# GENERALIZATIONS AND MINIMALISTIC REFINEMENTS OF THE $t$-BIRATIONAL SECTION CONJECTURE

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Abstract. In this note we give generalizations and present and prove "minimalistic" refinements of the $t$-birational Section Conjecture ($t$-BSC), cf. [Be], by doing both: First, by extending the class of base fields over which the $t$-BSC holds, and second, by proving refinements of the $t$-BSC which involve much less, that is *minimalistic,* Galois theoretical information.

## 1. Introduction/Motivation

For reader's sake and to make the presentation self contained (to some extent), we begin by recalling a few notations and Galois theoretical basics.

**Notation/Definition 1.0.** Throughout the paper, if not otherwise explicitly stated, we will use the following notations and definitions:

- $k$ is a field, $\overline{k}|k$ is a separable closure of $k$, and $\tilde{k}|k \hookrightarrow \overline{k}|k$ is a Galois subextension.
- $\ell \neq \mathrm{char}(k)$ is some odd prime number, fixed throughout.
- $X$ is a complete geometrically integral normal $k$-curve.
- $K = k(X)$ is its function field, hence $K|k$ a regular field extension.
- $\overline{X} = X \times_k \overline{k}$ is the base change, thus $\overline{X}$ is normal integral.
- $\overline{\pi}_1(X) := \pi_1(\overline{X})$ and $\overline{\pi}_1(K) := \pi_1(K\overline{k})$ are the geometric étale fundamental groups.

Hence get the canonical commutative diagram with exact rows:

$$(*)_k \qquad \begin{array}{cccccccccc} \pi_{K/k}: & 1 & \to & \overline{\pi}_1(K) & \xrightarrow{\overline{p}_K} & \pi_1(K) & \xrightarrow{p_K} & \pi_1(k) & \to & 1 \\ & & & \downarrow \overline{q}_X & & \downarrow q_X & & \| & & \\ \pi_{X/k}: & 1 & \to & \overline{\pi}_1(X) & \xrightarrow{\overline{p}_X} & \pi_1(X) & \xrightarrow{p_X} & \pi_1(k) & \to & 1 \end{array}$$

Let $\mathcal{S}(\pi_{X/k})$ and $\mathcal{S}(\pi_{K/k})$ denote, respectively, the sets of $\overline{\pi}_1(X)$-conjugacy classes of sections $s : \pi_1(k) \to \pi_1(X)$ of $\pi_{X/k}$, and $\overline{\pi}_1(K)$-conjugacy classes of sections $s : \pi_1(k) \to \pi_1(K)$ of $\pi_{K/k}$. Obviously, if $s_K \in \mathcal{S}(\pi_{K/k})$, then $s_X := \pi_K \circ s_K$ lies in $\mathcal{S}(\pi_{X/k})$.

Next let $k_t := k(t)$ be the rational function field. For the base change $X_t := X \times_k k_t$ and its function field $K_t := k_t(X_t)$ consider the resulting diagram $(*)_{k_t}$ over $k_t$ for $X_t$ and $K_t$ below:

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$$(*)_{k_t} \qquad \begin{array}{lccccccccc} \pi_{K_{t}/k_t}: & 1 & \to & \overline{\pi}_1(K_t) & \xrightarrow{\overline{p}_t} & \pi_1(K_t) & \xrightarrow{p_t} & \pi_1(k_t) & \to & 1 \\ & & & \downarrow \overline{q}_{X_t} & & \downarrow q_{X_t} & & \| & & \\ \pi_{X_{t}/k_t}: & 1 & \to & \overline{\pi}_1(X_t) & \xrightarrow{\overline{p}_{X_t}} & \pi_1(X_t) & \xrightarrow{p_{X_t}} & \pi_1(k_t) & \to & 1 \end{array}$$

Let $\mathcal{S}(\pi_{X_{t}/k_t})$ be the set of all the $\overline{\pi}_1(X_t)$-conjugacy classes of section $s_t : \pi_1(k_t) \to \pi_1(X_t)$ of $\pi_{X_t/k_t} : \pi_1(X_t) \to \pi_1(k_t)$, and $\mathcal{S}(\pi_{K_{t}/k_t})$ be the set of the $\overline{\pi}_1(K_t)$-conjugacy classes of sections $s_t : \pi_1(k_t) \to \pi_1(K_t)$ of $\pi_{K_t/k_t} : \pi_1(K_t) \to \pi_1(k_t)$. One has a functorial identification $\pi_1(X_t) = \pi_1(X) \times_{\pi_1(k)} \pi_1(k_t)$, hence $s \in \mathcal{S}(\pi_{X/k}) \hookrightarrow \mathcal{S}(\pi_{X_t/k_t})$ via $s \mapsto s_t := s \times_{\pi_1(k)} \mathrm{id}$.

**Definition.** Let $pr_k : \pi_1(k_t) \to \pi_1(k)$ and $pr_K : \pi_1(K_t) \to \pi_1(K)$ be the canonical projections, hence $pr_k \circ p_{K_t} = p_K \circ pr_K$. Given a section $s \in \mathcal{S}(\pi_{X/k})$, we say that $s$ is:

1) *birationally liftable,* if there is $s_K \in \mathcal{S}(\pi_{K/k})$ such that $s = q_X \circ s_K$.
2) $t$*-birationally liftable,* if there is $s_t \in \mathcal{S}(\pi_{K_t/k_t})$ such that $s \circ pr_k = pr_K \circ s_t$.

Since $X$ is a complete normal $k$-curve, the points $x \in X$ are in bijection with the $k$-valuation rings $\mathcal{O}_v \in \mathrm{Val}_k(K)$ of the $k$-valuations of $K$ via $\mathcal{O}_x = \mathcal{O}_v$. Hence $x \in X$ is closed if and only if $v \in \mathrm{Val}_k(K)$ is non-trivial iff $\kappa(x) = \kappa(v)$ is finite over $k$. Further, $x \in X(k)$ is $k$-rational iff $\kappa(x) = k = \kappa(v)$ iff $v$ is a $k$-rational valuation. One has: By functoriality of $\pi_1$, every $x \in X(k)$ gives rise naturally to some $s_x \in \mathcal{S}(\pi_{X/k})$. Second, given a $k$-rational $v \in \mathrm{Val}_k(K)$, let $\overline{v}|v$ be the prolongations of $v$ to $K^{\mathrm{sep}}|K$, and $T_v \lhd Z_v < \pi_1(K)$ be the inertia/decomposition groups. Then all $\overline{v}|v$ are $\overline{\pi}_1(K)$-conjugated, and so are $T_v \lhd Z_v$, and the canonical exact sequence

$$(\pi_v): \qquad 1 \to T_v \xrightarrow{\overline{p}_K} Z_v \xrightarrow{p_K} G_k \to 1 \quad \text{is split.}$$

Hence the set of conjugacy classes of the sections $s_v \in \mathcal{S}(\pi_{K/k})$ defined by a $k$-rational $v$ is in bijection with the conjugacy classes of splittings of the exact sequece $(\pi_v)$ above, hence with $H^1_{\mathrm{cont}}(G_k, T_v)$, the cohomology pointed set of $G_k$ with values in $T_v$. In particular, if $\mathrm{char}(k) = 0$, one has that $T_v = \widehat{\mathbb{Z}}(1)$, thus via Kummer Theory, one has $H^1_{\mathrm{cont}}(G_k, T_v) = \widehat{k^\times}$.

The section conjecture (SC) originates from Grothendieck [G1], [G2], see [GGA], and asserts:

**Grothendieck SC.** *Let $k|\mathbb{Q}$ be a finitely generated field and $X$ be a projective hyperbolic $k$-curve. Then all $s \in \mathcal{S}(\pi_{X/k})$ arise form $x \in X(k)$ as described above and $X(k) \to \mathcal{S}(\pi_{X/k})$ is a bijection.*

There are several variants of section conjectures as follows. The **birational section conjecture (BSC)** asserts that in the context of SC, letting $K = k(X)$ be the function field of $X$, all sections $s \in \mathcal{S}(\pi_{K/k})$ arise from $k$-rational valuations $v$ of $K|k$, thus from $k$-rational points $x \in X(k)$ as explained above. The $p$**-adic SC** and $p$**-adic BSC** are obtained by replacing the f.g. field $k|\mathbb{Q}$ by a $p$-adic field $k$, i.e., by a finite field extension $k|\mathbb{Q}_p$. Finally, in the context of Grothendieck SC, the $t$**-BSC** asserts that any section $s \in \mathcal{S}(\pi_{X/k})$ which is $t$-birationally liftable originates form a $k$-rational point $x \in X(k)$ as explained above, using the fact that $x \in X(k)$ gives rise canonically to the $k_t$-rational point $x_t := x \times_k k_t \in X_t(k_t)$ of $X_t = X \times_k k_t$, etc.

Concerning results, conditional/weaker forms of the SC are part of the *local theory* in anabelian geometry by Nakamura [Na], Tamagawa [Ta], Mochizuki [Mz1], see e.g. the survey articles Faltings [Fa], Szamuely [Sz]. One can say that SC is wide open, and there are only a few complete unconditional results concerning forms of the BSC, precisely: The $p$-adic BSC is known, see Koenigsmann [Ko1] for the case of curves and Stix [St1] for higher dimensional varieties. The

BSC is known for the generic curve $C_g$ over $\kappa(M_g)$ by Hain [Ha], and second, for the geometrically integral hyperbolic curves over totally real number fields $k$ by Stix [St2]. Finally, very recently, the $t$-BSC was proved *over all* $k|\mathbb{Q}$ *finitely generated* by Bresciani [Be].

To complete this short list of results, recall that the $p$-adic BSC for curves (for all $p$) and higher dimensional varieties (for $p > 2$) holds under Galois "minimalistic" hypotheses. For instance, if the $p$-adic field $k$ contains the $p^{\text{th}}$ roots of unity, then the $\mathbb{Z}/p$ *-metabelian Galois theory* encodes already the rational points of proper smooth $k$-varieties. See Pop [P1], [P2] and Lüdtke [Lu] for details and further more general facts.

The aim of this note is to both *generalize* the $t$-BSC in its initial form and define/introduce and prove "*Galois-minimalistic*" type results for the $t$-BSC over quite general base fields $k$, thus giving wide generalizations of the $t$-BSC over $k|\mathbb{Q}$ finitely generated.

An application/consequence of the methods developed in this note is the following.

**Theorem 1.1 (Generalized $t$-BSC).** *Let $k$ be a perfect not $\ell$-closed field for some given odd prime number $\ell \neq \mathrm{char}(k)$. Let $X$ be a complete integral normal $k$-curve, $K = k(X)$. Then every $t$-birationally liftable section $s \in \mathcal{S}(\pi_{K/k})$ is defined by a unique $k$-rational point $x_s \in X(k)$ in the way explained above. That is, the $t$-BSC holds over $k$.*

The above theorem is a relatively easy consequence of Theorem 1.5 below.

**Notation 1.2.** Let $k$, $\tilde{k}|k$, e.g. $\tilde{k} = \overline{k}$, $\ell$ and $X$, $K = k(X)$ be as at Notation/Definition 1.0 above. We set $\tilde{X} = X \times_k \tilde{k}$, $\tilde{K} := \tilde{k}K = k(X_{\tilde{k}})$ and let $\tilde{k}_t := \tilde{k}(t)$ be the rational function field. Define $\tilde{X}_t = X \times_k \tilde{k}_t$, $\tilde{K}_t = \tilde{K}(t) = \tilde{k}_t(\tilde{X}_t)$ correspondingly. Since $\tilde{K}|K$ and $\tilde{K}_t|k_t$ are Galois extensions, both $\overline{K}|\tilde{K}^c|\tilde{K}^a|\tilde{K}|K$ and $\overline{K}_t|\tilde{K}^c_c|\tilde{K}^a_t|\tilde{K}_t|K_t$ are Galois extensions of $K$, respectively $K_t$.

**Remark 1.3.** Considering the commutative diagrams below:

$$
\begin{array}{ccc}
\pi_1(K_t) & \underset{p_t}{\overset{s_t}{\rightleftarrows}} & \pi_1(k_t) \\
\downarrow q_K & & \downarrow q_k \\
\pi_1(K) & \underset{p}{\overset{s}{\rightleftarrows}} & \pi_1(k)
\end{array}
\qquad
\begin{array}{ccc}
G(\tilde{K}^c_t|K_t) & \underset{p^c_t}{\overset{s^c_t}{\rightleftarrows}} & G(\tilde{k}^c_t|k_t) \\
\downarrow q^c_K & & \downarrow q^c_k \\
G(\tilde{K}^c|K) & \underset{p^c}{\overset{s^c}{\rightleftarrows}} & G(\tilde{k}^c|k)
\end{array}
\qquad
\begin{array}{ccc}
G(\tilde{K}^a_t|K_t) & \underset{p^a_t}{\overset{s^a_t}{\rightleftarrows}} & G(\tilde{k}^a_t|k_t) \\
\downarrow q^a_K & & \downarrow q^a_k \\
G(\tilde{K}^a|K) & \underset{p^a}{\overset{s^a}{\rightleftarrows}} & G(\tilde{k}^a|k)
\end{array}
$$

with $s \in \mathcal{S}(\pi_{K/k})$ and $s_t \in \mathcal{S}(\pi_{K_t/k_t})$ being corresponding sections. The following hold:

(†) Every $s \in \mathcal{S}(\pi_{K/k})$ gives rise canonically to a section $s^c : \pi_1(k) \to G(\tilde{K}^c|K)$ of $p^c$ and $s^a : \pi_1(k) \to G(\tilde{K}^a|K)$ of $p^a$ as above such that $s^c$ is a lifting of $s^a$.

$(\dagger)_t$ Every $s_t \in \mathcal{S}(\pi_{K_t/k_t})$ gives rise canonically to a sections $s^c_t : G(\tilde{k}^c_t|k_t) \to G(\tilde{K}^c_t|K_t)$ of $p^c_t$ and $s^a_t : G(\tilde{k}^a_t|k_t) \to G(\tilde{K}^a_t|K_t)$ of $p^a_t$ as above such that $s^c_t$ is a lifting of $s^a_t$.

(‡) If $s_t \in \mathcal{S}(\pi_{K_t/k_t})$ is a $t$-birational lifting of some given $s \in \mathcal{S}(\pi_{K/k})$, then $s_t$ gives rise canonically to sections $s^c_t : G(\tilde{k}^c_t|k_t) \to G(\tilde{K}^c_t|K_t)$ of $p^c_t$ and $s^a_t : G(\tilde{k}^a_t|k_t) \to G(\tilde{K}^a_t|K_t)$ of $p^a_t$ which lift $s^c$ and $s^a$, respectively, i.e., one has the following:

$$q^c_K \circ s^c_t = s^c \circ q^c_k \quad \text{and} \quad q^a_K \circ s^a_t = s^a \circ q^a_k .$$

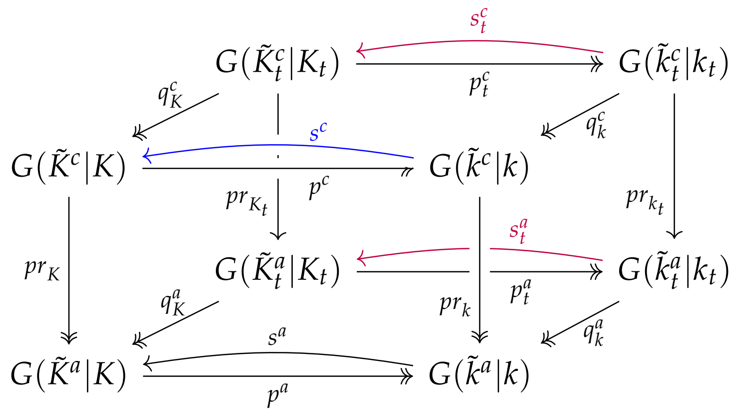
**Definition/Remark 1.4.** Let $s^a : G(\tilde{k}^a|k) \to G(\tilde{K}^a|K)$ be a section of $p^a : G(\tilde{K}^a|K) \to G(\tilde{k}^a|k)$.

1) We say that $s^a$ is $\tilde{k}|k$*-a.b.c. liftable,* if there is a section $s^c$ of $p^c$ which lifts $s^a$.

2) We say that $s^a$ is $\tilde{k}|k$-$t$-*a.b.c. liftable,* if there is a section $s^c_t$ of $p^c_t$ which lifts $s^a$.

We notice that, in particular, if $s^a$, $s^c$ and $s^c_t$ are sections as above, and $pr_K$, $pr_k$, $pr_{K_t}$, $pr_{k_t}$ are the canonical projections, one has commutative diagrams as follows:

$(*)_{\tilde{k}|k}$




The above Generalized $t$-BSC is a consequence of the following deeper fact.

**Theorem 1.5 ($\bar{k}|k$-Minimalistic $t$-BSC).** *Let $k$ be a perfect and not $\ell$-closed field for a fixed $\ell \neq \mathrm{char}(k)$. Let $X$ be a complete integral normal $k$-curve, $K = k(X)$, and $\tilde{k} = \bar{k}$, thus $G(\tilde{k}|k) = \pi_1(k)$ and $\tilde{K} = K\bar{k}$. Then every $\bar{k}|k$-$t$-a.b.c. liftable section $s^a : \pi_1(k) \to G(\tilde{K}^a|K)$ of $p^a : G(\tilde{K}^a|k) \to \pi_1(k)$ is defined by a unique $k$-rational point $x \in X(k)$ as explained above.*

**Remark 1.6.** We notice that Theorem 1.5 above implies the Generalized $t$-BSC above, hence the $t$-BSC in the classical context, where $k$ is of finite type over $\mathbb{Q}$. Namely, let $s \in \mathfrak{S}(\pi_{K/k})$ be given and $s_t : \pi_1(k_t) \to \pi_1(K_t)$ be a lifting of $s : \pi_1(k) \to \pi_1(K)$. Let $K_s \subset K^{\mathrm{sep}}$ be the fixed field of the image $s(\pi_1(k)) \subset \pi_1(K)$, and set $K_s = \cup_\alpha K_\alpha$ with $K_\alpha|K$ the inductive system of finite subextensions of $K_s|K$. Then the normalization $X_\alpha \to X$ of $X$ in the finite field extension $K_\alpha|K$ is a geometrically integral model of $K_\alpha|k$, and setting $X_{\alpha,t} := X_\alpha \times_k k_t$, $K_{\alpha,t} = k_t(X_{\alpha,t}) = K_\alpha(t)$, one has: The section $s : \pi_1(k) \to \pi_1(K)$ gives rise canonically to sections $s_\alpha : \pi_1(k) \to \pi_1(K_\alpha)$, because $s(\pi_1(k)) \subset \pi_1(K_\alpha)$. Second, if $K_{s_t} \subset K_t^{\mathrm{sep}}$ is the fixed field of the image $s_t(\pi_1(k_t)) \subset \pi_1(K_t)$, it follows that $K_{\alpha,t} \subset K_{s_t}$. Hence for every $s_\alpha$, the section $s_t : \pi_1(k_t) \to \pi_1(K_t)$ gives rise canonically to a lifting $s_{\alpha,t} : \pi_1(k_t) \to \pi_1(K_{\alpha,t})$.

To conclude, for every $K_\alpha$, consider the resulting $\tilde{K}_\alpha := K_\alpha\bar{k}$, and $\tilde{K}_{\alpha,t} := K_{\alpha,t}\bar{k}$. Then the section $s_\alpha$ gives rise to a section $s^a_\alpha : \pi_1(k) \to G(\tilde{K}^a_\alpha|K)$ of $p^a_\alpha : G(\tilde{K}^a_\alpha|K) \to \pi_1(k)$, which by the discussion above, is obviously $\bar{k}|k$-$t$-a.b.c. liftable. Hence by Theorem 1.5 above, $s^a_\alpha$ is defined by a unique closed point unique $x_\alpha \in X_\alpha(k)$. On the other hand, if $K_\alpha \subset K_\beta$, and $f_{\beta\alpha} : X_\beta \to X_\alpha$ is the canonical projection, then sorting through the definitions, one has: $x'_\alpha = f_{\beta\alpha}(x_\beta) \in X_\alpha(k)$ is a $k$-rational point of $X_\alpha$ which defines the section $s_\alpha$ as well. Hence by the uniqueness of the point $x_\alpha \in X_\alpha(k)$, one must have $x'_\alpha = x_\alpha$, i.e., $f_{\beta\alpha}(x_\beta) = x_\alpha$. Conclude that the compatible system $(x_\alpha)_\alpha$ of rational points defines the unique $k$-rational point $x_s \in X(k)$ which defines the $t$-birationally liftable section $s : \pi_1(k) \to \pi_1(K)$ we started with.

Finally, we present a refinement of the above Theorem 1.5, which is as follows.

**Hypothesis.** For $\ell \neq \mathrm{char}(k)$ odd, and $\tilde{k}|k$ Galois extension, consider the hypotheses:

(H) $\mu_\ell \subset \tilde{k}$ and $\tilde{k}^a|k$ is a infinite Galois extension.

(H0) Setting $\tilde{k} := k(\mu_\ell)$, the field extension $\tilde{k}|k$ satisfies hypothesis (H).

**Example 1.7.** For an odd prime number $\ell \neq \mathrm{char}(k)$, one has the following:

1) If $k$ is not $\ell$-closed, i.e., $\ell$ divides the degree $[\bar{k}:k]$, then $\bar{k}|k$ satisfies hypothesis (H).
2) The hypothesis (H0) is quite general, e.g., the infinite finitely generated fields, and more general, any Hilbertian field, etc., satisfy hypothesis (H0). And if $k$ satisfies (H0), one has:

   $(*)$ $\bar{k} = \cup_\alpha k_\alpha$ *inductively, where* $k_\alpha | k$ *are finite Galois extensions with* $k_\alpha^a | k$ *satisfying* (H).
3) Suppose that $\mu_\ell \subset k$. Then by mere definition, TFAE:

   (i) $k$ *satisfies hypothesis* (H0). (ii) $k^\times/\ell$ *is infinite.*

Recalling the notions of $\tilde{k}|k$-a.b.c. liftable sections and $\tilde{k}|k$-$t$-a.b.c. liftable sections, and the commutative diagram $(*)_{\tilde{k}|k}$ above, consider/define the following:

**Definition 1.8.** For closed points $x \in X$, set $k_x := \kappa(x) \cap \bar{k}$. For the $k$-valuation $v_x$ of $K$ with $\mathcal{O}_{v_x} = \mathcal{O}_x$, let $\tilde{Z}_x \subset G(\tilde{K}^a|K)$ be the decomposition groups of prolongations $\tilde{v}_x^a$ of $v_x$ to $\tilde{K}^a$. Let a section $s^a : G(\tilde{k}^a|l) \to G(\tilde{K}^a|K)$ of $p^a : G(\tilde{K}^a|K) \to G(\tilde{k}^a|k)$ be given. We say that:

1) A closed point $x \in X$ *defines* $s^a$ if $x \in X(k)$ is $k$-rational, and $s^a\big(G(\tilde{k}^a|k)\big) \subset \tilde{Z}_x$ for some decomposition group $\tilde{Z}_x \subset G(\tilde{K}^a|K)$ above $v_x$.
2) A closed point $x \in X$ *quasi-defines* $s^a$ if $k_x := \kappa(x) \cap \tilde{k} = k$, and $s^a\big(G(\tilde{k}^a|k)\big) \subset \tilde{Z}_x$ for some decomposition group $\tilde{Z}_x \subset G(\tilde{K}^a|K)$ above $v_x$.

   In particular, for $\tilde{k} = \bar{k}$, the notions "defines" and "quasi defines" are identical.

The above $\bar{k}|k$-minimalistic $t$-BSC is a consequence of the following deeper fact.

**Theorem 1.9** (**$\tilde{k}|k$-Minimalistic $t$-BSC**). *In the above notation, let* $s^a : G(\tilde{k}^a|k) \to G(\tilde{K}^a|K)$ *be a* $\tilde{k}|k$-$t$*-a.b.c. liftable section of* $p^a : G(\tilde{K}^a|K) \to G(\tilde{k}^a|k)$. *If* $\tilde{k}|k$ *satisfies hypothesis* (H)*, then the section* $s^a$ *is quasi-defined by a unique closed point* $x_{s^a} \in X$*, i.e.,* $\tilde{k} \cap \kappa(x_{s^a}) = k$.

**Corollary 1.10.** *If* $k = \bar{k}$*, then* $s^a$ *is defined by a unique* $k$*-rational point* $x_{s^a} \in X(k)$. *Hence Theorem 1.9 implies Theorem 1.5* (*Minimalistic* $t$*-BSC*)*, hence Theorem 1.1* (*Generalized* $t$*-BSC*).

## 2. Reviewing facts about recovering valuations

### 2.1. Basics of valuations theory.

For arbitrary fields $\Lambda$, let $\mathrm{Val}(\Lambda)$ be the set of (equivalence classes of) valuations $v$ of $\Lambda$. For $v \in \mathrm{Val}(\Lambda)$, let $\mathfrak{m}_v \subset \mathcal{O}_v$ be its valuation ideal/ring, $\Lambda v = \kappa(v) = \mathcal{O}_v/\mathfrak{m}_v$ its residue field, and $v\Lambda = \Lambda^\times/\mathcal{O}_v^\times$ the (canonical) value group of $v$. Recall that $\mathrm{Spec}(\mathcal{O}_v)$ is a chain w.r.t. inclusion, and for each $\mathfrak{m}_1 \in \mathrm{Spec}(\mathcal{O}_v)$, the localization $\mathcal{O}_1 := (\mathcal{O}_v)_{\mathfrak{m}_1}$ is a valuation ring with valuation ideal $\mathfrak{m}_1$. And if $v_1 \in \mathrm{Val}(\Lambda)$ is the corresponding valuation, then $\mathcal{O}_1 = \mathcal{O}_{v_1}$ and $\mathfrak{m}_1 = \mathfrak{m}_{v_1}$. Moreover, the rings $\mathcal{O}_1 \subset \Lambda$ with $\mathcal{O}_v \subset \mathcal{O}_1$ are the valuation rings of the form above, i.e., $\mathcal{O}_1 = (\mathcal{O}_v)_{\mathfrak{m}_1}$ for some $\mathfrak{m}_1 \in \mathrm{Spec}(\mathcal{O}_v)$ and $\mathcal{O}_0 = \mathcal{O}_v/\mathfrak{m}_1 \subset \Lambda v_1$ is a valuation ring on $\Lambda v_1$ with valuation ideal $\mathfrak{m}_0 = \mathfrak{m}/\mathfrak{m}_1$. Thus setting $\mathcal{V}_v(\Lambda) := \{v_1 \in \mathrm{Val}(\Lambda) \mid v_1 \leqslant v\}$ and $\mathcal{R}_{\mathcal{O}_v} := \{\mathcal{O}_1 \subset F \mid \mathcal{O}_v \subset \mathcal{O}_1\}$, one has canonical bijections:

$$\mathcal{V}_v(\Lambda) \to \mathcal{R}_{\mathcal{O}_v} \to \mathrm{Spec}(\mathcal{O}_v), \quad v_1 \mapsto \mathcal{O}_{v_1} \mapsto \mathfrak{m}_{v_1}.$$

Finally, $\mathrm{Val}\,(\Lambda)$ carries a natural partial ordering $\leqslant$ defined by the equivalent conditions:

$$v_1 \leqslant v_2 \;\text{ iff }\; \mathcal{O}_{v_1} \supset \mathcal{O}_{v_2} \;\text{ iff }\; \mathfrak{m}_{v_1} \subset \mathfrak{m}_{v_2} \;\text{ iff }\; \mathfrak{m}_{v_1} \subset \mathcal{O}_{v_2}.$$

We say that $v_1$ is a *coarsening* of $v_2$, respectively that $v_2$ is a *refinement* of $v_1$. In particular, if $v_1 \leqslant v_2$, then $\mathcal{O}_{v_0} := \mathcal{O}_v/\mathfrak{m}_{v_1}$ is a valuation ring of $\Lambda_0 := Ev_1$ having $\mathfrak{m}_{v_0} = \mathfrak{m}_{v_2}/\mathfrak{m}_{v_1}$ as valuation ideal, and obviously, $\Lambda_0 v_0 = \mathcal{O}_{v_0}/\mathfrak{m}_{v_0} = \mathcal{O}_v/\mathfrak{m}_v = \Lambda v$. Further, one has a canonical exact sequence of value groups $1 \to v_0\Lambda_0 \to v\Lambda \to v_1\Lambda \to 1$.

If $v_1 \leqslant v_2$, we denote $v_0 = v_2/v_1$ and call $v_0$ the (valuation theoretical) *quotient* of $v_2$ by $v_1$, and set $v_1 = v_0 \circ v_2$ and call $v_1$ the (valuation theoretical) *composition* of $v_0$ and $v_1$.

We also recall that $v_1 \leqslant v$ gives rise to the projection $v\Lambda = \Lambda^\times/\mathcal{O}_v^\times \twoheadrightarrow \Lambda^\times/\mathcal{O}_{v_1}^\times = v_1\Lambda$, which is order preserving, thus its kernel is a convex subgroup $\Delta_v$ of $v\Lambda$. And conversely, if $\Delta \leqslant v\Lambda$ is a convex subgroup, the $v\Lambda \to v\Lambda/\Delta_1$ is order preserving, giving rise to a valuation $v_\Delta \in \mathrm{Val}(\Lambda)$ with $v_\Delta \leqslant v$. Conclude that $\mathcal{V}_v(\Lambda)$ is in canonical bijective with the set of convex subgroups $\{\Delta \leqslant v\Lambda \mid \text{convex subgroup}\}$.

Last but not least, for $v_1, v_2 \in \mathrm{Val}\,(\Lambda)$ there is a well defined valuation $v = \min(v_1, v_2)$ in $\mathrm{Val}\,(\Lambda)$ whose valuation ring $\mathcal{O}_v$ is characterized as follows: $(\mathcal{O}_{v_1})_\mathfrak{m} = \mathcal{O}_v = (\mathcal{O}_{v_2})_\mathfrak{m}$ and $\mathfrak{m}_v = \mathfrak{m}$, where $\mathfrak{m} \in \mathrm{Spec}(\mathcal{O}_{v_1}) \cap \mathrm{Spec}(\mathcal{O}_{v_2})$ is the unique maximal element w.r.t. inclusion. Equivalently, $\mathfrak{m}$ is maximal in $\mathrm{Spec}(\mathcal{O}_{v_1}) \cap \mathrm{Spec}(\mathcal{O}_{v_2})$ satisfying $\mathfrak{m} \cap \mathcal{O}_{v_1}^\times = \emptyset = \mathcal{O}_{v_2} \cap \mathfrak{m}$.

Finally, every $v \in \mathrm{Val}\,(\Lambda)$ defines a field topology $\tau_v$ on $\Lambda$ (in which a basis of open neighborhoods of $0$ consists of the non-zero ideals of $\mathcal{O}_v$). Obviously, for $v_1, v_2 \in \mathrm{Val}\,(\Lambda)$ one has that $\tau_{v_1} = \tau_{v_2}$ iff $v_1$ and $v_2$ have a common non-trivial coarsening $v \leqslant v_1, v_2$ and if so, $\tau_{v_1} = \tau_v = \tau_{v_2}$. If this is the case, we say that $v_1, v_2$ are *dependent.* Complementary, we say that $v_1, v_2$ are *independent,* if $\tau_{v_1} \neq \tau_{v_2}$, or equivalently, the diagonal embedding $\Lambda \to (\Lambda, \tau_{v_1}) \times (\Lambda, \tau_{v_2})$ has a dense image. Notice that for $v_1, v_2 \in \mathrm{Val}\,(\Lambda)$, and $U_{v_i} \subset \Lambda$ non-empty $v_i$-open, $i = 1, 2$, the following are equivalent:

(i) $v_1$, $v_2$ are independent; (ii) $U_{v_1} - U_{v_2} = \Lambda$; (iii) $\Lambda^\times \subset U_{v_1} \cdot U_{v_2}$.

**Fact 2.1.** *In general, given $v_1, v_2 \in \mathrm{Val}(\Lambda)$ and $v := \min(v_1, v_2)$, set $U_{v_i} = 1 + \mathfrak{m}_{v_i}$, $i = 1, 2$ and $U_v = 1 + \mathfrak{m}_v$. The following hold:*

1) *If $v_1 \leqslant v_2$, one has $U_{v_1} \cdot U_{v_2} = U_{v_2}$, $\mathcal{O}_{v_1} \cdot \mathcal{O}_{v_2} = \mathcal{O}_{v_1}$, $U_{v_2} - U_{v_1} = \mathfrak{m}_{v_2}$.*

2) *If $v < v_1, v_2$ strictly, then $U_{v_1} \cdot U_{v_2} = \mathcal{O}_v^\times = \mathcal{O}_{v_1}^\times \cdot \mathcal{O}_{v_2}^\times$, $U_{v_1} - U_{v_2} = \mathcal{O}_v = \mathcal{O}_{v_1} - \mathcal{O}_{v_2}$.*

*Proof.* The assertions from 1) follow by mere definition.

To 2): By mere definitions, the quotient valuations $\overline{v}_i = v_i/v$, on the residue field $\Lambda v$ are independent. Hence setting $U_{\overline{v}_i} := 1 + \mathfrak{m}_{\overline{v}}$, $i = 1, 2$ one has that $\Lambda v^\times = U_{\overline{v}_1} \cdot U_{\overline{v}_2}$ by the discussion above. Further, the canonical exact sequence

$$(*) \qquad 1 \to U_v \to \mathcal{O}_v^\times \xrightarrow{\pi} \Lambda v^\times \to 1$$

defines exact sequences $1 \to U_v \to U_{v_i} \to U_{\overline{v}_i} \to 1$, thus an subsequence of $(*)$ above:

$$(**) \qquad 1 \to U_v \hookrightarrow U_{v_1} \cdot U_{v_2} \twoheadrightarrow U_{\overline{v}_1} \cdot U_{\overline{v}_2} \to 1,$$

in which the the first map is injective, and the second one is surjective. On the other hand, sine $\overline{v}_1, \overline{v}_2$ are independent on $\Lambda v$, one has $\Lambda v^\times = U_{\overline{v}_1} \cdot U_{\overline{v}_2}$. Hence since $U_{v_1} \cdot U_{v_2} \subset \mathcal{O}_v^\times$ and $\ker(\pi) = U_v$, we conclude that $(**)$ is exact, implying finally $U_{v_1} \cdot U_{v_2} = \mathcal{O}_v^\times$.

The proof of the assertion $U_{v_1} - U_{v_2} = \mathcal{O}_v$ is similar, being the additive variant. □

**Canonical $v$-valuation.** Let $\Omega|\Lambda$ be an arbitrary field extension and $w \in \mathrm{Val}(\Omega)$ and $v \in \mathrm{Val}(\Lambda)$ satisfy $w_\Lambda := w|_\Omega \geqslant v$. Equivalently, by general valuation theory, one has:

$$\mathcal{O}_w \cap \Lambda = \mathcal{O}_{w_\Lambda} \subset \mathcal{O}_v, \ \ (1+\mathfrak{m}_v) \cap \Lambda = 1 + \mathfrak{m}_{w_\Lambda} \supset 1 + \mathfrak{m}_v, \ \ \text{etc.}$$

In particular, by the above discussion about coarsening, $\mathcal{O}_v = (\mathcal{O}_{w_\Lambda})_\mathfrak{m}$ is the localization of $\mathcal{O}_{w_\Lambda}$ with respect to its prime ideal $\mathfrak{m}_v \in \mathrm{Spec}(\mathcal{O}_{w_\Lambda})$. Equivalently, setting $\Sigma_{w_\Lambda} := \mathcal{O}_{w_\Lambda} \setminus \mathfrak{m}_v$, one has that $\Sigma_{w_\Lambda}$ is a multiplicative system in $\mathcal{O}_{w_\Lambda}$ defining $\mathcal{O}_v$ as follows:

$$\mathcal{O}_v = (\mathcal{O}_{w_\Lambda})_{\mathfrak{m}_v} = \Sigma_{w_\Lambda}^{-1} \mathcal{O}_{w_\Lambda}.$$

**Lemma 2.2.** *$\mathcal{O}_0 = \Sigma_{w_\Lambda}^{-1}\mathcal{O}_w \subset \Omega$ is a valuation ring with valuation $w_0$ satisfying $w_0|_\Lambda = v$.*

*Proof.* Indeed, $\mathcal{O}_0 \cap \Lambda = \{\frac{a}{r} \in N \mid a \in \mathcal{O}_w,\, r \in \Sigma_{w_\Lambda}\}$ and we have to prove that $\mathcal{O}_0 \cap \Lambda = \mathcal{O}_v$. For the direct inclusion, let $x = \frac{a}{r} \in \Lambda$ with $a \in \mathcal{O}_w$, $r \in \Sigma_{w_\Lambda}$. Then $a = rx \in \Lambda$, thus concluding that $a \in \mathcal{O}_w \cap \Lambda = \mathcal{O}_{w_\Lambda} \subset \mathcal{O}_v$. Thus finally, $x = \frac{a}{r} \in \Sigma_{w_\Lambda}^{-1} = \mathcal{O}_v$. The converse implication is clear, because $\mathcal{O}_v = \Sigma_{w_\Lambda}^{-1}\mathcal{O}_{w_\Lambda} \subset \Sigma_{w_\Lambda}^{-1}\mathcal{O}_w = \mathcal{O}_0$ and $\mathcal{O}_{w_\Lambda} = \mathcal{O}_w \cap \Lambda$. □

Let $\mathfrak{m}_1 \in \mathrm{Spec}(\mathcal{O}_{w_0}) \subset \mathrm{Spec}(\mathcal{O}_w)$ be the (unique) prime ideal which is *minimal* satisfying $\mathfrak{m}_1 \cap \Lambda \supset \mathfrak{m}_v$. Then one has $\mathfrak{m}_v \subset \mathfrak{m}_1 \cap \Lambda \subset \mathfrak{m}_{w_0} \cap \Lambda = \mathfrak{m}_v$, thus $\mathfrak{m}_1 \cap \Lambda = \mathfrak{m}_v$. Hence we conclude that the valuation $w_v$ of the valuation ring $\mathcal{O}_1 = (\mathcal{O}_0)_{\mathfrak{m}_1}$ satisfies $w_v|_\Lambda = v$.

**Definition 2.3.** In the above notation and context, $w_v$ is the *canonical $v$-valuation of $\Omega$*. Thus $w_v$ is unique minimal with $w_v \leqslant w$, $w_v|_\Lambda = v$, that is, $\mathcal{O}_{w_v} \cap \Lambda = \mathcal{O}_v$, $\mathfrak{m}_{w_v} \cap \Lambda = \mathfrak{m}_v$.

Finally, let $(\Omega', w' \,|\, (\Omega, w)$ and $(\Lambda', v') \,|\, (\Lambda, v)$ be algebraic extensions of valued fields such that $\Lambda' \subset \Omega'$ and $w' \geqslant \mathfrak{v}'$, thus obviously, $w = w'|_\Lambda \geqslant v'|_\Lambda = v$. For short, we denote this situation by $(\Omega'|\Lambda', w'|v')\big|(\Omega|\Lambda, w|v)$. Obviously, $w' \geqslant v$ for the field extension $\Omega'|\Lambda$.

We conclude this discussion with the following (obvious) facts.

**Fact 2.4.** *In the above notation, the following hold:*

1) $\mathcal{O}_{w_v}^\times \cap \Lambda = \mathcal{O}_v^\times$ *and* $(1 + \mathfrak{m}_{w_v}) \cap \Lambda = 1 + \mathfrak{m}_v$.
2) *Let* $(\Omega'|\Lambda', w'|v')\big|(\Omega|\Lambda, v|v)$ *be as above, thus* $w' \geqslant v$ *for the field extension* $\Omega'|\Lambda$. *Then* $w'_v = w'_{v'}$ *and* $w'_v|_\Omega = w_v = w'_{v'}|_\Omega$.

*Proof.* Assertion 1) follows by mere definitions, etc. For assertion 2), recall that for any valuations $\tilde{w}'$ of $\Omega'$ and $\tilde{v}'|\tilde{v}$ of the algebraic extension $\Lambda'|\Lambda$ one has: $\tilde{w}'|_{\Lambda'} = v'$ iff $\tilde{w}'|_\Lambda = v$, etc. □

## 2.2. Basics of Hilbert decomposition theory, especially in $\mathcal{G}_F^a$.

Let $F'|E$ be an algebraic field extension, $v \in \mathrm{Val}\, E$ be a fixed valaution, and $\mathcal{V}_v(F')$ be the set of prolongations $w'|v$ of $v$ to $F|E$. Recall that $\mathrm{Val}_v(F')$ is a profinite topological space in the patch topology,[1] and moreover, if $F'|E$ is normal algebraic, the profinite group $G(F'|E) := \mathrm{Aut}_E(F')$ acts transitively and continuously on $\mathcal{V}_v(F')$ via $(w', g) \mapsto w'^g := w' \circ g^{-1} =: w''$. And if $T_{w'|v} \lhd Z_{w'|v}$ are the inertia/decomposition groups of $w'|v$, then $T_{w''|v} = g\, T_{w'|v}\, g^{-1}$ and $Z_{w''|v} = g\, Z_{w'|v}\, g^{-1}$, and for any $w' \in \mathcal{V}_v(F')$ fixed have:

$$\mathcal{V}_v(F') = G(F'|E)\cdot w' \cong Z_{w'|v}\backslash G(F'|E) \ \text{ as } G(F'|E)\text{-spaces, canonically.}$$

[1] Actually, $\mathrm{Val}_v(F')$ endowed with the patch topology is a profinite space even if $F'|E$ is not algebraic.

Further, the residue field extension $F'w'|Ev$ is a normal extension, and setting $G_{w'|v} := \mathrm{Aut}_{Ev}(F'w')$, one has the canonical exact sequence $1 \to T_{w'|v} \to Z_{w'|v} \to G_{w'|v} \to 1$.

Next let $v_1 < v$ in $\mathrm{Val}\,(E)$. There is a prolongation $w'_1|v_1$ of $v_1$ to $F'|E$ such that $w'_1 < w'$. Further, for any such $w'_1|v_1$ the following hold: First, $Z_{w'|v} \subset Z_{w'_1|v_1}$ and both $T_{w'_1|v_1} \lhd T_{w'|v}$ and $T_{w'_1|v_1} \lhd Z_{w'|v}$. Second, $w'_0 := w'/w'_1$ prolongs $v_0 := v/v_1$ to $F'w'$, and via the canonical exact sequence $1 \to T_{w'_1|v_1} \longrightarrow Z_{w'_1|v_1} \xrightarrow{\pi} G_{w'_1|v_1} \to 1$ the following hold:

$$Z_{w'_0|v_0} = \pi(Z_{w'|v}) = Z_{w'|v}/T_{w'_1|v_1} \text{ and } T_{w'_0|v_0} = \pi(T_{w'|v}) = T_{w'|v}/T_{w'_1|v_1},$$

giving rise to a commutative diagram exact sequences of the form:

$$(\dagger) \qquad \begin{array}{ccccccccc} 1 & \to & T_{w'_1|v_1} & \longrightarrow & Z_{w'_1|v_1} & \xrightarrow{\pi} & G_{w'_1|v_1} & \to & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \to & T_{w'_1|v_1} & \longrightarrow & Z_{w'|v} & \xrightarrow{\pi} & Z_{w'_0|v_0} & \to & 1. \end{array}$$

Finally, recall that if $w'|v$ is tame, i.e., $T_{w'|v}$ has order prime to $\mathrm{char}(Ev)$, one has that $T_{w'|v}$ is abelian, precisely, $T_{w'|v} = \mathrm{Hom}(w'F'/vE, \mu_{F'w'})$ with $\mu_{F'w'} \subset F'w'$ the group of roots of unity in $F'w'$. Further, the conjugation action of $Z_{w'|v}$ on $T_{w'|v}$ factors $Z_{w'|v} \twoheadrightarrow G_{w'|v}$, and $G_{w'|v}$ acts on $T_{w'|v} = \mathrm{Hom}(w'F'/vE, \mu_{F'w'})$ via the cyclotomic character of $G_{w'|v}$.

This being said, let $\ell > 2$ be a prime number fixed throughout the remaining of this section, and $F|E$ be a Galois field extension with $\mathrm{char}(E) \neq \ell$ and $\mu_\ell \subset F$. Let $F^c|F^a|F$ be the (maximal) $\mathbb{Z}/\ell$ abelian-by-central, respectively the (maximal) $\mathbb{Z}/\ell$ elementary abelian, extensions of $F$, and for the corresponding exact sequence of Galois groups

$$1 \to \Delta_F := G(F^c|F^a) \to \mathcal{G}^c_F := G(F^c|F) \to \mathcal{G}^a_F := G(F^a|F) \to 1,$$

denote $\mathcal{G}^c_F \ni \tilde\sigma \mapsto \tilde\sigma|_{F^a} =: \sigma \in \mathcal{G}^a_F$ the corresponding projection. Recall that by Kummer Theory, one has that $\mathcal{G}^a_F = \mathrm{Hom}(F^\times, \mu_\ell)$, and $\Delta_F$ is the maximal $\mathbb{Z}/\ell$ elementary abelian quotient of the absolute Galois group $G_{F^a}$ on which $\mathcal{G}^a_F$ acts trivially. Via Kummer Theory, one obtains $F^c|F$ as follows: $\mathcal{G}^a_F$ acts canonically on $F^{a\times}/\ell$, and let $A := (F^{a\times}/\ell)^{\mathcal{G}^a_F}$ be the subgroup of invariants; that is, $u \in F^a$ lies in $A$ iff $\forall\, \sigma \in \mathcal{G}^a_F\ \exists\, r_\sigma \in F^a$ such that $\sigma(u) = u r_\sigma^\ell$. Then one has $F^c = F^a[\sqrt[\ell]{A}]$. From this discussion immediately follows the following.

**Basic Fact.** *$F^c|E$ and $F^a|E$ are Galois extensions of $E$.*

One has the following basic facts (well known to experts, but I cannot give a precise reference).

**Fact 2.5.** *Let $F$ be a arbitrary field with $\mu_\ell \subset F$ if $\mathrm{char}(F) \neq \ell$. For a valuation $w \in \mathrm{Val}\,(F)$, let $w^a|w$ be a prolongation of $w$ to $F^a|F$, and $F^h$ be the Henselization. The following hold:*

1) *The compositum $F^hF^a$ equals the maximal $\ell$-elementary abelian extension $(F^h)^a$ of $F^h$.*
2) *The separable part of $F^aw^a|Fw$ is the maximal $\ell$-abelian extension of $Fw$.*

*Proof.* We prove the assertion along the following two reductions steps:
Step 1. The valuation $w$ has finite rank one. In particular, $F$ is dense in $F^h$.

Case a). $\ell = \mathrm{char}(F)$. Then the $\ell$-elementary abelian extension of both $F$ and $F^h$ are are composita of $\ell$-cyclic extensions, all of which being Artin-Schreier extensions. Let $F^h(x')|F^h$ with $x'^\ell - x' = a'$, $a' \in F^h$ by such an extension. Since $v$ has rank $1$, hence $F$ is dense in $F^h$, one can choose $a \in F$ such that $v^h(a' - a) > 0$. Then setting $a'' = a' - a \in F^h$, or equivalently, $a' = a'' + a$, one has: First, the Artin-Schreier equation $T^\ell - T = a''$ has a solution in $x'' \in F^h$ (because $v(a'') > 0$). Second, $x'$ is a solution of $T^\ell - T = a'' + a$ (by the additivity of $T^\ell - T$). Hence we conclude that $F^h(x') \subset F^h(x'' + x) \subset F^h F^a$.

Similarly, if $T^\ell - T = \overline{a}$ is an Artin-Schreier equations over $Fw$, and $a \in \mathcal{O}_w$ is a representative of $\overline{a} \in Fw$ and $x$ is a solution of the equation $T^\ell - T = a$, it follows that the reduction $\overline{x}$ of $x$ is a root of $T^\ell - T = \overline{a}$.

Case b). $\ell \neq \mathrm{char}(Fw)$. Proceed as above, but using Kummer type equations $U^\ell = a$, etc.

Case c). $\mathrm{char}(F) = 0$, $\mathrm{char}(Fw) = \ell$. Assertion 1) follows in the same way as in Case b). For assertion 2), recall that $\pi := \zeta_\ell - 1$ with $\zeta_\ell \in \mu_\ell$ primitive satisfies: $\ell = \pi^{\ell-1}\epsilon$ over $\mathbb{Z}[\mu_\ell]$ with $\epsilon \in 1 + \pi\mathbb{Z}[\mu_\ell] \subset \mathcal{O}_v^\times$ a principal $v$-unit. Then the Kummer equation $(U+1)^\ell = b$ has roots $u \in F^a$ for each $b \in F$, and can rewritten as follows: $u^\ell + \sum_{\ell > i > 0} \binom{\ell}{i} u^i + 1 = b$, thus diving by $\pi^\ell = \ell\pi\epsilon$ and setting $u = t\pi$, the equation satisfied by $t$ is:

$$(u) \qquad T^\ell + \epsilon^{-1} \textstyle\sum_{\ell > i > 0} \frac{1}{\ell}\binom{\ell}{i} \pi^{i-1} T^i = (b-1)/\pi^\ell .$$

In particular, choosing $b \in F$ such that $a = (b-1)/\pi^\ell$, i.e., $b = \pi^\ell a + 1$, it follows that the displayed equation $(u)$ above specializes to $T^\ell - T = \overline{a}$. That is, if $u \in F^a$ is a satisfies $(u+1)^\ell = b$, then $t = u/\pi$ is specializes to a solution of $T^\ell - T = \overline{a}$.

Step 2. The valuation $w$ has finite rank $d = \mathrm{rk}(v) = \mathrm{Kr.\,dim}(\mathcal{O}_v) < \infty$. We make induction on $d$. Namely, let $w_1 \leqslant w$ be the minimal non-trivial coarsening of $w$, and $w_0 = w/w_1$ the resulting valuation of the residue field $F_0 = Fw_1$. Then $w_1$ has rank one and $w_0$ has rank $d - 1 < d$. Hence by the induction hypothesis and Step 1, the assertions 1), 2) hold for both $w_1$ and $w_0$. From this instantly follows the same for $w$ ( by the fuctoriality of Hilbert Decomposition for valuations).

Step 3. Let $F = \cup_\alpha F_\alpha$ be the inductive union of its finitely generated subfields with $\mu_\ell \subset F_\alpha$ provided $\mathrm{char}(F) \neq \ell$. Then considering $F^a_\alpha | F_\alpha$, it follows that $F^a = \cup_\alpha F^a_\alpha$ and the extension of valued fields $F^a|F, w^a|w$ is the inductive limit of the system of valued fields $F^a_\alpha | F_\alpha, w^a_\alpha | w_\alpha$. And since $F^h = \cup_\alpha F^h_\alpha$ and $F^a w^a = \cup_\alpha F^a_\alpha w^a_\alpha$, by mere definitions one has that assertions 1), 2) from the Fact hold iff they hold for each $F^a_\alpha | F_\alpha$ endowed with $w^a_\alpha | w_\alpha$.

On the other hand, since $F_\alpha$ is finitely generated, the valuation $w_\alpha$ has finite rank (bounded by the Krull dimension of $F_\alpha$). Hence assertions 1), 2) hold for each $F^a|F$, $w^a|w$ by the discussion at Step 1. Hence conclude that assertions 1), 2) hold for $F^a|F, w^a|w$. □

Recall that via the canonical exact sequence $1 \to \mathcal{G}^a_F \xrightarrow{\imath} G(F^a|E) \xrightarrow{\pi} G(F|E) \to 1$ the group $G(F|E)$ **acts canonically** (by "conjugation") on subsets $\Sigma$ of the three groups above by

$$g(\Sigma) := g\,\Sigma\,g^{-1} \quad \text{for} \quad g \in G(F|E) \text{ and } \Sigma \subset G(F^a|F),\ G(F^a|E),\ G(F|E),$$

compatibly with the morphisms $\imath, \pi$. We fix the above notation for this action throughout.

Let $\mathcal{V} \subset \mathrm{Val}\,(E)$ be a non-empty set. For $v \in \mathcal{V}$, let $w^a|w|v$ be the prolongations of $v \in \mathcal{V}$ to $F^a|F|E$, and $\mathcal{V}_v(F) \subset \mathcal{V}(F)$ denote the prolongations of $v \in \mathcal{V}$ and of $\mathcal{V}$ to $F$. And to fix notation, recall that $G(F|E)$ acts on $\mathcal{V}_v(F)$ by $g(w) := w \circ g^{-1} =: w^g$.

By Hilbert decomposition theory for valuations, one has: Since $G(F^a|F)$ is abelian, it follows that $T^a_w := T_{w^a|w} \leqslant Z_{w^a|w} =: Z^a_w$ and $T_{w^a|v} \leqslant Z_{w^a|v}$ depend on $w$ only and not on the concrete prolongation $w^a|w$. And for $w \in \mathcal{V}_v(F)$, $g \in G(F|E)$ one has:

$$Z^a_{g(w)} = g\, Z^a_w\, g^{-1} = g(Z^a_w) \ \text{ and } \ Z_{g(w)|v} = g\, Z_{w|v}\, g^{-1} = g(Z_{w|v})\,.$$

Further, one has a canonical projection of topological $G(F|E)$-spaces:

$$\mathcal{Z}^a_v(F) := \{Z^a_w |\, w \in \mathcal{V}_v(F)\} \to \{Z_{w|v} |\, w \in \mathcal{V}_v(F)\} =: \mathcal{Z}_v(F)\,, \quad Z^a_w \mapsto Z_{w|v}\,.$$

**Definition/Remark 2.6.** In the above context and notation consider/define:

1) If $Z^a_w \neq 1$ for some $w \in \mathcal{V}_v(F)$, hence $Z^a_{w'} \neq 1$ for all $w' \in \mathcal{V}_v(F)$, and indicate this by writing $\mathcal{Z}^a_v(F) \neq 1$. And if $\mathcal{Z}^a_v(F) \neq 1$ for all $v \in \mathcal{V}$, we write $\mathcal{Z}^a_{\mathcal{V}}(F) \neq 1$.
2) We say that $v \in \mathcal{V}$ equals its *$F$-$\ell$-abelian core* if for any proper coarsening $v_1 < v$, the valuations $w^a_1 \in \mathrm{Val}_{v_1}(F^a)$ satisfy: *The separable part of $F^a w^a_1 \,|\, F w_1$ is non-trivial.*
   Further, we say that $\mathcal{V}$ equals its $F$-$\ell$-abelian core if each $v \in \mathcal{V}$ does so. For instance, this is the case is all $v \in \mathcal{V}$ have rank one and $F^a \neq F$.

- We notice that for every $v \in \mathrm{Val}\,(F)$ there is a valuation $v^0 \in \mathrm{Val}\,(F)$ which is maximal with the properties: $v^0 \leqslant v$ and $v^0$ equals it $F$-$\ell$-abelian core.

Let $\mathcal{V}$ be as above in Definition/Remark 2.6, and $v_1, v_2 \in \mathrm{Val}\,(E)$ be given, and $w^a_i | w_i | v_i$ be prolongations of $v_i$ to $F^a|F|E$, $i = 1,2$. Setting $v = \min(v_1, v_2)$ and $w = \min(w_1, w_2)$, it follows that $w|v$ prolongs $v$ to $F|E$, and setting $\overline{v}_i := v_i/v$, $\overline{w}_i := w_i/w$, $\overline{w}^a_i := w^a_i/w$, one has: $\overline{w}^a_i|\overline{w}_i|\overline{v}_i$ prolong $\overline{v}_i$ to $F^a w^a_i | F w_i | E v_i$ and further, $v_i = \overline{v}_i \circ v$, $w_i = \overline{w}_i \circ w$, $w^a_i = \overline{w}^a_i \circ w^a$ for $i = 1,2$. And one has a commutative diagram of exact sequences:

$$(\ddagger)\qquad \begin{array}{ccccccccc} 1 & \to & T_{w^a|v} & \longrightarrow & Z_{w^a|v} & \xrightarrow{\pi} & G_{w^a|v} & \to & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \to & T_{w^a|v} & \longrightarrow & Z_{w^a_i|v_i} & \xrightarrow{\pi} & Z_{\overline{w}^a_i|\overline{v}_i} & \to & 1, \qquad i = 1,2 \end{array}$$

**Fact 2.7.** *In the above notation, suppose that $\mathcal{V}$ equals its $F$-$\ell$-abelian core, $\mathcal{Z}^a_{\mathcal{V}}(F) \neq 1$, and any two distinct valuations $v_1, v_2 \in \mathcal{V}$ are not comparable. Then for any valuations $v$, $v_1$, $v_2 \in \mathcal{V}$ and $w \in \mathcal{V}_v(F)$, $w_i \in \mathcal{V}_{v_i}(F)$, $w^a_i \in \mathcal{V}_{v_i}(F^a)$, $i = 1,2$, the following hold:*

1) *Suppose that $w_1, w_2$ are not comparable, and set $w := \min(w_1, w_2) < w_1, w_2$. Then one has that $Z_{w^a_1|w_1} \cap Z_{w^a_2|w_2} = T_{w^a|w}$, and in particular, $Z_{w^a_1|w_1} \neq Z_{w^a_2|w_2}$. Therefore,*
   $$\mathcal{V}_v(F) \to \mathcal{Z}^a_v(F)\,,\ w \mapsto Z^a_w \ \textit{ is an isomorphism of topological } G(F|E)\textit{-spaces.}$$
2) *For $g \in G(F|E)$ one has: $g \in Z_{w|v}$ iff $g(Z^a_w) = Z^a_w$.*

*Hence items* 1), 2) *above give a group theoretical recipe to recover the $G(F|E)$-space isomorphism*

$$\mathcal{V}(F) \to \mathscr{Z}_{\mathcal{V}}(F|E) := \big\{ (Z^a_w, Z_{w|v_0}) \,|\, v \in \mathcal{V}(E), w \in \mathcal{V}_v(F) \big\}, \ \ w \mapsto (Z^a_w, Z_{w|v})$$

*from $G(F^a|E) \to G(F|E)$ endowed with $\mathcal{Z}^a_{\mathcal{V}}(F)$.*

*Proof.* We begin by proving the Lemma below, in which $\mathfrak{v}_1, \mathfrak{v}_2$ are abitrary valuations.

**Lemma 2.8.** *Let $N$ be a field with $\mu_\ell \subset N$ provided $\ell \neq \mathrm{char}(N)$ and $\mathcal{G}^a_N := G(N^a|N)$ be the Galois group of the maximal $\ell$-elementary abelian extension $N^a|N$. If $\mathfrak{v}_i \in \mathrm{Val}\,(N)$, $i = 1,2$ are independent valuations, their decomposition groups $Z^a_{\mathfrak{v}_i} \subset \mathcal{G}^a_N$ satisfy $Z^a_{\mathfrak{v}_1} \cap Z^a_{\mathfrak{v}_2} = 1$.*

*Proof of Lemma.* Set $U_i = 1 + \mathfrak{m}_{\mathfrak{v}_i}$, $i = 1,2$. We analyze separately the cases:

Case 1. $\mathrm{char}(N) \neq \ell$. By Hensel Lemma, for all $u_i \in U_i$, $i = 1,2$ one has: $T^\ell - u_i \in N[T]$ splits in linear factors over the Henselization of $N$ with respect to $\mathfrak{v}_i$. Therefore, $\mathfrak{v}_i$ is totally split in $N_i := N[\sqrt[\ell]{U_i}]$, and equivalently, $N_i$ is contained in the fixed field of $Z^a_{\mathfrak{v}_i} \subset \mathcal{G}^a_N$ in $N^a$. On the other hand, since $\mathfrak{v}_1, \mathfrak{v}_2$ are independent, one has $U_1 \cdot U_2 = N^\times$, hence $N^a = N_1 N_2$. Conclude by Kummer theory that $Z^a_{\mathfrak{v}_1} \cap Z^a_{\mathfrak{v}_2} = 1$.

Case 2. $\mathrm{char}(N) = \ell$. By Hensel Lemma, for all $u_i \in \mathfrak{m}_i$, $i = 1,2$ the Artin–Schreier polynomial $T^\ell - T - u_i \in N[T]$ splits in linear factors over the Henselization of $N$ w.r.t. $\mathfrak{v}_i$. Hence $N_i := N[\wp^{-1}(\mathfrak{m}_i)]$ is contained in the fixed field of $Z^a_{\mathfrak{v}_i} \subset \mathcal{G}^a_N$ in $N^a$. On the other hand, since $\mathfrak{v}_1, \mathfrak{v}_2$ are independent, one has $\mathfrak{m}_{\mathfrak{v}_1} + \mathfrak{m}_{\mathfrak{v}_2} = N$, hence $N^a = N_1 N_2$. Conclude by Artin–Schreier theory that $Z^a_{\mathfrak{v}_1} \cap Z^a_{\mathfrak{v}_2} = 1$.

This concludes the proof of Lemma 2.8. Returning to the proof of Fact 2.7, proceed as follows.

To 1): Since $w_1, w_2$ are not comparable, one has $w < w_1, w_2$ strictly, and $\overline{w}_i = w_i/w$ are two independent valuations on the residue field $Fw$. Further, by Fact 2.5, $F^a w^a | Fw$ is the maximal $\ell$-elementary abelian extension of $Fw$, hence $G_{w^a|w} := G(F^a w^a|Fw) = G\big((Fw)^a|Fw\big)$. Further, by the commutative diagram (‡) above, $Z_{\overline{w}^a_i|\overline{w}_i} = Z_{w^a_i|w_i}/T_{w^a|w} \subset G_{w^a|w}$ is the decomposition group of $\overline{w}^a_i|\overline{w}_i$ in $G_{w^a|w} = G\big((Fw)^a|Fw\big)$ for $i = 1,2$. In particular, since $\overline{w}_1, \overline{w}_2$ are independent valuations of $Fw$, it follows that by Lemma 2.8 above that $Z_{\overline{w}^a_1|\overline{w}_1} \cap Z_{\overline{w}^a_2|\overline{w}_2} = 1$. Hence since $T_{w^a|w} = \ker(Z_{w^a_i|w_i} \to Z_{\overline{w}^a_i|\overline{w}_i})$, finally get $Z_{w^a_1|w_1} \cap Z_{w^a_2|w_2} = T_{w^a|w}$.

To 2): By mere definitions one has that $\sigma \in Z_{w|v}$ iff $w^\sigma = w$. First, for the direct implication, if $\sigma \in Z_{w|v}$, then $w = w^\sigma$, thus $Z^a_w = Z^a_{w^\sigma} = \sigma(Z^a_w)$. For the converse implication, suppose that $w_1 := w \neq w^\sigma =: w_2$. Then by assertion 1) above one has $Z^a_{w_1} \neq Z^a_{w_1}$, that is $Z^a_w \neq Z^a_{w^\sigma}$.

Finally, the last assertion is an immediate consequence of the discussion above. □

2.3. **Commuting liftability.** See Topaz [To1] (and Pop[P1], section 3) for more details.

Let $F$ be a field with $\mathrm{char}(F) \neq \ell$, $\mu_\ell \subset F$, and $F^a|F$ be the maximal $\mathbb{Z}/\ell$ elementary abelian extension. For a valuation $w$ of $F$, set $F^D := F[\sqrt[\ell]{1+\mathfrak{m}_w}\,]$, $F^I := F[\sqrt[\ell]{\mathcal{O}^\times_w}\,]$. The groups $I_w \leqslant D_w$ below are called the *minimized inertia* and *decomposition groups* of $w$:

$$I_w := G(F^a|F^I) = \mathrm{Hom}(F^\times/\mathcal{O}^\times_w, \mu_\ell) \leqslant \mathrm{Hom}\,\big(F^\times/(1+\mathfrak{m}_w), \mu_\ell\big) = G(F^a|F^D) =: D_w.$$

We notice that the minimized inertia/decomposition groups behave under valued field extension as follows. Let $(F,w)\,|\,(N,\mathfrak{w})$ is an extension of valued fields, $\mu_\ell \subset N$, thus $\mathcal{O}^\times_{\mathfrak{w}} = \mathcal{O}^\times_w \cap N$ and $1+\mathfrak{m}_{\mathfrak{w}} = (1+\mathfrak{m}_w) \cap N$. Then by mere definitions one has:

**Fact 2.9 (Functoriality).** *The canonical projection $p^a : \mathcal{G}^a_F \to \mathcal{G}^a_N$ gives rise canonically to embeddings $p^a(I_w) \subset I_{\mathfrak{w}}$ and $p^a(D_w) \subset D_{\mathfrak{w}}$. Moreover, if $F$ and $N^a$ are linearly disjoint over $N$, i.e., $F \cap N^a = N$, then $p^a(I_w) = I_{\mathfrak{w}}$ and $p^a(D_w) = D_{\mathfrak{w}}$.*

**Fact 2.10.** *In the above notation, the following hold:*

1) $I_w \cong \mathrm{Hom}(wF/\ell, \mu_\ell)$ *and* $D_w/I_w \cong \mathrm{Hom}(Fw^\times/\ell, \mu_\ell)$. *Hence one has:*

$$I_w = 1 \text{ iff } wF \text{ is } \ell\text{-divisible, and } I_w = D_w \text{ iff } Fw^\times \text{ is } \ell\text{-divisible.}$$

2) *If* $\mathrm{char}(Fw) \neq \ell$, *then* $T^a_w = I_v \subset D_w = Z^a_w$. *Further,* $(Fw)^a = F^a w^a$, *thus* $\mathcal{G}^a_{Fw} = Z^a_w/T^a_w$.

3) *If* $\mathrm{char}(Fw) = \ell$, *then* $I_w \subset T^a_w$ *and* $D_w \subset Z^a_w$.

*Proof.* Everything follows by mere definitions, Pontryagin duality, and Kummer theory from the exact sequences $1 \to \mathcal{O}^\times_w \to F^\times \to wF \to 0$ and $1 \to (1+\mathfrak{m}_w) \to \mathcal{O}^\times_w \to Fw^\times \to 1$. □

In the above notation, for $\sigma \in \mathcal{G}_F^a$, let $\tilde{\sigma} \in \mathcal{G}_F^c$ denote preimages of $\sigma$, and for $\Sigma \subset \mathcal{G}_F^a$, let $\tilde{\Sigma} \subset \mathcal{G}_F^c$ denote the preimage of $\Sigma$. Recall the following canonical maps in this context:

- The bilinear map $\psi : \mathcal{G}_F^a \times \mathcal{G}_F^a \to \Delta_F$, defined by $(\sigma, \tau) \mapsto [\tilde{\sigma}, \tilde{\tau}]$.
- The linear map $\beta : \mathcal{G}_F^a \to \Delta_F$, $\sigma \mapsto \sigma^\beta := \tilde{\sigma}^\ell$.

**Definition/Remarks 2.11.** We next recall basics about *commuting liftability,* see Topaz [To1] for details. First $\sigma, \tau \in \mathcal{G}_F^a$ are called *independent,* if $\langle \sigma, \tau \rangle \cong (\mathbb{Z}/\ell)^2$. We say that / define:

1) Independent $\sigma, \tau$ are *commuting liftable* (c.l.) if $\sigma, \tau$ satisfy the equivalent conditions:

   (i) $\exists\ \tilde{\sigma}, \tilde{\tau}$ such that $[\tilde{\sigma}, \tilde{\tau}] \in \langle \sigma^\beta, \tau^\beta \rangle$; (ii) $\forall\ \tilde{\sigma}, \tilde{\tau}$ one has $[\tilde{\sigma}, \tilde{\tau}] \in \langle \tilde{\sigma}^\beta, \tilde{\tau}^\beta \rangle$.

2) An independent pair $\sigma, \tau \in \mathcal{G}_F^a$ is called *c.l. pair,* if $\sigma, \tau$ satisfy the equivalent conditions:

   (i) $\exists\ \tilde{\sigma}, \tilde{\tau}$ such that $[\tilde{\sigma}, \tilde{\tau}] \in \langle \sigma^\beta \rangle$; (ii) $\forall\ \tilde{\sigma}, \tilde{\tau}$ one has $[\tilde{\sigma}, \tilde{\tau}] \in \langle \tilde{\sigma}^\beta \rangle$.

   **Note** the following: Let $\sigma, \tau \in \mathcal{G}_F^a$ be independent and c.l. Then the following hold:

   a) If $\sigma_1, \tau_1 \in \langle \sigma, \tau \rangle$ are independent, then $\langle \sigma, \tau \rangle = \langle \sigma_1, \tau_1 \rangle$, and $\sigma_1, \tau_1$ is c.l.
   b) There exists $1 \neq \sigma_1 \in \langle \sigma, \tau \rangle$ such that $[\sigma_1, \tau_1] \in \langle \sigma_1^\beta \rangle$ for all $\tau_1 \in \langle \sigma, \tau \rangle$.
   c) For $k \in \mathbb{Z}$ with $(k, \ell) = 1$ one has: $\sigma^k, \tau^k$ are c.l. (pair, provided $\sigma, \tau$ is c.l. pair).
   d) One has: $\sigma, \tau$ and $\tau, \sigma$ are both c.l. pairs if and only if $[\tilde{\sigma}, \tilde{\tau}] = 1$.

3) Subgroups $I \leqslant D$ of $\mathcal{G}_F^a$ is a *c.l. (group) pair,* if $I \neq 1$, $D$ is non-cyclic, and all independent pairs $\sigma, \tau$ with $\sigma \in I$, $\tau \in D$ define c.l. pairs.
   In particular, if $\sigma, \tau \in I$ are independent, then by item d) above, one has that $[\tilde{\sigma}, \tilde{\tau}] = 1$.
   **Note** that if $\sigma, \tau \in \mathcal{G}_F^a$ define a c.l. pair, then in the notation from 2), b) above, one has:

$$I := \langle \sigma_1 \rangle \leqslant \langle \sigma, \tau \rangle := D \text{ is c.l. pair}.$$

4) For $I \leqslant D$ c.l. pair, the following hold:

   a) There exists a unique maximal $I_D \subset \mathcal{G}_F^a$ such that $I_D \leqslant D I_D$ is c.l. pair, hence $I \subset I_D$.
   b) There exists a unique maximal $D_I \subset \mathcal{G}_F^a$ such that $I \leqslant D_I$ is c.l. pair, hence $D \subset D_I$.

- Finally, a c.l. pair $I \leqslant D$ is called *maximal,* if $I = I_D$, $D = D_I$. We notice the following:
  Starting with a c.l. pair $I \leqslant D$, one has: $I_D \leqslant D_{I_D}$ and $I_{D_I} \leqslant D_I$ are maximal.

5) Let $\phi^a \in \mathrm{Aut}(\mathcal{G}_F^a)$ be the automorphism which lifts to an automorphism $\phi^c \in \mathrm{Aut}(\mathcal{G}_F^c)$. Then for every pair of subgroups $I \subset D \subset \mathcal{G}_F^a$ one has:

   a) $I \leqslant D$ is a maximal c.l. pair in $\mathcal{G}_F^a$ iff $\phi(I \leqslant D) := \big(\phi(I) \leqslant \phi(D)\big)$ is so.
   b) If $I \leqslant D$ is a maximal c.l. pair, then $\phi(I \leqslant D) = (I \leqslant D)$ iff $\phi(I) = I$ iff $\phi(D) = D$.

   (Indeed, $I \leqslant D$ is a maximal c.l. pair iff $\phi(I) \leqslant \phi(D)$ is a maximal c.l. pair, etc.)

The essential property of commuting liftability is that it is related in an intimate way to (arithmetically significant) valuations of $F$, see Theorem 2.15 below. But first recall the following basic fact, see e.g. the discussion in Pop [P1], Section 3, and Topaz [To1] for details:

**Fact 2.12.** *In the above notation, suppose that $wF$ not $\ell$-divisible, and $F^\times/(1 + \mathfrak{m}_w)$ non-cyclic, or equivalently, $I_w \neq 1$ and $D_w$ non-cyclic. The following hold:*

1) *$I_w \leqslant D_w$ is c.l., hence $I_w \leqslant I_{D_w}$ and $D_w \leqslant D_{I_w}$.*
2) *Moreover, if $w$ has rank one, then $I_{D_w} \leqslant D_w$ is a maximal c.l. pair. In particular, in this case, every group automorphism of $D_w$ defined by some $\sigma \in \mathcal{G}_F^a$ maps $I_{D_w}$ into itself.*

By work of Ware, Jacob, Arason–Elman–Jacob, Bogomolov, Koenigsmann, Bogomolov–Tschinkel, culminating with contributions of Topaz, [To1], where more literature can be found, one has the following fundamental facts. Recall that the set of (equivalence classes of) valuations of $F$ is partially ordered by $w_1 \leqslant w_2$ if the following equivalent conditions are satisfied:

$$\text{(i)}\ \mathcal{O}_{w_1} \supset \mathcal{O}_{w_2}\,; \quad \text{(i)}'\ \ \mathfrak{m}_{w_1} \subset \mathfrak{m}_{w_2}\,; \quad \text{(ii)}\ \mathcal{O}^{\times}_{w_1} \supset \mathcal{O}^{\times}_{w_2}\,; \quad \text{(ii)}'\ \ 1+\mathfrak{m}_{w_1} \subset 1+\mathfrak{m}_{w_2}.$$

Equivalently, the kernel $\Delta_{w_2/w_1} := \ker\big(w_2F = F^{\times}/\mathcal{O}^{\times}_{w_2} \twoheadrightarrow F^{\times}/\mathcal{O}^{\times}_{w_1} = w_1F\big)$ is a convex subgroup of $w_1F$, which is equals actually value group of $w_0 = w_2/w_1 \in \mathrm{Val}\,(Fw_1)$. In particular, one has the following obvious facts on the behavior of reduced inertia/decomposition groups:

**Fact 2.13.** *In the above context, let $w_1 \leqslant w_2$ be as above. Then $I_{w_1} \subset I_{w_2}$ and $D_{w_1} \supset D_{w_2}$.*

**Notations/Remark 2.14** (cf. Topaz [To1], §1.2 for some details)**.** In the above context, consider the following:

1) Let $\mathcal{W}_F$ be the set of valuations $w \in \mathrm{Val}\,(F)$ which for all $w_1 \in \mathrm{Val}(F)$ ***satisfy***:
   (i) Let $w_1 < w$ strictly. Then the value group of $w/w_1$ is not $\ell$-divisible, i.e., $I_{w/w_1} \neq 1$.
   (ii) Let $w < w_2$ strictly. Then $D_{w_2} = D_w$ implies $I_{w_2} = I_w$, i.e., $I_{w_2/w} = 1$.
- Notice that *every $w \in \mathcal{W}_F$ equals its $F$-$\ell$-abelian core.* Indeed, if $w \in \mathcal{W}_F$ and $w_1 < w$ strictly, then $I_{w/w_1} \neq 1$, implying that $F^a w_1|Fw_1$ is not purely inseparable.

2) Let $\mathcal{P}_F$ be the set of ***maximal*** c.l. pairs $I \leqslant D$ in $\mathcal{G}^a_F$ with $I \neq 1$, $D$ not cyclic, and denote:
$$\mathcal{I}_F := \{I \subset \mathcal{G}^a_F \mid \exists\, I \leqslant D \text{ in } \mathcal{D}_F\}, \quad \mathcal{D}_F := \{D \subset \mathcal{G}^a_F \mid \exists\, I \leqslant D \text{ in } \mathcal{D}_F\}.$$
- Notice that given $I \leqslant D$ in $\mathcal{P}_F$, *each $I$ and $D$ individually determine the c.l. pair $I \leqslant D$.* Indeed, by by Definition/Remarks 2.11, 4), one has both $D = D_I$ and $I = I_D$.
- In particular, both projection maps $\mathcal{P}_F \to \mathcal{I}_F, \mathcal{D}_F$, $I \leqslant D \mapsto I, D$ are bijective.

**Theorem 2.15** (cf. Topaz [To1], Thm 1, (1) & Thm 6, for $N = n = 1 = \mathbf{R}(1)$)**.** *The following hold:*

1) *For $w$ in $\mathcal{W}_F$, there is $I \leqslant D$ in $\mathcal{P}_F$ such that $D = D_w$, $1 \neq I_w \subset I$, and if so, $I/I_w$ is cyclic. Moreover, if $Fw^{\times}/\ell$ is not cyclic, then $D = D_w$, $I = I_w$.*
2) *For $I \leqslant D$ in $\mathcal{P}_F$, there is $w \in \mathcal{W}_F$ satisfying the condition from 1) above.*

**Remark 2.16.** Note that in Theorem 2.15 above, both $w$ in $\mathcal{W}_F$ and $I \leqslant D$ in $\mathcal{P}_F$ are *unique corresponding to each other.* **Notation**: $w \rightsquigarrow (I \leqslant D)^w \rightsquigarrow I^w, D^w$, resp. $I \leqslant D \rightsquigarrow I, D \rightsquigarrow w^I, w^D$.

Uniqueness of $I \leqslant D$: Let $w \rightsquigarrow I_i \leqslant D_i$, $i = 1,2$. Then $D_i = D_w$ and $I_i \leqslant D_w$, $i = 1,2$ are both c.l. pairs, hence so is $I_1 I_2 \leqslant D_w$. And $I_i \leqslant D_w$ being maximal implies $I_1 = I_1 I_2 = I_2$.

Uniqueness of $w$: By contradiction, let $(I \leqslant D) \rightsquigarrow w_1, w_2$, $w_1 \neq w_2$. Then $D_{w_i} = D$, $I_{w_i} \subset I$, and $w_1, w_2$ are not comparable by Notations/Remark 2.14, 1). Set $w_0 = \min(w_1, w_2)$, hence and $\overline{w}_i = w_i/w_0$ are non-trivial. Thus letting $\pi : Z^a_{w_0} \to G_{w^a_0|w_0}$ be the canonical projection, one has $1 \neq I_{\overline{w}_i} = I_{w_i}/I_{w_0} = \pi(I_{w_i})$, $i = 1,2$. On the other hand, one has

$$1 \neq I_{\overline{w}_i} \subset \pi(I) \subset \pi(D) = D_{\overline{w}_i} \subset Z^a_{\overline{w}_i} \subset G_{w^a_0|w_0}\,, \text{ hence } 1 \neq \pi(I) \subset Z^a_{\overline{w}_1} \cap Z^a_{\overline{w}_2}.$$

Since $\overline{w}_1, \overline{w}_2$ are independent, this is a contradiction by Fact 2.7.

2.4. **Commuting liftability and Galois action.** In the above context, let $F|E$ be a Galois extension with $\mu_\ell \subset F$ and Galois group $G(F|E)$. Recall that $G(F|E)$ acts on $\mathcal{W}_F$ by $g(w) = w \circ g^{-1}$, $g \in G(F|E)$ and on the spaces $\mathcal{P}_F, \mathcal{D}_F, \mathcal{I}_F$ by conjugation $g(I \leqslant D) = \big(g(I) \leqslant g(D)\big)$.

Further, the $G(F|E)$-actions are compatible with the previous constructions and introduced objects in the following sense:

- If $g(w) = w$, then $g\big(I_{\mathfrak{w}} \leqslant D_{\mathfrak{w}}\big) = \big(g(I_{\mathfrak{w}}) \leqslant g(D_{\mathfrak{w}})\big)$ and $g\big(I_{D_{\mathfrak{w}}} \leqslant D_{\mathfrak{w}}\big) = \big(g(I_{D_{\mathfrak{w}}}) \leqslant g(D_{\mathfrak{w}})\big)$.
- If $I \leqslant D \rightsquigarrow w$, then $g(I \leqslant D) = \big(g(I) \leqslant g(D)\big) \rightsquigarrow g(w)$.

Further, by mere definitions one has that $D_w \lhd Z_{w^a|v}$ and $I_w,\ I_{D_w} \lhd Z_{w^a|v}$. In particular, there is a unique maximal (normal) subgroup $D_{w|v} \lhd Z_{w^a|v}$ satisfying the following two conditions:

$$\text{(i)}\ \ D_{w|v} \cap Z^a_w = D_w\,; \qquad \text{(ii)}\ \ D_{w|v}/D_w = Z_{w|v}\,.$$

And obviously, $I_w, I_{D_w} \lhd D_{w|v}$ and $D_{w|v}$ fits in the exact sequence

$$1 \to D_w \to D_{w|v} \to Z_{w|v} \to 1\,,$$

which is in an obvious way a subsequence of $1 \to Z^a_w \to Z_{w^a|v} \to Z_{w|v} \to 1$. Further, if $\mathrm{char}(Fw) \neq \ell$, then $D_{w|v} = Z_{w^a|v}$ and $D_w = Z^a_w,\ I_w = T^a_w$.

**Definition/Notations 2.17.** In the above notation, we define and consider notation as follows:

1) We say that $D_{w|v}$ is the (relative) minimized decomposition group of $w^a|v$ in $F^a|E$.
2) Recalling Remark/Notation 2.14, we denote:
   a) $\mathcal{W}_{F|E} := \big\{ w|v \,\big|\, w \in \mathcal{W}_F,\ v = w|_E \big\}$.
   b) $\mathcal{D}_{F|E} := \big\{ D_{w|v} \subset G(F^a|E) \,\big|\, w|v \in \mathcal{W}_{F|E} \big\}$.

In the above notation and context, let $F'|F|E$ be Galois extensions with $F'|F$ finite and $\mu_\ell \subset F$, and $pr : G(F'|E) \to G(F|E)$ be the projection of Galois groups. For $I \leqslant D$ from $\mathcal{P}_F$ and $I \leqslant D \rightsquigarrow w \in \mathcal{W}_F$ relating to each other as in Theorem 2.15, set $v := w|_E$, thus $w|v \in \mathcal{W}_{F|E}$, and let $w'|w$ be a prolongation of $w$ to $F''|F$.

**Proposition 2.18.** *In the above notation, the following hold:*

1) **(Fact 2.7 revisited)**. *For $g \in G(F|E)$ and $I \leqslant D \rightsquigarrow w|v \rightsquigarrow D_{w|v}$, the following hold:*

$$g(I) = I \ \text{ iff }\ g(D) = D \ \text{ iff }\ g(D_{w|v}) = D_{w|v} \ \text{ iff }\ g(Z^a_w) = Z^a_w \ \text{ iff }\ g(w) = w \ \text{ iff }\ g \in Z_{w|v}.$$

2) **(Galois action)**. *$\mathcal{W}_F$, $\mathcal{W}_{F|E}$, $\mathcal{D}_{F|E}$, $\mathcal{P}_F$, $\mathcal{D}_F$, $\mathcal{I}_F$ are $G(F|E)$-spaces, and the maps*

$$\mathcal{W}_F \to \mathcal{W}_{F|E} \to \mathcal{D}_{F|E} \to \mathcal{P}_F \to \mathcal{D}_F, \mathcal{I}_F \quad w \mapsto w|v \mapsto D_{w|v} \mapsto I_D \leqslant D \mapsto I_D, D$$

*are $G(F|E)$-isomorphisms, where the last two projections are as defined in Remark/Notation 2.14, 2).*

*Proof.* To 1): First, $I \leqslant D \in \mathcal{P}_F$ and $w \in \mathcal{W}_F$ relate to each other iff $D = D_w$ and $I = I_{D_w}$. Next, by Remark 2.16, since $w \in \mathcal{W}_F$ equals its $F$-$\ell$-abelian core, the last three equivalences follow from Fact 2.7. Further, $g(w) = w$ iff $g(\mathcal{O}^\times_w) = \mathcal{O}^\times_w$ iff $g(1 + \mathfrak{m}_w) = 1 + \mathfrak{m}_w$. Hence by the definitions of $I_w \subset D_w$ and Kummer theory one has: $g(w) = w \Rightarrow g(F^I) = F^I,\ g(F^D) = F^D$, and therefore, $g(w) = w \Rightarrow g(I_w) = I_w,\ g(D_w) = D_w$. And further, by mere definitions, this implies $g(I_{D_w}) = I_{D_w}$. Hence it is left to show that $g(I) = I$ and/or $g(D) = D$ implies $g(w) = w$. First, since both $I$ and $D$ individually define $I \leqslant D$ uniquely, it is sufficient to prove one of the assertions, e.g., that $g(I) = I$ implies $g(w) = w$. This is more-or-less a reformulation of the last part of the proof of the Remark 2.16 above, along the following lines: First, we notice that $\mathcal{W}_F$ is invariant under automorphisms of $F$ (by mere definitions). Hence $w_1 := w \in \mathcal{W}_F$ iff $w_2 := g(w) \in \mathcal{W}_F$. And if so, by mere definitions on has $D_{w_2} = g(D) = D = D_{w_1}$, hence $I_{D_{w_2}} = g(I) = I = I_{D_{w_1}}$. Conclude that $w_2 = w_1$ by arguing as at the end of Remark 2.16.

To 2): Recall that by Remark 2.16, for $I \in \mathcal{I}_F$ given, there is a unique $D \subset \mathcal{G}^a_F$ with $I \leqslant D$ in $\mathcal{P}_F$. Hence the stabilizer $\mathrm{St}_{G(F|E}(I)$ of $I$ in $G(F|E)$ stabilizes $D$, i.e., stabilizes $I \leqslant D$. Since $w \in \mathcal{W}_F$ with $D = D_w$ is unique, we conclude: $\mathrm{St}_{G(F|E}(I) = \mathrm{St}_{G(F|E)}(w) = Z_{w|v}$ for the unique $w|v \in \mathcal{W}_{F|E}$ with $D_w = D$, $I_w \subset I$ and $I/I_w$ cyclic. Similarly, starting with $w|v \in \mathcal{W}_{F|E}$ and setting $D = D_{w|v} \cap \mathcal{G}^a_F \in \mathcal{I}_F$, it follows that $\mathrm{St}_{G(F|E)} = Z_{w|v}$, etc.

□

This being said, we notice though that Theorem 2.15 and Fact 2.18 above do not give conditions to ensure that the valuation $w$ has $\mathrm{char}(Fw) \neq \ell$. In the next section we discuss —among other things— this issue, which is essential for the proof of the main results of the paper.

## 3. Commuting liftability, field extensions, and sections

Let $E|L$ be a regular field extension, $\tilde{L}|L$ be a Galois extension, and $\tilde{E} := E\tilde{L}$ be the compositum of $E$ and $\tilde{L}$ over $L$ (which is well defined up to $L$-isomorphism, because $E|L$ was a regular field extension). In particular, $\tilde{E}|E$ is Galois such that the canonical projection map $\tilde{\imath} : G(\tilde{E}|E) \to G(\tilde{L}|L)$ is an isomorphism. For valuations $v \in \mathrm{Val}(E)$, let $\tilde{v}|v$ denote prolongations of $v$ to $\tilde{E}|E$, and $v_L := v|_L$ and $\tilde{v}_L := \tilde{v}|_{\tilde{L}}$ be the corresponding restrictions, thus in particular, $v_{\tilde{L}} = (\tilde{v}_{\tilde{L}})|_L$. Next suppose that $\mathrm{char}(L) \neq \ell$ and $\mu_\ell \subset \tilde{L}$. Recall that $\tilde{L}^c|\tilde{L}^a|\tilde{L}|L$ and $\tilde{E}^c|\tilde{E}^a|\tilde{E}|E$ are Galois extensions, and one has a commutative diagram with exact rows and surjective vertical morphisms, $\tilde{\imath} : G(\tilde{E}|E) \to G(\tilde{L}|L)$ being an isomorphism, where $\bullet$ stays for either $a$ or $c$ (similar to the ones in the Introduction):

$$(*)_{\tilde{E}|\tilde{L}} \qquad \begin{array}{ccccccccc} 1 & \to & \mathcal{G}^\bullet_{\tilde{E}} & \hookrightarrow & G(\tilde{E}^\bullet|E) & \twoheadrightarrow & G(\tilde{E}|E) & \to & 1 \\ & & \downarrow\!\!\!\downarrow \tilde{p}^\bullet & & \downarrow\!\!\!\downarrow p^\bullet & & \downarrow \tilde{\imath} & & \\ 1 & \to & \mathcal{G}^\bullet_{\tilde{L}} & \hookrightarrow & G(\tilde{L}^\bullet|L) & \twoheadrightarrow & G(\tilde{L}|L) & \to & 1 \end{array}$$

Next let $s^a : G(\tilde{L}^a|L) \to G(\tilde{E}^a|E)$ be a section of $p^a : G(\tilde{E}^a|E) \to G(\tilde{L}^a|L)$, which is a.b.c.-liftable, i.e., $s^a$ lifts to a section $s^c$ of $p^c : G(\tilde{E}^c|E) \to G(\tilde{L}^c|L)$. Consider the diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{G}^\bullet_{\tilde{E}} & \longrightarrow & G(\tilde{E}^\bullet|E) & \xrightarrow{p_E} & G(\tilde{E}|E) & \longrightarrow & 1 \\ & & \uparrow \downarrow\!\!\!\downarrow \tilde{p}^\bullet & & s^\bullet \uparrow \downarrow\!\!\!\downarrow p^\bullet & & \downarrow \tilde{\imath} & & \\ 1 & \longrightarrow & \mathcal{G}^\bullet_{\tilde{L}} & \longrightarrow & G(\tilde{L}^\bullet|L) & \xrightarrow{p_L} & G(\tilde{L}|L) & \longrightarrow & 1 \end{array}$$

**Claim.** *The restriction $s^\bullet|_{\mathcal{G}^\bullet_{\tilde{L}}}$ is a section of $\tilde{p}^\bullet$ and $p_E \circ s^\bullet \circ p_L^{-1}$ is the inverse map of $\tilde{\imath}$.*

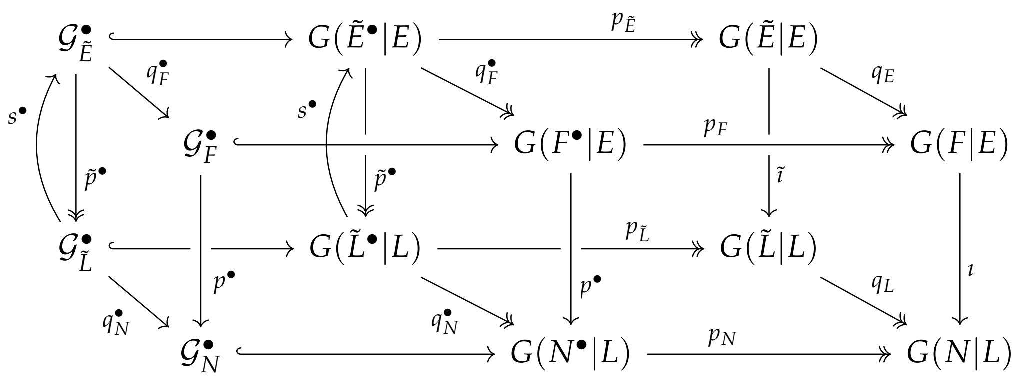
Indeed, let $\mathrm{im}(s^\bullet) \subset G(\tilde{E}^\bullet|E)$ be the image of $s^\bullet$. Then $\tilde{\imath} \circ p_E = p_L \circ \tilde{p}^\bullet$ and $\tilde{\imath}$ being an isomorphism implies that $p_E\big(s^\bullet(g)\big) = 1$ iff $p_L(g) = 1$. Hence $s^\bullet(g) \in \mathcal{G}^\bullet_{\tilde{E}}$ iff $p_E\big(s^\bullet(g)\big) = 1$ iff $p_E(g) = 1$ iff $g \in \mathcal{G}^\bullet_{\tilde{L}}$, concluding that $s^\bullet$ is a section of $pr^\bullet$. For the last assertions, one has: $h = p_E(g) \in G(\tilde{L}|L)$ iff $\tilde{\imath}^{-1}(h) = p_E \circ s^\bullet(g)$. Thus the Claim is proved. (Note that $p_L^{-1}$ is a multi-valued correspondence, but $p_E \circ s^\bullet \circ p_L^{-1}$ is indeed single valued, hence a map.)

### 3.1. $s^\bullet$-valuations arising from $\mathrm{Val}^1(L)$.

In the above context, let $N|L \hookrightarrow \tilde{L}|L$ be a *finite* Galois subextension of $\tilde{L}|L$ with $\mu_\ell \subset N$ and setting $F = NE$, consider $F|E \hookrightarrow \tilde{E}|E$ and the resulting projections of Galois groups

$q_E^\bullet : G(\tilde{E}^\bullet|E) \twoheadrightarrow G(F^\bullet|E)$, $q_F^\bullet : G(\tilde{E}^\bullet|F) \twoheadrightarrow G(F^\bullet|F)$ and $q_L^a : G(\tilde{L}^\bullet|L) \twoheadrightarrow G(N^\bullet|L)$, $q_N^a : G(\tilde{L}^\bullet|L) \twoheadrightarrow G(N^\bullet|N)$, and finally $p^\bullet : G(F^\bullet|E) \twoheadrightarrow G(N^\bullet|L)$, hence $p^\bullet : \mathcal{G}_F^\bullet \twoheadrightarrow \mathcal{G}_N^\bullet$.

In particular, given the a.b.c.-liftable section $s^a : G(\tilde{L}^a|L) \to G(\tilde{E}^a|L)$, for $\bullet = a, c$ one has a commutative diagrams of the form:




in which all maps are the canonical projections and $\tilde{\imath}$ and $\imath$ are isomorphisms.

Finally, let $L_1|L \hookrightarrow N|L \hookrightarrow \tilde{L}|L$ be a finite Galois subextensions of $\tilde{L}|L$ such that $\mu_\ell \subset L_1$, and for $\mathfrak{v} \in \mathrm{Val}(L)$, denote by $\tilde{\mathfrak{v}}|\mathfrak{w}|\mathfrak{v}_1|\mathfrak{v}$ the prolongations of $\mathfrak{v}$ to $\tilde{L}|N|L_1|L$.

**Notations/Remark 3.1.** . Let $\mathrm{Val}^1(L) \subset \mathrm{Val}(L)$ be the set of valuations $\mathfrak{v} \in \mathrm{Val}(L)$ satisfying:

(i) $\mathfrak{v}L = \mathbb{Z}$; (ii) $\mathrm{char}(L\mathfrak{v}) \neq \ell$; (iii) $L_1\mathfrak{v}_1{}^\times/\ell \neq 1$; (iv) $\tilde{\mathfrak{v}}|\mathfrak{v}$ are unramified.

We notice that if $\mathfrak{v} \in \mathrm{Val}^1(L)$, then $\mathfrak{w}N = \mathbb{Z} = \mathfrak{v}L$ and $N\mathfrak{w}^\times/\ell \neq 1$ for all $N|L$ as above. [*Proof.* Since $[N\mathfrak{w} : L_1\mathfrak{v}_1] \leqslant [N : L_1] < \infty$, by basic Galois theory, $L_1\mathfrak{v}_1{}^\times/\ell \neq 1$ implies $N\mathfrak{w}^\times/\ell \neq 1$ (because $\ell > 2$). Since $\tilde{\mathfrak{v}}|\mathfrak{v}$ unramified, $\mathfrak{v}L \subset \mathfrak{w}N \subset \tilde{\mathfrak{v}}\tilde{L} = \mathfrak{v}L$, thus $\mathfrak{v}L = \mathfrak{w}N$.]

For $\mathfrak{v} \in \mathrm{Val}^1(L)$, $\tilde{L}|N|L$ as above, let $\tilde{\mathfrak{v}}^\bullet|\mathfrak{w}^\bullet|\mathfrak{v}$ be the prolongations of $\tilde{\mathfrak{v}}|\mathfrak{w}|\mathfrak{v}$ to $\tilde{L}^\bullet|N^\bullet|L$. Since $\mathrm{char}(\tilde{L}\tilde{\mathfrak{v}}) = \mathrm{char}(N\mathfrak{w}) = \mathrm{char}(L\mathfrak{v}) \neq \ell$ and $\mathbb{Z} = \mathfrak{v}L = \mathfrak{w}N = \tilde{\mathfrak{v}}\tilde{L}$, by functoriality and basics of Hilbert decomposition theory, the following hold:

1) $T_{\tilde{\mathfrak{v}}^a|\mathfrak{v}} = T_{\tilde{\mathfrak{v}}^a|\tilde{\mathfrak{v}}} \cong \mathbb{Z}/\ell \cong T_{\mathfrak{w}^a|\mathfrak{w}} = T_{\mathfrak{w}^a|\mathfrak{v}}$ and $T_{\tilde{\mathfrak{v}}^c|\mathfrak{v}} = T_{\tilde{\mathfrak{v}}^c|\tilde{\mathfrak{v}}} \cong \mathbb{Z}/\ell^2 \cong T_{\mathfrak{w}^c|\mathfrak{w}} = T_{\mathfrak{w}^c|\mathfrak{v}}$. Further, denoting by $G_{\tilde{\mathfrak{v}}^\bullet|\mathfrak{v}}$ and $G_{\mathfrak{w}^\bullet|\mathfrak{v}}$ the Galois group of the residue field extensions $\tilde{L}^\bullet\tilde{\mathfrak{v}}^\bullet|L\mathfrak{v}$, respectively $N^\bullet\mathfrak{w}^\bullet|L\mathfrak{v}$, one has $Z_{\tilde{\mathfrak{v}}^\bullet|\mathfrak{v}} = T_{\tilde{\mathfrak{v}}^\bullet|\mathfrak{v}} \rtimes G_{\tilde{\mathfrak{v}}^\bullet|\mathfrak{v}}$ and $Z_{\mathfrak{w}^\bullet|\mathfrak{v}} = T_{\mathfrak{w}^\bullet|\mathfrak{v}} \rtimes G_{\mathfrak{w}^\bullet|\mathfrak{v}}$, the action being in both cases by the $\ell$-adic cyclotomic character of $G_{\tilde{\mathfrak{v}}^\bullet|\mathfrak{v}}$, respectively $G_{\mathfrak{w}^\bullet|\mathfrak{v}}$.

2) $q_L^\bullet : G(\tilde{L}^\bullet|L) \to G(N^\bullet|L)$ maps $T_{\tilde{\mathfrak{v}}^\bullet|\mathfrak{v}}$ isomorphically onto $T_{\mathfrak{w}^\bullet|\mathfrak{v}}$ and defines a surjective morphism of the residue Galois groups $q^\bullet_{\tilde{\mathfrak{v}}|\mathfrak{w}|\mathfrak{v}} : G_{\tilde{\mathfrak{v}}^\bullet|\mathfrak{v}} \twoheadrightarrow G_{\mathfrak{w}^\bullet|\mathfrak{v}}$ which is obviously compatible with the $\ell$-adic characters. Finally, the restriction $q_L^\bullet : Z_{\tilde{\mathfrak{v}}^\bullet|\mathfrak{v}} \twoheadrightarrow Z_{\mathfrak{w}^\bullet|\mathfrak{v}}$ is defined canonically by its restrictions to the inertia groups and the residue Galois groups.

Conclude: First, $\mu_\ell \subset N \Rightarrow G_{\tilde{\mathfrak{v}}^a|\mathfrak{w}}$ and $G_{\mathfrak{w}^a|\mathfrak{w}}$ act trivially on $T_{\tilde{\mathfrak{v}}^a|\mathfrak{w}} \cong \mathbb{Z}/\ell \cong T_{\mathfrak{w}^a|\mathfrak{w}}$. Hence $Z_{\tilde{\mathfrak{v}}^a|\mathfrak{w}}$ and $Z_{\mathfrak{w}^a|\mathfrak{w}}$ are abelian and $I_\mathfrak{w} = T_{\mathfrak{w}^a|\mathfrak{w}} \leqslant Z_{\mathfrak{w}^a|\mathfrak{w}} = D_\mathfrak{w}$ is a c.l. pair in $G(N^a|N)$. Second, for $\tilde{\sigma} \in T_{\tilde{\mathfrak{v}}^c|\mathfrak{w}}$ and $\tilde{\tau} \in Z_{\tilde{\mathfrak{v}}^c|\mathfrak{w}}$ one has $[\tilde{\sigma}, \tilde{\tau}] \in \langle \tilde{\sigma}^\ell \rangle$. Hence by abuse of language, we will say:

**Terminology**. *$T_{\tilde{\mathfrak{v}}^a|\mathfrak{w}} \leqslant Z_{\tilde{\mathfrak{v}}^a|\mathfrak{w}}$ is a **generalized-commuting c.l. pair** of subgroups of $G(\tilde{L}^a|N)$.*

**Key Lemma 3.2.** *In the above notation/context, set $I^a := q_F^a\big(s^a(T_{\tilde{\mathfrak{v}}^a|\mathfrak{w}})\big)$, $D^a := q_F^a\big(s^a(Z_{\tilde{\mathfrak{v}}^a|\mathfrak{w}})\big)$. Then $I^a \leqslant D^a$ is a c.l. pair in $\mathcal{G}_F^a \subset G(F^a|E)$ which is mapped by $p^a : G(F^a|E) \to G(N^a|L)$ onto* *the c.l. pair* $I_{\mathfrak{w}} = T_{\mathfrak{w}^a|\mathfrak{w}} \leqslant Z_{\mathfrak{w}^a|\mathfrak{w}} = D_{\mathfrak{w}}$ *in* $\mathcal{G}_N^a \subset G(N^a|L)$. *And the same holds, correspondingly, for the maximal c.l. pair* $I_{D_{\mathfrak{w}}} \leqslant D_{\mathfrak{w}}$ *in* $\mathcal{G}_N^a$.

*Proof.* In the above notation, recalling the remarks at 1) above, one has: The canonical projection $G(\tilde{E}^c|F) \twoheadrightarrow G(\tilde{E}^a|F)$ maps the subgroups $s^c(T_{\tilde{\mathfrak{v}}^c|\mathfrak{w}}) \leqslant s^c(Z_{\tilde{\mathfrak{v}}^c|\mathfrak{w}})$ of $G(\tilde{E}^c|F)$ onto the subgroups $s^a(T_{\tilde{\mathfrak{v}}^a|\mathfrak{w}}) \leqslant s^a(Z_{\tilde{\mathfrak{v}}^a|\mathfrak{w}})$ of $G(\tilde{E}^a|F)$. Hence recalling that $[\tilde{\sigma}, \tilde{\tau}] \in \langle \tilde{\sigma}^\ell \rangle$ for all $\tilde{\sigma} \in T_{\tilde{\mathfrak{v}}^c|\mathfrak{w}}$ and $\tilde{\tau} \in Z_{\tilde{\mathfrak{v}}^c|\mathfrak{w}}$, one has $[s^c(\tilde{\sigma}), s^c(\tilde{\tau})] \in \langle s(\tilde{\sigma})^\ell \rangle$ for all $\tilde{\sigma} \in T_{\tilde{\mathfrak{v}}^c|\mathfrak{w}}$ and $\tilde{\tau} \in Z_{\tilde{\mathfrak{v}}^c|\mathfrak{w}}$. Hence since the canonical projections $G(\tilde{E}^\bullet|F) \twoheadrightarrow G(F^\bullet|F)$ are surjective and that $q_N^\bullet = p^\bullet \circ q_F^\bullet \circ s^\bullet$, it follows that the subgroups $I^\bullet := q_F^\bullet\big(s^\bullet(T_{\tilde{\mathfrak{v}}^\bullet|\mathfrak{w}})\big)$, $D^\bullet = q_F^\bullet\big(s^\bullet(Z_{\tilde{\mathfrak{v}}^\bullet|\mathfrak{w}})\big)$ of $\mathcal{G}_F^\bullet = G(F^\bullet|F)$ satisfy:

a) $I^a \subset D^a$ are subgroups of $\mathcal{G}_F^a$. Further, $q_N^\bullet = p^\bullet \circ q_F^\bullet \circ s^\bullet$ implies $p^a(I^a) = T_{\mathfrak{w}^a|\mathfrak{w}}$ and $p^a(D^a) = Z_{\mathfrak{w}^a|\mathfrak{w}}$. Hence $I^a \neq 1$ and $D^a$ is not cyclic.

b) $I^c \subset D^c$ are subgroups of $\mathcal{G}_F^c$ which project onto $I^a \subset D^a$ under $G(F^c|F) \twoheadrightarrow G(F^a|F)$ and $\forall\, \sigma \in I^a$, $\tau \in D^a$ and any preimages $\tilde{\sigma} \in I^c$, $\tilde{\tau} \in D^c$ one has $[\tilde{\sigma}, \tilde{\tau}] \in \langle \tilde{\sigma}^\ell \rangle$.

Further, in the case when $I_{\mathfrak{w}} = T_{\mathfrak{w}^a|\mathfrak{w}} \leqslant Z_{\mathfrak{w}^a|\mathfrak{w}} = D_{\mathfrak{w}}$ is replaced by $I_{D_{\mathfrak{w}}} = I_{Z_{\mathfrak{w}^a|\mathfrak{w}}} \leqslant Z_{\mathfrak{w}^a|\mathfrak{w}} = D_{\mathfrak{w}}$, the assertion of Lemma follows along the same lines, so we omit the details. □

**Construction 3.3 ($\mathfrak{w}|\mathfrak{v} \rightsquigarrow I_{D_I} \leqslant D_I \rightsquigarrow w|v \in \mathcal{W}_{F|E}$).**
In notation from Lemma 3.2, let $I_{D_{I^a}} \leqslant D_{I^a}$ be the maximal c.l. pair in $\mathcal{G}_F^a$ attached to the c.l. pair $I := I^a \leqslant D^a =: D$ as in Definition/Remark 2.11, 4. Further, consider $I_{D_{\mathfrak{w}}} \leqslant D_{\mathfrak{w}} \rightsquigarrow \mathfrak{w} \in \mathcal{W}_N$ and $I_{D_{I^a}} \leqslant D_{I^a} \rightsquigarrow w \in \mathcal{W}_F$ as defined in Remark 2.16. Hence since $\mathfrak{v} = \mathfrak{w}|_L$ and $v = w|_E$, in the context of Remark/Notation 2.17, one has $\mathfrak{w}|\mathfrak{v} \in \mathcal{W}_{N|L}$ and $w|v \in \mathcal{W}_{F|E}$, and further:

1) $Z_{\mathfrak{w}^a|\mathfrak{v}} = D_{\mathfrak{w}|\mathfrak{v}} \in \mathcal{D}_{N|L}$, $Z_{\mathfrak{w}^a|\mathfrak{v}} \twoheadrightarrow Z_{\mathfrak{w}|\mathfrak{v}}$ and $D_{w|v} \in \mathcal{D}_{F|E}$, $D_{w|v} \to Z_{w|v}$.
2) By Fact 2.18 applied to $\mathfrak{w}|\mathfrak{v}$ and $w|v$, for $g \in G(F|E)$, $h \in G(N|L)$ the following hold:

$(*)_{\mathfrak{w}}$ $h(I_{D_{\mathfrak{w}}}) = I_{D_{\mathfrak{w}}}$ *iff* $h(D_{\mathfrak{w}}) = D_{\mathfrak{w}}$ *iff* $h(D_{\mathfrak{w}|\mathfrak{v}}) = D_{\mathfrak{w}|\mathfrak{v}}$ *iff* $h(Z^a_{\mathfrak{w}}) = Z^a_{\mathfrak{w}}$ *iff* $h(\mathfrak{w}) = \mathfrak{w}$ *iff* $h \in Z_{\mathfrak{w}|\mathfrak{v}}$.
$(*)_w$ $g(I_{D_{I^a}}) = I_{D_{I^a}}$ *iff* $g(D_{I^a}) = D_{I^a}$ *iff* $g(D_{w|v}) = D_{w|v}$ *iff* $g(Z^a_w) = Z^a_w$ *iff* $g(w) = w$ *iff* $g \in Z_{w|v}$.

**Proposition 3.4 (Fact 2.7 re-revisited).** *In the above notations, the following hold:*

1) *Setting* $w_N := w|_N$, *one has:* $\mathfrak{w} \leqslant w_N$, $Z^a_{\mathfrak{w}} = D_{w_N} = p^a(D_w)$, $T^a_{\mathfrak{w}} = p^a(I^a) \subset I_{w_N} = I_{D_{\mathfrak{w}}}$.
2) $\imath(Z_{w|v}) = Z_{\mathfrak{w}|\mathfrak{v}}$ *and* $\imath(D_{w|v}) = Z_{\mathfrak{w}^a|\mathfrak{v}}$.
3) *For for* $g \in G(F|E)$ *one has:*

$$g(I^a) = I^a \ \textit{iff}\ g(D_{I^a}) = D_{I^a} \ \textit{iff}\ g(D_{w|v}) = D_{w|v} \ \textit{iff}\ g(w) = w \ \textit{iff}\ g \in Z_{w|v}.$$

*Proof.* To 1): We first notice that the surjectivity of $p^a$ and $p^c$ imply that for every c.l. pair $I' \leqslant D'$ in $\mathcal{G}_F^a$ with $p^a(I') \neq 1$ and $p^a(D')$ non-cyclic, the image $p^a(I') \leqslant p^a(D')$ of $I' \leqslant D'$ under $p^a$ is a c.l. pair in $\mathcal{G}_N^a$. (Actually, by mere definitions one has: If $g, \tau$ is a c.l. pair in $\mathcal{G}_F^a$, then $p^a(g), p^a(\tau)$ is a c.l. pair in $\mathcal{G}_N^a$, provided $p^a(g), p^a(\tau)$ are independent in $\mathcal{G}_N^a$.) In particular, since both $I^a \subset I_{D_{I^a}}$, $p^a(I^a) = I_{\mathfrak{w}} = T^a_{\mathfrak{w}}$, and $D^a \subset D_{I^a}$, $p^a(D^a) = D_{\mathfrak{w}} = Z^a_{\mathfrak{w}}$, one has: $I_N := p^a(I_{D_{I^a}}) \leqslant p^a(D_{I^a}) =: D_N$ is a c.l. pair in $\mathcal{G}_N^a$ such that $I_{\mathfrak{w}} \subset I_N$, $D_{\mathfrak{w}} \subset D_N$. Thus $w_N = w|_N$ is non-trivial, because $I_N \neq 1$. Further, since $\mathrm{char}(N\mathfrak{w}) \neq \ell$, one has $D_{\mathfrak{w}} = Z^a_{\mathfrak{w}}$ (and $T^a_{\mathfrak{w}} = I_{\mathfrak{w}}$, provided $N\mathfrak{w}^\times/\ell$ non-cyclic), and since $\mathfrak{w}$ has rank one, one has that either $\mathfrak{w} \leqslant w_N$, or $\mathfrak{w}$ and $w_N$ are independent. By contradiction, suppose that $w_1 := \mathfrak{w}$ and $w_2 := w_N$ are independent. Then by Lemma 2.8, it follows that $Z^a_{w_1} \cap Z^a_{w_2} = 1$, hence $D_{w_1} \cap D_{w_2} = 1$ as well, because $D_{w_i} \subset Z^a_{w_i}$, $i = 1, 2$. This is a contradiction, because $w_1 = \mathfrak{w}$ discrete implies

$1 \neq D_{w_1}$, hence $1 \neq D_{w_1} = D_{\mathfrak{w}} = pr^a(D^a) \subset D_{w_2}$. Conclude that $w_1, w_2$ are comparable, hence $\mathfrak{w} = w_1 \leqslant w_2 = w_N$, because $w_1 = \mathfrak{w}$ has rank one. Therefore, see e.g., Topaz [To1], Lemma 4.1, one has: Since $w_1 \leqslant w_2$, i.e., $w_2$ is a refinement of $w_1$, one has $D_{w_2} \subset D_{w_1}$, $I_{w_2} \supset I_{w_1}$, etc. On the other hand, by functoriality of (reduced) Hilbert decomposition theory, $p^a(D_w) \subset D_{w_N} = D_{w_2}$. Hence since $D_{\mathfrak{w}} = p^a(D^a)$, putting everything together, we get:

$$D_{w_1} = D_{\mathfrak{w}} = p^a(D^a) \subset D_{w_N} = D_{w_2} \subset D_{\mathfrak{w}} = D_{w_1},$$

concluding that $D_{\mathfrak{w}} = D_{w_1} = D_{w_2} = p^a(D_w) = D_{w_N}$. Concerning the last assertion about (reduced) inertia, as mentioned above, one has that $p^a(I_w) \leqslant p^a(D_w)$ is a c.l. pair in $\mathcal{G}^a_N$. Recalling that $p^a(D^a) = D_{w_2} = D_{\mathfrak{w}}$, let $I_{D_{\mathfrak{w}}} \subset D_{\mathfrak{w}} = p^a(D^a)$ be maximal such that $I_{D_{\mathfrak{w}}} \leqslant D_{\mathfrak{w}}$ is a c.l. pair in $\mathcal{G}^a_N$. Since $p^a(I^a) \leqslant p^a(D^a)$, that is, $p^a(I^a) \leqslant D_{\mathfrak{w}}$ is a c.l. pair, it follows that $p^a(I^a) \subset I_{D_{\mathfrak{w}}}$ by the maximality of $I_{D_{\mathfrak{w}}} \subset D_{\mathfrak{w}}$ such that $I_{D_{\mathfrak{w}}} \leqslant D_{\mathfrak{w}}$ is a c.l. pair in $\mathcal{G}^a_N$.

To 2): Since $pr^a(D_w) = D_{\mathfrak{w}}$, taking into account assertions $(*)_{\mathfrak{w}}$ and $(*)_w$ above, by mere definitions one has $\imath(Z_{w|v}) \subset Z_{\mathfrak{w}|\mathfrak{v}}$. For the converse inclusion, proceed as follows: By assertion $(*)_{\mathfrak{w}}$, one has that $h \in Z_{\mathfrak{w}|\mathfrak{v}}$ iff $h(I_{D_{\mathfrak{w}}}) = I_{D_{\mathfrak{w}}}$. Let $h' \mapsto h$ under $G(N^a|L) \twoheadrightarrow G(N|L)$, and denote $g' = s^a(h')$. Then by mere definitions one has: $\tau \in D_{I^a}$ iff $\forall\, \sigma \in I^a = s^a(I_{D_{\mathfrak{w}}})$ have: $\sigma, \tau$ is a c.l. pair in $\mathcal{G}^a_F$. Obviously, since the inner conjugation by $g'$ is an automorphism of $\mathcal{G}^a_F$ which lifts to the inner conjugation in $\mathcal{G}^c_F$, the latter assertion is equivalent to $\sigma', \tau'$ being a c.l. pair in $\mathcal{G}^a_F$, where $\sigma' = g'\sigma\, g'^{-1}$ and $\tau' = g'\tau g^{-1}$. On the other hand, $\sigma \mapsto \sigma' := g'\sigma g'^{-1}$ is an automorphism of $I_{D_{\mathfrak{w}}}$ (because $g' I_{D_{\mathfrak{w}}}\, g'^{-1} = g'(I_{D_{\mathfrak{w}}}) = I_{D_{\mathfrak{w}}}$). Hence $\tau \in D_{I^a}$ iff $\tau' := g'\tau g'^{-1} \in D_{I^a}$. Thus $g' D_{I^a} g'^{-1} = D_{I^a}$, that is, $g'(D_{I^a}) = D_{I^a}$. Since $g \in Z_{\mathfrak{w}|\mathfrak{v}}$ was arbitrary and $g'(D_{I^a}) = D_{I^a}$, conclude by Fact 2.18 that $\imath^{-1}(\tau) \in Z_{w|v}$ under the isomorphism $\imath : G(F|E) \to G(N|L)$. Thus $\imath(Z_{w|v}) = Z_{\mathfrak{w}|\mathfrak{v}}$ as claimed.

Finally, the assertion $p^a(D_{w|v}) = D_{\mathfrak{w}^a|\mathfrak{v}} = Z_{\mathfrak{w}^a|\mathfrak{v}}$ follows by mere definitions from functoriality of Hilbert decomposition theory, using assertions 1) and the fact that $\imath(Z_{w|v}) = Z_{\mathfrak{w}|\mathfrak{v}}$.

To 3): By assertion $(*)_w$ above, it is enough to show that for $g \in G(F|E)$ one has: $g(I^a) = I^a$ iff $g(D_{I^a}) = D_{I^a}$. To fix notation, for every $g \in G(F|E)$, set $h := \imath(g)$, and for preimages $h^\bullet \in G(N^\bullet|L)$ of $h$, let $g^\bullet = s^\bullet(h^\bullet) \in G(F^\bullet|E)$ be the corresponding preimages of $g$.

- For the direct implication, let $g(I^a) = I^a$ for some $g \in G(F|E)$, and $g^\bullet$ be the liftings as defined above. Let $\sigma_{\mathfrak{w}} \in I_{\mathfrak{w}} = T^a_{\mathfrak{w}} \cong \mathbb{Z}/\ell$ be a generator and $\tilde{\sigma}_{\mathfrak{w}} \in T^c_{\mathfrak{w}} \cong \mathbb{Z}/\ell^2$ be a lifting of $\sigma_{\mathfrak{w}}$. In particular, $\sigma = s^a(\sigma_{\mathfrak{w}})$ generates $I^a = s^a(I_{\mathfrak{w}})$ and $\tilde{\sigma} := s^c(\tilde{\sigma}_{\mathfrak{w}})$ generates $I^c := s^c(T^c_{\mathfrak{w}})$. Further, $\tilde{\sigma}$ and $I^c \subset \mathcal{G}^c_F$ are liftings of $\sigma$, respectively $I^a$ under $\mathcal{G}^c_F \to \mathcal{G}^a_F$. This being said, the action of $g^\bullet$ on $I^\bullet = s(T^\bullet_{\mathfrak{w}})$ is induced by the action of $h^\bullet$ on $I^\bullet_{\mathfrak{w}}$. In particular, if $\tau \in D_I$ and $\tilde{\tau} \in \mathcal{G}^c_F$ is a lifting, then by the definition of $D_I$ one has: $[\tilde{\sigma}, \tilde{\tau}] = \tilde{\sigma}^{\ell n}$ for some $n \in \mathbb{N}$, thus:

$$[\tilde{\sigma}^{g^c}, \tilde{\tau}^{g^c}] = (\tilde{\sigma}^{\ell n})^{g^c} = (\tilde{\sigma}^{g^c})^{\ell n} = (*)$$

On the other hand, $g^a(I^a) = I^a$ and $I^a = \langle \sigma \rangle$ implies $g^a(\sigma) = \sigma^m$ for some $m \in \mathbb{N}$ with $(\ell, m) = 1$. Therefore, $\tilde{\sigma}^{g^c} = \tilde{\sigma}^m \sigma_0$ for some $\sigma_0 \in G(F^c|F^a)$, hence finally:

$$(*) = (\tilde{\sigma}^{g^c})^{\ell n} = (\tilde{\sigma}^m \sigma_0)^{\ell n} = \tilde{\sigma}^{\ell m n} \sigma_0^{\ell n} = \tilde{\sigma}^{\ell m n} \in \langle \tilde{\sigma}^\ell \rangle,$$

because $\sigma_0^\ell = 1$ by the fact that $G(F^c|F^a)$ is an $\ell$-elementary abelian group. Hence we conclude that $g^a(\sigma), g^a(\tau)$ is a c.l. pair in $\mathcal{G}^a_F$. Thus since $\tau \in D_{I^a}$ was arbitrary, and $g^a(I^a) = I^a$, it follows that $I^a \leqslant g^a(D_{I^a})$ is a c.l. pair in $\mathcal{G}^a_F$. Since $D_{I^a} \subset \mathcal{G}^a_F$ is (the unique) maximal such that

$I^a \leqslant D_{I^a}$ is c.l. in $\mathcal{G}^a_F$, one has $g^a(D_{I^a}) \subset D_{I^a}$. Let $\overline{g}^a$ be the inverse of $g^a$. Then $\overline{g}^a(I^a) = I^a$ and reasoning as above, one has that $\overline{g}^a(D_{I^a}) \subset D_{I^a}$. Conclude that $g^a(D_{I^a}) = D_{I^a}$.

- For the converse implication, let $I_{D^{I^a}} \leqslant D_{I^a} \rightsquigarrow w|v \in \mathcal{W}_{F|E}$ and $g \in G(F|E)$ satisfy $g(D_{I^a}) = D_{I^a}$. Then by assertion $(*)_w$ before Proposition 3.4, one has that $g \in Z_{w|v}$. In particular, $h = \imath(g) \in G(N|L)$ lies in $Z_{\mathfrak{w}|\mathfrak{v}}$, hence by functoriality of Hilbert decomposition theory one has $h(T^a_{\mathfrak{w}}) = T^a_{\mathfrak{w}}$ and $h(Z^a_{\mathfrak{w}}) = Z^a_{\mathfrak{w}}$. Hence taking into account that $I_{\mathfrak{w}} = T^a_{\mathfrak{w}}$, we get $g(I_{\mathfrak{w}}) = g(T^a_{\mathfrak{w}}) = T^a_{\mathfrak{w}} = I_{\mathfrak{w}}$. Therefore, since by definition we have $I^a = s^a(I_{\mathfrak{w}})$ and $g = \imath^{-1}(h)$ as well, it follows instantly by mere definitions of $\imath$, $s^a$ and $p^a$ that

$$g(I^a) = g\big(s^a(I_{\mathfrak{w}})\big) = s^a\big(h^a(I_{\mathfrak{w}})\big) = s^a(I_{\mathfrak{w}}) = I^a.$$

□

### 3.2. Canonical $s^\bullet$-valuations and their functorial behavior.

In the context of Proposition 3.4 above, recall that given valuations $\mathfrak{w}|\mathfrak{v}$ of $N|L$, $\mathfrak{v} \in \mathrm{Val}^1(L)$, via the section $s^\bullet$ of $p^\bullet$ one gets valuations $w|v$ of $F|E$ satisfying $w_N := w|_N \geqslant \mathfrak{w}$ and $v_L := w|_L \geqslant \mathfrak{v}$. In particular, the general Fact 2.4 above applies in this context, leads to:

1) The canonical $\mathfrak{w}$-valuations and/or $\mathfrak{v}$-valuations of $F$, which turn out to be equal $w_{\mathfrak{w}} = w_{\mathfrak{v}}$. Indeed, this follows by Fact 2.4, 3), because $N|L$ is an algebraic extension.
2) The canonical $\mathfrak{w}^a$- and $\mathfrak{w}$- and $\mathfrak{v}$-valuations of $F^a$ are equal $w^a_{\mathfrak{w}^a} = w^a_{\mathfrak{w}} = w^a_{\mathfrak{v}}$ and prolong $w_{\mathfrak{w}} = w_{\mathfrak{v}}$ to $F^a$. Indeed, this follows from Fact 2.4, 3), because $F^a|F$ is algebraic.
   And since $F^a|F|E$ are algebraic, the above valuations all have the same restriction to $E$, denoted $v_{\mathfrak{v}} := (w^a_{\mathfrak{w}^a})|_E = (w^a_{\mathfrak{w}})|_E = (w_{\mathfrak{w}})|_E$, etc.

**Definition 3.5.** In the above context, the valuations $w^a_{\mathfrak{w}^a} = w^a_{\mathfrak{w}} = w^a_{\mathfrak{v}}$ of $F^a$ and $w_{\mathfrak{w}} = w_{\mathfrak{v}}$ of $F$ and $v_{\mathfrak{v}}$ of $E$ are called *canonical $s^\bullet$-valuations* of $F^a|F|E$ (defined by $\mathfrak{w}^a|\mathfrak{w}|\mathfrak{v}$ via $s^\bullet$).

- We notice the following: Since $\mathrm{char}(N\mathfrak{w}) = \mathrm{char}(L\mathfrak{v}) \neq \ell$, $\mathfrak{v} \in \mathrm{Val}^1(L)$ and $(w_{\mathfrak{w}})|_N = \mathfrak{w}$, one has that $Fw_{\mathfrak{w}} \neq \ell$. In particular, one also has $D_{w_{\mathfrak{w}}} = Z^a_{w_{\mathfrak{w}}}$ and $I_{w_{\mathfrak{w}}} = T^a_{w_{\mathfrak{w}}}$, etc.

**Fact 3.6.** *The $s^\bullet$-canonical valuations $w_{\mathfrak{w}}|v_{\mathfrak{v}}$ arising from $\mathfrak{w}|\mathfrak{v}$, $\mathfrak{v} \in \mathrm{Val}^1(L)$ satisfy:*

(i) $\mathcal{O}^\times_{w_{\mathfrak{w}}} \supset \mathcal{O}^\times_w$, $1 + \mathfrak{m}_{w_{\mathfrak{w}}} \subset 1 + \mathfrak{m}_w$; (ii) $\mathcal{O}^\times_{w_{\mathfrak{w}}} \cap N = \mathcal{O}^\times_{\mathfrak{w}}$, $(1 + \mathfrak{m}_{w_{\mathfrak{w}}}) \cap N = 1 + \mathfrak{m}_{\mathfrak{w}}$.

*Therefore, by mere definitions one has:*

1) $s^a(I_{\mathfrak{w}}) \subset I_{w_{\mathfrak{w}}} \subset I_w$, $D_{w_{\mathfrak{w}}} = D_w \supset s^a(D_{\mathfrak{w}})$, *and* $D_{w_{\mathfrak{w}}|v_{\mathfrak{v}}} = D_{w|v} \supset s^a(D_{\mathfrak{w}|\mathfrak{v}})$.
2) $p^a(I_{w_{\mathfrak{w}}}) = I_{\mathfrak{w}}$, $p^a(D_{w_{\mathfrak{w}}}) = D_{\mathfrak{w}}$, *and* $p^a(D_{w_{\mathfrak{w}}|v_{\mathfrak{v}}}) = D_{\mathfrak{w}|\mathfrak{v}}$. *Thus* $\imath(Z_{w_{\mathfrak{w}}|v_{\mathfrak{v}}}) = Z_{\mathfrak{w}|\mathfrak{v}}$.

*Proof.* To 1): The inclusions $s^a(I_{\mathfrak{w}}) \subset I_{w_{\mathfrak{w}}} \subset I_w$ and $D_{w_{\mathfrak{w}}} \supset D_w \supset s^a(D_{\mathfrak{w}})$ are clear by mere definitions. For the converse inclusion $D_{w_{\mathfrak{w}}} \subset D_w$ one has: Since $I^a \subset I_{w_{\mathfrak{w}}}$ and $D_{w_{\mathfrak{w}}}$ acts on $I_{w_{\mathfrak{w}}}$ via the cyclotomic character, one has: $\forall\, \sigma \in I^a$ and $\forall\, \tau D_{w_{\mathfrak{w}}}$ one has: $\sigma, \tau$ is a c.l. pair in $\mathcal{G}^a_F$. Hence by definition one has $D_w = D_{I^a}$, one finally has $D_{w_{\mathfrak{w}}} \subset D_w$, as claimed.

To 2): The equality $p^a(D_{w_{\mathfrak{w}}}) = D_{\mathfrak{w}}$ follows from $p^a(D_w) = D_{\mathfrak{w}}$ cf. Proposition 3.4, 1) and the other equalities follow along the same lines using loc.cit. 2) and 3). □

Next, in the notation from the previous subsections, let $\tilde{L}|N_\alpha|L_1|L$, $\alpha \in I$, be the family of finite Galois subextensions of $\tilde{L}|L$ with $L_1 \subset N_\alpha$. We notice that $I$ is (filtered partially) ordered by $i \leqslant j$ iff $N_\alpha \subset N_\beta$. And for each $i \in I$ consider the resulting $F_\alpha := EN_\alpha$, thus the filtered family of finite Galois subextensions $\tilde{E}|F_\alpha|E_1|E$ if $\tilde{E}|E$. Let $p^\bullet_\alpha : G(F^\bullet_\alpha|E) \to G(N^\bullet_\alpha|L)$ be the resulting projections, and $s^\bullet_\alpha : G(N^\bullet_\alpha|L) \to G(F^\bullet_\alpha|E)$ be the resulting sections.

For $\mathfrak{v} \in \mathrm{Val}^1(L)$ and its prolongations $\tilde{\mathfrak{v}}|\mathfrak{w}_\alpha|\mathfrak{v}$ to $\tilde{L}|N_\alpha|L$, $i \in I$, let $w_\alpha|v_\alpha \in \mathcal{W}_{F_\alpha|E}$ be the valuation defined via $s^\bullet$ and $\mathfrak{v} \in \mathrm{Val}^1(L)$. In particular, the valuations $w_\alpha \geqslant \mathfrak{w}_\alpha$ give rise to the canonical $\mathfrak{w}_\alpha$-valuation $w_{\mathfrak{w}_\alpha}$ of $F_\alpha$. Further, for $N_\alpha \subset N_\beta$ and the resulting $F_\alpha \subset F_\beta$, etc., let $p^\bullet_{\beta\alpha} : G(F^\bullet_\beta|E) \to G(F^\bullet_\alpha|E)$ be the resulting canonical projections, thus $p^\bullet_\beta = p^\bullet_\alpha \circ p^\bullet_{\beta\alpha}$. And recalling the notation introduced in Construction 3.3, set $I^a_\alpha := s_\alpha(I_\mathfrak{w})$, $D^a_\alpha = s_\alpha(D_\mathfrak{w})$ and notice that $p^\bullet_\beta = p^\bullet_\alpha \circ p^\bullet_{\beta\alpha}$ implies:

$$(*) \qquad p^a_{\beta\alpha}(I^a_\beta) = I^a_\alpha, \quad p^a_{\beta\alpha}(D^a_\beta) = D^a_\alpha.$$

**Key Lemma 3.7 (Functoriality of $s^\bullet$-canonical valuations).**
*For $F_\alpha \subset F_\beta$, let $w_{\mathfrak{w}_\alpha} \mathrm{Val}(F_\alpha)$ and $w_{\mathfrak{w}_\beta} \in \mathrm{Val}(F_\beta)$ be the corresponding $s^\bullet$-canonical valuations. Then $w_{\mathfrak{w}_\beta}|_{F_\alpha} = w_{\mathfrak{w}_\alpha}$. In particular, $v_\mathfrak{v} = w_{\mathfrak{w}_\alpha}|_E$ is independent of $\alpha \in I$ and $\mathfrak{v} = v_\mathfrak{v}|_L$.*

*Proof.* Let $w'_\alpha := w_{\mathfrak{w}_\beta}|_{F_\alpha}$ and set $w^0 = \min(w'_\alpha, w_{\mathfrak{w}_\alpha}) \in \mathrm{Val}(F_\alpha)$.

Step 1. We claim that $w^0$ is non-trivial, or equivalently, $w'_i$ and $w_{\mathfrak{w}_\alpha}$ are not independent. Indeed, by Fact 3.6, 1), one has $I^a_\alpha = s^a_\alpha(I_{\mathfrak{w}_\alpha}) \subset I_{w_{\mathfrak{w}_\alpha}}$, and by assertion $(*)$ right before Key Lemma 3.7, one has $p^a_{\beta\alpha}(I^a_\beta) = I^a_\alpha \subset p^a_{\beta\alpha}(I_{w_{\mathfrak{w}_\beta}})$. Thus since $w'_\alpha = w_{\mathfrak{w}_\beta}|_{F_\alpha}$, by Fact 2.9, one has $p^a_{\beta\alpha}(I_{w_{\mathfrak{w}_\beta}}) \subset I_{w'_\alpha}$. Hence $1 \neq I^a_\alpha \subset I_{w'_\alpha} \cap I_{w_{\mathfrak{w}_\alpha}}$, hence by Fact 2.7, 1), it follows that $w'_\alpha$ and $w_{\mathfrak{w}_\alpha}$ are not independent, as claimed.

Step 2. First, $\mathfrak{w}^0 := w^0|_{N_\alpha}$ is non-trivial. Indeed, by Fact 2.9, one has $p^a_\alpha(I_{w^0}) \subset I_{\mathfrak{w}^0}$. Thus $I_{w^0} \supset I^a_\alpha$ implies $I_{\mathfrak{w}^0} \supset p^a_\alpha(I_{w^0}) \supset p^a_\alpha(I^a_\alpha) = I_{\mathfrak{w}_\alpha} \neq 1$, and therefore, $w^0$ is non-trivial. Second, since $w^0 \leqslant w_{\mathfrak{w}_\alpha}$, one has $\mathfrak{w}^0 = w^0|_{N_\alpha} \leqslant w_{\mathfrak{w}_\alpha}|_{N_\alpha} = \mathfrak{w}_\alpha$. Thus since $\mathfrak{w}_\alpha$ is discrete and $\mathfrak{w}^0 \leqslant \mathfrak{w}_\alpha$ is nontrivial, one must have $\mathfrak{w}^0 = \mathfrak{w}_\alpha$. Finally, since $w^0 \leqslant w_{\mathfrak{w}_\alpha}$ and $w^0|_{N_\alpha} = \mathfrak{w}_\alpha$, by the definition of the canonical $\mathfrak{w}_\alpha$ valuation $w_{\mathfrak{w}_\alpha}$ one must have $w^0|_{N_\alpha} = \mathfrak{w}_\alpha$. Therefore, one finally must have $w^0 = \min(w'_\alpha, w_{\mathfrak{w}_\alpha}) = w_{\mathfrak{w}_\alpha}$, concluding that $(w_{\mathfrak{w}_\beta})|_{F_\alpha} = w'_\alpha \geqslant w_{\mathfrak{w}_\alpha}$.

Step 3. Finally, given that $w_{\mathfrak{w}_\beta}|_{F_\alpha} = w'_\alpha \geqslant w_{\mathfrak{w}_\alpha}$, let $w'_\beta \leqslant w_{\mathfrak{w}_\beta}$ be the minimal coarsening of $w_{\mathfrak{w}_\beta}$ such that $w'_\beta|_{F_\alpha} = w'_\alpha$, that is, $w'_\beta|_{F_\alpha} = w_{\mathfrak{w}_\alpha}$. Then by the discussion above and Fact 2.4, 2) one has both: First, $w'_\beta|_{N_\beta}$ is the prolongation is $w'_\alpha = w_{\mathfrak{w}_\alpha}$ to $F_\beta$, thus $w'_\beta|_{N_\alpha} = \mathfrak{w}_\alpha$, and second, $w'_\beta \leqslant w_\beta$. Thus by the definition of the canonical $\mathfrak{w}_\beta$-valuation $w_{\mathfrak{w}_\beta}$ one has that $w_{\mathfrak{w}_\beta} \leqslant w'_\beta$, thus finally concluding that $w'_\beta = w_{\mathfrak{w}_\beta}$, hence $w_{\mathfrak{w}_\alpha} = w_{\mathfrak{w}_\beta}|_{F_\alpha}$. □

In order to avoid overloaded notation, we introduce the following:

**Notation 3.8.** Given $N_\alpha|L \hookrightarrow F_\alpha|E$, we denote $\tilde{w}_\alpha := w_{\mathfrak{w}_\alpha}$, thus $\tilde{w}_\alpha|_E = v_\mathfrak{v}$ for all $\alpha \in I$.

An important consequence of the Key Lemma 3.7 above is as follows.

- $\widetilde{\mathcal{O}} := \cup_\alpha \mathcal{O}_{\tilde{w}_\alpha} \subset \tilde{E}$ is a valuation ring satisfying $\widetilde{\mathcal{O}} \cap F_\alpha = \mathcal{O}_{\tilde{w}_\alpha}$ for all $i \in I$.

In particular, if $\tilde{w}_{\tilde{\mathfrak{v}}}$ denotes the valuation of $\widetilde{\mathcal{O}}$, i.e., $\widetilde{\mathcal{O}} = \mathcal{O}_{\tilde{w}_{\tilde{\mathfrak{v}}}}$, the following hold: First, since $\tilde{L} = \cup_\alpha N_\alpha$, $\tilde{E} = \cup_\alpha F_\alpha$ and $\mathfrak{w}_\alpha = \tilde{w}_\alpha|_{N_\alpha}$, one has $\tilde{\mathfrak{v}} = \tilde{w}_{\tilde{\mathfrak{v}}}|_{\tilde{L}}$, and further, $\tilde{w}_{\tilde{\mathfrak{v}}}|_E = v_\mathfrak{v} = \tilde{w}_\alpha|_E$, $\tilde{w}_{\tilde{\mathfrak{v}}}|_L = \tilde{w}_\alpha|_L = \mathfrak{w}_\alpha|_L = \mathfrak{v}$, $\alpha \in I$. Moreover, by Fact 3.6, 2) one has:

$$s^a_\alpha(I_{\mathfrak{w}_\alpha}) = I^a_\alpha \subset I_{D_{\tilde{w}_\alpha}} \text{ and } s_\alpha(D_{\mathfrak{w}_\alpha}) \subset D_{\tilde{w}_\alpha}),\ \alpha_\alpha \in I.$$

Therefore, taking into account that $G(L^a|L) = \varprojlim_\imath G(N^a_\alpha|L)$ and $G(E^a|E) = \varprojlim_\imath G(F^a_\alpha|E)$ and further, $s^a(I_{\tilde{\mathfrak{v}}}) = \varprojlim_\imath I^a_\alpha$, $s^a(D_{\tilde{\mathfrak{v}}}) = \varprojlim_\imath D^a_\alpha$, etc., by "taking limits" and taking into account that $I_{\mathfrak{w}_\alpha} = T^a_{\mathfrak{w}_\alpha}$, $D_{\mathfrak{w}_\alpha} = Z^a_{\mathfrak{w}_\alpha}$ and $D_{\tilde{w}^a_\alpha|v_{\mathfrak{v}}} = Z_{\tilde{w}^a_\alpha|v_{\mathfrak{v}}}$, on gets the following:

**Fact 3.9 (Fact 3.6 revisited).** *The $s^\bullet$-canonical valuations $\tilde{w}_{\tilde{\mathfrak{v}}}$ arising from $\mathfrak{v} \in \mathrm{Val}^1(L)$ in the way explained above have $\mathrm{char}(\tilde{F}\tilde{w}_{\tilde{\mathfrak{v}}}) \neq \ell$ and further satisfy the following:*

1) $\tilde{w}_{\tilde{\mathfrak{v}}}|_{F_\alpha} = \tilde{w}_\alpha$, $\tilde{w}_{\tilde{\mathfrak{v}}}|_E = v_{\mathfrak{v}}$, $\tilde{w}_{\tilde{\mathfrak{v}}}|_{N_\alpha} = \mathfrak{w}_\alpha$, $\tilde{w}_{\tilde{\mathfrak{v}}}|_L = \mathfrak{v}$. *Hence* $\mathrm{char}(\tilde{E}\tilde{w}_{\tilde{\mathfrak{v}}}) \neq \ell$*, thus concluding:*
$$I_{\tilde{w}_{\tilde{\mathfrak{v}}}} = T^a_{\tilde{w}_{\tilde{\mathfrak{v}}}}, \quad D_{\tilde{w}_{\tilde{\mathfrak{v}}}} = Z^a_{\tilde{w}_{\tilde{\mathfrak{v}}}}, \quad D_{\tilde{w}^a_{\tilde{\mathfrak{v}}}|v_{\mathfrak{v}}} = Z_{\tilde{w}^a_{\tilde{\mathfrak{v}}}|v_{\mathfrak{v}}}.$$
2) $s^a(T^a_{\tilde{\mathfrak{v}}}) \subset T^a_{\tilde{w}_{\tilde{\mathfrak{v}}}}$, $s^a(Z^a_{\tilde{v}}) \subset Z^a_{\tilde{w}_{\tilde{\mathfrak{v}}}}$ *and* $s^a(Z_{\tilde{\mathfrak{v}}^a|\mathfrak{v}}) \subset Z_{\tilde{w}^a_{\tilde{\mathfrak{v}}}|v_{\mathfrak{v}}})$ *and further,* $p^a(T^a_{\tilde{w}_{\tilde{\mathfrak{v}}}}) = T^a_{\tilde{\mathfrak{v}}}$, $p^a(Z^a_{\tilde{w}_{\tilde{\mathfrak{v}}}}) = Z^a_{\tilde{v}}$, *and* $p^a(Z_{\tilde{w}^a_{\tilde{\mathfrak{v}}}|v_{\mathfrak{v}}}) = Z_{\tilde{\mathfrak{v}}^a|\mathfrak{v}}$. *Thus* $\imath(Z_{\tilde{w}_{\tilde{\mathfrak{v}}}|v_{\mathfrak{v}}}) = Z_{\tilde{\mathfrak{v}}|\mathfrak{v}}$.

*Proof.* Beweis klar! □

## 4. Proof of Theorem 1.9 ( $\tilde{k}|k$-Minimalistic $t$-BSC)

### 4.1. **Preparation for the proof of Main Theorem 1.9.**

Let $\tilde{k}|k$ be a field extension satisfying Hypothesis (H), $K = k(X)$ be the function field of a geometrically integral $k$-curve $X$. This gives rise to a concrete case of the more general situation from Section 3 as follows. Let $L := k(t) =: k_t$ the rational function field in the variable $t$ over $k$ and $E := K(t) =: K_t$ be the compositum of $K = k(X)$ and $L = k(t)$ over $k$. Then $E|L$, that is, $K_t|k_t$ is a regular field extension (because $K|k$ was so). Thus setting $\tilde{k}_t := \tilde{k}(t)$ and $\tilde{K} := K\tilde{k}$, $\tilde{K}_t := \tilde{K}(t)$, one has $\tilde{E} = E\tilde{L} = \tilde{K}_t$, etc. For the resulting embeddings of Galois field extensions $\tilde{K}_t|K_t \hookleftarrow \tilde{k}_t|k_t \hookleftarrow \tilde{k}|k$ and $\tilde{K}^a_t|K_t \hookleftarrow \tilde{k}^a_t|k_t \hookleftarrow \tilde{k}|k$, let $G(\tilde{K}_t|K_t) \xrightarrow{\imath} G(\tilde{k}_t|k_t) \to G(\tilde{k}|k)$ be the canonical isomorphisms of Galois groups, respectively $G(\tilde{K}^\bullet_t|K_t) \twoheadrightarrow G(\tilde{k}^\bullet_t|k_t) \twoheadrightarrow \tilde{k}^\bullet|k$ the resulting surjective morphisms of Galois groups, where $\bullet$ stays for $a$ or $c$.

Finally, setting $k_1 := k_t(\mu_\ell)$, consider the family of finite Galois subextensions $k_\alpha|k$, $\alpha \in I$ of $\tilde{k}|k$ with $k_1 \subset k_\alpha$, partially ordered by: $\alpha \leqslant \beta$ iff $k_\alpha \subset k_\beta$. We are in the context if Section 3 with $N_\alpha = k_{\alpha,t} := k_\alpha(t) \subset \tilde{k}_t = \tilde{L}$ and $F_\alpha = EN_\alpha = k_{\alpha,t} \subset \tilde{k}_t$, getting isomorphic projective systems of finite groups $G(k_\beta|k) \twoheadrightarrow G(k_\alpha|k)$, $G(N_\beta|L) \twoheadrightarrow G(N_\alpha|L)$, $G(F_\beta|E) \twoheadrightarrow G(F_\alpha|E)$ for $\alpha \leqslant \beta$, having the canonical isomorphisms $G(\tilde{E}|E) \xrightarrow{\imath} G(\tilde{L}|L) \to G(\tilde{k}|k)$ as limit.

Concerning valuations: Recall that all $k$-valuations $\mathfrak{v} \in \mathrm{Val}_k(k_t)$ are discrete, being either the $p(t)$-adic valuations $\mathfrak{v} = v_{k,p}$ with $p = p(t) \in k[t]$ the monic irreducible polynomials, or $\mathfrak{v} = v_\infty$ with uniformizing parameter $\pi_\infty = \frac{1}{t}$. Further, $k_t\mathfrak{v}|k$ is a finite field extension, hence $\mathrm{char}(k_t\mathfrak{v}) \neq \ell$, and therefore, $\mathrm{Val}_k(k_t) \subset \mathrm{Val}^1(k_t)$. For $\mathfrak{v} \in \mathrm{Val}_k(k_t)$ consider the prolongations $\mathfrak{w}^a_\alpha|\mathfrak{w}_\alpha|\mathfrak{v}$ of $\mathfrak{v}$ to $k^a_{\alpha,t}|k_{\alpha,t}|k_t$ with limit $\tilde{\mathfrak{v}}^a|\tilde{\mathfrak{v}}|\mathfrak{v}$ as prolongations of $\mathfrak{v}$ to $\tilde{k}^a_t|\tilde{k}_t|k_t$. Similarly, with $F_\alpha := K_{\alpha,t} := Kk_{\alpha,t}$ and $v \in \mathrm{Val}_k(K_t)$, consider its prolongations $w^a_\alpha|w_\alpha|v$ to $K^a_{\alpha,t}|K_{\alpha,t}|K_t$ and $\tilde{v}^a|\tilde{v}|v$ prolonging $v$ to $\tilde{K}^a_t|\tilde{K}_t|K_t$. This being said, $\tilde{L}|L = \tilde{k}|k_t$, $\tilde{E}|E = \tilde{K}_t|K_t$ introduced/defined above are as in the previous section.

Next, let $s^a : G(\tilde{k}^a_t|k_t) \to G(\tilde{K}^a_t|K_t)$ be a liftable section of the canonical (surjective) projection $p^a : G(\tilde{K}^a_t|K_t) \twoheadrightarrow G(\tilde{k}^a_t|k_t)$, i.e., there is a section $s^c : G(\tilde{k}^c_t|k_t) \to G(\tilde{K}^c_t|K_t)$ of the canonical (surjective) projection $p^c : G(\tilde{K}^c_t|K_t) \to G(\tilde{k}^c_t|k_t)$. Then recalling the canonical isomorphism
$$\imath : G(\tilde{K}_t|K_t) \to G(\tilde{k}_t|k_t) \quad \text{defined by} \quad \tilde{K}_t|K_t \hookleftarrow \tilde{k}_t|k_t,$$

one has the following:

**Fact 4.1 (Fact 3.9 revisited).** *In the above context, for* $\mathfrak{v} \in \mathrm{Val}_k(k_t)$ *and its prolongation* $\tilde{\mathfrak{v}}|\mathfrak{v}$ *to* $\tilde{k}_t|k_t$ *, consider the corresponding inertia/decomposition groups* $T^a_{\tilde{\mathfrak{v}}} \subset Z^a_{\tilde{\mathfrak{v}}} \subset Z_{\tilde{\mathfrak{v}}^a|\mathfrak{v}} \subset G(\tilde{k}^a_t|k_t)$. *Then there is a unique valuation* $\tilde{w}_{\tilde{\mathfrak{v}}} \in \mathrm{Val}(\tilde{K}_t)$ *such that the following hold:*

1) $\tilde{w}_{\tilde{\mathfrak{v}}}|_{\tilde{k}_t} = \tilde{\mathfrak{v}}$*, thus* $\tilde{w}_{\tilde{\mathfrak{v}}}$ *is trivial on* $k$*, i.e.,* $\tilde{w}_{\tilde{\mathfrak{v}}} \in \mathrm{Val}_k(\tilde{K}_t)$ *and* $v_{\mathfrak{v}} := \tilde{w}_{\tilde{\mathfrak{v}}}|_{K_t} \in \mathrm{Val}_k(K_t)$.
2) $s^a(T^a_{\tilde{\mathfrak{v}}}) \subset T^a_{\tilde{w}_{\tilde{\mathfrak{v}}}}$ , $s^a(Z^a_{\tilde{\mathfrak{v}}}) \subset Z^a_{\tilde{w}_{\tilde{\mathfrak{v}}}}$ , *and* $s^a(Z_{\tilde{\mathfrak{v}}^a|\mathfrak{v}}) \subset Z_{\tilde{w}^a_{\tilde{\mathfrak{v}}}|v}$ , *and further,*
   $p^a(T^a_{\tilde{w}_{\tilde{\mathfrak{v}}}}) = T^a_{\tilde{\mathfrak{v}}}$, $p^a(Z^a_{\tilde{w}_{\tilde{\mathfrak{v}}}}) = Z^a_{\tilde{v}}$, *and* $p^a(Z_{\tilde{w}^a_{\tilde{\mathfrak{v}}}|v_{\mathfrak{v}}}) = Z_{\tilde{\mathfrak{v}}^a|\mathfrak{v}}$. *Thus* $\imath(Z_{\tilde{w}_{\tilde{\mathfrak{v}}}|v_{\mathfrak{v}}}) = Z_{\tilde{\mathfrak{v}}|\mathfrak{v}}$.

*In particular, every liftable section* $s^a : G(\tilde{k}^a_t|k_t) \to G(\tilde{K}^a_t|K_t)$ *of the canonical (surjective) projection* $p^a : G(\tilde{K}^a_t|K_t) \twoheadrightarrow G(\tilde{k}^a_t|k_t)$ *gives rise to an injective map*

$$\varphi : \mathrm{Val}_k(\tilde{k}_t) \to \mathrm{Val}_k(\tilde{K}_t), \quad \tilde{\mathfrak{v}} \mapsto \tilde{w}_{\tilde{\mathfrak{v}}},$$

*such that the* $k$*-valuations* $\tilde{\mathfrak{v}}$ *and* $\tilde{w}_{\tilde{\mathfrak{v}}}$ *satisfy the conditions* 1), 2) *above.*

*Proof.* Beweis, klar! □

4.2. **Places via** $\tilde{k}|k$-$t$-*a.b.c.* **liftable sections.**

If not otherwise explicitly stated, through out this subsection, the notation is that from Theorem 1.9, that is: $s^a_K : G(\tilde{k}^a|k) \to G(\tilde{K}^a|K)$ is a $\tilde{k}|k$-$t$-a.b.c. liftable section of the canonical projection $p^a_K : G(\tilde{K}^a|K) \to G(\tilde{k}^a|k)$, and $s^\bullet_t : G(\tilde{k}^\bullet_t|k) \to G(\tilde{K}^\bullet_t|K_t)$ be $\tilde{k}|k$-$t$-a.b.c. liftings of $s^a_K$ to sections of $p^\bullet_{K_t} : G(\tilde{K}^\bullet_t|K_t) \to G(\tilde{k}^\bullet_t|k_t)$ for $\bullet$ equal to $a$ and $c$. In particular, recalling the notations and the commutative diagram introduced before Theorem 1.9, one has the following:

$$(\ddagger) \qquad \begin{array}{ccccccccc} G(K^c_t|K_t) & \xrightarrow{p^c_{K_t}} & G(k^c_t|k_t) & \xrightarrow{p^c_{k_t}} & G(k^c|k) & \qquad & G(K^c_t|K_t) & \xleftarrow{s^c_t} & G(k^c_t|k_t) \\ \downarrow q^c_{K_t} & & \downarrow q^c_{k_t} & & \downarrow q^c_k & & \downarrow q^c_{K_t} & & \downarrow q^c_{k_t} \\ G(K^a_t|K_t) & \xrightarrow{p^a_{K_t}} & G(k^a_t|k_t) & \xrightarrow{p^a_{k_t}} & G(k^a|k) & & G(K^a_t|K_t) & \xleftarrow{s^a_t} & G(k^a_t|k_t) \end{array}$$

Here, $s^a_t$ is the section of $p^a_{K_t}$ liftable to $s^c_t$, i.e., $s^a_t \circ q^a_{k_t} = q^c_{K_t} \circ s^c_t$.

**Notation/Remark.** Denote by $v_{\tilde{k},\infty}$, $v_{\tilde{K},\infty}$ the $\frac{1}{t}$-adic valuations of $\tilde{k}_t$, $\tilde{K}_t$, thus $v_{\tilde{k},\infty} = v_{\tilde{K},\infty}|_{\tilde{k}_t}$. We notice the following: Let $v \in \mathrm{Val}(\tilde{K}_t)$ be a given valuation. Then $v = v_{\tilde{K},p}$ with $p \in \tilde{k}[t]$ monic irreducible iff $v_{\tilde{K}} := v|_{\tilde{K}}$ is trivial, $v|_{\tilde{k}_t}$ is non-trivial, and $v \neq v_{\tilde{K},\infty}$.

[For reader's sake we present the quite obvious proof. First, the direct implication is clear, because $\tilde{k} = \overline{k} \cap \tilde{K}$ implies: $p \in \tilde{k}[t]$ is irreducible iff $p$ is irreducible over $\tilde{K}$. For the covers implication proceed as follows: Since $v|_K$ is trivial, $v$ is a $\tilde{K}$-valuation of $\tilde{K}_t = \tilde{K}(t)$, and $v \neq v_{\tilde{K},\infty}$ implies that $v = v_{\tilde{K},q}$ is the $q$-adic $\tilde{K}$-valuation for some monic irreducible polynomial $q \in \tilde{K}[t]$. Since $v|_{\tilde{k}_t}$ is nontrivial, there exists a unique monic irreducible $p \in \tilde{k}[t]$ such that $v|_{\tilde{k}_t} = v_{\tilde{k},p}$, and since $\tilde{k} = \overline{k} \cap \tilde{K}$ by hypothesis, one has that $p \in \tilde{K}[t]$ is irreducible. On the other hand, since $v_{\tilde{k},p} = v_{\tilde{K},q}|_{\tilde{k}_t}$, one must have $v_{\tilde{K},q}(p) = v_{\tilde{k},p}(p) > 0$. Hence $q|p$ in $\tilde{K}[t]$, thus $p = q$ (because both $p, q$ are irreducible monic).]

**Key Lemma 4.2.** *In the above notation from Fact 4.1, denote* $\tilde{K} = K\tilde{k}$ *and for* $\tilde{\mathfrak{v}} \in \mathrm{Val}_k(\tilde{k}_t)$ *and* $\tilde{w}_{\tilde{\mathfrak{v}}} \in \mathrm{Val}_k(\tilde{K}_t)$ *set* $\tilde{\mathfrak{w}} := \tilde{w}_{\tilde{\mathfrak{v}}}|_{\tilde{K}} \in \mathrm{Val}_k(K)$. *Then there is* $\tilde{v}|\mathfrak{v}$ *such that* $\tilde{\mathfrak{w}} := \tilde{w}_{\tilde{\mathfrak{v}}}|_{\tilde{K}} \in \mathrm{Val}(K)$ *is non-trivial, and non-trivial* $\tilde{\mathfrak{w}}$ *and* $\mathfrak{w} := \tilde{\mathfrak{w}}|_K = \tilde{w}_{\tilde{\mathfrak{v}}}|_K$ *satisfy:*

1) $\tilde{\mathfrak{w}} \in \mathrm{Val}_k(K)$ *depends on* $s_K^a$ *only and not on the specific* $\tilde{\mathfrak{v}} \in \mathrm{Val}_k(\tilde{k}_t)$ *used to define it.*

2) $\tilde{K}\tilde{\mathfrak{w}}\,|\,k$ *is algebraic,* $K\mathfrak{w} \cap \tilde{k} = k_t\mathfrak{v} \cap \tilde{k}$, *and* $K\mathfrak{w}|k$ *and* $\tilde{k}|k$ *are linearly disjoint over* $k$.

*Proof.* First, if $\tilde{\mathfrak{v}} \in \mathrm{Val}_k(\tilde{k}_t)$, then either $\tilde{\mathfrak{v}} = v_\infty$ or $\mathfrak{v} = v_{k,p} = v_p$ is the $p$-adic valuation for a unique $p \in \tilde{k}[t]$ monic irreducible and the degree $d_{\tilde{\mathfrak{v}}}$ is $d_{\tilde{\mathfrak{v}}} := [\tilde{k}_t\tilde{\mathfrak{v}} : \tilde{k}]$. Obviously, and one has:

$$d_{v_\infty} = 1 \text{ and } d_{v_p} = [\tilde{k}_t v_p : k] = \deg(p) \text{ is the degree of } p \in \tilde{k}[t].$$

And the same holds, correspondingly, for the $K$-valuations $\tilde{v} \in \mathrm{Val}_K(\tilde{K}_t)$.

Let $\Sigma := \mathrm{Val}_k(\tilde{k}_t)$, $\Sigma' := \{v_\infty\} \cup \{v_p \in \mathrm{Val}_k(\tilde{k}_t) \,|\, (\ell, d_{v_p}) = 1\}$ and $\Sigma'' := \Sigma\backslash\Sigma'$, thus obviously, $\Sigma = \Sigma' \cup \Sigma'' = \mathrm{Val}_k(\tilde{k}_t)$ as a disjoint union. And define $\Sigma'_K, \Sigma''_K \subset \mathrm{Val}_K(\tilde{K}_t)$ correspondingly. We notice that since $k \subset K$ is relatively algebraically closed, it follows that every monic irreducible polynomial $p \in \tilde{k}[t]$ is monic irreducible in $\tilde{K}[t]$. Hence if $v_{\tilde{K},p}$ is the prolongation of $v_p \in \mathrm{Val}_k(\tilde{k}_t)$ to $\tilde{K}_t$, then one has:

$$(*) \qquad d_{v_p} = [\tilde{k}_t v_p : \tilde{k}] = \deg(p) = [\tilde{K}_t v_{\tilde{K},p} : \tilde{K}] = d_{v_{\tilde{K},p}},$$

implying that $\Sigma' \subset \Sigma'_K$ and $\Sigma'' \subset \Sigma''_K$. Further, let $p \in \tilde{K}[t]\backslash\tilde{k}[t]$ be monic irreducible. Then $v_{\tilde{K},p}$ is trivial on $\tilde{k}_t$, implying that $\Sigma' = \mathrm{Val}_k(\tilde{k}_t) \cap \Sigma'_K$ and $\Sigma'' = \mathrm{Val}_k(\tilde{k}_t) \cap \Sigma''_K$.

Further, given $p \in \tilde{k}[t]$ monic irreducible, and $\alpha_p \in \tilde{k}_t^a$ with $\alpha_p^\ell = p$, the following hold:

a) $v_p$ is ramified in $\tilde{k}_t[\alpha_p]\,|\,\tilde{k}_t$, and $\tilde{\mathfrak{v}} \in \mathrm{Val}_k(\tilde{k}_t)$, $\tilde{\mathfrak{v}} \neq v_p, v_\infty$, are unramified in $\tilde{k}_t[\alpha_p]\,|\,\tilde{k}_t$.

b) $v_\infty$ is ramified in $\tilde{k}_t[\alpha_p]\,|\,\tilde{k}_t$ iff $\ell \nmid d_p$.

Recall the exact sequence $1 \to \tilde{k}_t^\times \xrightarrow{\imath} \tilde{k}^\times \oplus_{\tilde{\mathfrak{v}}\in\Sigma} \tilde{\mathfrak{v}}\tilde{k}_t \xrightarrow{\deg} \mathbb{Z} \to 0$ with $\imath(f) = a_f \oplus_{\tilde{\mathfrak{v}}} \tilde{\mathfrak{v}}(f)$, $a_f$ the leading coefficient of $f$ and $\deg = \sum_{\tilde{\mathfrak{v}}} d_{\tilde{\mathfrak{v}}}$, and tensoring with $\mathbb{Z}/\ell$, on gets an exact the exact sequence $1 \to \tilde{k}_t^\times/\ell \to \tilde{k}^\times/\ell \oplus_{\tilde{\mathfrak{v}}\in\Sigma} \tilde{\mathfrak{v}}\tilde{k}_t/\ell \to \mathbb{Z}/\ell \to 0$. Using the latter exact sequence, by Hilbert decomposition theory and Kummer theory the following hold:

**Fact 4.3.** *The following hold:*

(I) *In the above notation, setting* $k_t^0 := \tilde{k}^a\tilde{k}_t$, $k'_t := \tilde{k}_t[\alpha_{v_p}]_{v_p\in\Sigma'}$, $k''_t := \tilde{k}_t[\alpha_{v_p}]_{v_p\in\Sigma''}$, *one has:*

1) *The fields* $k_t^0$, $k'_t$, $k''_t$ *are linearly disjoint over* $\tilde{k}_t$, *and* $\tilde{k}_t^a|\tilde{k}_t$ *is the compositum* $\tilde{k}_t^a = k_t^0 k'_t k''_t$. *Hence the Galois groups* $G^0 = G(k_t^0|\tilde{k}_t) = \mathcal{G}_{\tilde{k}}^a$, $G' = G(k'_t|\tilde{k}_t)$ *and* $G'' = G(k''_t|\tilde{k}_t)$ *satisfy:*

$$\textit{The canonical projection } \mathcal{G}_{k_t}^a \to G^0 \times G' \times G'' \textit{ is an isomorphism.}$$

2) *Concerning generation of* $G'$ *and* $G''$ *one has:*

a) *Given a fix generator* $\tau_\infty \in T_{v_\infty}^a$ *there are unique inertia generators* $(\tau_{\mathfrak{v}} \in T_{\mathfrak{v}}^a)_{\mathfrak{v}\in\Sigma'}$ *which topologically generate* $G'$ *and satisfy the unique prorelation* $\prod_{\mathfrak{v}\in\Sigma'} \tau_{\mathfrak{v}} = 1$.

b) $G''$ *is profinite-freely generated by any system of inertia generators* $(\tau_{\mathfrak{v}} \in T_{\mathfrak{v}}^a)_{\mathfrak{v}\in\Sigma''}$.

(II) *The same holds, correspondingly, for* $K_t$, *and the sets of* $K$*-valuations* $\Sigma'_K, \Sigma''_K \subset \mathrm{Val}_K(K_t)$. *Further since* $\Sigma' = \mathrm{Val}_k(k_t) \cap \Sigma'_K$ *and* $\Sigma'' = \mathrm{Val}_k(k_t) \cap \Sigma''_K$, *and one has:*

$$k^0 \subset K^0,\ k'_t \subset K'_t,\ k''_t \subset K''_t, \text{ and } k^0 = K^0 \cap k_t^a,\ k'_t = K'_t \cap k_t^a,\ k''_t = K''_t \cap k_t^a.$$

Proceed along the following steps:

**Claim 1**. In the above notation, there is $\tilde{\mathfrak{v}} \in \Sigma'$ with $\tilde{\mathfrak{w}} := \tilde{w}_{\tilde{\mathfrak{v}}}|_K$ not-trivial.

Indeed, *by contradiction,* suppose that the assertion of Claim 1 does not hold, that is, for every $\tilde{\mathfrak{v}} \in \Sigma'$ the resulting $\tilde{w}_{\tilde{\mathfrak{v}}}$ is trivial on $K$. Thus the map $\varphi : \Sigma' \to \mathrm{Val}_k(\tilde{K}_t)$, $\tilde{\mathfrak{v}} \mapsto \tilde{w}_{\tilde{\mathfrak{v}}}$ defined in Fact 4.1 has image $\mathcal{V}_k \subset \mathrm{Val}_K(K_t)$ such that, by Fact 4.1, 1), $\tilde{w}_{\tilde{\mathfrak{v}}}|_{\tilde{k}_t} = \tilde{\mathfrak{v}}$ for all $\tilde{\mathfrak{v}} \in \mathrm{Val}_k(\tilde{k}_t)$. Therefore, if $\tilde{\mathfrak{v}} = v_\infty$, then $\tilde{w}_{\tilde{\mathfrak{v}}} = v_{\tilde{K},\infty}$, and if $\tilde{\mathfrak{v}} = v_p$ with $p \in \tilde{k}[t]$ monic irreducible, then $\tilde{w}_{\tilde{\mathfrak{v}}} = v_{\tilde{K},p}$ is the $p$-adic valuation of $\tilde{K}_t$. Thus by $(*)$ above, $[\tilde{k}_t\tilde{\mathfrak{v}} : \tilde{k}] = d_p = [\tilde{K}_t v_{\tilde{K},p} : \tilde{K}]$, concluding that $\mathcal{V}_k \subset \Sigma'_K$. Hence the map below is a bijection

$$\varphi : \Sigma' \to \mathcal{V}_k \subset \Sigma'_K, \ \ \tilde{\mathfrak{v}} \mapsto \tilde{w}_{\tilde{\mathfrak{v}}} \ \text{ with } \mathcal{V}_k \subset \Sigma'_K \text{ stictly,}$$

which via $s^a_t : G(\tilde{k}^a_t|k_t) \to G(\tilde{K}^a_t|K_t)$ and $p^a_{K_t} : G(\tilde{K}^a_t|K_t) \to G(\tilde{k}^a_t|k_t)$, is compatible with inertia and decomposition groups, that is: For $\tilde{\mathfrak{v}} \leftrightarrow \tilde{w}_{\tilde{\mathfrak{v}}}$ one has $s^a_t(T^a_{\tilde{\mathfrak{v}}}) = T^a_{\tilde{w}_{\tilde{\mathfrak{v}}}}$, $s^a_t(Z^a_{\tilde{\mathfrak{v}}}) \subset Z^a_{\tilde{w}_{\tilde{\mathfrak{v}}}}$, $p^a_{K_t}(Z^a_{\tilde{w}_{\tilde{\mathfrak{v}}}}) = Z^a_{\tilde{\mathfrak{v}}}$, and the residue fields satisfy $\tilde{K}_t\tilde{w}_{\tilde{\mathfrak{v}}} = K\tilde{k}_t\tilde{\mathfrak{v}}$. Let $\tau_\infty \in T_{v_\infty}$ be a fixed generator, hence $\tau_{\tilde{K},\infty} := s^a(\tau_\infty) \in T_{v_{\tilde{K},\infty}}$ generates $T_{v_{\tilde{K},\infty}}$ and $p^a_{K_t}(\tau_{\tilde{K},\infty}) = \tau_\infty$. Further, let $(\tau_{\tilde{\mathfrak{v}}} \in T^a_{\tilde{\mathfrak{v}}})_{\tilde{\mathfrak{v}}\in\Sigma'}$ and $(\tau_{\tilde{\mathfrak{v}}} \in T^a_{\tilde{\mathfrak{v}}})_{\tilde{\mathfrak{v}}\in\Sigma''}$ and $(\tau_{\tilde{w}} \in T^a_{\tilde{w}})_{\tilde{w}\in\Sigma'_K}$, $(\tau_{\tilde{w}} \in T^a_{\tilde{w}})_{\tilde{w}\in\Sigma''_K}$ be systems of inertia generators as in Fact 4.3, 2) with $\tau_{v_\infty} = \tau_\infty$, $\tau_{v_{K,\infty}} = \tau_{K,\infty}$.

**Conclude** that $\big(s^a(\tau_{\tilde{v}})\big)_{\tilde{v}\in\Sigma'} = (\tau_{\tilde{w}_{\tilde{\mathfrak{v}}}})_{\tilde{w}_{\tilde{\mathfrak{v}}}\in\mathcal{W}}$ is a proper subsystem of $(\tau_{\tilde{w}})_{\tilde{w}\in\Sigma'_K}$ such that

$$\textstyle\prod_{\tilde{w}_{\tilde{\mathfrak{v}}}\in\mathcal{W}}\tau_{\tilde{w}_{\tilde{\mathfrak{v}}}} = \prod_{\tilde{\mathfrak{v}}\in\Sigma'}s^a_t(\tau_{\tilde{\mathfrak{v}}}) = s^a_t(\prod_{\tilde{\mathfrak{v}}\in\Sigma'}\tau_{\tilde{\mathfrak{v}}}) = s^a_t(1) = 1.$$

Hence we reached a contradiction, and Claim 1 is proved.

**Claim 2**. The non-trivial valuation $\tilde{\mathfrak{w}} := \tilde{w}_{\tilde{\mathfrak{v}}}|_K$ from Claim 1 does not dependent of $\tilde{\mathfrak{v}}$.

Indeed, recall the inclusion $\varphi : \mathrm{Val}_k(\tilde{k}_t) \hookrightarrow \mathrm{Val}_k(\tilde{K}_t)$, $\tilde{\mathfrak{v}} \mapsto \tilde{w}_{\tilde{\mathfrak{v}}}$ from Fact 4.1, and by loc.cit., 2), one has $p^a(Z^a_{\tilde{w}_{\tilde{\mathfrak{v}}}}) = Z^a_{\tilde{\mathfrak{v}}} \subset \mathcal{G}^a_{\tilde{k}_t}$ and $p^a(Z_{\tilde{w}^a_{\tilde{\mathfrak{v}}}|v_{\mathfrak{v}}}) = Z_{\tilde{\mathfrak{v}}^a|\mathfrak{v}} \subset G(\tilde{k}^a_t|k_t)$. That implies in terms of decomposition fields $k_{\tilde{w}} := \tilde{K}_t\tilde{w}_{\tilde{\mathfrak{v}}} \cap \tilde{k}^a = \tilde{k}_t\tilde{\mathfrak{v}} \cap \tilde{k}^a =: k_{\tilde{\mathfrak{v}}}$ as finite extension of $\tilde{k}$. Next recall the commutative diagrams:

$$\begin{array}{ccc} G(\tilde{K}^a_t|K_t) & \xrightarrow{p^a_t} & G(\tilde{k}^a_t|k_t) \\ \downarrow q^a_{K_t} & & \downarrow q^a_{k_t} \\ G(\tilde{K}^a|K) & \xrightarrow{p^a} & G(\tilde{k}^a|k) \end{array} \qquad \begin{array}{ccc} G(\tilde{K}^a_t|K_t) & \xleftarrow{s^a_t} & G(\tilde{k}^a_t|k_t) \\ \downarrow q^a_{K_t} & & \downarrow q^a_{k_t} \\ G(\tilde{K}^a|K) & \xleftarrow{s^a} & G(\tilde{k}^a|k) \end{array}$$

which give rise to commutative diagrams for the inertia/decomposition groups:

$$\begin{array}{ccc} Z_{\tilde{w}^a_{\tilde{\mathfrak{v}}}|v_{\mathfrak{v}}} & \xrightarrow{p^a_t} & Z_{\tilde{\mathfrak{v}}^a|\mathfrak{v}} \\ \downarrow q^a_{K_t} & & \downarrow q^a_{k_t} \\ G(\tilde{K}^a|K) & \xrightarrow{p^a} & G(\tilde{k}^a|k) \end{array} \qquad \begin{array}{ccc} Z_{\tilde{w}^a_{\tilde{\mathfrak{v}}}|v_{\mathfrak{v}}} & \xleftarrow{s^a_t} & Z_{\tilde{\mathfrak{v}}^a|\mathfrak{v}} \\ \downarrow q^a_{K_t} & & \downarrow q^a_{k_t} \\ G(\tilde{K}^a|K) & \xleftarrow{s^a} & G(\tilde{k}^a|k) \end{array}$$

Let $Z^a_{\tilde{\mathfrak{w}}} \subset \mathcal{G}^a_{\tilde{K}}$ be the decomposition group of a non-trivial $k$-valuation $\tilde{\mathfrak{w}} = \tilde{w}_{\tilde{\mathfrak{v}}}|_{\tilde{K}}$. Then by Hilbert decomposition theory one has that $q^a_{K_t}(Z_{\tilde{w}^a_{\tilde{\mathfrak{v}}}|v_{\mathfrak{v}}}) \subset Z_{\tilde{\mathfrak{w}}|\mathfrak{w}}$. Hence since $q^a_{K_t} \circ s^a_t = s^a \circ q^a_{k_t}$, and taking into account that $q^a_{k_t}(Z_{\tilde{\mathfrak{v}}^a|\mathfrak{v}}) = G(\tilde{k}^a|k_{\tilde{\mathfrak{v}}})$, one has finally commutative diagrams:

$$\begin{array}{ccc} Z_{\tilde{w}^a_{\tilde{\mathfrak{v}}}|v_{\mathfrak{v}}} & \xrightarrow{p^a_t} & Z_{\tilde{\mathfrak{v}}^a|\mathfrak{v}} \\ \downarrow q^a_{K_t} & & \downarrow q^a_{k_t} \\ Z_{\tilde{\mathfrak{w}}^a|\mathfrak{w}} & \xrightarrow{p^a} & G(\tilde{k}^a|k_{\tilde{\mathfrak{v}}}) \end{array} \qquad \begin{array}{ccc} Z^a_{\tilde{w}^a_{\tilde{\mathfrak{v}}}|v_{\mathfrak{v}}} & \xleftarrow{s^a_t} & Z_{\tilde{\mathfrak{v}}^a|\mathfrak{v}} \\ \downarrow q^a_{K_t} & & \downarrow q^a_{k_t} \\ Z_{\tilde{\mathfrak{w}}^a|\mathfrak{w}} & \xleftarrow{s^a} & G(\tilde{k}^a|k_{\tilde{\mathfrak{v}}}) \end{array}$$

that is, $s^a\big(G(\tilde{k}^a|k_{\tilde{\mathfrak{v}}})\big) \subset Z_{\tilde{\mathfrak{w}}^a|\mathfrak{w}}$. In particular, since $\tilde{k}^a|k$ satisfies Hypothesis (H), i.e., $\tilde{k}^a|k$ has infinite degree, and $k_{\tilde{\mathfrak{v}}}|k$ has finite degree by the discussion above, one has the following:

$(*)_{s^a}$ $\qquad \operatorname{im}(s^a) \cap Z_{\tilde{\mathfrak{w}}^a|\mathfrak{w}} \supset s^a\big(G(\tilde{k}^a|k_{\tilde{\mathfrak{v}}}) \cap Z_{\tilde{\mathfrak{w}}^a|\mathfrak{w}}$ are open subgroups of $\operatorname{im}(s^a)$.

In particular, if $\tilde{\mathfrak{v}}' \in \operatorname{Val}_k(\tilde{k}_t)$ and the resulting $\tilde{w}'_{\tilde{\mathfrak{v}}'} \in \operatorname{Val}_k(K_t)$ is such that $\tilde{\mathfrak{w}}' := \tilde{w}'_{\tilde{\mathfrak{v}}'}|_K$ is non-trivial, then the corresponding $k_{\tilde{\mathfrak{v}}'}|k$ is finite. Hence $G(\tilde{k}_{\tilde{\mathfrak{v}}'}|k) \subset G(\tilde{k}|k)$ is open and $s^a\big(G(\tilde{k}^a|k_{\tilde{\mathfrak{v}}'})\big) = \operatorname{im}(s^a) \cap Z_{\tilde{\mathfrak{w}}'^a|\mathfrak{w}'}$. Therefore, $G_{\tilde{\mathfrak{v}},\tilde{\mathfrak{v}}'} := G(\tilde{k}|k_{\tilde{\mathfrak{v}}}) \cap G(\tilde{k}_{\tilde{\mathfrak{v}}'}|k) \subset G(\tilde{k}|k)$ is an open subgroup as well, and we conclude: $s^a(G_{\tilde{\mathfrak{v}},\tilde{\mathfrak{v}}'}) \subset \operatorname{im}(s^a)$ is open, thus infinite, and one has:

$$s^a(G_{\tilde{\mathfrak{v}},\tilde{\mathfrak{v}}'}) \subset s^a\big(G(\tilde{k}^a|k_{\tilde{\mathfrak{v}}})\big) \cap s^a\big(G(\tilde{k}^a|k_{\tilde{\mathfrak{v}}'})\big) \subset Z_{\tilde{\mathfrak{w}}^a|\mathfrak{w}} \cap Z_{\tilde{\mathfrak{w}}'^a|\mathfrak{w}'}$$

is an infinite group. On the other hand, the $k$-valuations of $\tilde{K}$ are discrete, because $K = k(X)$ is the function field of a $k$-curve $X$. Therefore, the non-equivalent $k$-valuation of $K$ are independent. Thus we conclude that $\tilde{\mathfrak{w}} = \tilde{\mathfrak{w}}'$ by Lemma 2.8.

Hence Claim 2 is proved.

**Claim 3.** $K\mathfrak{w}|k$ and $\tilde{k}|k$ are linearly disjoint over $k$.

Indeed, identify $G(\tilde{K}|K) =: G := G(\tilde{k}|k)$ under $G(\tilde{K}|K) \xrightarrow{\imath} G(\tilde{k}|k)$, and recall that $k_{\tilde{\mathfrak{v}}}|k \hookrightarrow \tilde{k}|k$ is a finite subextension, where $k_{\tilde{\mathfrak{v}}} := K\mathfrak{w} \cap \tilde{k}$. By Hilbert decomposition theory, $G$ acts transitively on the set $\mathcal{V}_{\mathfrak{w}}$ of prolongation $\tilde{\mathfrak{w}}'|\mathfrak{w}$ of $\mathfrak{w}$ to $\tilde{K}|K$ by $\tilde{\mathfrak{w}}^\sigma = \tilde{\mathfrak{w}} \circ \sigma$, and $Z_{\tilde{\mathfrak{w}}|\mathfrak{w}}$ is the stabilizer of $\tilde{\mathfrak{w}}$. Setting $\tilde{\mathfrak{w}}' := \tilde{\mathfrak{w}}^\sigma$ and $\tilde{\mathfrak{v}}' := \tilde{\mathfrak{v}}^\sigma$, one has: First, if $\sigma^a \mapsto \sigma$ under $G(\tilde{k}^a|k) \to G$, then $\sigma(k_{\tilde{\mathfrak{v}}}) = k_{\tilde{\mathfrak{v}}'}$, thus $G(\tilde{k}|k_{\tilde{\mathfrak{v}}'}) = G(\tilde{k}|k_{\tilde{\mathfrak{v}}})^{\sigma^a}$ is open subgroup of $G(\tilde{k}|k)$. Second, if $\sigma^a_K \mapsto \sigma$ under $G(\tilde{K}^a|k) \to G$, then $Z_{\tilde{\mathfrak{w}}'^a|\mathfrak{w}} = Z_{\tilde{\mathfrak{w}}^a|\mathfrak{w}}^{\sigma^a_K}$ inside $G(\tilde{K}^a|K)$. In particular, choosing $\sigma^a_K = s(\sigma^a)$, thus $\sigma^a_K \in \operatorname{im}(s^a)$, the following hold:

a) $\operatorname{im}(s^a) \subset G(\tilde{K}^a|K)$ is invariant under the $\sigma^a$-conjugation.

b) $s^a\big(G(\tilde{k}^a|k_{\tilde{\mathfrak{v}}'})\big) \subset \operatorname{im}(s^a)$ is open, and so is $s^a\big(G(\tilde{k}^a|k_{\tilde{\mathfrak{v}}})\big) \subset \operatorname{im}(s^a)$.

c) $s^a\big(G(\tilde{k}^a|k_{\tilde{\mathfrak{v}}'}) = s^a\big(G(\tilde{k}^a|k_{\tilde{\mathfrak{v}}})\big)^{\sigma^a} \subset Z^{\sigma^a}_{\tilde{\mathfrak{w}}^a|\mathfrak{w}} = Z_{\tilde{\mathfrak{w}}'^a|\mathfrak{w}}$, because $s^a\big(G(\tilde{k}^a|k_{\tilde{\mathfrak{v}}})\big) \subset Z_{\tilde{\mathfrak{w}}^a|\mathfrak{w}}$.

Conclude: $G_{\tilde{\mathfrak{v}},\tilde{\mathfrak{v}}'} := s^a\big(G(\tilde{k}^a|k_{\tilde{\mathfrak{v}}'})\big) \cap s^a\big(G(\tilde{k}^a|k_{\tilde{\mathfrak{v}}})\big) \subset \operatorname{im}(s^a)$ is open in $\operatorname{im}(s^a)$, hence infinite, and

$$1 \neq G_{\tilde{\mathfrak{v}},\tilde{\mathfrak{v}}'} \subset s^a\big(G(\tilde{k}^a|k_{\tilde{\mathfrak{v}}'})\big) \cap s^a\big(G(\tilde{k}^a|k_{\tilde{\mathfrak{v}}})\big) \subset Z_{\tilde{\mathfrak{w}}'^a|\mathfrak{w}} \cap Z_{\tilde{\mathfrak{w}}^a|\mathfrak{w}}.$$

Hence arguing as in the proof of Claim 2, one gets $\tilde{\mathfrak{w}} = \tilde{\mathfrak{w}}'$. Equivalently, $\sigma \in Z_{\tilde{\mathfrak{w}}|\mathfrak{w}}$, thus finally, implying that $K\mathfrak{w} \cap \tilde{k} = k$, as claimed.

This concludes the proof of Key Lemma 4.2. □

## 4.3. **Concluding the proof of Theorem 1.5.**

In the context/notation of Theorem 1.5, let $s^a : G(\tilde{k}^a|k) \to G(\tilde{K}|K)$ be a $\boldsymbol{t}$-bitationally liftable section of $p^a : G(\tilde{K}^a|K) \to G(\tilde{k}|k)$. Then by Key Lemma 4.2, there is a unique non-trivial valuation $\tilde{\mathfrak{w}} \in \operatorname{Val}_k(K)$ which together with its restriction $\mathfrak{w} := \tilde{\mathfrak{w}}|_K$ satisfy:

$(*)$ $K\mathfrak{w}|k$ and $\tilde{k}|k$ are linearly disjoint over $k$.

Since $K = k(X)$ with $X$ a complete $k$-curve and $\mathfrak{w} \in \operatorname{Val}_k(K)$, it follows that $\mathfrak{w}$ has a center $x_{\mathfrak{w}} \in X$ such that $\mathcal{O}_{\mathfrak{w}} = \mathcal{O}_{x_{\mathfrak{w}}}$ and $\mathfrak{m}_{\mathfrak{w}} = \mathfrak{m}_{x_{\mathfrak{w}}}$, and similarly, $\tilde{\mathfrak{w}}$ has a center $x_{\tilde{\mathfrak{w}}} \in \tilde{X} = X_{\tilde{k}}$ such that $x_{\tilde{\mathfrak{w}}} \mapsto x_{\mathfrak{w}}$ under the canonical projection $\tilde{X} \to X$. In particular, $\kappa_{x_{\mathfrak{w}}} = K\mathfrak{w}$, and therefore, by $(*)$ above, it follows that $\kappa_{x_{\mathfrak{w}}}|k$ and $\tilde{k}|k$ are linearly disjoint over $k$. And $\tilde{\mathfrak{w}}|\mathfrak{w}$ are defined by the points $x_{\tilde{\mathfrak{w}}}|x_{\mathfrak{w}}$ as required in Theorem 1.5. Finally, the points $x_{\tilde{\mathfrak{w}}}|x_{\mathfrak{w}}$ are unique with the property that $\mathrm{im}(s^a) \cap Z^a_{x_{\tilde{\mathfrak{w}}}|x_{\mathfrak{w}}} \neq 1$ by the uniqueness part of the Key Lemma 4.2.

This concludes the proof of Theorem 1.5.

## 5. Final comments/Open questions

Naturally, the elephant in the room is whether the Section Conjecture holds in the geometric case, i.e., form geometrically integral normal $k$-curves $X$, where $k$ is a not $\ell$-closed for some $\ell \neq \mathrm{char}(k)$. Here is a short list of questions which might be addressed with methods similar to the ones developed in this manuscript. Here, the notations are as in sections 3 and 4 above.

0) Prove all the above results for $\ell = 2$, provided $\mathrm{char} \neq 2$ (after replacing $\mu_\ell$ by $\mu_4$).

1) Suppose that $\mu_{2\ell} \subset \tilde{k}$ and $\tilde{k}^\times/\ell$ infinite. Does the $\tilde{k}|k$-$\boldsymbol{t}$-BSC hold?

2) Replacing $\mathbb{P}^1_t$ (in the $\boldsymbol{t}$-BST) by a $k$-curve or a $k$-variety $Z$, formulate & prove the $Z$-BSC.

3) Let $k$ be Hilbertian, $X$ be proper smooth $k$-variety. Does the $t$-BSC hold for $K = k(X)$ ?

4) Let $k$ be as above. Does the BSC hold for $K_t|k_t$, e.g., for $k = k_0(u)$, $k_0 = \overline{k}_0$ ?
   - This would sharpen Bogomolov–Rovinsky–Tschinkel [BRT] over $k := k_0(t,u)$, $k_0 = \overline{k}_0$.

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