

CHARACTERIZING FINITELY GENERATED FIELDS BY A SINGLE FIELD AXIOM

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Peter Roquette zu seinem 93. Geburtstag gewidmet

1. INTRODUCTION

First-order logic naturally applies to the study of fields. Consequently, it is of interest to investigate the expressive power of first-order logic in natural classes of fields. This is well-understood in the cases of algebraically closed fields, real-closed fields and p -adically closed fields. Namely, every such field K is elementary equivalent to its “constant field” κ – the relative algebraic closure of the prime field in K –, and its first-order theory is decidable.

This article is concerned with fields which are at the centre of (birational) arithmetic geometry, namely the finitely generated fields K , which are the function fields of integral \mathbb{Z} -schemes of finite type. For such a field K , let $\text{td}(K)$ be the absolute transcendence degree, i.e., the transcendence degree over its prime field κ_0 , and define the *Kronecker dimension* of K by $\dim(K) := \text{td}(K)$ if $\text{char}(K) > 0$, and $\dim(K) := \text{td}(K) + 1$ if $\text{char}(K) = 0$.

The *Elementary Equivalence versus Isomorphism Problem*, for short EEIP, asks whether the elementary theory $\mathfrak{Th}(K)$ of a finitely generated field K (always in the language of rings) encodes the isomorphism type of K in the class of all finitely generated fields. This question goes back to the 1970s and seems to have first been posed explicitly in [P1], with the work of RUMELY [Ru], DURET [Du] and PIERCE [Pi] notable predecessors.

On the other hand, through the work of RUMELY [Ru], much more than the EEIP is known for global fields: namely, the existence of uniformly definable Gödel functions proved in that article implies that each global field K is axiomatizable by a single sentence θ_K^{Ru} in the class of global fields, i.e. θ_K^{Ru} holds in a global field L if and only if $L \cong K$. This was extended and sharpened by the second author in [P2], by showing that for every finitely generated field K of Kronecker dimension $\dim(K) \leq 2$ there exists a sentence θ_K such that θ_K holds in a finitely generated field L iff $L \cong K$ as fields.

In this note we establish the analogue of this stronger property for all finitely generated fields K , thus in particular completely resolving the EEIP; in characteristic two, though, our proof is conditional, requiring a version of resolution of singularities in algebraic geometry, which we call above \mathbb{F}_2 . (See Section 2 for the version of resolution that we need.)

Theorem 1.1. *Let K be a finitely generated field. If $\text{char}(K) = 2$ and $\dim(K) > 3$, assume that resolution of singularities above \mathbb{F}_2 holds. Then there exists a sentence θ_K in the language of rings such that any finitely generated field L satisfies θ_K if and only if $L \cong K$.*

Our approach follows an idea of SCANLON in [Sc], and thereby establishes an even stronger statement, giving information about the class of definable sets in finitely generated fields. Specifically, it shows that the class of definable sets is as rich as possible. One way of

making this precise (cf. [AKNS, Lemma 2.17]) is the following statement. (See [Sc, Section 2] or [AKNS, Section 2] for a discussion of the notion of bi-interpretability.)

Theorem 1.2. *Let K be an infinite finitely generated field. If $\text{char}(K) = 2$ and $\dim(K) > 3$, assume that resolution of singularities above \mathbb{F}_2 holds. Then K is bi-interpretable with \mathbb{Z} (where both K and \mathbb{Z} are considered as structures in the language of rings).*

Note that while this completely characterizes the definable sets in K , certain questions of uniformity across the class of finitely generated fields are left open, see e.g. [Po, Question 1.8].

The chief technical result on which the theorems above build, and indeed the result that occupies the bulk of this article, concerns a definability statement regarding *prime divisors* of finitely generated fields. In order to state this, we recall some terminology. A *prime divisor* of a finitely generated field K is any discrete valuation v whose residue field Kv is finitely generated and has $\dim(Kv) = \dim(K) - 1$. It is well-known that v being a prime divisor of K is equivalent to the latter condition $\dim(Kv) = \dim(K) - 1$ on its own, see e.g. [EP, Theorem 3.4.3]. A prime divisor is called *geometric* if $\text{char}(K) = \text{char}(Kv)$ and *arithmetic* otherwise. Throughout, we freely identify valuations v with their valuation rings \mathcal{O}_v , and in particular do not distinguish between equivalent valuations.

Since the cases $\dim(K) \leq 2$ were treated already in [P2] and [Ru], we will work under the following hypothesis:

$$(H_d) \quad \begin{cases} - K \text{ is finitely generated with } \dim(K) = d \geq 3. \\ - \text{ If } \text{char}(K) = 2 \text{ and } d > 3, \text{ resolution of singularities holds above } \mathbb{F}_2. \end{cases}$$

Theorem 1.3. *For every d , the geometric prime divisors of fields satisfying (H_d) are uniformly first-order definable. In other words, there exists a formula $\text{val}_d(X, \underline{Y})$ in the language of rings such that for every field K satisfying (H_d) and every geometric prime divisor \mathcal{O} of K there exists a tuple \underline{y} in K such that*

$$\mathcal{O} = \{x \in K : K \models \text{val}_d(x, \underline{y})\},$$

and conversely, for every tuple \underline{y} , the subset of K defined above is either a geometric prime divisor or empty.

1.1. Short historical note and the genesis of this article. The first step in the resolution of the strong form of the EEIP as mentioned in Theorem 1.1 above is RUMELY’s work [Ru], which itself builds on previous ideas of J. ROBINSON. The next major step toward the resolution of the strong EEIP was the introduction of the “Pfister form machinery” in [P1], followed by the work of POONEN [Po], providing (among other things) uniform first-order formulas to define the maximal global subfields of finitely generated fields, and SCANLON [Sc], which reduces the strong EEIP to first-order defining the prime divisors of finitely generated fields, and finally the introduction of the higher LGPs in [P2], as a tool for recovering the prime divisors. The resolution of the strong form of the EEIP as mentioned in Theorem 1.1 was originally achieved independently by the co-authors of this note, with various restrictions on the characteristic being removed in quick succession, see the preprints at [P3, P4, Di1, Di2]. The present paper represents a synthesis of the two separate approaches. The proof builds on and expands the above ideas and tools, but it is not a straightforward extension of the methods of [Ru], [P2], especially because the higher LGPs involved, cf. [K-S], [Ja], lead to some additional complications compared to the Brauer–Hasse–Noether LGP for global fields,

respectively Kato’s LGP in the case Kronecker dimension two. Finally, in this note the authors do not discuss the natural question of the complexity of the formulas describing prime divisors, thus the sentences characterizing the isomorphism type. This aspect of the problem is subject of further research. It would also be interesting to treat the EEIP for fields which are finitely generated over natural base fields such as \mathbb{C} , \mathbb{R} and \mathbb{Q}_p , cf. [P-P].

1.2. Acknowledgements. The writing of this work was completed while the first author was a postdoctoral fellow, and the second author a research professor, in a programme hosted by the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 2020 semester, which was supported by the US National Science Foundation under Grant No. DMS-1928930.

2. PRELIMINARIES: COHOMOLOGICAL LOCAL–GLOBAL PRINCIPLES (LGP)

The proof for the definability of prime divisors is based on local–global principles for certain cohomology groups over fields which were introduced in [Ka2]. These extend the well-known Brauer–Hasse–Noether LGP, in particular the injectivity of the map

$$\mathrm{Br}(K) \rightarrow \bigoplus_v \mathrm{Br}(K_v),$$

where K is a global field, the sum is over all places v of K , and K_v is the completion at v .

For arbitrary positive integers $n = mp^r$ with $(p, m) = 1$, set $\mathbb{Z}/n(i) = \mu_m^{\otimes i} \oplus W_r \Omega_{\log}^i[-i]$ for arbitrary twists $i \in \mathbb{Z}$, where $W_r \Omega_{\log}$ is the logarithmic part of the de Rham–Witt complex on the étale site on char = p , see ILLUSIE [III] for details. With these notations, one has:

$$\mathrm{H}^1(K, \mathbb{Z}/n(0)) = \mathrm{Hom}_{\mathrm{cont}}(G_K, \mathbb{Z}/n), \quad \mathrm{H}^2(K, \mathbb{Z}/n(1)) = {}_n\mathrm{Br}(K),$$

suggesting that the cohomology groups $\mathrm{H}^{i+1}(K, \mathbb{Z}/n(i))$ have a particular arithmetical significance. Noticing that K is a global field precisely if $\dim(K) = 1$, hence the Brauer–Hasse–Noether local-global principle is an LGP for $\mathrm{H}^2(K, \mathbb{Z}/n(1))$, KATO proposed that for arithmetically significant fields K with $\dim(K) = d$, e.g. for finitely generated fields, there should exist similar LGPs for $\mathrm{H}^{d+1}(K, \mathbb{Z}/n(d))$. We refer the reader to KATO’s original paper [Ka2] for details, and present only facts in special cases relevant for our needs.

Throughout the paper $\mathbf{n} = 2$.¹

For arbitrary fields F and $i \geq 0$, we denote $\mathrm{H}^{i+1}(F) := \mathrm{H}^{i+1}(F, \mathbb{Z}/n(i))$, and let $\mathrm{K}_i^M(F)$ be the i^{th} Milnor K-group of F .² By the work on the Milnor Conjectures by Kato, Gabber, Voevodsky, Rost and others, the canonical Kummer isomorphism $\mathrm{K}_1^M(F)/n \rightarrow \mathrm{H}^1(F, \mathbb{Z}/n(1))$ gives rise (via cup product) to canonical isomorphisms:

$$\mathrm{K}_i^M(F)/n \rightarrow \mathrm{H}^i(F, \mathbb{Z}/n(i)), \quad \{a_i, \dots, a_1\}/n \mapsto a_i \cup \dots \cup a_1,$$

where we write a_i for the image of $\{a_i\}/n$ under the Kummer isomorphism. In particular, recalling that $n = 2$, for $i \geq 0$ one has:

¹ Most of the facts in the remainder of this section hold for prime powers $n = \ell^e$, provided K contains μ_n .

² Note that some authors, e.g. [EKM], use the notation $\mathrm{H}^i(F) := \mathrm{H}^i(F, \mathbb{Z}/n(i))$.

- If $\text{char}(F) \neq n$, then $\mu_n \subset L$, $\text{Aut}(\mu_n) = 1$, therefore $\mu_n \cong \mathbb{Z}/n$ canonically and thus $\mathbb{Z}/n(i+1) = \mathbb{Z}/n(i)$, hence there is a canonical isomorphism:

$$\mathbb{K}_{i+1}^M(F)/n \rightarrow \mathbb{H}^{i+1}(F, \mathbb{Z}/n(i+1)) = \mathbb{H}^{i+1}(F), \quad \{a_{i+1}, \dots, a_1\}/n \mapsto a_{i+1} \cup \dots \cup a_1$$

- If $\text{char}(F) = n$, viewing $\mathbb{K}_i^M(F^s)/n$ as a G_F -module, one has canonical isomorphisms

$$\mathbb{H}^1(F, \mathbb{K}_i^M(F^s)/n) \rightarrow \mathbb{H}^1(F, \mathbb{H}^i(F^s, \mathbb{Z}/n(i))) \rightarrow \mathbb{H}^{i+1}(F)$$

There are two fundamental operations relating groups \mathbb{H}^i of different fields.

- For an arbitrary field extension $E|F$, we have the restriction map $\text{res}_{E|F} : \mathbb{H}^{i+1}(F) \rightarrow \mathbb{H}^{i+1}(E)$. We also denote this by res_w when E is the henselization or completion of F with respect to a valuation w .
- If w is a discrete valuation on F , and under some additional mild conditions if $\text{char}(Fw) = n$, we have the boundary map $\partial_w : \mathbb{H}^{i+1}(F) \rightarrow \mathbb{H}^i(Fw)$, which factors through the group $\mathbb{H}^{i+1}(F_w)$ over the completion F_w via the restriction map.³

The higher dimensional generalizations of the Brauer–Hasse–Noether LGP as proposed by KATO are about (complexes of cohomology groups involving) $\mathbb{H}^{i+1}(\kappa(x))$ for $0 \leq i \leq d$, the above res_w and ∂_w , and the residue fields $\kappa(x)$ of points $x \in X$ with $\text{codim}(x) = d - i$ of models X for the finitely generated field K ; in particular, the generic point $x = \eta_X \in X$ has $\text{codim}(\eta_X) = 0$, $\kappa(\eta_X) = K$, and the cohomology group involved is $\mathbb{H}^{d+1}(K)$. In the seminal paper [Ka2], KATO proves: Let K have $\dim(K) = 2$, and X be a proper connected regular scheme over \mathbb{Z} with function field $\kappa(X) = K$. Then setting $X^1 = \{x \in X \mid \text{codim}_X(x) = 1\}$, the local rings $\mathcal{O}_{X,x} = \mathcal{O}_{w_x}$ are the valuation rings of prime divisors w_x of K . Let K_x be the completion of K at w_x , and $\text{res}_x = \text{res}_{w_x}$, $\partial_x = \partial_{w_x}$ be correspondingly defined. Then one has:

Theorem 2.1 (KATO [Ka2], Corollary, p.145). *Suppose that K has no orderings. Then setting $\text{res}_X := \bigoplus_x \text{res}_x$ and $\partial_X := \bigoplus_x \partial_x$, the maps below*

$$\iota_X : \mathbb{H}^3(K) \xrightarrow{\text{res}_X} \bigoplus_{x \in X^1} \mathbb{H}^3(K_x) \xrightarrow{\partial_X} \bigoplus_{x \in X^1} \mathbb{H}^2(\kappa(x))$$

are well-defined, res_X is injective, and ∂_X is an isomorphism.

There is major progress on Kato’s conjectures hinted at above, see KERZ-SAITO [K-S] and JANSEN [Ja], where further literature can be found. We mention below only instances of the local-global principles under discussion which will be used in the sequel.

First, let K be finitely generated of Kronecker dimension d , $k_1 \subset K$ a relatively algebraically closed global subfield, $\mathbb{P}(k_1)$ the set of places of k_1 . Let k_{1v} be the henselization of k_1 at v if v is a finite place, the real closure of k_1 at v if v is a real place, and the algebraic closure of k_1 if v is a complex place. Let $K_v = Kk_{1v}$ be the compositum of K and k_{1v} . Then considering the canonical restriction map $\text{res}_v : \mathbb{H}^{i+1}(K) \rightarrow \mathbb{H}^{i+1}(K_v)$ for every $v \in \mathbb{P}(k_1)$, the following holds:

³ See e.g. [Ka2] for details. We will not explicitly use ∂_w later on.

Theorem 2.2 (JANNSEN [Ja], Theorem 0.4⁴). *If $n \neq \text{char}(K)$, the map $\iota_{k_1} := \bigoplus_v \text{res}_v$ below*

$$\iota_{k_1} : H^{d+1}(K) \rightarrow \bigoplus_{v \in \mathbb{P}(k_1)} H^{d+1}(K_v)$$

is well-defined and injective.

Next let R be either a finite field, or the valuation ring of a henselization of a global field k at a finite place $v \in \mathbb{P}(k)$ such that n is invertible in R (or equivalently, the residue characteristic of R is not equal to n). Let X be a proper regular connected flat R -variety, $K = \kappa(X)$ be the field of rational functions on X , $d = \dim X = \dim K$, and set $X^1 := \{x \in X \mid \text{codim}_X(x) = 1\}$. Then $\mathcal{O}_{X,x} = \mathcal{O}_{w_x}$ with $x \in X^1$ are the valuation rings of the prime divisor w_x of K having centers in X^1 on X . If K_x is the completion of K at w_x , one has the canonical maps $\text{res}_x = \text{res}_{w_x}$, $\partial_x = \partial_{w_x}$ introduced above:

$$\partial_x : H^{d+1}(K) \xrightarrow{\text{res}_x} H^{d+1}(K_x) \xrightarrow{\partial_x} H^d(\kappa(x))$$

Theorem 2.3 (KERZ-SAITO [K-S], Theorem 8.1). *In the above notation, setting $\text{res}_X := \bigoplus_x \text{res}_x$ and $\partial_X := \bigoplus_x \partial_x$, the maps below*

$$\iota_X : H^{d+1}(K) \xrightarrow{\text{res}_X} \bigoplus_{x \in X^1} H^{d+1}(K_x) \xrightarrow{\partial_X} \bigoplus_{x \in X^1} H^d(\kappa(x))$$

are well-defined, and res_X is injective.

By work of SUWA [Su] and JANNSEN [Ja], the above also works in characteristic two, albeit conditionally: Following [Ja, Definition 4.18], we say that *resolution of singularities holds above \mathbb{F}_2* if for every finite field extension $\mathbb{F} \mid \mathbb{F}_2$ the following hold: (i) For any proper integral \mathbb{F} -variety X , there is a proper birational morphism $\tilde{X} \rightarrow X$, where \tilde{X} is a smooth (or equivalently regular) \mathbb{F} -variety. (ii) For any affine smooth \mathbb{F} -variety U , there is an open immersion $U \hookrightarrow X$, where X is a projective smooth \mathbb{F} -variety, and the reduced complement of U in X is a simple normal crossings divisor.

Resolution of singularities is well known for surfaces, and is proved in dimension three by COSSART-PILTANT [C-P]. Further, if resolution of singularities above \mathbb{F}_2 holds, then any finitely generated field of characteristic two has a smooth proper model over \mathbb{F}_2 .

With our terminology in place, the following theorem is proved for $\dim(K) \leq 3$ in [Su, p. 270], and for $\dim(K)$ arbitrary in [Ja, Theorem 0.10], provided resolution of singularities holds above \mathbb{F}_2 .

Theorem 2.4. *Let K be a finitely generated field with $\text{char}(K) = 2$, and if $d = \dim(K) > 3$, suppose that resolution of singularities holds above \mathbb{F}_2 . Let X be a projective smooth model for K . Then the maps ι_X , $\text{res}_X := \bigoplus_x \text{res}_x$, and $\partial_X := \bigoplus_x \partial_x$ below are well-defined*

$$\iota_X : H^{d+1}(K, \mathbb{Z}/n(d)) \xrightarrow{\text{res}_X} \bigoplus_{x \in X^1} H^{d+1}(K_x, \mathbb{Z}/n(d)) \xrightarrow{\partial_X} \bigoplus_{x \in X^1} H^d(\kappa(x), \mathbb{Z}/n(d-1))$$

and res_X is injective.

⁴Jannsen uses completions at the places v instead of henselizations, real closures or algebraic closures. This result using completions implies the version stated.

3. CONSEQUENCES/APPLICATIONS OF THE LOCAL-GLOBAL PRINCIPLES

In this section we give a few consequences of the cohomological local-global principles mentioned above. These are about the local-global behavior of Pfister forms, which are at the core of first-order definability of prime divisors.

Let namely F be an arbitrary field, and $\mathbf{a} := (a_i, \dots, a_0)$ be a system of non-zero elements in F . The $(i+1)$ -fold Pfister form defined by $\mathbf{a} = (a_i, \dots, a_0)$ is the following quadratic form (see [EKM, 9.B]):

- If $\text{char}(F) \neq 2$, then setting⁵ $\langle\langle a \rangle\rangle = x_1^2 - ax_2^2$ for $a \in F^\times$, the $(i+1)$ -fold Pfister form defined by \mathbf{a} is $\langle\langle a_i \rangle\rangle \otimes \dots \otimes \langle\langle a_0 \rangle\rangle$. Hence $q_{\mathbf{a}} := \langle\langle \mathbf{a} \rangle\rangle = \sum_{\chi} a_{\chi} x_{\chi}^2$ has coefficients $a_{\chi} := \prod_j (-a_j)^{\chi(j)}$ and variables x_{χ} indexed by the maps $\chi : \{0, \dots, i\} \rightarrow \{0, 1\}$.
- If $\text{char}(F) = 2$, then setting $\langle\langle a \rangle\rangle := x_1^2 + x_1x_2 + ax_2^2$, the $(i+1)$ -fold Pfister form defined by \mathbf{a} is $\langle\langle \mathbf{a} \rangle\rangle = \langle\langle a_i \rangle\rangle \otimes \dots \otimes \langle\langle a_1 \rangle\rangle \otimes \langle\langle a_0 \rangle\rangle$. Hence if $a_{\chi} := \prod_j a_j^{\chi(j)}$ for $\chi : \{1, \dots, i\} \rightarrow \{0, 1\}$, one has by definition that $q_{\mathbf{a}} := \langle\langle \mathbf{a} \rangle\rangle = \sum_{\chi} a_{\chi} (x_{1\chi}^2 + x_{1\chi}x_{2\chi} + a_0x_{2\chi}^2)$.

It is well known, see [EKM, Corollary 9.10], that the $(i+1)$ -fold Pfister forms $q_{\mathbf{a}}$ defined above satisfy: $q_{\mathbf{a}}$ is isotropic iff $q_{\mathbf{a}}$ is hyperbolic. Further, recalling that $H^{i+1}(F) := H^{i+1}(F, \mathbb{Z}/n(i))$ as introduced above, by [EKM], Section 16,⁶ to every Pfister form $q_{\mathbf{a}} = \langle\langle \mathbf{a} \rangle\rangle$ or $q_{\mathbf{a}} = \langle\langle \mathbf{a} \rangle\rangle$, one can attach in a canonical way a cohomological invariant

$$e(q_{\mathbf{a}}) \in H^{i+1}(F)$$

by the rule: $e(q_{\mathbf{a}}) := a_{i \cup \dots \cup a_0}$ if $\text{char}(F) \neq 2$, respectively $e(q_{\mathbf{a}}) = a_{i \cup \dots \cup a_1 \cup \chi_{a_0}}$ if $\text{char}(F) = 2$, where $\chi_{a_0} \in H^1(G_F, \mathbb{Z}/2)$ is defined via Artin–Schreier Theory by $a_0 \in F/\wp(F)$. We recall the following fundamental fact on the behavior of $e(q_{\mathbf{a}})$. Here, assertion 1) relates to the Milnor Conjectures (cf. [EKM, Fact 16.2]), although its proof much precedes the full resolution of the latter (see [EL, Ka]), assertion 2) follows by mere definition, and assertion 3) follows from the fact that the projective quadric described by a Pfister form is smooth using [BLR, Corollary 3.6.10].

Fact 3.1. *Let $q_{\mathbf{a}}$ be a Pfister form over F , $E|F$ be a field extension, $q_{\mathbf{a},E}$ be $q_{\mathbf{a}}$ viewed over E , and $\text{res}_{E|F} : H^{i+1}(F) \rightarrow H^{i+1}(E)$ be the canonical restriction map. The following hold:*

- 1) *The Pfister form $q_{\mathbf{a}}$ is isotropic over F if and only if $e(q_{\mathbf{a}}) = 0$ in $H^{i+1}(F)$.*
- 2) *One has $e(q_{\mathbf{a},E}) = \text{res}(e(q_{\mathbf{a}}))$ under $\text{res}_{E|F} : H^{i+1}(F) \rightarrow H^{i+1}(E)$.*
- 3) *Let w be a rank-1 valuation on F . Then $q_{\mathbf{a}}$ is isotropic over the henselization with respect to w if and only if it is isotropic over the completion with respect to w .*

We conclude this preparation with the following facts scattered throughout the literature (although some of them might be new in the generality presented here); variants of these will be used later. For the reader's sake we give the (straightforward) full proofs.

Proposition 3.2. *Let F be henselian with respect to a non-trivial valuation w which is non-dyadic, i.e., if $\text{char}(F) = 0$, then $\text{char}(Fw) \neq 2$. Let $\varepsilon = (\varepsilon_r, \dots, \varepsilon_0)$ be w -units in F , $\bar{\varepsilon}$ be the image of ε under the residue map $\mathcal{O}_w^\times \rightarrow Fw$. In the above notation, the following hold:*

- 1) *Suppose that $w(\varepsilon_1 - 1) > 0$. Then $q_{\varepsilon_1, \varepsilon_0}$ is isotropic over F , thus q_{ε} is isotropic over F .*

⁵ Some other sources prefer the convention $\langle\langle a \rangle\rangle = x_1^2 + ax_2^2$.

⁶ There is a typo in [EKM], the right object to work with in the notation of [EKM] would be $H^{i+1,i}(F)$.

- 2) Let $\boldsymbol{\pi} = (\pi_s, \dots, \pi_1) \in F^s$ be such that $w(\pi_s), \dots, w(\pi_1)$ are \mathbb{F}_2 -independent in $wF/2$. Then $q_{\bar{\boldsymbol{\varepsilon}}}$ is isotropic over Fw iff $q_{\boldsymbol{\varepsilon}}$ is isotropic over F iff $q_{\boldsymbol{\pi}, \boldsymbol{\varepsilon}}$ is isotropic over F .
- 3) Suppose that $\dim(F) = r$, and $q_{\boldsymbol{\varepsilon}}$ is isotropic over $F_v = Fk_{1v}$ for all real places $v \in \mathbb{P}(k_1)$ of the constant field $k_1 \subset F$ (if there are any such places). Then $q_{\boldsymbol{\varepsilon}}$ is isotropic over F .

Proof. Let $N = 2^{r+1}$, hence $q_{\boldsymbol{\varepsilon}} = q_{\boldsymbol{\varepsilon}}(\mathbf{x})$ is a quadratic form in N variables $\mathbf{x} = (x_1, \dots, x_N)$. Since w is non-dyadic, $V(q_{\boldsymbol{\varepsilon}}) \hookrightarrow \mathbb{P}_{\mathcal{O}_w}^{N-1}$ is a smooth \mathcal{O}_w -subvariety of $\mathbb{P}_{\mathcal{O}_w}^{N-1}$, with special fiber the projective smooth Kw -variety $V(q_{\bar{\boldsymbol{\varepsilon}}}) \hookrightarrow \mathbb{P}_{Fw}^{N-1}$. Hence by Hensel's Lemma, one has:

(*) *The specialization map on rational points $V(q_{\boldsymbol{\varepsilon}})(F) \rightarrow V(q_{\bar{\boldsymbol{\varepsilon}}})(Fw)$ is surjective.*

To 1): Let $q_{\varepsilon_0} := \langle\langle \varepsilon_0 \rangle\rangle$ if $\text{char} \neq 2$, $q_{\varepsilon_0} = \langle\langle \varepsilon_0 \rangle\rangle$ if $\text{char} = 2$. If q_{ε_0} is isotropic, so is $q_{\varepsilon_1, \varepsilon_0}$, and there is nothing to prove. If q_{ε_0} is not isotropic, then $q_{\varepsilon_0}(x_1, x_2)$ is the norm form of the unramified quadratic extension $F_{\varepsilon_0}|F$ defined by ε_0 . Since $\varepsilon_1 \in 1 + \mathfrak{m}_w$, by Hensel's Lemma, the simple solution $(1, 0)$ of $q_{\bar{\varepsilon}_0}(x_1, x_2) = \bar{1}$ lifts to a solution of $q_{\varepsilon_0}(x_1, x_2) = \varepsilon_1$ in F . This forces $q_{\varepsilon_1, \varepsilon_0}$ to be isotropic.

To 2): Setting $\mathbf{x}_{\chi} = (x_{\chi, i})_{i \leq N}$, $\pi_{\chi} = \prod_i \pi_i^{\chi(i)}$ for $\chi : \{1, \dots, s\} \rightarrow \{0, 1\}$, $\mathbf{y} = (\mathbf{x}_{\chi})_{\chi}$, one has $q_{\boldsymbol{\pi}, \boldsymbol{\varepsilon}}(\mathbf{y}) = \sum_{\chi} \pi_{\chi} q_{\boldsymbol{\varepsilon}}(\mathbf{x}_{\chi})$. By (*) above, $q_{\bar{\boldsymbol{\varepsilon}}}$ is isotropic over Fw iff $q_{\boldsymbol{\varepsilon}}$ is isotropic over F , and if so, $q_{\boldsymbol{\pi}, \boldsymbol{\varepsilon}}$ is isotropic over F . For the converse, let $q_{\bar{\boldsymbol{\varepsilon}}}$ be anisotropic. Then $w(q_{\boldsymbol{\varepsilon}}(\boldsymbol{\nu})) \in 2 \cdot wF$ for all $\boldsymbol{\nu} \neq 0$ in F^N , and for $\boldsymbol{\mu} = (\boldsymbol{\nu}_{\chi})_{\chi} \neq \mathbf{0}$, one has: Since $w(\pi_{\chi}) = \sum_i \chi(i)w(\pi_i)$, and $w(q_{\boldsymbol{\varepsilon}}(\boldsymbol{\nu}_{\chi})) \in 2 \cdot wF$, and $(w(\pi_i))_i$ are independent in $wF/2$, it follows that the summands in $q_{\boldsymbol{\pi}, \boldsymbol{\varepsilon}}(\boldsymbol{\mu}) = \sum_{\chi} \pi_{\chi} q_{\boldsymbol{\varepsilon}}(\boldsymbol{\nu}_{\chi})$ have distinct values. Hence $q_{\boldsymbol{\pi}, \boldsymbol{\varepsilon}}(\boldsymbol{\mu}) \neq 0$, thus $q_{\boldsymbol{\pi}, \boldsymbol{\varepsilon}}$ is anisotropic.

To 3): Since w is non-trivial, by Abhyankar's inequality (see [EP, Theorem 3.4.3]), $e := \dim(Fw) < r = \dim(F)$.

Case 1. $\text{char}(Fw) = p > 0$. Then $e = \dim(Fw) = \text{td}(Fw)$, and $q_{\bar{\boldsymbol{\varepsilon}}}$ is a quadratic form in 2^{r+1} variables over Fw . Since $e < r$, and Fw is a C_e -field, $q_{\bar{\boldsymbol{\varepsilon}}}(\mathbf{x}) = 0$ has non-trivial solutions in Fw , i.e., $V(q_{\bar{\boldsymbol{\varepsilon}}})(Fw) \neq \emptyset$, hence by (*) above, $V(q_{\boldsymbol{\varepsilon}})(F) \neq \emptyset$. Thus $q_{\boldsymbol{\varepsilon}}$ is isotropic over F .

Case 2. $\text{char}(Fw) = 0$. By Hensel's Lemma, there are fields of representatives $E \subset F$ for Fw , any such E is relatively algebraically closed in F , thus has constant field k_1 , and $\dim(E) = e = \dim(Fw)$. Let $\boldsymbol{\gamma} = (\gamma_r, \dots, \gamma_0) \in E^{r+1}$ be the of lifting $\bar{\boldsymbol{\varepsilon}} = (\bar{\varepsilon}_r, \dots, \bar{\varepsilon}_0)$. Then $\varepsilon_i = \gamma_i \epsilon'_i$ with $\epsilon'_i \in F$ and $w(\epsilon'_i - 1) > 0$. By Hensel's Lemma, ϵ'_i is a square in F , thus $q_{\boldsymbol{\varepsilon}} \approx q_{\boldsymbol{\gamma}}$ over F . Hence $V(q_{\boldsymbol{\varepsilon}})(F_v) \neq \emptyset$ implies $V(q_{\boldsymbol{\gamma}})(F_v) \neq \emptyset$ for all real places $v \in \mathbb{P}(k_1)$. If v is a non-real place of k_1 , k_{1v} has no orderings, hence E_v has no orderings, thus $\text{cd}(E_v) \leq \dim(E_v) + 1 < r + 1$. Hence $H^{r+1}(E_v) = 0$, implying $e(q_{\boldsymbol{\gamma}, E_v}) = 0$, thus $e(q_{\boldsymbol{\gamma}, F_v}) = 0$, because by Fact 3.1, 2), one has $e(q_{\boldsymbol{\gamma}, F_v}) = \text{res}_{F_v|E_v}(e(q_{\boldsymbol{\gamma}}))$. Hence $e(q_{\boldsymbol{\gamma}, F_v}) = 0$ for all $v \in \mathbb{P}(k_1)$. Finally, since cohomology is compatible with inductive limits, by JANNSEN's Theorem 2.2, $H^{r+1}(F) \rightarrow \prod_v H^{r+1}(F_v)$ is injective. Hence $e(q_{\boldsymbol{\gamma}}) = 0$, thus $q_{\boldsymbol{\gamma}}$ is isotropic over F by Fact 3.1. Hence so is $q_{\boldsymbol{\varepsilon}}$. \square

A) Prime divisors via anisotropic k_1 -nice Pfister forms. We now state a technical condition for the Pfister forms we are going to work with. This technical condition in particular serves to ensure that orderings and dyadic places can always be eliminated from our subsequent considerations.

Definition 3.3. Let K be a field satisfying Hypothesis (H_d) from the Introduction, and $q_{\mathbf{a}}$ be a $(d+1)$ -fold Pfister form $q_{\mathbf{a}}$ defined by $\mathbf{a} := (a_d, \dots, a_1, a_0)$ with $a_i \in K^{\times}$.

- 1) $q_{\mathbf{a}}$ is called *nice* if there is a global subfield $k_1 \subset K$ such that a_1 and a_0 are elements of k_1 , and the 2-fold Pfister form $q_1 := q_{a_1, a_0}$ defined by (a_1, a_0) satisfies:
 - (*) If $v \in \mathbb{P}(k_1)$ is real, or dyadic, or $v(a_0) \neq 0$, or $v(a_1) < 0$, then q_1 is isotropic over k_{1v} .
- 2) In the above notation, we will also say that $q_{\mathbf{a}}$ is k_1 -nice. We will usually tacitly assume that k_1 is relatively algebraically closed in K .

Note that being nice is not an isometry invariant of Pfister forms, so strictly speaking it is a property of the concrete presentation; this should not lead to confusion.

Due to the results of the previous section, we now have the following local–global principle for isotropy of nice Pfister forms.

Proposition 3.4. *Let K satisfying Hypothesis (H_d) , and $k_1 \subset K$ be a global subfield. Then for every anisotropic k_1 -nice Pfister form $q_{\mathbf{a}}$ there is a geometric prime divisor w of K such that $q_{\mathbf{a}}$ is anisotropic over the completion K_w of K at w . Further, $w(a_0) = 0$, $w(a_1) \geq 0$, and $w(a_i) \neq 0$ for some $i = 1, \dots, d$.*

Proof. By Fact 3.1 1) above, proving that $q_{\mathbf{a}}$ is anisotropic over K_w is equivalent to proving that $e(q_{\mathbf{a}}) \neq 0$ over K_w under the restriction map $\text{res}_w : H^{d+1}(K) \rightarrow H^{d+1}(K_w)$. Noticing that $e(q_{\mathbf{a}}) \neq 0$ in $H^{d+1}(K)$, denote $q_1 := q_{a_1, a_0}$, and proceed as follows:

Case 1). If $\text{char}(K) = n$, then choosing a smooth projective \mathbb{F}_2 -model X for K , by JANSEN’S and SUWA’S Theorem 2.4 above, there is a prime divisor w of K , say $w = w_x$ for some point $x \in X^1$, such that $\text{res}_w(e(q_{\mathbf{a}})) \neq 0$ in $H^{d+1}(K_w)$. On the other hand, if $q_{\mathbf{a}, w}$ is the Pfister form $q_{\mathbf{a}}$ viewed over K_w , then $\text{res}_w(e(q_{\mathbf{a}})) = e(q_{\mathbf{a}, w})$ by Fact 3.1, 2). Hence $e(q_{\mathbf{a}, w}) \neq 0$, implying that $q_{\mathbf{a}, w}$ is anisotropic over K_w , or equivalently, $q_{\mathbf{a}}$ is anisotropic over K_w . Finally, if the restriction v of w to k_1 is non-trivial, one has $k_{1v} \subset K_w$, and $q_1 := q_{a_1, a_0}$ is anisotropic over k_{1v} . Hence $w(a_0) = v(a_0) = 0$ and $w(a_1) = v(a_1) \geq 0$ by the fact that q_1 is k_1 -nice, and even $w(a_1) > 0$ by Proposition 3.2, 3).

Case 2). If $\text{char}(K) \neq n$, we apply JANSEN’S Theorem 2.2 above, so there is $v \in \mathbb{P}(k_1)$ such that $e(q_{\mathbf{a}}) \neq 0$ in $H^{d+1}(K_v)$. Hence if $q_{\mathbf{a}, v}$ is the Pfister form $q_{\mathbf{a}}$ viewed over K_v , then $\text{res}_v(e(q_{\mathbf{a}})) = e(q_{\mathbf{a}, v})$, implying that $e(q_{\mathbf{a}, v}) \neq 0$ in $H^{d+1}(K_v)$. Equivalently, $q_{\mathbf{a}, v}$ is anisotropic over K_v , hence its Pfister subform $q_1 := q_{a_1, a_0}$ is anisotropic over $k_{1v} \subset K_v$, thus by condition i) of Definition 3.3 above, v is a finite non-dyadic place of k_1 . In particular, letting $R \subset k_{1v}$ be the henselization of \mathcal{O}_v , it follows that $n = 2$ is invertible in R . Let X_v be any projective R -model of K_v . Then using *prime to n -alterations*, see [ILO], Exposé X, Theorem 2.4, there is a projective regular R -variety \tilde{X} and a projective surjective R -morphism $\tilde{X} \rightarrow X_v$ defining a finite field extension $\tilde{K} | K_v$ of degree prime to n . In particular, the restriction of $e(q_{\mathbf{a}, v})$ in $H^{d+1}(\tilde{K})$ is non-zero. Hence by KERZ–SAITO Theorem 2.3 above, there exists $\tilde{x} \in \tilde{X}^1$ such that setting $\tilde{w} := w_{\tilde{x}}$, one has $\text{res}_{\tilde{w}}(e(q_{\mathbf{a}, v})) \neq 0$ in $H^{d+1}(\tilde{K}_{\tilde{w}})$. Hence letting $q_{\mathbf{a}, \tilde{w}}$ be the Pfister form $q_{\mathbf{a}}$ viewed over $\tilde{K}_{\tilde{w}}$, one has $e(q_{\mathbf{a}, \tilde{w}}) = \text{res}_{\tilde{w}}(e(q_{\mathbf{a}, v})) \neq 0$ in $H^{d+1}(\tilde{K}_{\tilde{w}})$, concluding by Fact 3.1 that $q_{\mathbf{a}, \tilde{w}}$ is anisotropic over $\tilde{K}_{\tilde{w}}$. Let $w := \tilde{w}|_K$. Then since $\tilde{K} | K$ is an algebraic extension, and \tilde{w} is a prime divisor of \tilde{K} , it follows that $w = \tilde{w}|_K$ is a prime divisor of K . Further, since $\tilde{K}_{\tilde{w}}$ is complete with respect to \tilde{w} , it follows that $\tilde{K}_{\tilde{w}}$ contains the completion K_w of K with respect to w . Hence $q_{\mathbf{a}, \tilde{w}}$ being anisotropic over $\tilde{K}_{\tilde{w}}$ implies that $q_{\mathbf{a}}$ is anisotropic of $K_w \subset \tilde{K}_{\tilde{w}}$.

Finally, since $q_{\mathbf{a}}$ is isotropic over $K_v = Kk_{1v}$ for all real or dyadic places $v \in \mathbb{P}(k_1)$, w is not dyadic. Hence by Proposition 3.2, 3), it follows that $w(a_i) \neq 0$ for some $i = 1, \dots, d$. \square

B) Abundance of anisotropic k_1 -nice Pfister forms

We saw in the last section that anisotropy of nice Pfister forms over the fields we are concerned with is always explained by completions of geometric prime divisors. In this section, we conversely prove that given any geometric prime divisor, there are many nice Pfister forms that remain anisotropic over its completion. Our actual result, Proposition 3.8 below, is more complicated to state, because we want to realize additional restrictions on the Pfister form, and work with a pair of fields L/K instead of a single field K .

Lemma 3.5. *Let l_1/k_1 be a finite separable extension of global fields, and $P \subset \mathbb{P}_{\text{fin}}(k_1)$ a finite set of finite places of k_1 . Then there exist $a_0, a_1 \in k_1^\times$ such that the Pfister form $q = q_{a_1, a_0}$ over k_1 is nice and remains anisotropic over l_1 . We can further achieve that $v(a_1) = v(a_0) = 0$ for all $v \in P$.*

Proof. We may enlarge P to contain all dyadic places of k_1 . There are infinitely many finite places of k_1 which split completely in l_1 . Pick one such place w which is not in P . Using weak approximation, choose $a_0 \in k_1^\times$ such that $w(a_0) = 0$ and the reduction of the polynomial $X^2 - X - a_0$ in k_1w is irreducible, and further $v(a_0) = 0$ for all $v \in P$.

Let $l' = k_1(\alpha_0)$, with α_0 a root of $X^2 - X - a_0$. Pick a place $w' \in \mathbb{P}_{\text{fin}}(k_1)$ distinct from w and not in P which splits completely in l' . Using strong approximation (see [CF, II.§15]), let $a_1 \in k_1^\times$ such that $w(a_1) = 1$, a_1 is positive at all real places of k_1 , a_1 is a norm of the local extension $l'_v|k_{1v}$ for all the finitely many $v \in \mathbb{P}_{\text{fin}}(k_1)$ for which $v(a_0) \neq 0$, $v(a_1) = 0$ for all $v \in P$, and $v(a_1) \geq 0$ at all $v \in \mathbb{P}_{\text{fin}}(k_1) \setminus \{w'\}$. Then q_{a_1, a_0} is anisotropic over l_1 since it is anisotropic over the completion of k_1 with respect to w by construction. It is easy to check that q_{a_1, a_0} is nice. \square

Lemma 3.6. *Let K satisfy Hypothesis (H_d) , and w be a geometric prime divisor of K . There are global subfields $k_1 \subset K$, and k_1 -algebraically independent elements $\mathbf{u} = (u_i)_{1 \leq i < d}$ of K such that w is trivial on $k_1(\mathbf{u})$ and Kw is finite separable over $k_1(\mathbf{u})$. Moreover, if $u_d \in K$ has $w(u_d) = 1$, then (u_d, \mathbf{u}) is a separating transcendence basis of $K|k_1$.*

Proof. Since w is geometric, K and Kw have the same prime field κ_0 , and are separably generated over κ_0 . Proceed as follows: (i) If $\text{char}(K) = 0$, let $(u_i)_{1 \leq i < d}$ be any w -units which lift a transcendence basis of Kw . (ii) If $\text{char}(K) > 0$, let $(u_i)_{0 \leq i < d}$ be w -units which lift a separating transcendence basis of Kw . Let $k_1 \subset K$ be the constant field in case (i), and the relative algebraic closure of $\kappa_0(u_1)$ in K in case (ii), and set $\mathbf{u} = (u_i)_{1 \leq i < d}$ in both cases. Then w is trivial on $k_1(\mathbf{u})$, and the residue of \mathbf{u} in Kw is a separating transcendence basis of Kw over k_1 . Finally, if $K|k_1(u_d, \mathbf{u})$ is not separable, then $\text{char}(K) > 0$ divides $w(u_d)$, as follows easily from the theory of algebraic function fields. \square

Definition 3.7. Let K satisfy Hypothesis (H_d) , $k_1 \subset K$ be a global subfield, and $\mathbf{t} = (t_i)_{1 \leq i < d}$ be k_1 -algebraically independent elements of K . A k_1, \mathbf{t} -test form for an element $a_d \in K^\times$ is any k_1 -nice Pfister form $q_{\mathbf{a}}$ defined by $\mathbf{a} = (a_d, a_{d-1}, \dots, a_1, a_0)$, where $(a_i)_{1 \leq i < d} = \mathbf{t} - \boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon} = (\epsilon_i)_i \in k_1^{d-2}$ satisfies: All ϵ_i , $1 < i < d$, are v -units for all $v \in \mathbb{P}_{\text{fin}}(k_1)$ with $v(a_1) > 0$.

Proposition 3.8. *Let K satisfy Hypothesis (H_d) , $L|K$ be a finite extension, and \mathcal{V}_w be the unramified prolongations $w_L|w$ of a geometric prime divisor w of K to L . Let $k_1 \subset K$ be*

a global subfield, $\mathbf{t} = (t_i)_{1 \leq i < d}$ be k_1 -algebraically independent elements of K such that w is trivial on $k_1(\mathbf{t})$, and $Kw|k_1(\mathbf{t})$ is finite separable. Then there is a Zariski open dense subset $U_{w,\mathbf{t},L} \subset k_1^{d-2}$ satisfying: For every $\epsilon = (\epsilon_i)_{1 \leq i < d} \in U_{w,\mathbf{t},L}$, there are $a_1, a_0 \in k_1$ such that setting $(a_i)_{1 \leq i < d} = \mathbf{t} - \epsilon$, one has: If $w(a_d)$ is odd, the k_1, \mathbf{t} -test form $q_{\mathbf{a}}$ for a_d defined by $\mathbf{a} = (a_d, \dots, a_0)$ is anisotropic over L_{w_L} for all $w_L \in \mathcal{V}_w$.

Proof. Let $l|Kw$ be a finite separable extension containing $L_{w_L}|Kw$ for all $w_L \in \mathcal{V}_w$. Then the normalization morphism $S \rightarrow S_{w_L} \rightarrow S_w \rightarrow \mathbb{A}_{\mathbf{t}}$ of $\mathbb{A}_{\mathbf{t}} = \text{Spec } k_1[\mathbf{t}]$ in the finite separable field extensions $l \leftarrow L_{w_L} \leftarrow Kw \leftarrow k_1(\mathbf{t})$ are finite generically separable covers, thus étale above a Zariski open dense subset $U_l \subset \mathbb{A}_{\mathbf{t}}$. Hence for $\epsilon := (\epsilon_i)_{1 \leq i < d} \in U_{w,\mathbf{t},L} := U_l(k_1)$, the points $s_{\epsilon} \mapsto s_{w_L} \mapsto s_w \mapsto \epsilon$ above ϵ are smooth, and one has: $\pi := (a_i)_{1 \leq i < d} := (t_i - \epsilon_i)_{1 \leq i < d}$ is a regular system of parameters at $s_{\epsilon} \mapsto s_w \mapsto s_w \mapsto \epsilon$, and the residue field extensions $k_1 = \kappa(\epsilon) \hookrightarrow \kappa(s_w) \hookrightarrow \kappa(s_{w_L}) \hookrightarrow \kappa(s_{\epsilon}) =: l_{\epsilon}$ are finite separable. In particular (see e.g. [Ei, Proposition 10.16]), the completion of the local ring $\mathcal{O}_{s_{\epsilon}}$ is the ring of formal powers series $\widehat{\mathcal{O}}_{s_{\epsilon}} = l_{\epsilon}[[\pi]]$ in the variables $\pi = (a_{d-1}, \dots, a_2)$ over l_{ϵ} . Hence one has $k_1(\pi)$ -embeddings

$$l = \text{Quot}(\mathcal{O}_{s_{\epsilon}}) \hookrightarrow \text{Quot}(\widehat{\mathcal{O}}_{s_{\epsilon}}) = \text{Quot}(l_{\epsilon}[[a_2, \dots, a_{d-1}]]) \hookrightarrow l_{\epsilon}((a_2)) \dots ((a_{d-1})) =: \hat{l}.$$

Let $P \subset \mathbb{P}(k_1)$ be any finite set of finite places such that all $(\epsilon_i)_{1 \leq i < d}$ are P -units, and for $l_1 := l_{\epsilon}$, consider $a_1, a_0 \in k_1$ as in Lemma 3.5. Then setting $\mathbf{a} := (a_d, \dots, a_0)$ with $(a_i)_{0 \leq i < d}$ as introduced above, and $a_d \in K$ with $w(a_d)$ odd, we claim that the k_1, \mathbf{t} -test form $q_{\mathbf{a}}$ satisfies the requirements of Proposition 3.8.

Indeed, q_{a_1, a_0} is anisotropic over $l_1 = l_{\epsilon}$, by the choice of $a_1, a_0 \in k_1$. Hence q_{π, a_1, a_0} is anisotropic over \hat{l} , by Proposition 3.2, 2), thus anisotropic over $l \subset \hat{l}$. Finally, if $u \in K$ has $w(u) = 1$, then $w_L(u) = 1$, and $L_{w_L} \subset (L_{w_L})((u)) \subset l((u))$. Set $\epsilon = (\pi, a_1, a_0)$. Since $w(a_d) = w_L(a_d)$ is odd, $q_{\mathbf{a}} = q_{a_d, \epsilon}$ is anisotropic over $l((u))$, by Proposition 3.2, 2). Thus finally, $q_{\mathbf{a}}$ is anisotropic over $L_{w_L} \subset l((u))$. \square

C) A strengthening of Proposition 3.4

In this section, we prove a strengthening of Proposition 3.4 under stronger hypotheses. Specifically, we obtain a geometric prime divisor over whose completion a given Pfister form is anisotropic, with additional information on the leading coefficient.

For the proof of the following proposition, we need the notion of the patch (or constructible) topology on the space of valuations of some field. Recall, for instance from [HK, Section 1], that this consists of the set of all valuation rings (or equivalence classes of valuations) v of the field, with the coarsest topology that makes sets of the form $\mathcal{V}_{\Sigma} := \{v \mid v(a) \geq 0 \forall a \in \Sigma\}$ open-closed, where Σ are arbitrary finite sets of non-zero field elements.

Lemma 3.9. *Let F be a field and E/F a finite extension. Then the set of (equivalence classes of) valuations w of F such that E embeds over F into the henselization of F with respect to w is open in the patch topology.*

Proof. Suppose that w is a valuation on F and E a finite subextension of the henselization F_w/F ; we also write w for the prolongation of w to F_w and its restriction to E . By Theorem 1.2 of [KN], E is generated over F by an element η whose minimal polynomial $p \in F[t]$ has coefficients that are integral with respect to w , and $w(\eta) = w(p'(\eta)) = 0$. Since F_w/F is an immediate extension, we can find an integral element $x \in F$ with the same residue as η , so that $w(p'(x)) = 0$ and $w(p(x)) > 0$ since $p(\eta) = 0$.

For any other valuation w' on F such that $p \in \mathcal{O}_{w'}[t]$, $w'(p(x)) > 0$ and $w'(p'(x)) = 0$, the polynomial p has a zero in the henselization $F_{w'}$, and therefore E embeds into $F_{w'}$. This set of valuations w' is open in the patch topology, proving the claim. \square

Proposition 3.10. *Suppose that K satisfies Hypothesis (H_d). Let $L|K$ be finite separable, and $a_d \in K$. Suppose that there are a global subfield $k_1 \subset K$, and k_1 -algebraically independent elements $\mathbf{u} = (u_i)_{1 \leq i < d}$ of K , such that setting $\mathbf{t} = \mathbf{u}^2 - \mathbf{u}$, there is a k_1, \mathbf{t} -test form $q_{\mathbf{a}}$ for a_d which is anisotropic over the fields $L_{\theta} := L(\alpha)$ with $\alpha^2 - \alpha = a_d/\theta$ for all $\theta = (a_{d-1} \cdots a_1)^N$, $N > 0$. Then there exists a prime divisor w_L of $L|k_1(\mathbf{t})$ with $w_L(a_d) > 0$ odd, and $q_{\mathbf{a}}$ anisotropic over L_{w_L} .*

In the proof it will be necessary to consider the henselization of L with respect to w_L instead of the completion, which we still notate L_{w_L} . By Fact 3.1, 3), for rank-1 valuations moving between henselization and completion does not affect isotropy of the quadratic form $q_{\mathbf{a}}$.

Proof. Since $q_{\mathbf{a}}$ is an anisotropic k_1 -nice Pfister form over $\tilde{K} := L_{\theta}$, by Proposition 3.4, there is a prime divisor \tilde{w} of \tilde{K} such that $q_{\mathbf{a}}$ is anisotropic over the completion $\tilde{K}_{\tilde{w}}$.

Claim 1. $\mathbf{a} = (a_d, \dots, a_0)$ satisfies: $\tilde{w}(a_i) \geq 0$ for $d > i \geq 0$, and $\tilde{w}(a_d) > 0$, $\tilde{w}(a_d) \geq \tilde{w}(\theta^2)$.

Proof of Claim 1. For a contradiction, suppose that $\tilde{w}(a_i) < 0$ for some i . First, if $v := \tilde{w}|_{k_1}$ is non-trivial, then $k_{1v} \subset \tilde{K}_{\tilde{w}}$, hence $q_{\mathbf{a}}$ anisotropic over $\tilde{K}_{\tilde{w}}$ implies that q_{a_1, a_0} is anisotropic over k_{1v} . Since $q_{\mathbf{a}}$ is k_1 -nice, Proposition 3.4 implies that $v(a_1) > 0$ and $v(a_0) = 0$. Thus $a_i = t_i - \epsilon_i$ with $\epsilon_i \in k_1$ all v -units, implying $\tilde{w}(\epsilon_i) = v(\epsilon_i) = 0$. Hence since $\tilde{w}(a_i) < 0$, one has $\tilde{w}(t_i) < 0$ and $t_i = u_i^2 - u_i$ in K implies that $\tilde{w}(u_i) < 0$. Therefore, $a_i = u_i^2 - u_i - \epsilon_i = u_i^2 a'_i$ with $a'_i = 1 - 1/u_i + \epsilon_i/u_i^2$ a principal \tilde{w} -unit. Then $q_{a'_i, a_0}$ is isotropic over $\tilde{K}_{\tilde{w}}$ by Proposition 3.2, 1), hence so are q_{a_i, a_0} and $q_{\mathbf{a}}$ – contradiction! Second, if $v := \tilde{w}|_{k_1}$ is trivial, $\tilde{w}(\epsilon_i) = 0$, hence $\tilde{w}(a_i) < 0$ implies $\tilde{w}(t_i) < 0$, getting a contradiction as above. In particular, $\tilde{w}(\theta) = N \sum_{0 < i < d} \tilde{w}(a_i) \geq 0$, and $\tilde{w}(\theta) > 0$ iff $\tilde{w}(a_i) > 0$ for some $0 < i < d$. It remains to show that $\tilde{w}(a_d) \geq \tilde{w}(\theta^2)$. By contradiction, let $\tilde{w}(a_d) < \tilde{w}(\theta^2)$. Then $\alpha^2 - \alpha = a_d/\theta^2$ in \tilde{K} implies $\tilde{w}(\alpha) < 0$. Hence $a_d = (\alpha\theta)^2(1 - 1/\alpha) = u^2\eta$ with $u = \alpha\theta$ and $\eta = 1 - 1/\alpha$ a principal \tilde{w} -unit, getting a contradiction as above. Finally, if $\tilde{w}(\theta) > 0$, then $\tilde{w}(a_d) > 0$ by the fact that $\tilde{w}(a_d) \geq \tilde{w}(\theta)$. And if $\tilde{w}(\theta) = 0$, then a_i are w -units for $0 \leq i < d$, and since $q_{\mathbf{a}}$ is anisotropic over \tilde{K} , by Proposition 3.2, 3), it follows that $\tilde{w}(a_d) \neq 0$, hence $\tilde{w}(a_d) \geq \tilde{w}(\theta^2)$ implies $\tilde{w}(a_d) > 0$. Claim 1 is proved.

Next for integers $N > 0$, let $\mathcal{V}_{\mathbf{a}, N}$ be the set of a valuations w on L satisfying the conditions:

- i) $q_{\mathbf{a}}$ is anisotropic over the henselization L_w .
- ii) $w(a_i) \geq 0$ for $0 \leq i < d$, and $w(a_d) > Nw(a_i)$ for $0 \leq i < d$.

We claim that $\mathcal{V}_{\mathbf{a}, N}$ is closed in the patch topology. Indeed, let $w' \in \text{Val}_L$ lie in the closure of $\mathcal{V}_{\mathbf{a}, N}$ in Val_L . Then by definition of the patch topology, w' satisfies condition (ii). For condition (i), by contradiction, suppose that $q_{\mathbf{a}}$ is isotropic over the henselization $L_{w'}$. Then there is a finite subextension $L'|L$ of $L_{w'}$ such that $q_{\mathbf{a}}$ is isotropic over any field extension of L containing L' . Hence by Lemma 3.9, w' has a neighborhood $\mathcal{V}_{L'}$ such that for all $w \in \mathcal{V}_{L'}$ one has $L' \hookrightarrow L_w$. Hence choosing $w \in \mathcal{V}_{L'} \cap \mathcal{V}_{\mathbf{a}, N}$, it follows that $q_{\mathbf{a}}$ is isotropic over L_w , contradiction!

Let $\theta = (a_0 \cdots a_{d-1})^N$. Then Claim 1 applied for θ implies that each $\mathcal{V}_{\mathbf{a}, N}$ is non-empty: Namely, the \tilde{w} from Claim 1 in that case satisfies $\tilde{w}(a_i) \geq 0$ for all $0 < i < d$, and further,

$\tilde{w}(a_d) \geq 2N(\tilde{w}(a_{d-1}) + \dots + \tilde{w}(a_0))$ together with $w(a_d) > 0$ implies that $\tilde{w} \in \mathcal{V}_{\mathbf{a}, N}$. Because $\mathcal{V}_{\mathbf{a}, N+1} \subset \mathcal{V}_{\mathbf{a}, N}$, it follows from compactness of the valuation space that $\mathcal{V}_{\mathbf{a}} := \bigcap_N \mathcal{V}_{\mathbf{a}, N}$ is non-empty, and by mere definitions, every $w \in \mathcal{V}_{\mathbf{a}}$ satisfies: $q_{\mathbf{a}}$ is anisotropic over $L_{w_{\mathbf{a}}}$, and $w(a_i) \geq 0$, $w(a_d) > Nw(a_i)$ for all $N > 0$ and $0 \leq i < d$. Finally, let $\mathfrak{p} \subset \mathcal{O}_w$ be the minimal prime ideal with $a_d \in \mathfrak{p}$, and $\mathcal{O}_{w_L} = (\mathcal{O}_w)_{\mathfrak{p}}$ be the corresponding valuation ring of L ; this is the coarsest coarsening of w with $w_L(a_d) > 0$. Then $w(a_d/a_i^N) \geq 0$ for all $N > 0$ implies $a_i \notin \mathfrak{p}$, hence a_i are w_L -units for $0 \leq i < d$; second, one has an inclusion of henselizations $L_{w_L} \subset L_w$. In particular, $q_{\mathbf{a}}$ is not isotropic over the henselization L_{w_L} .

Claim 2. w_L is trivial on $k_1(\mathbf{t})$, hence w_L is a prime divisor of $L|k_1(\mathbf{t})$.

Proof of Claim 2. By contradiction, suppose that w_L is not trivial on $k_1(\mathbf{t})$. Let $F \subset L_{w_L}$ be the relative algebraic closure of $k_1(\mathbf{t})$ in L_{w_L} , set $\boldsymbol{\varepsilon} := (a_{d-1}, \dots, a_1, a_0)$, and $w := (w_L)|_F$. Then $q_{\boldsymbol{\varepsilon}}$ is defined over F , and w is a non-trivial henselian valuation of F such that all entries a_i of $\boldsymbol{\varepsilon}$ are w -units. In particular, since w is non-trivial, by Abhyankar's inequality one has $\dim(Fw) < \dim(F) \leq d-1$, hence by Proposition 3.2, 3), it follows that $q_{\boldsymbol{\varepsilon}}$ is isotropic over F , hence over L_{w_L} , because $F \subset L_{w_L}$. Since $q_{\boldsymbol{\varepsilon}}$ is a Pfister subform of $q_{\mathbf{a}}$, it follows that $q_{\mathbf{a}}$ is isotropic over L_{w_L} , contradiction! Claim 2 is proved.

We conclude the proof of Proposition 3.10 as follows: By contradiction, suppose that $w_L(a_d)$ is even. Equivalently, one can write $a_d = u^2 a'_d$ with $u, a'_d \in L_{w_L}$ and a'_d some w_L -unit. Then setting $\mathbf{a}' := (a'_d, a_{d-1}, \dots, a_0)$, it follows that $q_{\mathbf{a}} \approx q_{\mathbf{a}'}$ over $F := L_{w_L}$, and $q_{\mathbf{a}'}$ is anisotropic over F , because $q_{\mathbf{a}}$ is so. Since $w := w_L$ is non-trivial, one has $\dim(Fw) < \dim(F) = d$. Hence by Proposition 3.2, 3), $q_{\mathbf{a}'}$ is isotropic, contradiction! \square

4. UNIFORM DEFINABILITY OF THE GEOMETRIC PRIME DIVISORS OF K

In this section we show that geometric prime divisors of finitely generated fields are uniformly first order definable. This relies in an essential way on the consequences of the cohomological principles presented in the previous section, and on the (obvious) fact that for an n -fold Pfister form $q_{\mathbf{a}}$, whether that $q_{\mathbf{a}}$ is (an)isotropic, or universal, over K and/or $\tilde{K} = K[\sqrt{-1}]$ is expressed by formulae in which the n entries in $\mathbf{a} = (a_n, \dots, a_1)$ are the only free variables. Further, the Kronecker dimension $\dim(K)$, the relatively algebraically closed global subfields $k_1 \subset K$ of finitely generated fields K , and algebraic independence over such fields k_1 are uniformly first order definable (by POP [P1], POONEN [Po]). Finally recall that if $\dim(K) = 1$, i.e., K is a global field, the prime divisors of K are uniformly first-order definable by the formulae \mathbf{val}_1 given by RUMELY [Ru] – in particular, this holds for subfields $k_1 \subset K$ as above; and if $\dim(K) = 2$, the *geometric* prime divisors of $K|k_1$ are uniformly first-order definable by the formulae \mathbf{val}_2 given by POP [P2]. Hence it is left to consider the case $\dim(K) > 2$, i.e. K satisfying Hypothesis (H_d) from Introduction.

Notations 4.1. Let K satisfy Hypothesis (H_d) from Introduction. For $a_d \in K^\times$ consider:

- relatively algebraically closed global subfields $k_1 \subset K$, and $a_1, a_0 \in k_1^\times$ such that q_{a_1, a_0} is a k_1 -nice Pfister form.
- k_1 -algebraically independent elements $\mathbf{u} = (u_i)_{1 < i < d}$ of K , and set $\mathbf{t} = (t_i)_i = \mathbf{u}^2 - \mathbf{u}$, and let $k_{\mathbf{t}} \subset K$ be the relative algebraic closure of $k_1(\mathbf{t})$ in K .
- $\boldsymbol{\varepsilon} = (\varepsilon_i)_{1 < i < d} \in k_1^{d-2}$ systems of elements of k_1 which are v -units for all v with $v(a_1) > 0$.
- Setting $\mathbf{a} = (a_d, \dots, a_1, a_0)$, where $a_i = t_i - \varepsilon_i$ for $1 < i < d$, consider $q_{\mathbf{a}}$.

e) For $\theta, \tau \in K$, set $K_\tau := K(\beta)$, $\beta^2 - \beta = \tau^2/a_d$, $K_\theta = K(\alpha)$, $\alpha^2 - \alpha = a_d/\theta^2$, $K_{\tau,\theta} := K_\theta K_\tau$.

Proposition 4.2. *In the above notation, for $\tau \in K$ the following are equivalent:*

- (i) *There exists $q_{\mathbf{a}}$ as in Notations 4.1 which is anisotropic over $K_{\tau,\theta}$ for all $\theta \in k_{\mathbf{t}}$.*
- (ii) *There is a geometric prime divisor w of $K|k_1(\mathbf{t})$ with $w(\tau^2) > w(a_d) > 0$ and $w(a_d)$ odd.*

Proof. To (i) \Rightarrow (ii): This follows from Proposition 3.10 by setting $L = K_\tau$. Indeed, if $q_{\mathbf{a}}$ is anisotropic over $L_\theta = K_{\theta,\tau}$ for all $\theta \in k_{\mathbf{t}}$, then $q_{\mathbf{a}}$ is anisotropic over L_θ for all $\theta = (a_{d-1} \dots a_1)^N$, $N > 0$. Hence if w_L is the valuation of L given in loc.cit., then $w_L(a_d) > 0$ is odd. By contradiction, assume $w_L(\tau^2) \leq w_L(a_d)$, hence $w_L(\tau^2) < w_L(a_d)$, because $w_L(a_d)$ is odd. Then $w_L(\tau^2/a_d) < 0$, hence $w_L(\beta) < 0$, so $a'_d := 1 - 1/\beta$ is a principal w_L -unit, thus $q_{a'_d, a_0}$ is isotropic over L_{w_L} by Proposition 3.2, 1). Since $a_d = (a_d\beta/\tau)^2(1 - 1/\beta)$, one has $q_{a_d, a_0} \approx q_{a'_d, a_0}$ over L_{w_L} , hence q_{a_d, a_0} is isotropic over L_{w_L} , hence $q_{\mathbf{a}}$ is isotropic over L_{w_L} , contradiction!

To (ii) \Rightarrow (i): This follows from Proposition 3.8 as follows. Namely given the prime divisor w , proceeding as in Lemma 3.6, choose $k_1 \subset K$ and \mathbf{u} such that w is trivial on $k_1(\mathbf{u})$ and $Kw|k_1(\mathbf{u})$ is finite separable. Set $\mathbf{t} := \mathbf{u}^2 - \mathbf{u}$. Then $k_1(\mathbf{u})|k_1(\mathbf{t})$ is a finite abelian extension, hence $Kw|k_1(\mathbf{t})$ is finite separable. Hence by Proposition 3.8, there are $\epsilon = (\epsilon_i)_{1 < i < d} \in k_1^{d-2}$ and $a_1, a_0 \in k_1$ such that setting $a_i = t_i - \epsilon_i$, $1 < i < d$, the quadratic form $q_{\mathbf{a}}$ defined by $\mathbf{a} = (a_d, \dots, a_0)$ is a k_1, \mathbf{t} -test form for a_d which is anisotropic over K_w , in particular, $w(a_d) > 0$ is odd. Then $\alpha, \beta \in K_w$ by the fact that $w(\tau^2/a_d), w(a_d/\theta^2) > 0$. \square

Next for $\mathbf{t} = (t_i)_{1 < i < d}$ as in Notations 4.1 above, recall that $k_{\mathbf{t}} \subset K$ is uniformly first-order definable in terms of \mathbf{t} by POONEN's result. Since $\text{td}(K|k_{\mathbf{t}}) = 1$, one has $K = k_{\mathbf{t}}(C)$ for a (unique) projective normal $k_{\mathbf{t}}$ -curve C , and the prime divisors w of $K|k_{\mathbf{t}}$ are in bijection with the closed points $P \in C$ via $\mathcal{O}_w = \mathcal{O}_P$. By Riemann-Roch (for the projective normal $k_{\mathbf{t}}$ -curve C), for every $P \in C$ and $m \gg 0$, there are functions $f \in K$ with $(f)_\infty = mP$, hence $a_d := 1/f$ has $P \in C$ as its unique zero, and $v_P(a_d) = m$. Further, $\mathfrak{a} := \{\tau \in K \mid v_P(\tau^2) > v_P(a_d)\} \subset \mathcal{O}_P$ is an ideal, and $\mathcal{O}_P = \{a \in K \mid a \cdot \mathfrak{a} \subset \mathfrak{a}\}$. Hence noticing that we can always choose $m \gg 0$ odd, one has the following:

Recipe 4.3. One gets the valuation rings of all the geometric prime divisors w of K along the following steps:

- 1) Consider the uniformly first-order definable $k_1, \mathbf{t} = (t_i)_{1 < i < d}$, and $k_1 \subset k_{\mathbf{t}} \subset K$ as above.
- 2) For all $\mathbf{a} = (a_d, \dots, a_1, a_0)$ as in Notations 4.1, define:

$$\mathfrak{a} := \{\tau \in K \mid q_{\mathbf{a}} \text{ is not isotropic over } K_{\tau,\theta} \ \forall \theta \in k_{\mathbf{t}}\}, \quad \mathcal{O}_{\mathfrak{a}} := \{a \in K \mid a \cdot \mathfrak{a} \subset \mathfrak{a}\}$$

- 3) Check whether $\mathcal{O}_{\mathfrak{a}}$ is a valuation ring. If so, $w_{\mathcal{O}_{\mathfrak{a}}}$ is a geometric prime divisor of $K|k_{\mathbf{t}}$.
- 4) Finally, all the valuation rings of geometric prime divisors of K arise in the form above.

Conclude that the geometric prime divisors of K are uniformly first-order definable.

5. PROOF OF THE MAIN THEOREM

We will now prove that every field satisfying Hypothesis (H_d) is bi-interpretable with the ring \mathbb{Z} , building on the uniform definability of the geometric prime divisors. The insight that this is possible is due to SCANLON [Sc] (more precisely one can use [Sc, Theorem 4.1], because the part of the proof needed here is not affected by the gap in the recipe of the definability

of prime divisors in that paper). For the convenience of the reader, we instead build on the later [AKNS], where the bi-interpretability result is established for finitely generated integral domains (as well as some other rings).

Proposition 5.1. *Let K satisfying Hypothesis (H_d) , \mathcal{T} denote a transcendence basis of K , and $R_{\mathcal{T}}$ be the integral closure in K of the subring generated by \mathcal{T} . Then the rings $R_{\mathcal{T}}$ are uniformly first order definable finitely generated domains.*

Proof. First, let $R_{\mathcal{T}}^0 := \mathbb{F}_p[\mathcal{T}]$ if $\text{char}(K) = p$, and $R_{\mathcal{T}}^0 := \mathbb{Z}[\mathcal{T}]$ if $\text{char}(K) = 0$. By [Ei], Corollary 13.13, and Prop. 13.14, $R_{\mathcal{T}}$ is a finite $R_{\mathcal{T}}^0$ -module, hence a finitely generated ring.

Next let $\kappa \subset K$ be the constant field of K , and recall that both: By POONEN [Po], κ is uniformly first order definable, and if $\text{char}(K) = 0$, or equivalently κ is a number field, by RUMELY [Ru], the ring of integers $A = \mathcal{O}_{\kappa}$ is uniformly first-order definable. To fix notations, let $A = \kappa$ if $\text{char}(K) > 0$, respectively $A = \mathcal{O}_{\kappa}$ if $\text{char}(K) = 0$, and \mathcal{D}_{κ} be the non-archimedean places of κ , including the trivial valuation. Let $e = \text{td}(K) = \text{td}(K|\kappa)$, i.e., $e = \dim(K) - \dim(\kappa)$. Recall that the geometric prime e -divisors of K are valuations \mathfrak{w} of K trivial on κ with value group $\mathfrak{w}K \cong \mathbb{Z}^e$ lexicographically ordered. Equivalently, $\mathfrak{w} = w_e \circ \dots \circ w_1$, where w_{i+1} is a geometric prime divisor of $\kappa(w_i)$, $1 \leq i < e$. In particular, by induction on i , it follows that the set $\mathcal{D}^e(K|\kappa)$ of geometric prime e -divisors $\mathfrak{w} = w_e \circ \dots \circ w_1$ of K is a first-order definable family by POP [P2] if $\dim(K) = 2$, and Theorem 1.3 above if $\dim(K) > 2$. Further, the residue fields $\kappa_{\mathfrak{w}} := K\mathfrak{w}$ are finite extensions of κ , and the non-archimedean places $v \in \mathcal{D}_{\kappa_{\mathfrak{w}}}$ of $\kappa_{\mathfrak{w}}$, hence the integral closures $A_{\mathfrak{w}}|A$ of A in $\kappa_{\mathfrak{w}}$ are uniformly first-order definable by [Ru]. For every $\mathfrak{w} \in \mathcal{D}^e(K|\kappa)$ and every prime divisor $v \in \mathcal{D}_{\kappa_{\mathfrak{w}}}$, we set $\mathfrak{w}_v := v \circ \mathfrak{w}$, and for a given transcendence basis \mathcal{T} of $K|\kappa$, denote:

$$\mathcal{W}(K) = \{\mathfrak{w}_v \mid \mathfrak{w} \in \mathcal{D}^e(K|\kappa), v \in \mathcal{D}_{\kappa_{\mathfrak{w}}}\}, \quad \mathcal{W}_{\mathcal{T}}(K) = \{\mathfrak{w}_v \in \mathcal{V}(K) \mid \mathfrak{w}_v(\mathcal{T}) \geq 0\}.$$

Claim. One has $R_{\mathcal{T}} = \bigcap_{\mathfrak{w}_v \in \mathcal{W}_{\mathcal{T}}(K)} \mathcal{O}_{\mathfrak{w}_v}$. Hence $R_{\mathcal{T}}$ is first order definable.

Proof of the claim. Let $R_{0,\mathcal{T}} := A[\mathcal{T}] \subset \kappa[\mathcal{T}]$, and $R_{\mathcal{T}}$ be the integral closures of $R_{0,\mathcal{T}}$ in a finite extension $K|K_0$, where $K_0 = \kappa(t)$. Defining $\mathcal{W}_{\mathcal{T}}(K_0) \subset \mathcal{W}(K_0)$ for K_0 instead of K , it follows that $\mathcal{W}_{\bullet}(K)$ consists exactly of the prolongations of the members of $\mathcal{W}_{\bullet}(K_0)$ to K . Hence one has that $R'_{\mathcal{T}} := \bigcap_{\mathfrak{w}_v \in \mathcal{W}_{\mathcal{T}}(K)} \mathcal{O}_{\mathfrak{w}_v}$, $R'_{0,\mathcal{T}} := \bigcap_{\mathfrak{w}_v \in \mathcal{W}_{\mathcal{T}}(K_0)} \mathcal{O}_{\mathfrak{w}_v}$ are integrally closed, and $R'_{\mathcal{T}}$ is the integral closure of $R'_{0,\mathcal{T}}$ in K . Since R is the integral closure of $R_{0,\mathcal{T}}$ in K , it is sufficient to prove the claim in the case $K = K_0 = \kappa(\mathcal{T})$, $R_{\mathcal{T}} = R_{0,\mathcal{T}} = A[\mathcal{T}]$. Let $\mathfrak{w} = w_e \circ \dots \circ w_1$. Then one has $\mathcal{O}_{\mathfrak{w}_v} \subset \mathcal{O}_{\mathfrak{w}} \subset \mathcal{O}_{w_1}$, and therefore, $R'_{\mathcal{T}} \subset \bigcap_{w_1 \in \mathcal{D}_{\mathcal{T}}^1(K|\kappa)} \mathcal{O}_{w_1}$. On the other hand, since $\mathcal{D}_{\mathcal{T}}^1(K|\kappa)$ are of the form $\kappa[\mathcal{T}]_{(p)}$ with $p \in \kappa[\mathcal{T}]$ all the prime polynomials, it follows that $\kappa[\mathcal{T}] = \bigcap_{w_1 \in \mathcal{D}_{\mathcal{T}}^1(K|\kappa)} \mathcal{O}_{w_1}$, hence $R'_{\mathcal{T}} \subset \kappa[\mathcal{T}]$. Further, if $f(\mathcal{T}) \in A[\mathcal{T}]$, and $\mathfrak{w}_v(\mathcal{T}) \geq 0$, it follows that $\mathfrak{w}_v(f(\mathcal{T})) \geq 0$, hence $f(\mathcal{T}) \in R'_{\mathcal{T}}$, and finally, $R_{\mathcal{T}} \subset R'_{\mathcal{T}}$. To prove the converse inclusion $R_{\mathcal{T}} \supset R'_{\mathcal{T}}$, we show that if $f(\mathcal{T}) \notin R_{\mathcal{T}} = A[\mathcal{T}]$, then $f(\mathcal{T}) \notin R'_{\mathcal{T}}$. Indeed, if $f(\mathcal{T}) \notin A[\mathcal{T}]$, one must have $\text{char}(K) = 0$, hence $A = \mathcal{O}_{\kappa}$, and there exist $v \in \mathbb{P}_{\text{fin}}(\kappa)$ such that $f(\mathcal{T}) \notin \mathcal{O}_v[\mathcal{T}]$. Setting $f(\mathcal{T}) = \sum_{\mathbf{i}} a_{\mathbf{i}} \mathcal{T}^{\mathbf{i}}$, choose $a \in \kappa$ such that $v(a) = \min_{\mathbf{i}} v(a_{\mathbf{i}})$, and set $f(\mathcal{T}) = a(f_0(\mathcal{T}) + f_1(\mathcal{T}))$ with all the non-zero coefficients $b_{\mathbf{i}}$ of $f_0(\mathcal{T})$ having $v(b_{\mathbf{i}}) = 0$, and all the coefficients $b_{\mathbf{j}}$ of $f_1(\mathcal{T})$ having $v(b_{\mathbf{j}}) > 0$. Then for “sufficiently general” primitive roots of unity $\zeta = (\zeta_1, \dots, \zeta_e)$ of order prime to $\text{char}(\kappa v)$, one has: $v(f_0(\zeta)) = 0$, $v(f_1(\zeta)) > 0$, hence $v(f(\zeta)) = v(a) < 0$. Thus choosing \mathfrak{w} such that $\mathcal{T} \mapsto \zeta$ under $\mathcal{O}_{\mathfrak{w}} \rightarrow K\mathfrak{w}$, one has $\mathfrak{w}_v(\mathcal{T}) \geq 0$, but $\mathfrak{w}_v(f(\mathcal{T})) < 0$, implying that $f(\mathcal{T}) \notin R'_{\mathcal{T}}$, as claimed. The claim is proved.

This concludes the proof of Proposition 5.1. □

Remark 5.2. The first-order definition from the proof of Proposition 5.1 is easily seen to be uniform for fixed d , i.e. allowing for variables for the elements of \mathcal{T} , the defining formula can be chosen not to vary for all fields K satisfying Hypothesis (H_d) .

We are now ready to prove the bi-interpretability theorem: a field K satisfying Hypothesis (H_d) is bi-interpretable with \mathbb{Z} , where both K and \mathbb{Z} are considered as structures in the language of rings. We refer the reader to [AKNS, Section 2] for an introduction to the notion of bi-interpretability.

Proof of the bi-interpretability theorem. Let K be a field satisfying (H_d) , and $R \subseteq K$ the definable subring from Proposition 5.1 for some choice of transcendence basis \mathcal{T} . Since R is a finitely generated integral domain, it is bi-interpretable with the ring \mathbb{Z} by Theorem 3.1 of [AKNS].

The field K is interpretable in R as a localization, cf. [AKNS, Examples 2.9 (4)]. Then K is definably isomorphic to the interpreted copy of K in the definable subset $R \subseteq K$, namely by assigning to each $x \in K$ the class of pairs $(a, b) \in R \times (R \setminus \{0\})$ with $x = a/b$, and likewise R is definably isomorphic to the copy of R defined in the interpreted copy of K , namely by identifying $r \in R$ with the pair $(r, 1)$ (thought of as standing for $\frac{r}{1}$ in $\text{Frac}(R) = K$). Thus K is even bi-interpretable with R , and therefore transitively bi-interpretable with \mathbb{Z} . □

The existence of a characteristic sentence for K , i.e. the resolution of the strong form of the EEIP, now follows from [AKNS, Proposition 2.28].

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