_i \leq \rho < r_{i+1}} \sum_{1 \leq \mu \leq d_i} (\theta_{\mu,\rho} + 1)(p^\rho - p^{\rho-1}) \\ &= \sum_{1 \leq i \leq N_0} \sum_{r_i \leq \rho < r_{i+1}} (\iota_\rho + 1)(p^\rho - p^{\rho-1}) \\ &= \sum_{1 \leq \rho \leq e} (\iota_\rho + 1)(p^\rho - p^{\rho-1}) \\ &= \deg(\mathfrak{D}_{L|K}). \end{aligned}$$

We thus conclude the proof of Key Lemma 3.2 by applying Kato's criterion Fact 2.1.

D) *Characteristic  $p$  global Oort Conjecture*

**Theorem 3.7.** (Characteristic  $p$  global Oort conjecture) *In the notations from the Key Lemma 3.2, let  $Y \rightarrow X$  be a (ramified) Galois cover of complete smooth  $k$ -curves having only cyclic groups as inertia groups, and set  $\mathcal{X}_\mathfrak{o} := X \times_k \mathfrak{o}$ . Then there exists a  $G$ -cover of complete smooth  $\mathfrak{o}$ -curves  $\mathcal{Y}_\mathfrak{o} \rightarrow \mathcal{X}_\mathfrak{o}$  with special fiber  $Y \rightarrow X$  such that the generic fiber  $\mathcal{Y}_{\hat{\kappa}} \rightarrow \mathcal{X}_{\hat{\kappa}}$  of  $\mathcal{Y}_\mathfrak{o} \rightarrow \mathcal{X}_\mathfrak{o}$  has no essential ramification.*

*Proof.* First, as in the case of the classical Oort Conjecture, the local-global principle for lifting (ramified) Galois covers, see GARUTI [5], §3, as well as SAIDI [24], §1.2, where the proofs of Propositions 1.2.2 and 1.2.4 are very detailed, reduces the proof of the Theorem 3.7 to the corresponding local problem over  $\mathfrak{o}$ . Further, exactly as in the case of the classical local Oort Conjecture, the local problem is equivalent to the case where the inertia groups are cyclic  $p$ -groups. One concludes by applying the Key Lemma 3.2.  $\square$

#### 4. PROOF OF THEOREM 1.1

##### A) Generalities about covers of $\mathbb{P}^1$

**Notations 4.1.** We begin by introducing notations concerning families of covers of curves which will be used throughout this section. Let  $S$  be a separated, integral normal scheme, e.g.,  $S = \text{Spec } A$  with  $A$  and integrally closed domain, and  $\mathbf{k} := \kappa(S)$  its field of rational functions. Let  $\mathbf{k}(t) \hookrightarrow F$  be a finite extension of the rational function field  $\mathbf{k}(t)$ .

1)  $\mathbb{P}_{t,S}^1 = \text{Proj } \mathbb{Z}[t_0, t_1] \times S$  is the  $t$ -projective line over  $S$ , where  $t = t_1/t_0$  is the canonical parameter on  $\mathbb{P}_{t,S}^1$ . In particular,  $\mathbb{P}_{t,S}^1$  is the gluing of its canonical affine lines over  $S$ , namely  $\mathbb{A}_{t,S}^1 := \text{Spec } \mathbb{Z}[t] \times S$  and  $\mathbb{A}_{t^{-1},S}^1 := \text{Spec } \mathbb{Z}[t^{-1}] \times S$ .

2) Let  $\mathbf{k}(t) \hookrightarrow F$  be a finite extension, and  $\mathcal{Y}_{t,S} \rightarrow \mathbb{A}_{t,S}^1$  and  $\mathcal{Y}_{t^{-1},S} \rightarrow \mathbb{A}_{t^{-1},S}^1$  the corresponding normalizations in  $\mathbf{k}(t) \hookrightarrow F$ . Then the normalization  $\mathcal{Y}_S \rightarrow \mathbb{P}_{t,S}^1$  of  $\mathbb{P}_{t,S}^1$  in  $\mathbf{k}(t) \hookrightarrow F$  is nothing but the gluing of  $\mathcal{Y}_{t,S} \rightarrow \mathbb{A}_{t,S}^1$  and  $\mathcal{Y}_{t^{-1},S} \rightarrow \mathbb{A}_{t^{-1},S}^1$ .

3) For every  $\mathfrak{p} \in S$  we denote by  $\bar{\mathfrak{p}} \hookrightarrow S$  the closure of  $\mathfrak{p}$  in  $S$  (endowed with the reduced scheme structure). We denote by  $\mathcal{O}_{\mathfrak{p}} := \mathcal{O}_{S,\mathfrak{p}}$  the local ring at  $\mathfrak{p} \in S$ . We set  $S_{\mathfrak{p}} := \text{Spec } \mathcal{O}_{\mathfrak{p}}$  and consider the canonical morphism  $S_{\mathfrak{p}} \hookrightarrow S$ . We notice that  $\mathfrak{p} \hookrightarrow S$  is both the generic fiber of  $\bar{\mathfrak{p}} \hookrightarrow S$  and the special fiber of  $S_{\mathfrak{p}} \hookrightarrow S$  at  $\mathfrak{p}$ . We get corresponding base changes:

$$\mathcal{Y}_{\bar{\mathfrak{p}}} \rightarrow \mathbb{P}_{t,\bar{\mathfrak{p}}}^1, \quad \mathcal{Y}_{S_{\mathfrak{p}}} \rightarrow \mathbb{P}_{t,S_{\mathfrak{p}}}^1, \quad \mathcal{Y}_{\mathfrak{p}} \rightarrow \mathbb{P}_{t,\mathfrak{p}}^1$$

where  $\mathcal{Y}_{\mathfrak{p}} \rightarrow \mathbb{P}_{t,\mathfrak{p}}^1$  is both the generic fiber of  $\mathcal{Y}_{\bar{\mathfrak{p}}} \rightarrow \mathbb{P}_{t,\bar{\mathfrak{p}}}^1$  and the special fiber of  $\mathcal{Y}_{S_{\mathfrak{p}}} \rightarrow \mathbb{P}_{t,S_{\mathfrak{p}}}^1$ .

4) Finally, affine schemes will be sometimes replaced by the corresponding rings. Concretely, if  $S = \text{Spec } A$ , and  $\mathbf{k} = \text{Quot}(A)$ , for a finite extension  $\mathbf{k}(t) \hookrightarrow F$  one has/denotes:

- a) The  $t$ -projective line over  $A$  is  $\mathbb{P}_{t,A}^1 = \text{Spec } A[t] \cup \text{Spec } A[t^{-1}]$ , and the normalization  $\mathcal{Y}_A \rightarrow \mathbb{P}_{t,A}^1$  of  $\mathbb{P}_{t,A}^1$  in  $\mathbf{k}(t) \hookrightarrow F$  is obtained as the gluing of  $\text{Spec } \mathcal{R}_t \rightarrow \text{Spec } A[t]$  and  $\text{Spec } \mathcal{R}_{t^{-1}} \rightarrow \text{Spec } A[t^{-1}]$ , where  $\mathcal{R}_t$ , respectively  $\mathcal{R}_{t^{-1}}$ , are the integral closures of  $A[t]$ , respectively of  $A[t^{-1}]$ , in the field extension  $\mathbf{k}(t) \hookrightarrow F$ .
- b) For  $\mathfrak{p} \in \text{Spec}(A)$  one has/denotes:  $\mathcal{Y}_{A/\mathfrak{p}} \rightarrow \mathbb{P}_{t,A/\mathfrak{p}}^1$  and  $\mathcal{Y}_{A_{\mathfrak{p}}} \rightarrow \mathbb{P}_{t,A_{\mathfrak{p}}}^1$  are the base changes of  $\mathcal{Y}_A \rightarrow \mathbb{P}_{t,A}^1$  under  $A \rightarrow A/\mathfrak{p}$ , respectively  $A \hookrightarrow A_{\mathfrak{p}}$ ; and finally, the fiber  $\mathcal{Y}_{\kappa_{\mathfrak{p}}} \rightarrow \mathbb{P}_{t,\kappa_{\mathfrak{p}}}^1$  of  $\mathcal{Y}_A \rightarrow \mathbb{P}_{t,A}^1$  is both, the special fiber of  $\mathcal{Y}_{A_{\mathfrak{p}}} \rightarrow \mathbb{P}_{t,A_{\mathfrak{p}}}^1$  and the generic fiber of  $\mathcal{Y}_{A/\mathfrak{p}} \rightarrow \mathbb{P}_{t,A/\mathfrak{p}}^1$ .

In the above notations, suppose that  $A = \mathcal{O}$  is a local ring with maximal ideal  $\mathfrak{m}$  and residue field  $\kappa_{\mathfrak{m}}$ . Let  $\mathcal{O}_v$  be a valuation ring of  $\mathbf{k}$  dominating  $\mathcal{O}$  and having  $\kappa_{\mathfrak{m}} = \kappa_v$ . We denote by  $\mathcal{Y}_{\mathcal{O}} \rightarrow \mathbb{P}_{t,\mathcal{O}}^1$  and  $\mathcal{Y}_{\mathcal{O}_v} \rightarrow \mathbb{P}_{t,\mathcal{O}_v}^1$  the corresponding normalizations of the the corresponding projective lines. The canonical morphism  $\text{Spec } \mathcal{O}_v \rightarrow \text{Spec } \mathcal{O}$  gives canonically

a commutative diagrams dominant morphisms of the form:

$$\begin{array}{ccccc} \mathcal{Y}_{\mathcal{O}_v} & \rightarrow & \mathbb{P}_{t, \mathcal{O}_v}^1 & & \mathcal{Y}_{\mathbf{k}} \rightarrow \mathcal{Y}_{\mathcal{O}_v} \leftarrow \mathcal{Y}_v \\ \downarrow & & \downarrow & & \downarrow \cong \quad \downarrow \quad \downarrow \\ \mathcal{Y}_{\mathcal{O}} & \rightarrow & \mathbb{P}_{t, \mathcal{O}}^1 & & \mathcal{Y}_{\mathbf{k}} \rightarrow \mathcal{Y}_{\mathcal{O}} \leftarrow \mathcal{Y}_{\mathbf{m}} \end{array}$$

We denote by  $\eta_{\mathbf{m}} \in \mathbb{P}_{t, \mathbf{m}}^1$  the generic point of the special fiber of  $\mathbb{P}_{t, \mathcal{O}}^1$ , and by  $\eta_{\mathbf{m}, i} \in \mathcal{Y}_{\mathbf{m}}$  the generic points of the special fiber of  $\mathcal{Y}_{\mathcal{O}}$ . Correspondingly,  $\eta_v \in \mathbb{P}_{t, v}^1$  is the generic point of the special fiber of  $\mathbb{P}_{t, \mathcal{O}_v}^1$ , and  $\eta_{v, j} \in \mathcal{Y}_v$  are the generic points of the special fiber of  $\mathcal{Y}_{\mathcal{O}_v}$ . We notice that  $\mathbb{P}_{t, v}^1 \rightarrow \mathbb{P}_{t, \mathbf{m}}^1$  is an isomorphism (because  $\kappa_{\mathbf{m}} = \kappa_v$ ), and by the valuation criterion for completeness, for every  $\eta_{\mathbf{m}, i}$  there exists some  $\eta_{v, j}$  such that  $\eta_{v, j} \mapsto \eta_{\mathbf{m}, i}$  under  $\mathcal{Y}_v \rightarrow \mathcal{Y}_{\mathbf{m}}$ . The local ring  $\mathcal{O}_{\eta_v}$  of  $\eta_v \in \mathbb{P}_{t, v}^1$  is the valuation ring of the so called **Gauss valuation**  $v_t$  of  $\mathbf{k}(t)$ , thus  $\mathcal{O}_{\eta_{v, j}}$  are the valuation rings of the prolongations  $v_j$  of  $v_t$  to  $F$ . Finally, for every complete  $k$ -curve  $C$  we denote by  $g_C$  the geometric genus of  $C$ .

**Lemma 4.2.** *In the above notations, let  $Y_{\mathbf{m}, 1} \rightarrow \mathcal{Y}_{\mathbf{m}, 1}$  be the normalization of  $\mathcal{Y}_{\mathbf{m}, 1}$ . Suppose that  $[\kappa(\eta_{\mathbf{m}, 1}) : \kappa_{\mathbf{m}}(t)] \geq [F : \mathbf{k}(t)]$  and that  $\mathcal{Y}_{\mathbf{k}}$  is smooth. Then the following hold:*

- 1) *The special fibers  $\mathcal{Y}_{\mathbf{m}}$  and  $\mathcal{Y}_v$  are reduced and irreducible.*
- 2) *If  $g_{Y_{\mathbf{m}, 1}} \geq g_{\mathcal{Y}_{\mathbf{k}}}$ , then  $\mathcal{Y}_{\mathcal{O}_v} \rightarrow \mathbb{P}_{t, \mathcal{O}_v}^1$  is a cover of smooth  $\mathcal{O}_v$ -curves.*

*Proof.* To 1): Let  $\eta_{v, 1} \mapsto \eta_{\mathbf{m}, 1}$ . Then  $\kappa_{\eta_{\mathbf{m}, 1}} \hookrightarrow \kappa_{\eta_{v, 1}}$ , thus  $[\kappa_{\eta_{\mathbf{m}, 1}} : \kappa_{\mathbf{m}}(t)] \leq [\kappa_{\eta_{v, 1}} : \kappa_v(t)]$ . Hence using the fundamental equality and the hypothesis one gets that

$$[\kappa(\eta_{v, 1}) : \kappa_v(t)] \geq [\kappa(\eta_{\mathbf{m}, 1}) : \kappa_{\mathbf{m}}(t)] \geq [F : \mathbf{k}(t)] = \sum_j [\kappa_{v_j} : \kappa_v(t)] e(v_j|v_t) \delta(v_j|v_t),$$

where  $e(\cdot|\cdot)$  is the ramification index and  $\delta(\cdot|\cdot)$  is the Ostrowski defect. Hence we conclude that  $w_1$  is the only prolongation of  $v_t$  to  $F$ , and  $e(v_1|v_t) = 1 = \delta(v_1|v_t)$ .

To 2): Since  $\mathcal{Y}_v$  is reduced and irreducible, by ROQUETTE [22], Satz I, it follows that the Euler characteristics of the special fiber  $\mathcal{Y}_v$  and that of the generic fiber  $\mathcal{Y}_{\mathbf{k}}$  of  $\mathcal{Y}_{\mathcal{O}_v}$  are equal:

$$\chi(\mathcal{Y}_{\mathbf{k}}|\mathbf{k}) = \chi(\mathcal{Y}_v|\kappa_v).$$

Since  $\mathcal{Y}_v$  dominates  $\mathcal{Y}_{\mathbf{m}}$ , one has  $\kappa(\mathcal{Y}_{\mathbf{m}}) \hookrightarrow \kappa(\mathcal{Y}_v)$ , hence one has  $g_{\mathcal{Y}_{\mathbf{m}}} \leq g_{\mathcal{Y}_v}$ . Thus 2) implies:

$$1 - g_{Y_{\mathbf{m}, 1}} \leq 1 - g_{\mathcal{Y}_{\mathbf{k}}} = \chi(\mathcal{Y}_{\mathbf{k}}|\mathbf{k}) = \chi(\mathcal{Y}_v|\kappa_v) \leq \chi(Y_v|\kappa_v) \leq \chi(Y_{\mathbf{m}, 1}|\kappa_{\mathbf{m}}) = 1 - g_{Y_{\mathbf{m}, 1}}.$$

Hence all the above are equalities, thus finally one has that  $(\mathcal{Y}_v|\kappa_v) \leq \chi(Y_v|\kappa_v)$ . Therefore, the normalization  $Y_v \rightarrow \mathcal{Y}_v$  is an isomorphism, and  $\mathcal{Y}_{\mathcal{O}_v}$  is smooth.  $\square$

In the context above, let  $A = \mathcal{O}$  be a *valuation ring* and  $v$  be its valuation. Let  $v := v_0 \circ v_1$  be the valuation theoretical composition of two valuations, say with valuation rings  $\mathcal{O}_1 \subset \mathbf{k}$ , respectively  $\mathcal{O}_0 \subset \mathbf{k}_0$ , where  $\mathbf{k}_0 := \mathbf{k}v_1$  is the residue field of  $v_1$ . Then  $\mathbf{k}v := \mathbf{k} := \mathbf{k}_0v_0$  is the residue field of both  $v$  and  $v_0$ . Let  $t \in F$  is a fixed function, and  $t_0 := tv_{0, t}$  be the residue of  $t$  with to the Gauss valuation  $v_{0, t}$  on  $\mathbf{k}(t)$ . Suppose that the following hold:

i) The special fiber  $\mathcal{Y}_{1, s}$  of the normalization  $\mathcal{Y}_1 \rightarrow \mathbb{P}_{t, \mathcal{O}_1}^1$  of  $\mathbb{P}_{t, \mathcal{O}_1}^1$  in  $\mathbf{k}(t) \hookrightarrow F$  is irreducible. Thus  $v_{1, t}$  has a unique prolongation  $w_1$  to  $F$ , and  $F_0 := Fw_1 = \kappa(\mathcal{Y}_{1, s})$ .

ii) The special fiber  $\mathcal{Y}_{0, s}$  of the normalization  $\mathcal{Y}_0 \rightarrow \mathbb{P}_{t, \mathcal{O}_0}^1$  of  $\mathbb{P}_{t, \mathcal{O}_0}^1$  in  $\mathbf{k}_0(t_0) \hookrightarrow F_0$  is irreducible. Thus  $v_{0, t_0}$  has a unique prolongation  $w_0$  to  $F_0$ , and  $F_0w_0 = \kappa(\mathcal{Y}_{0, s})$ .

**Lemma 4.3.** (Transitivity of smooth covers) *In the above notations, suppose that the hypotheses i), ii) are satisfied. Set  $w := w_0 \circ w_1$ , and let  $\mathcal{Y} \rightarrow \mathbb{P}_{t,\mathcal{O}}^1$  be the normalization of  $\mathbb{P}_{t,\mathcal{O}}^1$  in  $\mathbf{k}(t) \hookrightarrow F$ . Then  $w$  is the unique prolongation of  $v_t$  to  $F$ , and the following hold:*

- 1) *The base change of  $\mathcal{Y}$  under  $\mathcal{O} \hookrightarrow \mathcal{O}_{v_1}$  is  $\mathcal{Y}_1 = \mathcal{Y} \times_{\mathcal{O}} \mathcal{O}_{v_1}$  canonically, thus  $\mathcal{Y}_{1,s} = \mathcal{Y}_{\mathfrak{m}_1}$  is the fiber of  $\mathcal{Y}$  at the valuation ideal  $\mathfrak{m}_1 \in \text{Spec } \mathcal{O}$  of  $v_1$ .*
- 2) *Let  $\mathcal{Y}_{\mathcal{O}_0} \rightarrow \mathbb{P}_{t,\mathcal{O}_0}^1$  be the base change of  $\mathcal{Y} \rightarrow \mathbb{P}_{t,\mathcal{O}}^1$  under the  $\mathcal{O} \twoheadrightarrow \mathcal{O}_0$ . Then  $\mathcal{Y}_{\mathfrak{m}_1}$  is the generic fiber of  $\mathcal{Y}_{\mathcal{O}_0}$  and  $\mathcal{Y}_0 \rightarrow \mathbb{P}_{t,\mathcal{O}_0}^1$  is the normalization of  $\mathcal{Y}_{\mathcal{O}_0} \rightarrow \mathbb{P}_{t,\mathcal{O}_0}^1$ .*

*In particular,  $\mathcal{Y}$  is a smooth  $\mathcal{O}$ -curve if and only if  $\mathcal{Y}_1$  is a smooth  $\mathcal{O}_1$  curve and  $\mathcal{Y}_0$  is a smooth  $\mathcal{O}_0$ -curve.*

*Proof.* Klar, by the discussion above, and ROQUETTE [22], Satz I, combined with the fact that a projective curve is smooth if and only if its arithmetic genus equal its geometric genus.  $\square$

## B) A specialization result

We begin by recalling the following two well known facts. The first one is by KATZ (and GABBER) [10]: Let  $k$  be an algebraically closed field with  $\text{char}(k) = p$ . Then the localization at  $t = 0$  defines a bijection between the finite Galois  $p$ -power degree covers of  $\mathbb{P}_{t,k}^1$  unramified outside  $t = 0$  and the finite Galois  $p$ -power extensions  $k[[t]] \hookrightarrow k[[z]]$ , and this bijection preserves the ramification data. Thus given a cyclic  $\mathbb{Z}/p^f$ -cover  $k[[t]] \hookrightarrow k[[z]]$ , there exists a unique cyclic  $\mathbb{Z}/p^f$  cover of complete smooth curves  $Y_k \rightarrow \mathbb{P}_{t,k}^1$  which is branched only at  $t = 0$  (thus totally branched there) such that  $k[[t]] \hookrightarrow k[[z]]$  is the extension of local rings of  $Y \rightarrow \mathbb{P}_{t,k}^1$  above  $t = 0$ . We will say that  $Y \rightarrow \mathbb{P}_{t,k}^1$  is the **KG-cover** for  $k[[t]] \hookrightarrow k[[z]]$ . The second fact is the local-global principle for the Oort Conjecture, see e.g. GARUTI [5], §3, SAIDI [24], §1.2, especially Proposition 1.2.4, which among other things imply:

**LGP 4.4.** *Let  $k[[t]] \hookrightarrow k[[z]]$  be a  $\mathbb{Z}/p^f$ -extension and  $Y_k \rightarrow \mathbb{P}_{t,k}^1$  be its KG-cover. Further let  $W(k) \hookrightarrow R$  be a finite extension of  $W(k)$ . Then the  $\mathbb{Z}/p^f$ -extension  $k[[t]] \hookrightarrow k[[z]]$  has a smooth lifting over  $R$  if and only if the  $\mathbb{Z}/p^f$ -cover  $Y_k \rightarrow \mathbb{P}_{t,k}^1$  has a smooth lifting over  $R$ .*

Next let  $f$  be a fixed positive integer, and consider a finite sequences of positive numbers  $\mathbf{\iota} := (\iota_1 \leq \dots \leq \iota_f)$  satisfying:  $1 \leq \iota_1$  is prime to  $p$ , and  $\iota_{\nu+1} = p\iota_{\nu} + \epsilon$  with  $\epsilon \geq 0$  and  $\epsilon$  prime to  $p$  if  $\epsilon > 0$ . For such a sequence  $\mathbf{\iota}$ , let  $|\mathbf{\iota}| = \iota_1 + \dots + \iota_f$  and consider  $\mathbf{P}_{\mathbf{\iota}} = (P_1, \dots, P_f)$  a sequence of generic polynomials  $P_{\nu} = P_{\nu}(t^{-1})$  of degrees  $\deg(P_{\nu}) = \iota_{\nu}$  for  $1 \leq \nu \leq f$ . In other words, all the coefficients  $a_{\nu,\rho}$ ,  $1 \leq \nu \leq f$ ,  $1 \leq \rho \leq \iota_{\nu}$  of the polynomials  $P_{\nu}$  are independent free variables over  $\mathbb{F} := \overline{\mathbb{F}}_p$ . Let  $\mathbf{A}_{\mathbf{\iota}} := \mathbb{F}[(a_{\nu,\rho})_{\nu,\rho}]$  be the corresponding polynomial ring and  $\mathbb{A}^{|\mathbf{\iota}|} = \text{Spec } \mathbf{A}_{\mathbf{\iota}}$  the resulting affine space over  $\mathbb{F}$ .

For every  $x \in \mathbb{A}^{|\mathbf{\iota}|}$  let  $k_x$  be any algebraically closed field extension of  $\mathbb{F}$ , and  $\bar{x} \in \mathbb{A}^{|\mathbf{\iota}|}(k_x)$  be a  $k_x$ -rational point of  $\mathbb{A}^{|\mathbf{\iota}|}$  defined by a  $\mathbb{F}$ -embedding  $\phi_x : \kappa_x \hookrightarrow k_x$ . Let  $\mathbf{p}_{\mathbf{\iota},x} = (p_{1,x}, \dots, p_{f,x})$  and  $\mathbf{p}_{\mathbf{\iota},\bar{x}} = (p_{1,\bar{x}}, \dots, p_{f,\bar{x}})$  be the images of  $\mathbf{P}_{\mathbf{\iota}}$  over  $\kappa_x$ , respectively  $k_x$ . Then one has virtually by definitions that  $p_{\nu,\bar{x}} = \phi_x(p_{\nu,x})$ , thus  $\mathbf{p}_{\mathbf{\iota},\bar{x}} = \phi_x(\mathbf{p}_{\mathbf{\iota},x})$ . In particular, if  $\deg(P_{\nu}) = \deg(p_{\nu})$  for all  $\nu$ , then  $\mathbf{p}_{\mathbf{\iota},x}$  gives rise to a cyclic extension  $k_x[[t]] \hookrightarrow k_x[[z_x]]$  of degree  $p^f$  and upper jumps  $\mathbf{\iota} = (\iota_1, \dots, \iota_f)$ , and canonically to its KG-cover  $Y_{k_x} \rightarrow \mathbb{P}_{t,k_x}^1$ .

**Definition 4.5.** For  $\mathbb{F} \hookrightarrow k_x$  as above, let  $k_x[[t]] \hookrightarrow k_x[[z_x]]$  be a cyclic  $\mathbb{Z}/p^f$ -extension and  $Y_{k_x} \rightarrow \mathbb{P}_{t,k_x}^1$  be its KG-cover. We say that  $k_x[[t]] \hookrightarrow k_x[[z_x]]$  is an  $\mathbf{\iota}$ -extension at  $x \in \mathbb{A}^{|\mathbf{\iota}|}$  and that  $Y_{k_x} \rightarrow \mathbb{P}_{t,k_x}^1$  is an  $\mathbf{\iota}$ -KG-cover at  $x \in \mathbb{A}^{|\mathbf{\iota}|}$ , if  $k_x[[t]] \hookrightarrow k_x[[z_x]]$  has  $\mathbf{\iota} = (\iota_1, \dots, \iota_f)$  as upper ramification jumps.

**Notations 4.6.** We denote by  $\Sigma_{\mathbf{z}} \subset \mathbb{A}^{|\mathbf{z}|}$  the set of all  $x \in \mathbb{A}^{|\mathbf{z}|}$  which satisfy: There exists some mixed characteristic valuation ring  $R_x$  with residue field  $k_x$  such that some  $\mathbf{z}$ -KG-cover  $Y_{k_x} \rightarrow \mathbb{P}_{t, k_x}^1$  has a smooth lifting over  $R_x$ .

**Proposition 4.7.** *In Notations 4.6, suppose that  $\Sigma_{\mathbf{z}} \subseteq \mathbb{A}^{|\mathbf{z}|}$  is Zariski dense. Then there exists an algebraic integer  $\pi_{\mathbf{z}}$  such that for every algebraically closed field  $k$  of characteristic  $\text{char}(k) = p$  one has: Every  $\mathbf{z}$ -KG-cover  $Y_k \rightarrow \mathbb{P}_{t, k}^1$  has a smooth lifting over  $W(k)[\pi_{\mathbf{z}}]$ .*

*Proof.* The proof is quite involved, and has two main steps as follows:

**Step 1.** *Proving that the generic point  $\eta_{\mathbf{z}} \in \mathbb{A}^{|\mathbf{z}|}$  lies in  $\Sigma_{\mathbf{z}}$*

Let  $\mathcal{U}$  be an ultrafilter on  $\Sigma$  which contains all the Zariski open subsets of  $\Sigma$ . (Since  $\Sigma$  is Zariski dense in the irreducible scheme  $\mathbb{A}^{|\mathbf{z}|}$ , any Zariski open subset of  $\Sigma$  is dense as well, thus ultrafilter  $\mathcal{U}$  exist.) Let  $k_x \rightarrow \Theta_x \subset R_x$  be any set of representatives for for  $R_x$ . Consider the following ultraproducts index by  $\Sigma$ :

$${}^*k := (\prod_x k_x) / \mathcal{U} \rightarrow {}^*\Theta := (\prod_x \Theta_x) / \mathcal{U} \subset {}^*W := (\prod_x W(k_x)) / \mathcal{U} \hookrightarrow (\prod_x R_x) / \mathcal{U} =: {}^*R.$$

By general model theoretical principles, it follows that  ${}^*R$  is a valuation ring having residue field equal to  ${}^*k$ , and  ${}^*\Theta \subset {}^*R$  is a system of representatives for the residue field  ${}^*k$  of  ${}^*R$ .

Next, coming to geometry, by general model theoretical principles, it follows that the family of  $\mathbb{Z}/p^e$ -covers  $Y_x \rightarrow \mathbb{P}_{t, k_x}^1$  with upper ramification jumps  $\mathbf{z} = (\iota_1, \dots, \iota_f)$  gives rise to a  $\mathcal{U}$ -generic  $\mathbb{Z}/p^f$ -cover  $Y_{*k} \rightarrow \mathbb{P}_{t, *k}^1$  of complete smooth  ${}^*k$ -curves with upper ramification jumps  $\mathbf{z}$ . Precisely, setting  ${}^*p_{\nu} := (p_{\nu, x})_x / \mathcal{U}$ , the system of polynomials  ${}^*\mathbf{p}_{\mathbf{z}} = ({}^*p_1(t^{-1}), \dots, {}^*p_f(t^{-1}))$  defines the local extension  ${}^*k[[t]] \hookrightarrow {}^*k[[z]]$  of  $Y_{*k} \rightarrow \mathbb{P}_{t, *k}^1$  at  $t = 0$ . Moreover, consider the  ${}^*k$ -rational point  ${}^*\phi_{\eta_{\mathbf{z}}}$  of  $\mathbb{A}^{|\mathbf{z}|}$  defined by

$$(*) \quad {}^*\phi_{\eta_{\mathbf{z}}} : \kappa_{\eta_{\mathbf{z}}} \rightarrow {}^*k, \quad {}^*\phi_{\eta_{\mathbf{z}}}(a_{\nu, \rho}) := (\phi_x(a_{\nu, \rho}))_x / \mathcal{U}.$$

Then  ${}^*\mathbf{p}_{\mathbf{z}} = {}^*\phi_{\eta_{\mathbf{z}}}(\mathbf{P}_{\mathbf{z}})$ , which means that  ${}^*\phi_{\eta_{\mathbf{z}}}$  is the  ${}^*k$ -rational point of  $\mathbb{A}^{|\mathbf{z}|}$  defining  ${}^*\mathbf{p}_{\mathbf{z}}$ .

Again, by general model theoretical principles for ultraproducts of (covers of) curves, the family of the  $\mathbb{Z}/p^f$ -covers  $\mathcal{Y}_{R_x} \rightarrow \mathbb{P}_{t, R_x}^1$  with special fiber  $Y_x \rightarrow \mathbb{P}_{t, k_x}^1$  gives rise to a  $\mathbb{Z}/p^f$ -cover  $\mathcal{Y}_{*R} \rightarrow \mathbb{P}_{t, *R}^1$  of complete smooth  ${}^*R$ -curves, with  $Y_{*k} \rightarrow \mathbb{P}_{t, *k}^1$  as special fiber.

Let  $\mathbb{A}^{|\mathbf{z}|} \hookrightarrow \mathbb{P}^{|\mathbf{z}|}$  be the canonical embedding of the affine  $\mathbb{F}$ -space  $\mathbb{A}^{|\mathbf{z}|} := \text{Spec } \mathbb{F}[(a_{\nu, \rho})_{\nu, \rho}]$  into the corresponding projective  $\mathbb{F}$ -space  $\mathbb{P}^{|\mathbf{z}|} := \text{Proj } \mathbb{F}[t_0, (t_{\nu, \rho})_{\nu, \rho}]$  via the  $t_0$ -dehomogenization  $a_{\nu, \rho} = t_{\nu, \rho} / t_0$ . Letting  $\mathbf{Z}_0 := \mathbb{Z}_p^{\text{nr}}$  be the maximal unramified extension, and

$$\mathbb{A}_{\mathbf{Z}_0}^{|\mathbf{z}|} = \text{Spec } \mathbf{Z}_0[(a_{\nu, \rho})_{\nu, \rho}] \quad \text{and} \quad \mathbb{P}_{\mathbf{Z}_0}^{|\mathbf{z}|} = \text{Proj } \mathbf{Z}_0[t_0, (t_{\nu, \rho})_{\nu, \rho}],$$

the embedding  $\mathbb{A}^{|\mathbf{z}|} \hookrightarrow \mathbb{P}^{|\mathbf{z}|}$  is the special fiber of  $\mathbb{A}_{\mathbf{Z}_0}^{|\mathbf{z}|} \hookrightarrow \mathbb{P}_{\mathbf{Z}_0}^{|\mathbf{z}|}$ . Notice that  ${}^*\phi_{\eta_{\mathbf{z}}} : A_{\mathbf{z}} \rightarrow {}^*k$  gives rise via  ${}^*k \rightarrow {}^*\Theta$  canonically to an embedding of  $\mathbf{Z}_0$ -algebras defined by

$${}^*\phi_{\mathbf{Z}_0} : A_{\mathbf{z}, \mathbf{Z}_0} := \mathbf{Z}_0[(a_{\nu, \rho})_{\nu, \rho}] \hookrightarrow {}^*R, \quad a_{\nu, \rho} \mapsto {}^*\phi_{\eta_{\mathbf{z}}}(a_{\nu, \rho}).$$

Let  $V$  be a projective normal  $\mathbf{Z}_0$ -scheme with function field  $\kappa(V)$  embeddable in  $\text{Quot}({}^*R)$ , say via  $\kappa(V) \hookrightarrow \text{Quot}({}^*R)$ , such that the following are satisfied:

1) Let  $\mathfrak{p} \in V$  be the center of  ${}^*w$  of  ${}^*R$  on  $V$  induced by  $\kappa(V) \hookrightarrow {}^*R$ , and  $\mathcal{O}_{\mathfrak{p}}$  the local ring of  $\mathfrak{p}$ . Then the  $\mathbb{Z}/p^f$ -cover of complete smooth  ${}^*R$ -curves  $\mathcal{Y}_{*R} \rightarrow \mathbb{P}_{t, *R}^1$  is defined over  $\mathcal{O}_{\mathfrak{p}}$ .



2) The image of  ${}^*\phi_{\mathbf{z}_0} : A_{\mathbf{z}, \mathbf{z}_0} \rightarrow {}^*R$  is contained in the image of  $\kappa(V) \hookrightarrow {}^*R$ , and the resulting embedding  $A_{\mathbf{z}, \mathbf{z}_0} \hookrightarrow \kappa(V)$  is defined by some proper morphism

$$V \rightarrow \mathbb{P}_{\mathbf{z}_0}^{|\mathbf{z}|}.$$

We notice that condition 1) means that there exists a  $\mathbb{Z}/p^f$ -cover of complete smooth  $\mathcal{O}_{\mathbf{p}}$ -curves  $\mathcal{Y}_{\mathcal{O}_{\mathbf{p}}} \rightarrow \mathbb{P}_{t, \mathcal{O}_{\mathbf{p}}}^1$  such that  $\mathcal{Y}_{*R} \rightarrow \mathbb{P}_{t, *R}^1$  is the base change of  $\mathcal{Y}_{\mathcal{O}_{\mathbf{p}}} \rightarrow \mathbb{P}_{t, \mathcal{O}_{\mathbf{p}}}^1$  under  $\mathcal{O}_{\mathbf{p}} \hookrightarrow {}^*R$ . In particular, if  $\mathcal{O}_{\mathbf{p}} \rightarrow \kappa_{\mathbf{p}}$  is the residue field of  $\mathcal{O}_{\mathbf{p}}$ , then the special fiber  $Y_{\mathbf{p}} \rightarrow \mathbb{P}_{t, \mathbf{p}}^1$  of  $\mathcal{Y}_{\mathcal{O}_{\mathbf{p}}} \rightarrow \mathbb{P}_{t, \mathcal{O}_{\mathbf{p}}}^1$  is a  $\mathbb{Z}/p^f$ -cover of complete smooth  $\kappa_{\mathbf{p}}$ -curves whose base change under  $\kappa_{\mathbf{p}} \hookrightarrow {}^*k$  is canonically isomorphic to  $Y_{*k} \rightarrow \mathbb{P}_{t, *k}^1$ . In other words, the embedding  ${}^*\phi_{\eta_{\mathbf{z}}} : A_{\mathbf{z}} \hookrightarrow {}^*k$  defined by (\*) above factors through  $A_{\mathbf{z}} \hookrightarrow \kappa_{\eta_{\mathbf{z}}} \hookrightarrow \kappa_{\mathbf{p}}$ , and  $\mathbf{p} \in V$  is mapped to the generic point  $\mathbf{p} \mapsto \eta_{\mathbf{z}}$  of the special fiber  $\eta_{\mathbf{z}} \in \mathbb{P}^{|\mathbf{z}|} \hookrightarrow \mathbb{P}_{\mathbf{z}_0}^{|\mathbf{z}|}$  under  $V \rightarrow \mathbb{P}_{\mathbf{z}_0}^{|\mathbf{z}|}$ .

Recall that  $\bar{\mathbf{p}} \subset V$  the Zariski closure of  $\mathbf{p}$  in  $V$  viewed as a closed  $\mathbf{Z}_0$ -subscheme of  $V$  endowed with the reduced scheme structure. Since  $\mathbf{p} \mapsto \eta_{\mathbf{z}}$ , one has that  $\kappa_{\eta_{\mathbf{z}}} \hookrightarrow \kappa_{\mathbf{p}}$ , hence  $\kappa_{\mathbf{p}}$  has characteristic  $p$ , and  $\mathbf{p}$  lies in the special fiber  $V_{\mathbb{F}}$  of  $V$ . We conclude that  $\bar{\mathbf{p}} \subset V_{\mathbb{F}}$ .

Next, if  $\mathbf{p}$  has codimension  $> 1$ , let  $\tilde{V} \rightarrow V$  be the normalization of the blowup of  $V$  along the closed  $\mathbf{Z}_0$ -subscheme  $\bar{\mathbf{p}}$ . Let  $E_1, \dots, E_r \subset \tilde{V}$  be the finitely many irreducible components of the preimage of the exceptional divisor of the blowup. Then the generic points  $\mathbf{r}_i$  of the  $E_i$ ,  $i = 1, \dots, r$  are precisely the points of codimension one of  $\tilde{V}$  which map to  $\mathbf{p}$  under  $\tilde{V} \rightarrow V$ , and  $\cup_i E_i$  is the preimage of  $\bar{\mathbf{p}}$  in  $V$ . Further, if  $\mathbf{r} = \mathbf{r}_i$  is fixed, and  $\tilde{\mathcal{O}}$  is the local ring of  $\mathbf{r} \in \tilde{V}$  and  $\kappa_{\mathbf{r}}$  is its residue field, it follows that  $\mathcal{O}_{\mathbf{p}} \hookrightarrow \tilde{\mathcal{O}}$  and  $\kappa_{\mathbf{p}} \hookrightarrow \kappa_{\mathbf{r}}$  canonically. Recall that by the property 1) above,  $\mathcal{Y}_{\mathcal{O}_{\mathbf{p}}} \rightarrow \mathbb{P}_{t, \mathcal{O}_{\mathbf{p}}}^1$  is a  $\mathbb{Z}/p^f$ -cover of smooth  $\mathcal{O}_{\mathbf{p}}$ -curves with special fiber  $Y_{\mathbf{p}} \rightarrow \mathbb{P}_{t, \mathbf{p}}^1$ , whose base change under  $\kappa_{\mathbf{p}} \hookrightarrow {}^*k$  is  $Y_{*k} \rightarrow \mathbb{P}_{t, *k}^1$ . Therefore, the base change  $\mathcal{Y}_{\tilde{\mathcal{O}}} \rightarrow \mathbb{P}_{t, \tilde{\mathcal{O}}}^1$  of  $\mathcal{Y}_{\mathcal{O}_{\mathbf{p}}} \rightarrow \mathbb{P}_{t, \mathcal{O}_{\mathbf{p}}}^1$  defined by the inclusion  $\mathcal{O}_{\mathbf{p}} \hookrightarrow \tilde{\mathcal{O}}$  is a  $\mathbb{Z}/p^f$  cover of proper smooth  $\tilde{\mathcal{O}}$ -curves whose special fiber  $Y_{\mathbf{r}} \rightarrow \mathbb{P}_{t, \mathbf{r}}^1$  is the base change of  $Y_{\mathbf{p}} \rightarrow \mathbb{P}_{t, \mathbf{p}}^1$  under  $\kappa_{\mathbf{p}} \hookrightarrow \kappa_{\mathbf{r}}$ . Hence choosing any  $\kappa_{\mathbf{p}}$ -embedding  $\kappa_{\mathbf{r}} \hookrightarrow {}^*k$ , we get that the special fiber  $Y_{\mathbf{r}} \rightarrow \mathbb{P}_{t, \mathbf{r}}^1$  becomes  $Y_{*k} \rightarrow \mathbb{P}_{t, *k}^1$  under  $\kappa_{\mathbf{r}} \hookrightarrow {}^*k$ .

Hence by replacing  $V$  by  $\tilde{V}$  if necessary, we can suppose that  $\mathbf{p} \in V$  has codimension one, or equivalently, that  $\bar{\mathbf{p}} \subset V_{\mathbb{F}}$  is an irreducible component of  $V_{\mathbb{F}}$ .

By de Jong's theory of alterations DE JONG [?dJ], Theorem 6.5, there exists a finite extension of discrete valuation rings  $\mathbf{Z}_0 \hookrightarrow \mathbf{Z} := \mathbf{Z}_0[\pi_0]$  with  $\pi_0$  any uniformizing parameter of  $\mathbf{Z}$  and a dominant generically finite proper morphism  $W \rightarrow V$  of projective  $\mathbf{Z}_0$ -schemes with  $W$  *strictly semi-stable* over  $\mathbf{Z}$ , i.e., the generic fiber of  $W$  is a smooth projective variety over  $\text{Quot}(\mathbf{Z})$ , the special fiber  $W_{\mathbb{F}}$  is reduced and satisfies: If  $W_{\mathbb{F}, j}$ ,  $j \in J$  is any set of  $|J|$  distinct irreducible components of  $W_{\mathbb{F}}$ , then  $\cap_j W_{\mathbb{F}, j}$  is a smooth subscheme of  $W$  of codimension  $|J|$ . Hence the sequence of dominant proper morphisms of projective  $\mathbf{Z}_0$ -schemes

$$W \rightarrow V \rightarrow \mathbb{P}_{\mathbf{z}_0}^{|\mathbf{z}|}$$

satisfies: Let  $\mathbf{q} \in W$  denote a fixed preimage of  $\mathbf{p}$ . Then  $\mathbf{q}$  has codimension one, because  $\mathbf{p}$  does so. Further, the local ring  $\mathcal{O}_{\mathbf{q}}$  of  $\mathbf{q} \in W$  as a point of  $W$  dominates the local ring  $\mathcal{O}_{\mathbf{p}}$  of  $\mathbf{p} \in V$ , thus one has a canonical inclusion  $\mathcal{O}_{\mathbf{p}} \hookrightarrow \mathcal{O}_{\mathbf{q}}$  which gives rise to a canonical inclusion of the residue fields  $\kappa_{\eta_{\mathbf{z}}} \hookrightarrow \kappa_{\mathbf{p}} \hookrightarrow \kappa_{\mathbf{q}}$  corresponding to  $\mathbf{q} \rightarrow \mathbf{p} \rightarrow \eta_{\mathbf{z}}$ . Recall that  $\mathcal{Y}_{\mathcal{O}_{\mathbf{p}}} \rightarrow \mathbb{P}_{t, \mathcal{O}_{\mathbf{p}}}^1$  is a  $\mathbb{Z}/p^f$ -cover of smooth  $\mathcal{O}_{\mathbf{p}}$ -curves with special fiber  $Y_{\mathbf{p}} \rightarrow \mathbb{P}_{t, \mathbf{p}}^1$  whose base change under  $\kappa_{\mathbf{p}} \hookrightarrow {}^*k$  is  $Y_{*k} \rightarrow \mathbb{P}_{t, *k}^1$ . Let  $\mathcal{Y}_{\mathcal{O}_{\mathbf{q}}} \rightarrow \mathbb{P}_{t, \mathcal{O}_{\mathbf{q}}}^1$  be the base change of  $\mathcal{Y}_{\mathcal{O}_{\mathbf{p}}} \rightarrow \mathbb{P}_{t, \mathcal{O}_{\mathbf{p}}}^1$

under  $\mathcal{O}_{\mathfrak{p}} \hookrightarrow \mathcal{O}_{\mathfrak{q}}$ . Then  $\mathcal{Y}_{\mathcal{O}_{\mathfrak{q}}} \rightarrow \mathbb{P}_{t, \mathcal{O}_{\mathfrak{q}}}^1$  is a  $\mathbb{Z}/p^f$ -cover of proper smooth  $\mathcal{O}_{\mathfrak{q}}$ -curves whose special fiber  $Y_{\mathfrak{q}} \rightarrow \mathbb{P}_{t, \mathfrak{q}}^1$  is the base change of  $Y_{\mathfrak{p}} \rightarrow \mathbb{P}_{t, \mathfrak{p}}^1$  under  $\kappa_{\mathfrak{p}} \hookrightarrow \kappa_{\mathfrak{q}}$ . Again, choosing any  $\kappa_{\mathfrak{p}}$ -embedding of  $\kappa_{\mathfrak{q}} \hookrightarrow {}^*k$ , one gets that the base change of the special fiber  $Y_{\mathfrak{q}} \rightarrow \mathbb{P}_{t, \mathfrak{q}}^1$  under  $\kappa_{\mathfrak{q}} \hookrightarrow {}^*k$  becomes  $Y_{*k} \rightarrow \mathbb{P}_{t, *k}^1$ . This means that the embedding  ${}^*\phi_{\eta_{\mathfrak{z}}} : A_{\mathfrak{z}} \hookrightarrow {}^*k$  defined at (\*) above factors through  $A_{\mathfrak{z}} \hookrightarrow \kappa_{\eta_{\mathfrak{z}}} \hookrightarrow \kappa_{\mathfrak{p}} \hookrightarrow \kappa_{\mathfrak{q}}$ , reflecting the fact that  $\mathfrak{q} \mapsto \mathfrak{p} \mapsto \eta_{\mathfrak{z}}$ . In other words, there exists a  ${}^*k$ -rational point  ${}^*\phi_{\mathfrak{q}} : \kappa_{\mathfrak{q}} \rightarrow {}^*k$  such that the given  ${}^*k$ -rational point  ${}^*\phi_{\eta_{\mathfrak{z}}} : \kappa_{\eta_{\mathfrak{z}}} \rightarrow {}^*k$  defined by  ${}^*\phi_{\eta_{\mathfrak{z}}} : A_{\mathfrak{z}} \rightarrow \text{Spec } {}^*k$  is of the form

$$(**) \quad {}^*\phi_{\eta_{\mathfrak{z}}} = {}^*\phi_{\mathfrak{q}} \circ (\kappa_{\eta_{\mathfrak{z}}} \hookrightarrow \kappa_{\mathfrak{q}}).$$

**Step 2.** *Finishing the proof of Proposition 4.7*

Let  $\lambda := \kappa(W)$  denote the function field of  $W$ , and  $F := \kappa(\mathcal{Y}_{\mathcal{O}_{\mathfrak{q}}})$  be the function field of  $\mathcal{Y}_{\mathcal{O}_{\mathfrak{q}}}$ . Then  $\mathcal{Y}_{\mathcal{O}_{\mathfrak{q}}} \rightarrow \mathbb{P}_{t, \mathcal{O}_{\mathfrak{q}}}^1$  has as generic fiber a  $\mathbb{Z}/p^f$ -cover of complete smooth  $\lambda$ -curves  $\mathcal{Y}_{\lambda} \rightarrow \mathbb{P}_{t, \lambda}^1$ , and gives rise to a  $\mathbb{Z}/p^f$  extension of function field in one variable  $\lambda(t) \hookrightarrow F$ . Since  $\mathcal{O}_{\mathfrak{q}}$  is a (discrete) valuation ring, and  $\mathcal{Y}_{\mathcal{O}_{\mathfrak{q}}} \rightarrow \mathbb{P}_{t, \mathcal{O}_{\mathfrak{q}}}^1$  is a cover of smooth  $\mathcal{O}_{\mathfrak{q}}$ -curves, it follows by the discussion in subsection A), that  $\mathcal{Y}_{\mathcal{O}_{\mathfrak{q}}} \rightarrow \mathbb{P}_{t, \mathcal{O}_{\mathfrak{q}}}^1$  is precisely the normalization of  $\mathbb{P}_{t, \mathcal{O}_{\mathfrak{q}}}^1$  in the function field extension  $\lambda(t) \hookrightarrow F$ . Notice that  $\mathbb{P}_{t, \lambda}^1$  is the generic fiber of  $\mathbb{P}_{t, W}^1$ , and consider

$$\mathcal{Y}_W \rightarrow \mathbb{P}_{t, W}^1$$

the normalization of  $\mathbb{P}_{t, W}^1$  in the field extension  $\lambda(t) \hookrightarrow F$ . We notice that the base change of  $\mathcal{Y}_W \rightarrow \mathbb{P}_{t, W}^1$  under  $\text{Spec } \mathcal{O}_{\mathfrak{q}} \hookrightarrow W$  is precisely  $\mathcal{Y}_{\mathcal{O}_{\mathfrak{q}}} \rightarrow \mathbb{P}_{t, \mathcal{O}_{\mathfrak{q}}}^1$ .

**Lemma 4.8.** *Let  $x \in \mathbb{A}^{|\mathfrak{z}|}$  be such that the image  $\mathbf{p}_{\mathfrak{z}, x} = (p_{1,x}, \dots, p_{f,x})$  of  $\mathbf{P}_{\mathfrak{z}} = (P_1, \dots, P_f)$  under  $A_{\mathfrak{z}} \rightarrow \kappa_x$  satisfies  $\deg(p_{\nu, x}) = \deg(P_{\nu})$  for all  $\nu = 1, \dots, f$ . Let  $y \in \bar{\mathfrak{q}}$  be a preimage of  $x \in \mathbb{A}^{|\mathfrak{z}|} \subset \mathbb{P}^{|\mathfrak{z}|}$  under  $\bar{\mathfrak{q}} \rightarrow \bar{\mathfrak{p}} \rightarrow \mathbb{P}^{|\mathfrak{z}|}$  and  $\mathcal{O}_y$  be a valuation ring dominating  $\mathcal{O}_y$  with  $\kappa_y = \kappa_x$ . Then  $\mathcal{Y}_{\mathcal{O}_y} \rightarrow \mathbb{P}_{t, \mathcal{O}_y}^1$  is a cover of smooth curves.*

*Proof.* Recall that  $\mathcal{Y}_{\mathcal{O}_y} \rightarrow \mathbb{P}_{t, \mathcal{O}_y}^1$  is the base change of  $\mathcal{Y}_W \rightarrow \mathbb{P}_{t, W}^1$  under the canonical embedding  $\text{Spec } \mathcal{O}_y \hookrightarrow W$ , and in particular,  $\mathcal{Y}_{\mathcal{O}_y} \rightarrow \mathbb{P}_{t, \mathcal{O}_y}^1$  is the normalization of  $\mathbb{P}_{t, \mathcal{O}_y}^1$  in the field extension  $\lambda(t) \hookrightarrow F$ . Since  $y \in \mathfrak{q}$ , and the geometric fiber  $\mathcal{Y}_{\mathfrak{q}} \rightarrow \mathbb{P}_{t, \mathfrak{q}}^1$  of  $\mathcal{Y}_{\bar{\mathfrak{q}}} \rightarrow \mathbb{P}_{t, \bar{\mathfrak{q}}}^1$  is a  $\mathbb{Z}/p^f$ -cover of smooth complete curves, the same holds correspondingly, if one replaces  $\mathcal{O}_y = \mathcal{O}_{W, y}$  by  $\mathfrak{o}_y := \mathcal{O}_{\bar{\mathfrak{q}}, y} = \mathcal{O}_y/\mathfrak{q}$ , and  $\lambda(t) \hookrightarrow F$  by  $\kappa_{\mathfrak{q}}(t) \hookrightarrow F_{\mathfrak{q}}$ , where  $F_{\mathfrak{q}} := \kappa(\mathcal{Y}_{\mathfrak{q}})$  is viewed as function field over  $\kappa_{\mathfrak{q}}$ . Recall that the local extension  $\kappa_{\mathfrak{q}}[[t]] \hookrightarrow \kappa_{\mathfrak{q}}[[z_{\mathfrak{q}}]]$  of  $\mathcal{Y}_{\mathfrak{q}} \rightarrow \mathbb{P}_{t, \mathfrak{q}}^1$  at  $t = 0$  is defined by the image  $\mathbf{p}_{\mathfrak{z}, \mathfrak{q}}$  of  $\mathbf{P}_{\mathfrak{z}}$  under the canonical embedding  $A_{\mathfrak{z}} \hookrightarrow \kappa_{\eta_{\mathfrak{z}}} \hookrightarrow \kappa_{\mathfrak{q}}$ . On the other hand, if  $\mathfrak{o}_x$  denotes the local ring of  $x \in \mathbb{A}^{|\mathfrak{z}|} \subset \mathbb{P}^{|\mathfrak{z}|}$  then  $A_{\mathfrak{z}} \subset \mathfrak{o}_x$  and  $\mathfrak{o}_y$  dominates  $\mathfrak{o}_x$ . Hence  $A_{\mathfrak{z}} \hookrightarrow \mathfrak{o}_y$  and therefore,  $\mathbf{p}_{\mathfrak{z}, \mathfrak{q}}$  is defined over  $\mathfrak{o}_y$ . Further, by the commutativity of the diagrams

$$\begin{array}{ccc} A_{\mathfrak{z}} & \hookrightarrow & \mathfrak{o}_y & & \mathbf{P}_{\mathfrak{z}} & \mapsto & \mathbf{p}_{\mathfrak{z}, \mathfrak{q}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \kappa_x & \hookrightarrow & \kappa_y & & \mathbf{p}_{\mathfrak{z}, x} & \mapsto & \mathbf{p}_{\mathfrak{z}, y} \end{array}$$

it follows that the image of  $\mathbf{p}_{\mathfrak{z}, \mathfrak{q}}$  under the residue homomorphism  $\mathfrak{o}_y \rightarrow \kappa_y$  equals the image of  $\mathbf{p}_{\mathfrak{z}, x}$  under  $\kappa_x \hookrightarrow \kappa_y$ . Thus by the functoriality of the Artin–Schreier–Witt theory, it follows that every irreducible component of the special fiber of  $\mathbb{P}_{t, \mathfrak{o}_y}^1$  dominates the KG-cover of  $\mathbb{P}_{t, \kappa_y}^1$  defined by  $\mathbf{p}_{\mathfrak{z}, y}$ . Since  $\deg(p_{\nu, y}) = \deg(p_{\nu, x}) = \deg P_{\nu}$  for all  $\nu$ , the latter cover

must have degree  $p^f$  and upper ramification jumps  $\mathbf{v} = (v_1, \dots, v_f)$ . In particular, we can apply Lemma 4.2, and conclude that the special fibers  $\mathcal{Y}_y$  and  $\mathcal{Y}_v$  are reduced and irreducible.

In order to conclude, we notice that by the discussion above, the normalization  $Y_y \rightarrow \mathcal{Y}_y$  dominates the  $\mathbf{v}$ -KG-cover of  $\mathbb{P}_{t,y}^1$  defined by  $\mathbf{p}_{\mathbf{v},y}$ . Since every  $\mathbf{v}$ -KG-cover has as genus a constant depending on  $\mathbf{v}$  only, thus including the generic fiber it follows that  $g_{Y_y} \geq g_{\mathcal{Y}_y}$ . We thus conclude the proof of Lemma 4.8 by applying Lemma 4.2.  $\square$

Coming back to the proof of Proposition 4.7 we proceed as follows. Let  $k$  be any algebraically closed field with  $\text{char}(k) = p$ , and  $Y_k \rightarrow \mathbb{P}_{t,k}^1$  be an  $\mathbf{v}$ -KG-cover, say with local ring extension  $k[[t]] \hookrightarrow k[[z]]$  at  $t = 0$  defined by  $\mathbf{p}_{\mathbf{v}} = (p_1, \dots, p_f)$ .

In notations as introduced right before Lemma 4.8, let  $x \in \mathbb{A}^{|\mathbf{v}|}$  and  $\phi_x : \kappa_x \rightarrow k$  be such that  $\phi_x(\mathbf{p}_{\mathbf{v},x}) = \mathbf{p}_{\mathbf{v}}$ . Since  $\bar{\mathfrak{q}} \rightarrow \bar{\mathfrak{p}} \rightarrow \mathbb{P}^{|\mathbf{v}|}$  is dominant and proper, there exists a preimage  $y \in \bar{\mathfrak{q}}$  of  $x$  such that  $\kappa_x \hookrightarrow \kappa_y$  is finite. Since  $k$  is algebraically closed, there is a  $\kappa_x$ -embedding  $\phi_y : \kappa_y \hookrightarrow k$  such that  $\phi_x = \phi_y \circ (\kappa_x \hookrightarrow \kappa_y)$ . In particular, if  $\mathbf{p}_{\mathbf{v},y}$  is the image of  $\mathbf{p}_{\mathbf{v},x}$  under  $\kappa_x \hookrightarrow \kappa_y$ , then  $\mathbf{p}_{\mathbf{v}} = \phi_y(\mathbf{p}_{\mathbf{v},y})$ .

Let  $W_{\mathbb{F},j}$ ,  $j \in J$ , be the irreducible components of  $W_{\mathbb{F}}$  which contain  $y$ , and  $W_J := \bigcap_j W_{\mathbb{F},j}$ . Then  $W_J$  is a smooth  $\mathbb{F}$ -subvariety  $W_J \subset W_{\mathbb{F}}$ , and the following hold, see e.g., DE JONG [?dJ], section 2.16 and explanations thereafter: Let  $\mathcal{O}_y$  be the local ring of  $y \in W$ . There exists a system of regular parameters  $(u_1, \dots, u_N)$  of  $\mathcal{O}_y$  which satisfy:

- i)  $u_j$  defines locally at  $y$  the equation of  $W_{\mathbb{F},j}$  and  $\pi_0 = u_1 \dots u_{n_y}$ .
- ii)  $(u_j)_{n < j \leq N}$  give rise to a regular system of parameters at  $y \in W_J$  in  $\mathcal{O}_y / (u_1, \dots, u_{n_y})$ .

Let  $\mathfrak{r} := (u_1 - u_i)_i \subset \mathcal{O}_y$  be the ideal generated by all the  $u_1 - u_i$ ,  $1 \leq i \leq N$ . Then  $\mathfrak{r}$  is a regular point with  $(u_1 - u_i)_i$  a regular system of parameters, and

$$\mathbf{Z}_y := \mathcal{O}_y / \mathfrak{r}$$

is a discrete valuation ring having  $\pi_y := u_1 \pmod{\mathfrak{r}}$  as uniformizing parameter, and  $\pi_y^{n_y} = \pi_0$ . In particular, if  $\pi_0$  was an algebraic integer, then so is  $\pi_y$ .

Let  $v_1$  be a valuation of  $F$  with center  $\mathfrak{r}$ , and residue field equal to  $\kappa_{\mathfrak{r}} = \text{Quot}(\mathbf{Z}_y)$ . And further, let  $v_0$  be the canonical valuation of  $\mathcal{O}_0 := \mathbf{Z}_y$ . Then the valuation ring  $\mathcal{O}_v$  of the valuation  $v := v_0 \circ v_1$  dominates  $\mathcal{O}_y$  and has  $\kappa_v = \kappa_y$ . Hence by Lemma 4.8 above, it follows that  $\mathcal{Y}_{\mathcal{O}_v} \rightarrow \mathbb{P}_{t,\mathcal{O}_v}^1$  is a  $\mathbb{Z}/p^f$ -cover of smooth  $\mathcal{O}_v$ -curves. Hence by Lemma 4.3, it follows that  $\mathcal{Y}_{\mathbf{Z}_y} \rightarrow \mathbb{P}_{t,\mathbf{Z}_y}^1$  is a  $\mathbb{Z}/p^f$ -cover of smooth  $\mathbf{Z}_y$ -curves.

Let  $n_{\mathbf{v}} = \text{l.c.m.}(n_y)_y$ , and notice that  $n_{\mathbf{v}}$  is bounded by  $n!$ , where  $n = \dim(W) - 1$ . Choose a fixed algebraic integer  $\pi_0$  such that  $\mathbf{Z} = \mathbf{Z}_0[\pi_0]$ , and let  $\pi_{\mathbf{v}}$  be defined by  $\pi_{\mathbf{v}}^{n_{\mathbf{v}}} = \pi_0$ . Then there are canonical embeddings  $\mathbf{Z}_y \hookrightarrow W(\bar{\kappa}_y)[\pi_y] \hookrightarrow W(k)[\pi_{\mathbf{v}}]$ , and the base change of  $\mathcal{Y}_{\mathbf{Z}_y} \rightarrow \mathbb{P}_{t,\mathbf{Z}_y}^1$  under  $\mathbf{Z}_y \hookrightarrow W(k)[\pi_{\mathbf{v}}]$  is a  $\mathbb{Z}/p^f$ -cover of smooth  $W(k)[\pi_{\mathbf{v}}]$ -curves

$$\mathcal{Y}_{W(k)[\pi_{\mathbf{v}}]} \rightarrow \mathbb{P}_{t,W(k)[\pi_{\mathbf{v}}]}^1$$

with special fiber the  $\mathbf{v}$ -KG-cover  $Y_k \rightarrow \mathbb{P}_{t,k}^1$  of the given cyclic  $\mathbb{Z}/p^f$ -extension  $k[[t]] \hookrightarrow k[[z]]$ .

This concludes the proof of Proposition 4.7.  $\square$

### C) *The strategy of proof for Theorem 1.1*

We begin by recalling that there are several forms of the Oort Conjecture (OC) which are all equivalent, see e.g. SAIDI [24], §3.1, for detailed proofs.

Let  $k$  be an algebraically closed field with  $\text{char}(k) = p > 0$ . Let  $W(k)$  be the ring of Witt vectors over  $k$ , and  $W(k) \hookrightarrow R$  denote finite extension of discrete valuation rings. We consider the following two situations, which are related to two variants of OC:

- a)  $Y \rightarrow X$  is a finite (ramified)  $G$ -cover of complete smooth  $k$ -curves such that the inertia groups at all closed points  $y \in Y$  are cyclic.
- b)  $\mathcal{X}_R$  is a complete smooth  $R$ -curve with special fiber  $X$ , and  $Y \rightarrow X$  is as a (ramified)  $G$ -cover of complete smooth curves as in case a) above.

We say that *OC holds over  $R$*  in case a) or b), if there exists a  $G$ -cover of complete smooth  $R$ -curves  $\mathcal{Y}_R \rightarrow \mathcal{X}_R$ , with  $\mathcal{X}_R$  the given one in case b), having the  $G$ -cover  $Y \rightarrow X$  as special fiber. And given a cyclic extension  $k[[t]] \hookrightarrow k[[z]]$ , we say that the *local OC holds over  $R$*  for  $k[[t]] \hookrightarrow k[[z]]$ , if there exists a smooth lifting  $R[[T]] \hookrightarrow R[[Z]]$  of  $k[[t]] \hookrightarrow k[[z]]$ .

**Fact 4.9.** (SAIDI [24], §3.1) The following hold:

- 1) *Local global principle for OC.* Let  $Y \rightarrow X$  be a finite  $G$ -cover with cyclic inertia groups, and for  $y \mapsto x$ , let  $k[[t_x]] \hookrightarrow k[[t_y]]$  be the corresponding extension of local rings. Let  $\mathcal{X}_R$  be some complete smooth  $R$ -curve with special fiber  $X$ . Then the following are equivalent:
  - i) There is a  $G$ -cover of complete smooth  $R$ -curves  $\mathcal{Y}_R \rightarrow \mathcal{X}_R$  with special fiber  $Y \rightarrow X$ .
  - ii) For all  $y \mapsto x$ , the local cyclic extension  $k[[t_x]] \hookrightarrow k[[t_y]]$  has a smooth lifting over  $R$ .
- 2) *Equivalent forms of OC.* The following assertions are equivalent:
  - a) OC holds for all  $G$ -covers  $Y \rightarrow X$  as in case a), or b).
  - b) OC holds for  $G$  cyclic and  $X = \mathbb{P}_{t,k}^1$ .
  - c) OC holds for  $G$  cyclic and  $X = \mathbb{P}_{t,k}^1$  and  $Y \rightarrow \mathbb{P}_{t,k}^1$  branched at  $t = 0$  only.
  - d) The local OC holds.
  - e) Any of the assertions above, but restricted to cyclic  $p$ -groups as inertia groups.

Thus in order to prove Theorem 1.1 from Introduction, we can proceed as follows: Let  $Y \rightarrow X$  be a given  $G$ -cover of projective smooth  $k$ -curves, with branch locus  $\Sigma \subset X$ . Then for a given algebraic integer  $\pi$  and  $R := W(k)[\pi]$ , and a smooth model  $\mathcal{X}_R$  of  $X$  over  $R$  one has: The OC holds for  $Y \rightarrow X$  over  $R$  iff the local OC holds for the local cyclic extension  $k[[t_x]] \hookrightarrow k[[t_y]]$  over  $R$  for all  $x \in \Sigma$ . Further, the local OC holds for a fixed local cyclic extension  $k[[t_x]] \hookrightarrow k[[t_y]]$  over  $R$  if and only if the local OC holds over  $R$  for the  $p$ -power sub-extension  $k[[t_x]] \hookrightarrow k[[z_x]]$  of  $k[[t_x]] \hookrightarrow k[[t_y]]$ . Thus the global assertion of Theorem 1.1 is equivalent to the local assertion for cyclic  $p$ -power extensions  $k[[x]] \hookrightarrow k[[z]]$ .

We tackle the case of  $p$ -power cyclic extensions  $k[[t]] \hookrightarrow k[[z]]$  as follows.

Step 1. Let  $\mathbf{v} = (v_1, \dots, v_e)$  be a fixed upper ramification jumps sequence. By Key Lemma 3.2 and Theorem 3.7 there exists some  $N$  and sequences  $\mathbf{v}_\mu = (v_1, \dots, v_{e_\mu})$ ,  $1 \leq \mu \leq N$ , depending on  $\mathbf{v}$  only, such that the following hold: Let  $\mathfrak{o}$  be a complete discrete valuation ring over with residue field  $k$ , and  $x_1, \dots, x_N \in \mathfrak{m}_{\mathfrak{o}}$  be distinct points. Then for every  $\mathbf{v}$ -KG-cover  $Y \rightarrow \mathbb{P}_{t,k}^1$  there exists a  $\mathbb{Z}/p^e$ -cover of projective smooth  $\mathfrak{o}$ -curves  $\mathcal{Y}_{\mathfrak{o}} \rightarrow \mathbb{P}_{t,\mathfrak{o}}^1$  satisfying:

- a) The special fiber of  $\mathcal{Y}_{\mathfrak{o}} \rightarrow \mathbb{P}_{t,\mathfrak{o}}^1$  is the given  $\mathbf{v}$ -KG-cover  $Y \rightarrow \mathbb{P}_{t,k}^1$ .
- b) The generic fiber  $\mathcal{Y}_{\hat{k}} \rightarrow \mathbb{P}_{t,\hat{k}}^1$  of  $\mathcal{Y}_{\mathfrak{o}} \rightarrow \mathbb{P}_{t,\mathfrak{o}}^1$  is branched above  $x_1, \dots, x_N$  only.<sup>3</sup>

<sup>3</sup> N.B.,  $\mathcal{Y}_{\hat{k}} \rightarrow \mathbb{P}_{t,\hat{k}}^1$  is not always an  $\mathbf{v}$ -KG-cover!

c) The upper ramification jumps above each  $x_\mu$  are  $\mathbf{v}_\mu := (\nu_1, \dots, \nu_{e_\mu})$ ,  $\mu = 1, \dots, N$ .

Step 2. Let  $\hat{k} \hookrightarrow l$  be an algebraic closure, and  $\mathcal{Y}_l \rightarrow \mathbb{P}_{t,l}^1$  be the base change of  $\mathcal{Y}_{\hat{k}} \rightarrow \mathbb{P}_{t,\hat{k}}^1$ . Then  $\mathcal{Y}_l \rightarrow \mathbb{P}_l^1$  is a  $\mathbb{Z}/p^e$ -cover of projective smooth curves with no essential ramification.

**Hypothesis 4.10.** In the Notations 4.6, suppose that for every  $\mathbf{v}_\mu = (\nu_1, \dots, \nu_{e_\mu})$  with  $\mu = 1, \dots, N$ , the subset  $\Sigma_{\mathbf{v}_\mu} \subset \mathbb{A}^{|\mathbf{v}_\mu|}$  is Zariski dense.

For every fixed  $\mu = 1, \dots, N$ , consider the  $\mathbf{v}_\mu$ -KG-cover  $Y_\mu \rightarrow \mathbb{P}_{t,l}^1$  of the local  $\mathbb{Z}/p^{e_\mu}$ -extension  $l[[t_\mu]] \hookrightarrow l[[z_\mu]]$ , where  $z_\mu$  and  $t_\mu$  are local parameters at  $y_\mu \mapsto x_\mu$ . Then by Proposition 4.7 applied for each  $\mathbf{v}_\mu = (\nu_1, \dots, \nu_{e_\mu})$ , there exists some algebraic integer  $\pi_\mu$  such that  $\mathbf{v}_\mu$ -KG-cover  $Y_\mu \rightarrow \mathbb{P}_{t,l}^1$  has a smooth lifting over  $W(l)[\pi_\mu]$ . Thus by the local-global principle Fact 4.9, it follows that if  $\pi_{\mathbf{v}}$  is an algebraic integer such that  $\pi_\mu \in W(l)[\pi_{\mathbf{v}}]$  for all  $\mu = 1, \dots, N$  then  $Y_l \rightarrow \mathbb{P}_{t,l}^1$  has a smooth lifting  $\mathcal{Y}_{\mathcal{O}_1} \rightarrow \mathbb{P}_{t,\mathcal{O}_1}^1$  over  $\mathcal{O}_1 := W(l)[\pi_{\mathbf{v}}]$ .

Let  $v_1$  be the canonical valuation of  $\mathcal{O}_1$ , and  $v_0$  be the (unique) prolongation of the valuation of  $\mathfrak{o}$  to  $l$ , say having valuation ring  $\mathcal{O}_0$ . Then the base change  $\mathcal{Y}_{\mathcal{O}_0} \rightarrow \mathbb{P}_{t,\mathcal{O}_0}^1$  of the  $\mathbb{Z}/p^e$ -cover of complete curves  $\mathcal{Y}_{\mathfrak{o}} \rightarrow \mathbb{P}_{t,\mathfrak{o}}^1$  under  $\mathfrak{o} \hookrightarrow \mathcal{O}_0$  is a  $\mathbb{Z}/p^e$ -cover of complete smooth  $\mathcal{O}_0$ -curves with generic fiber  $\mathcal{Y}_l \rightarrow \mathbb{P}_{t,l}^1$ . And the  $\mathbb{Z}/p^e$ -cover of complete smooth  $l$ -curves  $\mathcal{Y}_l \rightarrow \mathbb{P}_{t,l}^1$  is the special fiber of the  $\mathbb{Z}/p^e$ -cover of smooth  $\mathcal{O}_1$ -curves  $\mathcal{Y}_{\mathcal{O}_1} \rightarrow \mathbb{P}_{t,\mathcal{O}_1}^1$ . The setting  $v := v_0 \circ v_1$  and letting  $\mathcal{O}$  be the valuation ring of  $v$ , it follows by Lemma 4.3 that there exists a smooth lifting of  $Y \rightarrow \mathbb{P}_{t,k}^1$  to a  $\mathbb{Z}/p^e$ -cover of smooth  $\mathcal{O}$ -curves  $\mathcal{Y}_{\mathcal{O}} \rightarrow \mathbb{P}_{t,\mathcal{O}}^1$ .

Since the  $\mathbf{v}$ -KG-cover  $Y \rightarrow \mathbb{P}_{t,k}^1$  we started with was arbitrary, it follows that the Hypothesis 4.10 implies that  $\Sigma_{\mathbf{v}} = \mathbb{A}^{|\mathbf{v}|}$ . Hence by Proposition 4.7 we conclude that:

*Hypothesis 4.10 implies the existence of an algebraic integer  $\pi_{\mathbf{v}}$  such that every  $\mathbf{v}$ -KG-cover  $Y \rightarrow \mathbb{P}_{t,k}^1$  has a smooth lifting over  $W(k)[\pi_{\mathbf{v}}]$ .*

#### D) Concluding the proof of the Oort Conjecture

By the observation above, the proof of the Oort Conjecture is reduced to showing that the Hypothesis 4.10 holds for every system of upper ramification indices  $\mathbf{v} = (\nu_1, \dots, \nu_e)$  which has no essential jump indices, i.e.,  $p\nu_{\rho-1} \leq \nu_\rho < p\nu_{\rho-1} + p$  for  $\rho = 1, \dots, e$ . Via the local-global principle Fact 4.9, this fact is equivalent to a (very) special case of the local Oort Conjecture, which follows from a result recently announced by OBUS–WEWERS [15], Theorem 1.7. Here is the special case needed here:

**Key Lemma 4.11.** (SPECIAL CASE OF OBUS–WEWERS) *In notations and context as above, let  $k[[t]] \hookrightarrow k[[z]]$  be cyclic extension of degree  $p^e$  which has no essential ramification. Then the local Oort Conjecture holds for  $k[[t]] \hookrightarrow k[[z]]$ , i.e.,  $k[[t]] \hookrightarrow k[[z]]$  has a smooth lifting over some finite extension  $R$  of  $W(k)$  to a smooth cyclic cover  $R[[T]] \hookrightarrow R[[Z]]$ .*

*Proof.* Recall that Theorem 1.7 from OBUS–WEWERS [15] asserts that the local Oort conjecture holds for cyclic extensions  $k[[t]] \hookrightarrow k[[z]]$  of degree  $p^e$ , provided the upper ramification jumps  $\nu_1 \leq \dots \leq \nu_e$  satisfy the following:

**Hypothesis 4.12.** For every  $3 \leq \nu < e$ , there is no integer  $a \in \mathbb{Z}$  such that:

$$\nu_{\nu+1} - p\nu_\nu < pa \leq (\nu_{\nu+1} - p\nu_\nu) \frac{\nu_{\nu+1}}{\nu_{\nu+1} - \nu_\nu}.$$

Notice that if  $k[[t]] \hookrightarrow k[[z]]$  has no proper essential ramification jumps, thus by definition  $\iota_{\nu+1} \leq p\iota_\nu + p - 1$  for all  $1 \leq \nu < e$ , then the hypothesis (\*) above is satisfied. Indeed, we first notice that  $p\iota_\nu \leq \iota_{\nu+1}$  implies that  $a$  must be positive. Hence setting  $\delta := \iota_{\nu+1} - p\iota_\nu$  and  $u := \iota_\nu$ , the second inequality becomes  $pa(\delta + (p-1)u) \leq \delta(\delta + pu)$ , which is equivalent to  $p(p-1)au \leq \delta(\delta + pu - pa)$ . Hence taking into account that  $\delta \leq p-1$ , we get:  $p(p-1)au \leq (p-1)(p-1 + pu - ap)$ , thus dividing by  $(p-1)$ , we get:  $pau \leq p-1 + pu - ap$ , or equivalently,  $p(a-1)(u+1) + 1 \leq 0$ , which does not hold for any positive integer  $a$ .  $\square$

This concludes the proof of Theorem 1.1.

**Remark 4.13.** The Hypothesis 4.12 above is automatically satisfied for  $e \leq 3$ , thus in particular, Theorem 1.7 of OBUS –WEWERS [15] proves the local Oort Conjecture *unconditionally* for  $e \leq 3$ . To see how restrictive Hypothesis 4.12 becomes for  $3 < e$ , one can proceed as follows: Setting  $\delta := \iota_{\nu+1} - p\iota_\nu$  and  $u := \iota_\nu$ , we have that  $\iota_{\nu+1} = \delta + pu$  for some  $0 \leq \delta$ . If  $0 < \delta$ , then  $\delta$  is prime to  $p$  hence writing  $\delta = rp - \eta$  with  $1 \leq \eta \leq p-1$ , Hypothesis 4.12 for  $a := r$  implies that:

*At least one of the inequalities  $\delta < rp \leq \delta(\delta + pu)/(\delta + (p-1)u)$  does not hold.*

Since the first inequality holds, the second inequality must not hold. Therefore we must have  $rp > \delta(\delta + pu)/(\delta + (p-1)u)$ . Since the RHS equals  $\delta + \delta u/(\delta + (p-1)u)$ , and  $pr - \delta = \eta$ , the above inequality is equivalent to  $\eta > \delta u/(\delta + (p-1)u)$ , hence to  $\eta\delta + \eta(p-1)u > \delta u$ , and finally to  $\eta^2(p-1) > (\delta - \eta(p-1))(u - \eta)$ . Thus since  $\delta - \eta(p-1) = (r - \eta)p$ , the last inequality becomes  $\eta^2(p-1) > p(r - \eta)(u - \eta)$ . Since  $p > p-1$ , the last inequality implies  $\eta^2 > (r - \eta)(u - \eta)$ . On the other hand, one has  $1 \leq \eta < p$  and  $p^2 \leq u$  for  $1 < \nu$ , thus  $\eta^2 < u - \eta$ . Therefore, in order to satisfy the inequality, one must have  $r \leq \eta$ . Therefore, for every  $1 < \nu < e$  the Hypothesis 4.12 is equivalent to:

(\*) *Setting  $\iota_{\nu+1} = p\iota_\nu + pr_\nu - \eta_\nu$  with  $0 \leq \eta_\nu < p$ , one must have  $0 \leq r_\nu \leq \eta_\nu$ .*

It seems to me that this is a better/easier formulation than the original form of Hypothesis 4.12. Using the reformulation (\*), we also can see how restrictive the Hypothesis 4.12 is. Namely, if  $\iota_{\nu+1} \equiv -1 \pmod{p}$ , then  $\iota_{\nu+1} = p\iota_\nu + p - 1$ . In general, one has  $\iota_{\nu+1} \leq p\iota_\nu + (p-1)^2$  and this upper bound can be reached only if  $\iota_{\nu+1} \equiv 1 \pmod{p}$ . This shows that the Hypothesis 4.12 is quite restrictive indeed.

## REFERENCES

- [1] Bertin, J. and Mézard, A., *Déformations formelles des revêtements sauvagement ramifiés de courbes algébriques*. Invent. Math. **141** (2000), 195–238.  $\uparrow$
- [2] N. Bourbaki, *Algèbre commutative*. Hermann Paris 1964.  $\uparrow$
- [3] Chinburg, T. and Guralnick, R. and Harbater, D., *Oort groups and lifting problems*. Compositio Math., **144** (2008), 849–866.  $\uparrow 1$
- [4] ———, *The local lifting problem for actions of finite groups on curves*. Ann. Sci. École Norm. Sup. **44** no. 4 (2011), 537–605.  $\uparrow 1$
- [5] M. Garuti, *Prolongement des revêtements galoisiens en géométrie rigide*. Compositio Math., **104** (1996), 305–331.  $\uparrow 1, 3, 4$
- [6] ———, *Géométrie rigide et géométrie formal*. in: Courbes Semi-Stables et Groupe Fondamental en Géométrie Algébrique, Luminy, 1998; eds. J.-B. Bost, F. Loeser, M. Raynaud, Progr. Math. **187**, Birkhäuser Verlag, Basel, 1998, pp. 21–32.  $\uparrow 1$

- [7] ———, *Linear systems attached to cyclic inertia*. in: Arithmetic Fundamental Groups and Non-commutative Algebra, Berkeley, CA, 1999; eds: M. Fried, Y. Ihara, Proc. Sympos. Pure Math. **70**, AMS Providence, RI, 2002, pp. 367–376. ↑2
- [8] Green, B.-W. and Matignon, M., *Liftings of Galois covers of smooth curves*. Compositio Math., **113** (1998), 237–272. ↑1, 2
- [9] ———, *Order  $p$  automorphisms of the open disc of a  $p$ -adic field*. Journal AMS, **12** (1999), 269–303. ↑
- [10] K. Kato, *Vanishing cycles, ramification of valuations, and class field theory*. Duke Math. J. **55** (1987), 629–659. ↑2, 4
- [11] N. Katz, *Local-to-global extensions of representations of fundamental groups*. Ann. Inst. Fourier Grenoble **36** (1986), no. 4, 69–106. ↑
- [12] S. Lang, *Algebra*. Graduate Texts in Mathematics **211**, Springer-Verlag, New York, third edition, 2002. ↑2
- [13] A. Obus, *The (local) lifting problem for curves*. Galois-Teichmüller theory and Arithmetic Geometry, eds: Nakamura, Pop, Schneps, Tamagawa, MSJ Conference Proceedings, 2012; see: arXiv:1105.1530v1 [math.AG]. ↑1
- [14] Obus, A. and Pries, R., *Wild tame-by-cyclic extensions*. J. Pure Appl. Algebra, **214** (2010), 565–573. ↑2
- [15] Obus, A. and Wewers, S., *Cyclic extensions and the local lifting problem*. Manuscript 2011. ↑1, 4, 4, 4.13
- [16] F Oort, *Lifting algebraic curves, abelian varieties, and their endomorphisms to characteristic zero*. in: Algebraic Geometry, Bowdoin, 1985; ed. S. Bloch, Proc. Sympos. Pure Math., **46**, Part 2, AMS Providence, RI, 1987, pp. 165–195. ↑1
- [17] ———, *Some Questions in Algebraic Geometry*. Utrecht University, Math. Dept. Preprint Series, June 1995; available at: <http://www.staff.science.uu.nl/oort0109/>. ↑1
- [18] Oort, F. and Sekiguchi, T. and Suwa, N., *On the deformation from Artin-Schreier to Kummer*. Ann. Sci. École Norm. Sup., **22** (1989) 345–375. ↑1
- [19] P. Pries, *Families of wildly ramified covers of curves*. Amer. J. Math. **124** (2002), no. 4, 737–768. ↑
- [20] M. Raynaud,  *$p$ -groupes et réduction semi-stable des courbes*. The Grothendieck Festschrift Vol. III, Progr. Math. **88**, Birkhäuser Boston, Boston, MA, 1990; pp. 179–197. ↑
- [21] ———, *Specialization des revêtements en caractéristique  $p > 0$* . Ann. Sci. École Norm. Sup. **32** (1999), 87–126. ↑
- [22] P. Roquette, *Zur Theorie der Konstantenreduktion algebraischer Mannigfaltigkeiten*. ↑4, 4
- [23] ———, *Abschätzung der Automorphismenzahl von Funktionenkörper bei Primzahlcharakteristik*. Math. Z. **117** (1970), 157–163. ↑1
- [24] M. Saidi, *Fake liftings of Galois covers between smooth curves*. Preprint, available at <http://arxiv.org/abs/1010.1311v1>. ↑1, 3, 4, 4, 4.9
- [25] M. Schmid, *Zyklischen algebraische Funktionenkörper vom Grade  $p^m$  über endlichem Konstantenkörper der Charakteristik  $p$* . J. Reine Angew. Math. **176** (1936), 108–123. ↑
- [26] J.-P. Serre, *Local Fields*. Springer-Verlag, New York, 1979. ↑2
- [27] Sekiguchi, T. and Suwa, N., *On the unified Kummer–Artin–Schreier–Witt Theory*. Chuo Math. Preprint Series **41**, 1994. ↑1
- [28] ———, *On the unified Kummer–Artin–Schreier–Witt Theory*. Laboratoire de Mathématiques Pures de Bordeaux Preprint Series **111**, 1999. ↑1
- [29] L. Thomas. J. Théorie Nombres Bordeaux, **17** (2005), 689–720. ↑2
- [30] S. Wewers, *Deformation of tame admissible covers of curves*. in: Aspects of Galois Theory, Gainesville, Florida, 1996; ed. H. Völklein, LMS Lecture Note Series. **256**, Cambridge Univ. Press, Cambridge, 1999; pp. 239–282. ↑
- [31] ———, *Fiercely ramified extensions of  $p$ -adic fields with imperfect residue fields*. Preprint, available at <http://arxiv.org/abs/1104.3785v1>. ↑

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