CLASSICALLY PROJECTIVE GROUPS AND PSEUDO CLASSICALLY CLOSED FIELDS

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This is the content of the preprint with the same title published in May 1990 in the *Preprint Reihe Mathematik* of the Mathematics Institute, University of Heidelberg. The present variant is *almost* identical with the one from 1990. The main change is a correction in proof of Lemma 2.4, which was pointed out to me by Matthias Flach. I would also like to give my special thanks to P. Roquette and A. Prestel for several discussions we have had on this subject since the publication of the preprint. Finally, I would like to thank various people who showed interest in this preprint, among others B. Green, Z. Chatzidakis, F. Delon, Ido Efrat, D. Harbater, Yu. Ershov, G. Frey, D.-W. Geyer, Dan Haran, M. Jarden, E. Kani, J. Koenigsmann, F.-V. Kuhlmann, A. Macintyre, L. Moret-Bailly, L. Ribes, J. Schmidt, and many others. Concerning the present manuscript, on the one hand I have tried to take into account as many of their correction suggestions as possible; but on the other hand I did not want to change (too much) the concept of the original preprint version. The proofs are quite sketchy, and the motivation for the (new) concepts is mostly not to be found in the text. (Nevertheless, for the expert, who is familiar with previous work on PAC, PRC, and PpC fields and their Galois theory, all these facts might be quite enlightening...) As for the 10$^{\text{th}}$ anniversary of the 1990 Preprint, I am glad to say that the preprint was at least in a couple of cases inspiring... Detailed proofs and explanations are underway.

Introduction

Let K be a PAC (pseudo algebraically closed) field. Then the absolute Galois group G_K of K is projective by Ax [A], i.e., group extensions of G_K by a finite groups H

$$1 \to H \to F \xrightarrow{\pi} G_K \to 1$$

are split: There exists $\sigma : G_K \to F$ such that $\pi\sigma = id_{G_K}$. On the other hand, Gruenberg [G] showed that for a profinite projective group G, all group extensions of G by profinite groups H are split. Using Gruenberg's theorem Lubotzky – van den Dries [L–vdD] solved the *inverse absolute Galois problem for projective groups* as follows:

. *For projective profinte groups G there exist PAC fields K with $G_K \cong G$.*

Extending the category of PAC fields, one introduced the PRC (pseudo real closed) fields, the PpC (pseudo p-adically closed) fields and further, fields regularly closed with respect to finitely many henselisations and orderings, see Prestel [Pr2], Ershov [E], Jarden [J], Heinemann–Prestel [H–P], and to some extent studied their properties. The corresponding group theoretical side had its turn only after understanding what the (group theoretical)

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abstract characterization of the absolute Galois groups of such fields might be. In this process one was led to the concept of *real,* respectively *p-adically projective groups,* the absolute Galois group of PRC, respectively PpC field being real, respectively p-adically projective, see Basarab [Ba], [E] and Haran–Jarden [H–J1], [H–J2]. To prove the converse of this assertion, i.e., to solve the *inverse absolute Galois problem for real,* respectively *p-adically projective groups,* one has had to work much harder. After some partial results in [E] and [J], the full result was obtained by Haran–Jarden loc. cit.. This was done by developing some sophisticated concepts, namely the *Artin–Schreier structures* in the real case, respectively the Γ-*structures* in the p-adic case. See also Basarab loc.cit..

Adopting a different point of view, in the present paper we make an attempt to lay the foundations by which the kind of problems discussed above could be *uniformly treated.*

In Ch 1 we begin by studying what we have called *classically projective groups,* which is a special class of *relatively projective groups.* They generalize the real, resp. p-adically projective groups in a natural and non-trivial way. A classically projective group is a profinite group G endowed with a family $\mathcal{G}$ of classical subgroups which, in some sense, contains all the obstructions for the splitting of all group extensions of G by finite groups. We show that the classically projective groups are actually strongly relatively projective. This is Theorem 1.2. For the proof we use the results by Herfort–Ribes [H–R], and a characterization of "*big*" pro-solvable subgroups of usual profinite free products, see Pop [P2]. Further, dealing with strongly relatively projective groups we first prove the Separating Theorem 1.9. Finally, we give a good enough *generalization of Gruenberg's Theorem* which is our Theorem 1.10. To obtain a similar result in the case of real, resp. p-adically projective groups, Haran–Jarden developed in loc. cit. their theory of Artin-Schreier, respectively Γ-structures. Here we take a direct approach. One should also remark that in order to make the machinery developed by Haran–Jarden work, one first needs to prove that the family of obstructions $\mathcal{G}$ of the real, resp. p-adically projective group in discussion has a compact fundamental domain with respect to conjugation. This is true for real, resp. p-adically projective groups and also for separably generated groups, see loc. cit. and Haran [H1], but seems not to be generally true.

In Ch 2 we introduce and study the so called PCC (pseudo classically closed) fields which include the PAC, PRC and PpC fields. We first review some basic facts about the *classical spectra of a field,* and we prove mainly that they have good functorial and topological behavior, see Proposition 2.1, and Theorems 2.5, Theorem 2.7, as well as their corollaries. See also [Be], [B–S], [C–R], [Ro] and others. After discussing generalities about the pseudo closed fields, like the *Density and Uniqueness Theorem* 2.9 and the *Overfield Theorem* 2.12, in the last paragraph we use an idea from van den Dries [vdD] in order to construct pseudo closed fields starting with some "initial conditions", see Theorem 2.15. Similar constructions have been realized in the real and p-adic case, but using the very specific situation. See Haran–Jarden [H–J1], [H–J2] and others.

Finally, in Ch 3 we are concerned with the relationship between the relatively projective groups and the pseudo closed fields. Here we prove the main result of the paper, by showing that the inverse absolute Galois problem for classically projective groups has a solution in the category of pseudo classically closed fields:

A profinite group is classically projective if and only if it is isomorphic to the absolute Galois group of a pseudo classically closed field.

By Theorem 3.3, the assertion in one direction has a very general character. Namely, the absolute Galois group G_K of any pseudo closed field K is relatively projective. (But we cannot prove that G_K is strongly relatively projective, and it seems it is not always the case.) The opposite direction, namely the realization of a $\mathcal{G}$-projective group G as an absolute Galois group of a pseudo closed field is a much more difficult question. For such a realization some obvious restrictions on the family of obstructions $\mathcal{G}$ are necessary from the start like for instance: all finite subgroups in $\mathcal{G}$ must be isomorphic to $G_{\mathbb{R}}$, and the set all such groups must be τ^{st}-closed (why?) !... But near this kind of obstructions, which are more or less obvious, there exist ones of a much deeper and not yet understood nature, namely those making a profinite group into an absolute Galois group. Theorem 3.4 solves the inverse absolute Galois problem for a strongly relatively projective group if some initial *Galois approximation* does exist. Applying this and using Theorem 1.5, we get immediately the absolute Galois realization of classically projective groups.

For definitions and facts concerning boolean and quasi-boolean spaces, the étale topology τ^{et} and the strict topology τ^{st} on spaces of subgroups or subfields, and for questions concerning the functors "subgroups" $\mathbf{sg}(\,)$, respectively "subfields" $\mathbf{sf}(\,)$, the reader is referred to the Appendix.

1. Relatively projective groups

A) Classically projective groups

A classical field is any finite extension of $\mathbb{R}$ or of some $\mathbb{Q}_p$. A classical Galois group $\mathbb{G}$ is any profinite group which is isomorphic to the absolute Galois group of a classical field. We say that $\mathbb{G}$ is real if $\mathbb{G} \cong G_{\mathbb{R}}$, respectively p-adic if it is isomorphic to an open subgroup of $G_{\mathbb{Q}_p}$ for some prime number p. It is well known that all classical Galois groups $\mathbb{G}$ have finite corank, i.e., $\mathbb{G}$ has only finitely many open subgroups of index $\leq n$ for every positive bound n.

Definition. Let G be a profinite group.

- A subset of closed subgroups $\mathcal{G} = \mathcal{G}^G \subseteq \mathbf{sg}(G)$ is called simple classical if it is τ^{st}-closed and consists only of subgroup which are isomorphic to a given classical Galois group $\mathbb{G}$.
- A subset $\mathcal{G} \subseteq \mathbf{sg}(G)$ is called classical if it is finite union of simple classical subsets of $\mathbf{sg}(G)$.

Taking into account that a classical Galois group is not isomorphic to any of its proper subgroups (see for instance the discussion in the proof of Theorem 1.5), it follows by Appendix, 4.4, that any simple classical subset of $\mathbf{sg}(G)$ is τ^{et}-compact. Further, any classical family of subgroups of G is τ^{st}-closed. Let G be an arbitrary profinite group endowed with a subset $\mathcal{G} \subseteq \mathbf{sg}(G)$. Let further

$$(\epsilon) \qquad 1 \to H \longrightarrow F \xrightarrow{\pi} G \to 1$$

be a group extension of G by a profinite group H. We say that (ϵ) splits locally, if for all $\Gamma \in \mathcal{G}$, the corresponding "*local extensions*"

$$(\epsilon)_\Gamma \qquad 1 \to H \longrightarrow \pi^{-1}(\Gamma) \xrightarrow{\pi} \Gamma \to 1$$

are split, i.e., there exist (group theoretical) sections of π over each $\Gamma \in \mathcal{G}$. We say that (ϵ) splits globally if (ϵ) is split, i.e., there exists a right inverse σ of π.

A *local splitting* of (ϵ) is a subset $\mathcal{F} \subseteq \mathbf{sg}(G)$ which is mapped by π onto $\mathcal{G}$ and such that π is injective on all $\Phi \in \mathcal{F}$. Obviously, (ϵ) splits locally if and only if (ϵ) has local splittings. A global splitting of (ϵ) is any right inverse of π.

Definition. Let G be an arbitrary profinite group, and $\mathcal{G} \subseteq \mathbf{sg}(G)$ a subset.

- We say that G is $\mathcal{G}$-projective if any group extension of G by a finite group, which splits locally, splits globally.
- We say that G is classically projective if there exists a classical family of subgroups $\mathcal{G}$ of G such that G is $\mathcal{G}$-projective.

Our next task is to prove that classically projective groups is actually strongly relatively projective.

Definition. Let G be an arbitrary profinite group endowed with a family of subgroups $\mathcal{G} \subseteq \mathbf{sg}(G)$. We say that G is strongly $\mathcal{G}$-projective if for any group extension (ϵ) of G by a finite group as above one has: For every local splitting $\mathcal{F}$ of (ϵ) and any τ^{et}-open quasi-compact neighborhood $\mathcal{U}$ of $\mathrm{con}(\mathcal{F})$ in $\mathbf{sg}(F)$ (see Appendix, 4.10, 1)–3) for definitions), there exists a global splitting σ of π with $\sigma(\mathcal{G}) \subseteq \mathcal{U}$.

Intuitively, if G is strongly $\mathcal{G}$-projective, then any local splitting $\mathcal{F}$ of any (ϵ) can be better and better approximated by global splittings; or equivalently, the family of all global splittings is dense in the set of all local splittings.

Alternatively, one can define the relative projectivity by means of the so called *finite embedding problems.*

Definition. An embedding problem $\mathrm{EP} = (\gamma, \alpha, \mathcal{B})$ for $(G, \mathcal{G})$ consists of:

i) A diagram of homomorphisms of profinite groups of the form

$$\begin{array}{ccc} & & G \\ & & \downarrow{\scriptstyle\gamma} \\ B & \xrightarrow{\alpha} & A \end{array}$$

where α is surjective.

ii) A quasi τ^{st}-compact subset $\mathcal{B} \subseteq \mathbf{sg}(B)$ such that every $\gamma|_\Gamma$ with $\Gamma \in \mathcal{G}$ factors through $\alpha|_\Delta$ for some $\Delta \in \mathcal{B}$. Equivalently, for every $\Gamma \in \mathcal{G}$ there is $\Delta \in \mathcal{B}$ and $\beta_\Gamma : \Gamma \to \Delta$ such that $\gamma|_\Gamma = \alpha\, \beta_\Gamma$.

An embedding problem EP as above is called finite if B is a finite group.

A *solution* of EP is a group homomorphism $\beta : G \to B$ with $\alpha\beta = \gamma$. A *strong solution* of EP is a solution β satisfying $\beta(\mathcal{G}) \subseteq \mathrm{con}(\mathcal{B})$; see Appendix, 4.10, 1)–3) for definitions.

A straightforward verification using fiber products shows the following.

Basic Fact. *In the above notation, G is (strongly) $\mathcal{G}$-projective if and only if every finite embedding problem for $(G,\mathcal{G})$ has (strong) solutions.*

For two embeddings problems EP and EP^0 we say that $\mathrm{EP} \prec \mathrm{EP}^0$ if there exists a commutative diagram of the form

$$\begin{array}{ccc} B^0 & \xrightarrow{\alpha^0} & A^0 \\ {\scriptstyle \mathrm{pr}_B}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_B} \\ B & \xrightarrow{\alpha} & A \end{array}$$

with $\mathrm{pr}_B(\mathcal{B}^0) \subseteq \mathrm{con}(\mathcal{B})$. It is obvious that the relation $\prec$ defined above is a pre-order on the class of all embedding problems for $(G,\mathcal{G})$. Furthermore, if β^0 is a (strong) solution of EP^0, then $\mathrm{pr}_B \circ \beta^0$ is a (strong) solution for EP.

This observation is in particular very useful when $\mathcal{G}$ is quasi τ^{et}-compact (or τ^{st}-closed). Let namely EP be a finite embedding problem for $(G,\mathcal{G})$. Then using fiber products, one easily checks that there exists an open normal subgroup D_{EP} of G with the property: For any normal subgroup $D^0 \subseteq D_{\mathrm{EP}}$ of G, there exists a finite embedding problem $\mathrm{EP}^0 = (\gamma^0, \alpha^0, \mathcal{B}^0)$ for $(G,\mathcal{G})$ with $\mathrm{EP} \prec \mathrm{EP}^0$ and having the properties

1) γ^0 is the canonical projection $G \to G/D^0$.
2) α^0 is injective on each $\Delta^0 \in \mathcal{B}^0$.

Thus this allows to restrict the set of embedding problems we have to check in order show/check that $(G,\mathcal{G})$ is (strongly) relatively projective. Summarizing we have:

Fact 1.1. *Let $(G,\mathcal{G})$ be given, and suppose that $\mathcal{G}$ is quasi τ^{et}-compact. Then G is (strongly) $\mathcal{G}$-projective iff every finite $\mathrm{EP} = (\gamma, \alpha, \mathcal{B})$ for $(G,\mathcal{G})$ has (strong) solutions, provided*

1) *γ is the canonical projection $G \to G/\ker(\gamma)$.*
2) *α is injective on each $\Delta \in \mathcal{B}$.*

An embedding problem as above is called a special embedding problem.

Theorem 1.2. *Let G be relatively projective with respect to a classical family $\mathcal{G}$ of subgroups. Then G is strongly $\mathcal{G}$-projective.*

Proof. We begin with the following general fact, comp. Haran–Jarden [H–J2], Sect. 4.4.

Lemma 1.3. *Let G be an arbitrary profinite group, and $\mathcal{X} = (\mathbb{G}_k)_k$ a finite family of profinite groups of finite corank. For each k let $\mathcal{G}_k \subseteq \mathbf{sg}(G)$ consist of subgroups $\Gamma \cong \mathbb{G}_k$, and set $\mathcal{G} = \cup_k \mathcal{G}_k$. Suppose that G is $\mathcal{G}$-projective.*

Then any embedding problem $\mathrm{EP} = (\gamma, \alpha, \mathcal{B})$ for $(G,\mathcal{G})$ has solutions, provided A is finite and B is separable as a topological space.

Proof. The proof is almost identical with that in loc. cit., therefore we only give a sketch here. Let namely for each $\Gamma \in \mathcal{G}_k$ consider a fixed isomorphism $\tau_\Gamma : \mathbb{G}_k \to \Gamma$. Then giving an embedding problem for $(G,\mathcal{G})$ amounts to giving for each k and $\Gamma \in \mathcal{G}_k$ a homomorphism $\rho_\Gamma : \mathbb{G}_k \to B$ such that $\gamma\,\tau_\Gamma = \alpha\,\rho_\Gamma$.

Next let $\{C_n\}_n$ be a decreasing sequence of open subgroups of B satisfying the conditions $\cap_n C_n = \{1\}$ and $C_0 = \ker(\alpha)$. For $n > 0$ consider the canonical projections $B \xrightarrow{\alpha_n} B/C_n$, $B/C_n \xrightarrow{\alpha^n} A$, and $B/C_n \xrightarrow{\delta^n} B/C_{n-1}$, and set $\mathcal{B}_n = \alpha_n(\mathcal{B})$. Then $\mathrm{EP}_n = (\gamma, \alpha_n, \mathcal{B}_n)$ is a finite embedding problem for $(G,\mathcal{G})$. For every solution β of EP_n, we set $\mathcal{D}_{k,\beta} = \{\beta \circ \tau_\Gamma \mid \Gamma \in \mathcal{G}_k\}$.

Then $\mathcal{D}_{k,\beta}$ is a subset of the finite set $\mathrm{Hom}(\mathbb{G}_k, B/C_n)$. Let $\mathcal{T}_{k,n}$ denote the set of all $\mathcal{D}_{k,\beta}$ for β solution of EP_n. Obviously, $\delta^n(\mathcal{T}_{k,n}) \subseteq \mathcal{T}_{k,n-1}$. Therefore, denoting $\mathcal{T}_n = \prod_k \mathcal{T}_{k,n}$ we get a projective system of finite sets $\cdots \xrightarrow{\delta^3} \mathcal{T}_2 \xrightarrow{\delta^2} \mathcal{T}_1 \xrightarrow{\delta^1} \mathcal{T}_0$. For any $(\mathcal{D}_{k,\beta_n})_{k,n} \in \varprojlim \mathcal{T}_n$ one has $\delta^{n+1}(\mathcal{D}_{k,\beta_{n+1}}) = \mathcal{D}_{k,\beta_n}$. Hence, for every $\Gamma \in \mathcal{G}_k$, there exists $\Gamma_n \in \mathcal{G}_k$ with $\Gamma_1 = \Gamma$ and $\delta^{n+1}\beta_{n+1}\tau_{\Gamma_{n+1}} = \beta_n \tau_{\Gamma_n}$ for all n. Therefore, $(\beta_n \circ \tau_{\Gamma_n} : \mathbb{G}_k \to B/C_n)_n$ is a family of group homomorphisms which is compatible with the canonical projections δ^n. Thus there exists a homomorphism $\rho^1_\Gamma : \mathbb{G}_k \to B$ such that $\alpha_n \rho^1_\Gamma = \beta_n \circ \tau_\Gamma$ for all n. In particular, for $n = 1$ we get $\alpha_1 \rho^1_\Gamma = \beta_1 \circ \tau_\Gamma$. This being true for every $\Gamma \in \mathcal{G}$, it follows that the family $(\rho^1_\Gamma)_{\Gamma\in\mathcal{G}}$ gives rise to an embedding problem $\mathrm{EP}^1 = (\beta_1, \alpha_1, \mathcal{B}^1)$ for $(G, \mathcal{G})$. One proceeds by induction on n and obtains a sequence of group homomorphisms $\beta_n : G \to B/C_n$ which satisfy $\beta_n = \delta^{n+1}\beta_{n+1}$. Hence, $\beta = \varprojlim \beta_n$ is a solution of EP. $\square$

The next ingredient in our proof is also of general nature:

Lemma 1.4. *Let* $\mathbf{G}$ *be an arbitrary profinite group of finite corank having the* ℓ*-cohomological dimension* $\mathrm{cd}_\ell \mathbf{G} \geq 2$. *Then there exists a finite quotient* $\overline{\mathbf{G}}$ *of* $\mathbf{G}$ *with the property: If* $\widetilde{\mathbf{G}}$ *is any quotient of* $\mathbf{G}$ *which itself admits* $\overline{\mathbf{G}}$ *as quotient, then* $\mathrm{cd}_\ell \widetilde{\mathbf{G}} \geq 2$.

A quotient $\overline{\mathbf{G}}$ *of* $\mathbf{G}$ *with the property above is called an* ℓ*-big quotient of* $\mathbf{G}$.

Proof. Since $\mathrm{cd}_\ell \mathbf{G} \geq 2$ there exists an open normal subgroup D of $\mathbf{G}$ and a finite ℓ-torsion $\mathbf{G}/D$-module M such that denoting $\pi_D : \mathbf{G} \to G/D$ the canonical projection, the inflation map $\mathrm{Inf}(\pi_D) : H^2(\mathbf{G}/D, M) \to H^2(\mathbf{G}, M)$ is non-trivial. Let D_0 be the intersection of all open subgroups of $\mathbf{G}$ of index $\leq (\mathbf{G} : D)$. We claim that $\overline{\mathbf{G}} = \mathbf{G}/D_0$ does the job. Indeed, let $\phi : \mathbf{G} \xrightarrow{\widetilde{\phi}} \widetilde{\mathbf{G}} \xrightarrow{\overline{\phi}} \overline{\mathbf{G}}$ be surjective group homomorphisms. By the choice of D_0, it follows that any surjective homomorphism $\pi_D : \mathbf{G} \to \mathbf{G}/D$ factor through some surjective homomorphism $\overline{\mathbf{G}} \to \mathbf{G}/D$. Hence, $\mathrm{Hom}(\mathbf{G}, \mathbf{G}/D)$ and $\mathrm{Hom}(\overline{\mathbf{G}}, \mathbf{G}/D)$ have the same cardinality. On the other hand, since ϕ is surjective, the last set has the same cardinality as $\mathrm{Hom}(\mathbf{G}, \mathbf{G}/D) \circ \phi$. Therefore, the canonical projection π_D can be written $\pi_D = \phi_D\,\phi$ for some surjective $\phi_D : \overline{\mathbf{G}} \to \mathbf{G}/D$ and so, $\pi_D = \phi_D\,\overline{\phi}\,\widetilde{\phi}$. Now, since $\mathrm{Inf}(\pi_D)$ is non-trivial, it follows that the inflation map $H^2(\mathbf{G}/D, M) \to H^2(\widetilde{\mathbf{G}}, M)$ defined via $\phi_D\,\overline{\phi}$ is non-trivial. Thus $\mathrm{cd}_\ell \widetilde{\mathbf{G}} \geq 2$. $\square$

We now come to the proof of Theorem 1.2. Set $\mathcal{G} = \cup_k \mathcal{G}_k$ with $\mathcal{G}_k = \mathcal{G}_{\mathbb{G}_k}$, the simple classical subsets of $\mathbb{G}$. By a theorem of Tate, see e.g., [Ri] or [Se], one has: $\mathrm{cd}_\ell \mathbb{G} = 2$ for all non-real $\mathbb{G}$ and all prime numbers ℓ. For each k let us set:

i) $\overline{\mathbb{G}}_k \cong G_{\mathbb{R}}$, if $\mathbb{G}_k$ is real.

ii) $\overline{\mathbb{G}}_k$ a fixed ℓ-big quotient of $\mathbb{G}_k$ for ℓ in a fixed finite set of prime numbers X_k, if $\mathbb{G}_k$ is non-real.

Let $\gamma : G \to G/D$ be the canonical projection, and $\gamma(\mathcal{G})$ and Γ/D the images of $\mathcal{G}$, respectively $\Gamma \in \mathcal{G}$ by γ. For every $\Gamma \in \mathcal{G}_k$, let D_Γ be an open normal subgroup of G such that Γ/D_Γ has $\overline{\mathbb{G}}_k$ as quotient, and let $\mathcal{V}_{D_\Gamma}(\Gamma)$ be the τ^{st}-neighborhood of Γ defined by the D_Γ. Then, if $\Delta \in \mathcal{V}_{D_\Gamma}(\Gamma)$, we have $\Delta/D_\Gamma = \Gamma/D_\Gamma$, thus Δ/D_Γ has $\overline{\mathbb{G}}_k$ as quotient too. Using this and taking into account that $\mathcal{G}_k$ is τ^{st}-compact, it follows that there exist open normal subgroups D_k of G such that Γ/D_k has $\overline{\mathbb{G}}_k$ as quotient for all $\Gamma \in \mathcal{G}_k$. Finally, for any open normal subgroup $D \subset \cap_k D_k$ of G we obtain: Γ/D has $\overline{\mathbb{G}}_k$ as quotient for all k and $\Gamma \in \mathcal{G}_k$. Furthermore, since all $\mathcal{G}_k$ are τ^{st}-closed, after replacing D by a small enough open normal subgroup of G, we can also suppose that for all $\Gamma, \Delta \in \mathcal{G}$ we have: If $\Gamma/D = \Delta/D$ then Γ, Δ lie in the same simple set $\mathcal{G}_k$.

For each $\Gamma/D \in \gamma(\mathcal{G}_k)$, let Φ_{ϕ_k} be isomorphic copies of $\mathbb{G}_k$ indexed by the finite family of all homomorphisms $\phi_k : \mathbb{G}_k \to \Gamma/D$. Let $F_{|G/D|}$ be the profinite free product on the adjacent set $|G/D|$ of the finite group G/D. Further we set $B_D = F_{|G/D|} * \mathop{*}_{k,\phi_k} \Phi_{\phi_k}$ for the corresponding profinite free product of (profinite) groups. We define

$$\alpha_D : B_D \to G/D \quad \text{by} \quad |g| \mapsto g, \quad \alpha|_{\Phi_{\phi_k}} = \phi_k$$

It is clear that if $\Delta/D = \Gamma/D$, then the canonical projection $\gamma|_\Delta$ factors through $\alpha|_{\Phi_{\phi_k}}$ for some ϕ_k. Hence, setting $\mathcal{B}_D = \{\Phi_{\phi_k}\}_{k,\phi_k}$, we get an embedding problem for $(G, \mathcal{G})$ as follows:

$$(*) \qquad \mathrm{EP}_D = (\gamma, \alpha_D, \mathcal{B}_D)\,.$$

The embedding problem EP_D has the following universal property: If EP is any special embedding problem for $(G, \mathcal{G})$ having $\ker(\gamma) = D$, then $\mathrm{EP} \prec \mathrm{EP}_D$. Therefore, in order to show that all finite embedding problems for $(G, \mathcal{G})$ has (strong) solutions, it is sufficient to show that all embedding problems of the form EP_D have (strong) solutions.

On the other hand, B_D is separable. Hence by Lemma 1.3 above, EP_D has solutions β. We now show that any such solution β is actually a strong solution. For $\Gamma \in \mathcal{G}_k$, set $\widetilde{\Gamma} = \beta(\Gamma)$. Then $\widetilde{\Gamma}$ is a quotient of $\mathbb{G}_k$, and $\alpha(\widetilde{\Gamma}) = \Gamma/D$. Hence, by the definition of D we have:

1) $\widetilde{\Gamma}$ is real if $\mathbb{G}_k$ is so.
2) $\mathrm{cd}_\ell \widetilde{\Gamma} \geq 2$ for $\ell \in X_k$, if $\mathbb{G}_k$ is non-real.

Now suppose that for non-real $\mathbb{G}_k$, the set X_k contains at least 2 primes. Then by Herfort–Ribes [H–R] in the first case, and by Pop [P2] in the second case, it follows that $\widetilde{\Gamma}$ is contained in a conjugate of some Φ_{ϕ_l}. Therefore, β is a strong solution of EP_D. □

We finish these considerations by the following theorem, which is used for the realization of classically projective groups as absolute Galois groups of pseudo classically closed fields.

Theorem 1.5. *Let G be relatively projective with respect to a classical family $\mathcal{G}$ of subgroups of G. Let $\mathcal{G}_k = \mathcal{G}_{\mathbb{G}_k}$ denote the simple components of $\mathcal{G}$. Then there exists a homomorphism*

$$\varphi : G \to *_k\, \mathbb{G}_k$$

which maps every $\Gamma \in \mathcal{G}$ injectively into some conjugate of one of the groups $\mathbb{G}_k$ for some k (which is uniquely determined).

Proof. The proof is actually only an elaboration of what we have done above. First, let $\mathbb{G} \subseteq G_{\mathbb{Q}_p}$ be any p-adic classical Galois group. Then the Sylow p-group of $\mathbb{G}$ is not finitely generated, contrary to the q-Sylow groups of $\mathbb{G}$, $(q \neq p)$, which are generated by two elements, see [Se], II. Therefore, there exists a finite quotient, $\overline{\mathbb{G}}$ of $\mathbb{G}$ with the property: If a classical Galois group $\mathbb{G}^1$ has $\overline{\mathbb{G}}$ as quotient, then $\mathbb{G}^1 \subseteq G_{\mathbb{Q}_P}$ too.

Second, by results of Demuskin and refinements by Poitou (see for instance loc. cit. and [N]), for any classical Galois group $\mathbb{G} \subseteq G_{\mathbb{Q}_p}$, its index in $G_{\mathbb{Q}_p}$ is encoded in the structure of $\mathbb{G}$ and can be described as follows. For an arbitrary subgroup $\mathbf{G}$ of $G_{\mathbb{Q}_p}$, let $\mathbf{G}(p)$ denote its maximal pro-p-quotient. Set $\delta^1(\mathbf{G}) = \dim H^1(\mathbf{G}(p), \mathbb{F}_p)$ the minimal number of generators of $\mathbf{G}(p)$ (which can be infinite). Further we set $\delta^2(\mathbf{G}) = \dim H^2(\mathbf{G}(p), \mathbb{F}_p)$ the minimal number of relations defining $\mathbf{G}(p)$ as quotient of a profinite free pro-p-group. If $\mathrm{cd}_p \mathbf{G} = 2$, then $\delta^2(\mathbf{G})$ is either 1 or 0, corresponding to the fact that the fixed field of $\mathbf{G}$ does or does not contain the p-roots of unity. In both cases $d(\mathbf{G}) = \delta^1(\mathbf{G}) - \delta^2(\mathbf{G}) - 1$ equals the index $(G_{\mathbb{Q}_p} : \mathbf{G})$ of $\mathbf{G}$ in $G_{\mathbb{Q}_p}$. In particular, if $\mathrm{cd}_p\mathbf{G} = 2$, then d is finite if and only if $\mathbf{G}$ is open.

Combining these two observations, for every k we can choose $\overline{\mathbb{G}}_k$ such that with the notation from ii) above the following holds: If $\mathbb{G}_k \subseteq G_{\mathbb{Q}_p}$, then X_k also contains p and furthermore, if some Γ has $\overline{\mathbb{G}}_k$ as quotient then Γ is p-adic.

Let β be any solution of the embedding problem EP_D from the proof of Theorem 1.2. Then $\widetilde{\Gamma} = \beta(\Gamma)$ is necessarily a classical Galois group. Indeed, let $\Gamma \in \mathcal{G}_{\mathbb{G}_k}$ and $\widetilde{\Gamma}$ be contained in a conjugate of some $\Phi_{\phi_l} \cong \mathbb{G}_l$. Then by the choice of $\overline{\mathbb{G}}_l$, $\overline{\mathbb{G}}_k$, and of D, it follows that both $\mathbb{G}_l$ and $\mathbb{G}_k$ are contained in the same $G_{\mathbb{Q}_p}$. Since p lies in X_k, we get $\mathrm{cd}_p\widetilde{\Gamma} \geq 2$. On the other hand, since $\widetilde{\Gamma}$ is a homomorphic image of Γ, one has $\delta^1(\Gamma) \geq \delta^1((\widetilde{\Gamma})$, and so, $d(\widetilde{\Gamma}) \leq d(\Gamma)+1$. Therefore, $d(\widetilde{\Gamma})$ is finite, and so $\widetilde{\Gamma}$ is isomorphic to an open subgroup of $G_{\mathbb{Q}_p}$. It remains to show that for a proper choice of the big quotients $\overline{\mathbb{G}}_k$, the groups $\widetilde{\Gamma}$ and Γ are actually isomorphic. As we have already remarked, one always has $d(\widetilde{\Gamma}) \leq d(\Gamma) + 1$.

Let $\mathbb{G} = \mathbb{G}_k$ be fixed. It is clear that there exist only finitely many isomorphy classes of subgroups $\mathbf{G}$ of $G_{\mathbb{Q}_p}$ with $d(\mathbf{G}) \leq d(\mathbb{G}) + 1$.

Claim. *There exists a quotient $\overline{\mathbb{G}}$ with the property: If $\mathbf{G}$ is an open subgroup of $G_{\mathbb{Q}_p}$ such that $d(\mathbf{G}) \leq d(\mathbb{G}) + 1$, $\mathbf{G}$ is a quotient of $\mathbb{G}$, and $\overline{\mathbb{G}}$ also is a quotient of $\mathbf{G}$, then $\mathbb{G}$ and $\mathbf{G}$ are isomorphic.*

Proof. (of the Claim) It is sufficient to prove the claim for any particular $\mathbf{G}$. Suppose that all finite quotients $\overline{\mathbb{G}}$ are also quotients of $\mathbf{G}$. Since $\mathbf{G}$ and $\mathbb{G}$ have finite corank it follows that $\mathbb{G}$ itself is a quotient of $\mathbf{G}$. But then we get surjective homomorphisms $\mathbb{G} \to \mathbf{G} \to \mathbb{G}$. Since $\mathbb{G}$ has finite rank the above maps must be isomorphisms. □

Coming back to the proof of Theorem 1.5, we remark that for the above choice of X_k and $\overline{\mathbb{G}}_k$ for all k, it is clear that β is injective on each Γ. To conclude the proof of Theorem 1.5, we remark that with B_D as above there exist group homomorphisms $B_D \to *_k\, \mathbb{G}_k$ which map all Φ_{ϕ_k} isomorphically onto $\mathbb{G}_k$ and are trivial on $F_{|G/D|}$. □

B) Strongly relatively projective groups

Convention. Unless explicitly stated, when speaking about a strongly $\mathcal{G}$-projective group G we will always assume that $\mathcal{G}$ is a standard G-subspace of $\mathbf{sg}(G)$, i.e., $\mathcal{G}$ is quasi τ^{et}-compact, G-invariant and $\mathcal{G} = \mathcal{G}_{\max}$.

Fact 1.6. Let G be strongly $\mathcal{G}$-projective. Then every special embedding problem $\mathrm{EP} = (\gamma, \alpha, \beta)$ with A finite has strong solutions, i.e., there exists $\beta : G \to B$ with $\beta(\mathcal{G}) \subseteq \mathrm{con}(\mathcal{B})$.

Proof. For each $\gamma(\Gamma)$ and $a \in A$, choose pre-images $\Delta \in \mathrm{con}(\mathcal{B})$ and $b \in B$. The closed subgroup B^0 of B generated by these Δ and b is finitely generated, hence separable. Moreover, setting $\mathcal{B}^0 = \mathrm{con}(\mathcal{B}) \cap \mathbf{sg}(B^0)$ we get a special embedding problem $\mathrm{EP}^0 = (\gamma, \alpha|_{B^0}, \mathcal{B}^0)$ for $(G, \mathcal{G})$ satisfying $\mathrm{EP} \prec \mathrm{EP}^0$. Hence we can suppose that B is separable. Let $\{C_n\}_n$ be a decreasing sequence of open normal subgroups of B with $\cap_n C_n = \{1\}$ and $C_0 = \ker(\alpha)$. Let $B \xrightarrow{\alpha_n} B/C_n \xrightarrow{\delta^n} B/C_{n-1}$ denote the canonical projections. Then $\big(\gamma, \delta^1, \alpha_1(\mathcal{B})\big)$ is an embedding problem for $(G, \mathcal{G})$. Let β_1 be any strong solution of it. Then $\beta_1(\mathcal{G}) \subseteq \alpha_1\big(\mathrm{con}(\mathcal{B})\big)$. Hence, for any $\beta_1(\Gamma)$ there exists $\Delta \in \mathrm{con}(\mathcal{B})$ such that $\alpha_1(\Delta) = \beta_1(\Gamma)$. Let $\mathcal{B}^1 \subseteq \mathrm{con}(\mathcal{B})$

be the (finite) collection of all such Δ. Then $EP^1 = (\alpha_1, \beta_1, \mathcal{B}^1)$ is an embedding problem for $(G, \mathcal{G})$ and α_1 is injective on all $\Delta \in \mathcal{B}^1$. Now, starting with EP^1 we take any strong solution β^2 of $\big(\beta_1, \delta^2, \alpha_2(\mathcal{B}^1)\big)$ and so on... Inductively we get a sequence of homomorphisms $\{\beta_n\}_n$ with the properties:

$$\beta_n : G \to B/C_n \quad \beta_n = \delta^{n+1}\beta_{n+1}, \quad \beta_n(\mathcal{G}) \subseteq \alpha_n\big(\mathrm{con}(\mathcal{B})\big).$$

It is now obvious that $\beta = \varprojlim \beta_n : G \to B$ has the properties we want. □

Fact 1.7. Let G be strongly $\mathcal{G}$-projective. For an arbitrary open normal subgroup D of G let $\gamma : G \to G/D$ denote the canonical projection, and let $\gamma(\mathcal{G})$, Γ/D be the images of $\mathcal{G}$, respectively $\Gamma \in \mathcal{G}$ by γ. Let $\phi_{\Gamma/D} : \Delta_{\Gamma/D} :\to \Gamma/D$ be an isomorphic copy of Γ/D, and let $\mathcal{B}_D$ be the set of all $\Delta_{\Gamma/D}$. Now consider the usual profinite free product $B_D = F_{|G/D|} * \mathop{*}\limits_{\Delta_{\Gamma/D}} \Delta_{\Gamma/D}$ and the "canonical projection"

$$\alpha_D : B_D \to G/D \quad \text{by} \quad |g_D| \mapsto g_D, \quad \alpha|_{\Delta_{\Gamma/D}} = \phi_{\Gamma/D}$$

Then α_D maps $\mathcal{B}_D$ bijectively onto $\gamma(\mathcal{G})$, and it is injective on all $\Delta_{\Gamma/D} \in \mathcal{B}_D$. Therefore, $\mathrm{EP}_D = (\gamma, \alpha_D, \mathcal{B}_D)$ is an embedding problem for $(G, \mathcal{G})$ which satisfies the hypothesis of Fact 1.6. Hence, EP_D has strong solutions.

Interesting corollaries of Fact 1.7 are the Separating Theorem 1.8 and the Subgroup Theorem 1.9, see also Haran [H1], Theorem 5.1 for the last assertion. For the proofs we use properties of the usual profinite free products, like the results of Herfort–Ribes [H–R], and Binz–Neukirch–Wenzel [B–N–W].

Theorem 1.8 (Separating Theorem). *Let G be strongly $\mathcal{G}$-projective. The following hold:*

(1) *Any distinct elements of $\mathcal{G}$ have trivial intersection. Hence the residues set of $\mathcal{G}$ consists only of $\{1\}$ and so, $\mathcal{G}$ is separating.*
(2) *Any $\Gamma \in \mathcal{G}$ is self-normalizing, i.e., if $\Gamma^g = \Gamma$ for some $g \in G$, then $g \in \Gamma$.*

Proof. . To (1): Let Γ_1, Γ_2 be elements of $\mathcal{G}$ and suppose that there exists $g \neq 1$ in their intersection. For every open normal subgroup D of G not containing g let β be a strong solution of the embedding problem EP_D considered at Fact 1.7. Since $\alpha\beta(g) = \gamma(g) \neq 1$, it follows that $\beta(\Gamma_1) \cap \beta(\gamma_2)$ contains $\beta(g) \neq 1$. On the other hand, $\beta(\mathcal{G}) \subseteq \mathrm{con}(\mathcal{B})$. Hence by [H–R] we have: $\beta(\Gamma_1)$, $\beta(\Gamma_2)$ are contained in the same maximal element Δ of $\mathrm{con}(\mathcal{B})$. Hence $\gamma(\Gamma_1)$ and $\gamma(\Gamma_2)$ are both contained in some $\gamma(\Gamma) = \alpha(\Delta)$. Since D was arbitrary, it follows that there exists some $\Gamma \in \mathcal{G}$ containing both Γ_1 and Γ_2. Hence $\Gamma_1 = \Gamma_2$.

The proof of (2) follows the same pattern. □

Remark. 1) Let G be strongly $\mathcal{G}$-projective. From the theorem above it follows that $\mathcal{G}$ is *a posteriori* separating in the sense of Haran [H1], Definition 3.1. Hence, our notion of strong relative projectivity coincides with the relative projectivity from [H1]. Therefore, the supplementary conditions imposed by the definition in [H1] on $\mathcal{G}$ are redundant.

2) Let G be strongly $\mathcal{G}$-projective and suppose that $\{1\}$ does not lie in the τ^{st}-closure of $\mathcal{G}$. Then by Appendix, Fact 4.4 and 4.5, it follows that $\mathcal{G}$ is τ^{et}-compact. In particular, this is the case for $\mathcal{G}_{\max}$, where $\mathcal{G}$ is a classical family of subgroups of G.

Without proof we mention here the Subgroup Theorem, see Haran [H1], Theorem 5.1. It is the group theoretical counterpart of field theoretic assertion Theorem 2.12, and its proof follows easily using Fact 1.1 and 1.7, and the structure theorem of open subgroups of usual profinite free products from [B–N–W]. For a different proof idea see Haran [H1].

Theorem 1.9 (**Subgroup Theorem**). *Let G be strongly $\mathcal{G}$-projective. Then for every closed subgroup $H \subseteq G$, the family $\mathcal{H} = \{\Gamma \cap H \mid \Gamma \in \mathcal{G}, \Gamma \cap H \neq 1\}$ is a standard H-subspace of $\mathbf{sg}(H)$, and H is strongly $\mathcal{H}$-projective.*

We are now going to introduce the notion of a cover for $(G, \mathcal{G})$. The covers for $(G, \mathcal{G})$ are group extensions (ϵ) of G endowed with local splittings $\mathcal{F}$ of a very special nature.

Definition. Let G be an arbitrary profinite group, and $\mathcal{G} \subseteq \mathbf{sg}(G)$ a quasi τ^{et}-compact subset. A cover $(F, \mathcal{F})$ for $(G, \mathcal{G})$ is a group extension of the form $1 \to H \to F \xrightarrow{\pi} G \to 1$ endowed with a τ^{et}-compact local splitting $\mathcal{F} \to \mathcal{G}$ satisfying the following:

i) $\mathcal{F}$ is mapped by π bijectively onto $\mathcal{G}$, and the orbit space $\mathcal{F}^\wedge$ of $\mathcal{F}$ is mapped by π bijectively onto the orbit space $\mathcal{G}^\wedge$ of $\mathcal{G}$.
ii) For every $\Phi, \Phi_1 \in \mathcal{F}$ and $1 \neq h \in \ker(\pi)$, one has $\Phi^h \cap \Phi_1 = \{1\}$.

We remark that with the notations from Appendix, 4.10, C), the "canonical cover" associated to D, say $\pi = \pi_D : (F_D, \mathcal{F}) \to (G, \mathcal{G})$, is a cover for $(G, \mathcal{G})$.

Behold, we now come to (one of) the promised generalization of Gruenberg's [G] result, which is the following

Theorem 1.10. *Let G be strongly $\mathcal{G}$-projective with $\mathcal{G}$ τ^{et}-compact and G-invariant. Then for every cover*

$$\pi : (F, \mathcal{F}) \to (G, \mathcal{G}),$$

there exists a section $\sigma : G \to F$ of π such that $\sigma(\mathcal{G}) \subseteq \mathcal{F}^{\ker(\pi)}$. Moreover, for any $\Phi \in \operatorname{con}(\mathcal{F})$, either $\Phi \subseteq \sigma(\mathcal{G})$ or $\Phi \cap \sigma(G) = \{1\}$.

Proof. . The proof is quite technical. For the beginning, we introduce the notion of a *sub-cover,* as follows: Let G be an arbitrary group, and $\mathcal{G} \subseteq \mathbf{sg}(G)$ a quasi τ^{et}-compact subspace. A sub-cover $(E, \mathcal{E})$ of a cover $\pi : (F, \mathcal{F}) \to (G, \mathcal{G})$, written $(E, \mathcal{E}) \prec (F, \mathcal{F})$, consists of a subgroup $E \subseteq F$, and a subset $\mathcal{E} \subseteq \mathcal{F}^{\ker(\pi)}$ of subgroups of E such that $\rho = \pi|_E : (E, \mathcal{E}) \to (G, \mathcal{G})$ itself is a cover for $(G, \mathcal{G})$. We say that a cover for $(G, \mathcal{G})$ is *minimal* if it has no proper sub-covers.

Lemma 1.11. *Let $(G, \mathcal{G})$ be as above. They every cover $\pi : (F, \mathcal{F}) \to (G, \mathcal{G})$ contains minimal sub-covers.*

Proof. . We will apply Zorn Lemma. It is clear that $\prec$ is a pre-order on the set of all sub-covers of $(F, \mathcal{F})$. Let $(E_\lambda, \mathcal{E}_\lambda)_\lambda$ be a totally ordered subfamily of sub-covers of $(F, \mathcal{F})$. For each λ set $\mathcal{D}_\lambda = \mathcal{E}_\lambda^{\ker(\rho_\lambda)}$. Then $\rho_\lambda = \pi|_{E_\lambda}$ has the properties:

1) ρ_λ maps $\mathcal{D}_\lambda$ onto $\mathcal{G}$ and $\mathcal{D}_\lambda^\wedge$ bijectively onto $\mathcal{G}^\wedge$.
2) $\mathcal{D}_\lambda$ is τ^{et}-compact and $\ker(\rho_\lambda)$ acts regularly on it.

We set $E = \cap_\lambda E_\lambda$, $\mathcal{D} = \cap_\lambda \mathcal{D}_\lambda$ and $\rho = \pi|_E$. It is clear that ρ has the properties 1), 2) above. Therefore, $\mathcal{D}$ contains a τ^{et}-compact fundamental domain $\mathcal{E}$ with respect to the action of $\ker(\rho)$. One verifies without difficulties that $(E, \mathcal{E})$ is a sub-cover of $(F, \mathcal{F})$. □

Lemma 1.12. *Under the hypothesis from Lemma above, suppose that $\mathcal{G}$ has the properties 1), 2) from the Separating Theorem. Let $\pi : (F, \mathcal{F}) \to (G, \mathcal{G})$ be a cover for $(G, \mathcal{G})$. Then* *for every $\Delta \in \mathrm{env}(\mathcal{F})$ the pre-image of $\pi(\Delta)$ in $\mathrm{con}(\mathcal{F})$ is exactly $\Delta^{\ker(\pi)}$. In particular, if $\mathcal{G}$ is a G-space, then $\mathcal{F}^{\ker(\pi)}$ is an F-space.*

Proof. . Let us denote $H = \ker(\pi)$. For a fixed $\Phi_0 \in \mathcal{F}$, consider $\pi(\Phi^f) = \pi(\Phi_0)$ for some $\Phi \in \mathcal{F}$ and $f \in F$. Setting $\Gamma = \pi(\Phi)$, $\Gamma_0 = \pi(\Phi_0)$ one has $\Gamma^{\pi(f)} = \Gamma_0$. Hence, by the definition of cover, Φ and Φ_0 are conjugated in F, say $\Phi_0 = \Phi^{f_0}$. This in turn implies $\Gamma_0 = \Gamma^{\pi(f)} = \Gamma^{\pi(f_0)}$. Since Γ is self-normalizing it follows that $\pi(f) = \pi(f_0)g^0$ for some $g^0 \in \Gamma$. Hence, $f = hf_0f^0$ for f^0 the pre-image of g^0 in Φ and some $h \in H$. With these data we now get: $\Phi^f = \Phi^{hf_0f^0} = \Phi_0^h$ and so, the pre-image of Γ_0 in $\mathcal{F}^F$ is Φ_0^H.

Now take any $\Delta, \Delta_1 \in \mathrm{con}(\mathcal{F})$ such that $\pi(\Delta) = \pi(\Delta_1)$. If $\Phi, \Phi_1 \in \mathcal{F}^F$ which contain Δ, respectively Δ_1, it follows that $\Gamma = \pi(\Phi)$ and $\Gamma_1 = \pi(\Phi_1)$ lie in $\mathcal{G}$ and have non-trivial intersection. Hence they are equal and so, $\Phi_1 = \Phi^h$ for some $h \in H$. From this the desired result follows. □

Coming back to the proof of Theorem 1.10. Set $H = \ker(\pi)$. We first prove the existence of right inverses σ of π satisfying $\sigma(\mathcal{G}) \subseteq \mathrm{con}(\mathcal{F})$. By the Lemma above, we can suppose that $(F, \mathcal{F})$ is a minimal cover for $(G, \mathcal{G})$. Let C be any open normal subgroup of F. Then $H \cap C$ is a normal subgroup of F open in H. Consider the canonical projections

$$F \xrightarrow{\widehat{\pi}} \widetilde{F} = F/(H \cap C) \xrightarrow{\widetilde{\pi}} G$$

and denote $\widetilde{H} = \widehat{\pi}(H)$, $\widetilde{\mathcal{F}} = \widehat{\pi}(\mathcal{F})$ and $\widetilde{\Phi} = \widehat{\pi}(\Phi)$ for $\Phi \in \mathcal{F}$. By Lemma 1.13, for every $\Delta \in \mathrm{con}(\mathcal{F})$ the pre-image of $\pi(\Delta)$ in $\mathrm{con}(\mathcal{F})$ is exactly Δ^H. Using this fact, a simple verification shows for any $\widetilde{\Delta} \in (\widetilde{\mathcal{F}})$ the pre-image of $\widetilde{\pi}(\widetilde{\Delta})$ in $\mathrm{con}(\widetilde{\mathcal{F}})$ is exactly $\widetilde{\Delta}^{\widetilde{H}}$. Moreover, since $\widetilde{H} = \ker \widetilde{\pi}$ is finite and $\widetilde{H} \cap \widetilde{\Delta} = \{1\}$ for all $\widetilde{\Delta} \in \mathrm{con}(\widetilde{\mathcal{F}})$, it follows from the τ^{st}-compactness of $\mathrm{con}(\widetilde{\mathcal{F}})$ that there exist open normal subgroups $\widetilde{C}$ of $\widetilde{F}$ such that $\widetilde{C}\widetilde{\Delta} \cap \widetilde{H} = \{1\}$ for all $\widetilde{\Delta} \in \widetilde{\mathcal{F}}$.

Let $\mathcal{U} \subseteq \mathbf{sg}(\widetilde{F})$ consist of all subgroups $\widetilde{\Delta}_1$ of $\widetilde{F}$ which are contained in $\widetilde{C}\widetilde{\Delta}$ for some $\widetilde{\Delta}$. Then $\mathrm{con}(\mathcal{U}) = \mathcal{U}$ and $\mathcal{U}$ is a quasi τ^{et}-compact open neighborhood of $\mathrm{con}(\widetilde{\mathcal{F}})$. A simple verification shows that if $\tilde{C}$ is small enough, then the pre-image of every $\Gamma \in \mathcal{G}$ in $\mathcal{U}$ is actually exactly $\widetilde{\Phi}^{\widetilde{H}}$, for any pre-image $\widetilde{\Phi}$ of Γ. Therefore, if $\widetilde{\sigma}$ is any right inverse of $\widetilde{\pi}$ with $\widetilde{\sigma}(\mathcal{G}) \subseteq \mathcal{U}$, then $\widetilde{\sigma}(\mathcal{G}) \subseteq \widetilde{\mathcal{F}}^{\widetilde{H}}$.

Let E be the pre-image of $\widetilde{\sigma}(G)$ by $\hat{\pi}$, and $\mathcal{E}_1$ the pre-image of $\widetilde{\sigma}(\mathcal{G})$ in $\mathcal{F}^H$. We show that $\mathcal{E}_1$ contains a subfamily $\mathcal{E}$ such that $(E, \mathcal{E})$ is a cover for $(G, \mathcal{G})$. For this we first observe that E is an open subgroup of F (indeed, it is the pre-image by $\hat{\pi}$ of the open subgroup $\widetilde{\sigma}(G)$ of $\widetilde{F}$). Secondly, we remark that every $\Phi \in \mathcal{F}^H = \mathcal{F}^F$ is conjugated to some $\Delta \in \mathcal{E}_1$. Applying Appendix, 4.9, B), it follows that there exists an E-invariant τ^{et} compact subset $\mathcal{D}$ of $\mathcal{E}_1$ with the property: If $\mathcal{D}^h \cap \mathcal{D} \neq \emptyset$ for some $h \in F$ then h lies in E. From this it follows that $\rho = \pi|_E$ has the properties 1), 2) from the proof of the first Lemma above with respect to $\mathcal{D}$. One proceeds as in the proof of Lemma, and obtains a τ^{et}-compact subset $\mathcal{E} \subseteq \mathcal{D}$ such that $(E, \mathcal{E})$ is a sub-cover of $(F, \mathcal{F})$. ¿From the minimality of $(F, \mathcal{F})$ it follows that $E = F$ and so, $\ker(\pi) = \{1\}$.

It remains to prove that for every $\Phi_1 \in \mathrm{con}(\mathcal{F})$, either $\Phi_1 \cap \sigma(G) = \{1\}$, or $\Phi_1 \subseteq \sigma(G)$. It is sufficient to consider the case $\Phi_1 = \Phi^h$ for some $\Phi \in \mathcal{F}$ and $h \in H$. Set $\Gamma = \pi(\Phi)$ and suppose $\Delta = \Phi^h \cap \sigma(G) \neq \{1\}$. Then $\Phi_0 = \sigma(\Gamma)$ contains Δ, hence $\Phi_0 \cap \Phi^h \neq \{1\}$. On the other hand, Φ^h and Φ_0 are both pre-images of Γ in $\mathrm{con}(\mathcal{F})$. By the definition of cover it follows that $\Phi_0 = \Phi^h$.

The proof of Theorem 1.10 is finished. □

2. Pseudo closed fields

A) Localities of a field

Definition. Let K be an arbitrary field. A locality of K any separable (but non-separably closed) overfield $\mathrm{L} \subseteq K^{\mathrm{alg}}$ of K which is real closed or henselian with respect to some non-trivial valuation.

We denote the family of all localities of K by $\mathbf{loc}(K)$.

We say that a locality of K is *classical,* if it is a real or p-adically closed field, p being an arbitrary rational prime number.

Each locality L of K defines a V-topology τ on K, and we denote by K^τ the separable closure of K in the τ-completion of K. One has: L is a minimal element of $\mathbf{loc}(K)$ if and only if $\mathrm{L} = K^\tau$.

A valuation v of a field K is called minimal (maximal) if it has no proper coarsennings (if it is not a proper coarsening of another valuation of K; or equivalently, if the residue field Kv of v is algebraic over a finite field). If v is maximal and K henselian with respect to it, then by general valuation theory, G_K is pro-solvable.

We now give some examples of prominent families of localities of a field.

The henselian Spectrum of a field

Let $\mathbf{lochens}(K)$ denote the space of all proper henselisations of K in its algebraic closure. Then G_K acts continuously on $\mathbf{lochens}(K)$ and the henselian spectrum of K is defined to be the orbit space of $\mathbf{lochens}(K)$ with respect to the conjugation. General valuation theory now gives: if $L|K$ is an algebraic extension of K, then $\mathbf{lochens}(L)$ consists of all non-separably closed compositae LK^h in a fixed algebraic closure of K.

The henselian spectrum of a field has a very complicated topology, and it is in general very difficult to describe it and its closure in $\mathbf{sf}(K)$.

The classical spectra of a field

For a family $\mathcal{X}$ of classical fields we denote by $\mathbf{loc}^{\mathcal{X}}(K)$ the set of all localities of K which are elementarily equivalent to elements in $\mathcal{X}$. If $\mathcal{X}$ contains just one element $\mathbb{K}$, then we write $\mathbf{loc}^{\mathbb{K}}(K)$ for $\mathbf{loc}^{\mathcal{X}}(K)$. It is clear that for every $\mathcal{X}$, the space $\mathbf{loc}^{\mathcal{X}}(K)$ is a G_K-invariant subspace of $\mathbf{loc}(K)$.

Proposition 2.1. *For every finite set $\mathcal{X}$ as above, the space $\mathbf{loc}^{\mathcal{X}}(K)$ is τ^{st}-closed in the space $\mathbf{sf}(K)$.*

Proof. . It is sufficient to prove the assertion for any particular $\mathbf{loc}^{\mathbb{K}}(K)$. Let an overfield $L \subset K^{\mathrm{alg}}$ of K lie in the τ^{st}-closure of $\mathbf{loc}^{\mathbb{K}}(K)$. For every finite Galois extension $K_1|K$, by our assumption on L it follows that

$$X_{K_1} = \{K \in \mathbf{loc}^{\mathbb{K}}(K) \mid K \cap K_1 = L \cap K_1\} \neq \emptyset.$$

Therefore, $\mathbf{U}' = (X_{K_1})_{K_1}$ is a filter base on $\mathbf{loc}^{\mathbb{K}}(K)$. Let $\mathbf{U}$ be a ultrafilter containing $\mathbf{U}'$. Then $(K^*, v^*) = \prod_K (K, v)/\mathbf{U}$ is elementary equivalent to $\mathbb{K}$. Moreover, the mapping

$L \to K^*$, $x \mapsto (x_K)/\mathbf{U}$ with $x_K = x$ if $x \in K \cap L$ and $x_K = 0$ otherwise, defines an embedding of L into K^*. Let Λ denote the relative algebraic closure of L in K^*. By the elementary theory of the classical fields, see for instance [Pr1] and [P–R], it follows that Λ is elementary equivalent to $\mathbb{K}$, i.e., it lies in $\mathbf{loc}^{\mathbb{K}}(K)$.

For every field R, let R^{abs} denote the absolute subfield of R. By loc. cit. it follows that $K^{\text{abs}} \in \mathbf{loc}^{\mathbb{K}}(K^{\text{abs}})$, and $K^{\text{alg}} = K(K^{\text{abs}})^{\text{alg}}$ for every K in $\mathbf{loc}^{\mathbb{K}}(K)$. Further, from the continuity of the canonical restriction map $\phi_K : \mathbf{sf}(K) \to \mathbf{sf}(K^{\text{abs}})$, it follows that L^{abs} lies in the τ^{st}-closure of $\mathbf{loc}^{\mathbb{K}}(K^{\text{abs}})$ in $\mathbf{sf}(\mathbb{Q})$. On the other hand, $\mathbf{loc}^{\mathbb{K}}(K^{\text{abs}})$ is τ^{st}-closed. Thus $L^{\text{abs}} \in \mathbf{loc}^{\mathbb{K}}(K^{\text{abs}})$. Since Λ^{abs} also lies in $\mathbf{loc}^{\mathbb{K}}(K^{\text{abs}})$, and further $L^{\text{abs}} \subseteq \Lambda^{\text{abs}}$, we obtain $L^{\text{abs}} = \Lambda^{\text{abs}}$.

On the other hand, G_L lies in the τ^{st}-closure of $\{G_K \mid K \in \mathbf{loc}^{\mathbb{K}}(K)\}$ in $\mathbf{sg}(G_K)$. Hence every finite quotient of G_L is a quotient of some $G_K \simeq G_{\mathbb{K}}$. Since the latter group has finite corank, it follows that G_L itself is a quotient of $G_{\mathbb{K}}$, i.e., there exists a surjective homomorphism $\varphi : G_{\mathbb{K}} \to G_L$. Thus we get a surjective homomorphism

$$\widetilde{\varphi} : G_{\mathbb{K}} \xrightarrow{\varphi} G_L \xrightarrow{\pi_L} G_{\Lambda^{\text{abs}}} \cong G_{\mathbb{K}}.$$

Since $G_{\mathbb{K}}$ has finite corank it follows that $\widetilde{\varphi}$ is an isomorphism. In particular, the canonical projection π_L is an isomorphism. Since $G_\Lambda \subseteq G_L$ is also mapped by π_L isomorphically onto $G_{L^{\text{abs}}} = G_{\Lambda^{\text{abs}}}$, it follows that $G_L = G_\Lambda$, hence $L = \Lambda$. □

Corollary 2.2. *The topologies τ^{et} and τ^{st} coincide on $\mathbf{loc}^{\mathbb{K}}(K)$ for every $\mathbb{K}$.*

Proof. By Appendix, 4.4, and Proposition 2.1, as $\mathbf{loc}^{\mathbb{K}}(K) = \mathbf{loc}^{\mathbb{K}}(K)_{\min}$. □

Remark. For a given finite family $\mathcal{X}$ of classical fields, the space $\mathbf{loc}^{\mathcal{X}}(K)$ does not have nice behavior as G_K-space. The reason for this is that the elements of $\mathbf{loc}^{\mathcal{X}}(K)$ are in some sense "too big". We are going to remedy this by working with *classical closures* instead of classical localities.

Definition. A classical locality of K is called classical closure of K if it is minimal with respect to the inclusion in the set of all classical localities of K.

Theorem 2.3. *For any classical locality* L *of K there exists a unique classical closure* L_0 *of K which is contained in* L.

Proof. If $\mathrm{L} \equiv \mathbb{R}$ then everything is clear. Suppose $\mathrm{L} \equiv \mathbb{K}$, with $\mathbb{K}$, a classical p-adic field. Let $\Lambda \subseteq \mathrm{L}$ be a classical locality of K. We show that Λ is p-adically closed and $\mathrm{L}|\Lambda$ is a finite extension. It is clear that Λ is non-real, hence it is p'-adically closed with respect to a finite rational prime number p'. By [P–R], §3, one has: L^{abs} is p-adically closed, and Λ^{abs} is p'-adically closed. Since $\Lambda \subseteq \mathrm{L}$, it follows that $\Lambda^{\text{abs}} \subseteq \mathrm{L}^{\text{abs}}$. Hence $p = p'$, and $[\mathrm{L}^{\text{abs}} : \Lambda^{\text{abs}}]$ is finite. But then $\mathrm{L}_1 = \Lambda\mathrm{L}^{\text{abs}}$ is p-adically closed and $\mathrm{L}_1^{\text{abs}} = \mathrm{L}^{\text{abs}}$. Therefore, $\mathrm{L} = \mathrm{L}_1$, and $[\mathrm{L} : \Lambda] = [\mathrm{L}^{\text{abs}} : \Lambda^{\text{abs}}]$.

¿From this we deduce that any strictly descending sequence $(\mathrm{L}_\lambda)_\lambda$ of classical localities of K which are contained in L is finite and has the same length as $(\mathrm{L}_\lambda^{\text{abs}})_\lambda$. Hence L contains classical closures Λ_0 of K. For any such classical closure (Λ, v_Λ), by [P1], (1.10), we get: The restriction of v_{L} to Λ is exactly v_Λ. Denote by v their common restriction to K. Let (K^h, v^h) be the unique henselisation of (K, v) inside L. Then (K^h, v^h) is also contained in every classical closure of K which lies in L. Hence the problem is reduced to the case where K is henselian with respect to a p-adic valuation v. We now claim:

Lemma 2.4. *Let K be a henselian field with respect to some p-adic valuation, and $\Lambda_1, \Lambda_2 \subseteq \mathrm{L}$ be p-adically closed over-fields of K. Then $\Lambda = \Lambda_1 \cap \Lambda_2$ is a p-adically closed field too.*

Proof. We use an elaborated version of the characterization of p-adically closed fields from [P–R], §3, which reads as follows:

Theorem. *Let K be a formally p-adic henselian field. Then K is p-adically closed if and only if for every $t \in K$ and any natural number n there exist $a_n \in K^{\mathrm{abs}}$ and $x_n \in K$ such that $t = a_n x_n^n$. Moreover, if K is p-adically closed, then the mapping*

$$K^\times \to \widehat{K} = \varprojlim K^\times/n \cong \varprojlim K^{\mathrm{abs}\times}/n$$

is a group homomorphism, and $K^\times/K^{\mathrm{abs}\times}$ is a $\mathbb{Q}$-vector space.

Since Λ contains K, by the assumptions on K it follows that Λ is henselian, and $\Lambda^{\mathrm{abs}} = \Lambda_1^{\mathrm{abs}} \cap \Lambda_2^{\mathrm{abs}}$. By the above Theorem, to show that Λ is p-adically closed, it is sufficient to prove that $\Lambda^\times/\Lambda^{\mathrm{abs}\times}$ is divisible. For $t \in \Lambda^\times$ and $n \geq 1$, set $t = b_n x_n^n = c_n y_n^n = d_n z_n^n$ in L, Λ_1, and Λ_2 respectively. Then $(b_n)_n, (c_n)_n$, and $(d_n)_n$ define the same element $\widehat{t}$ of $\widehat{\mathrm{L}}$; further, if $\widehat{\Lambda}'_i$ is the image of $\widehat{\Lambda}_i$ in $\widehat{\mathrm{L}}$, then $\widehat{t}$ lies in $\widehat{\Lambda}'_1 \cap \widehat{\Lambda}'_2$. Finally, let $\widehat{\Lambda}'$ be the image of $\widehat{\Lambda^{\mathrm{abs}}}$ in $\widehat{\mathrm{L}}$. Then by local class field theory, in $\widehat{\mathrm{L}}$ it holds $(\widehat{\Lambda}'_1 \cap \widehat{\Lambda}'_2 : \widehat{\Lambda}') < \infty$.

Therefore, there exists $d \geq 1$ such that $\widehat{t}^d$ lies in $\widehat{\Lambda}'$. Equivalently, there exists $(a_n)_n$ in $\widehat{\Lambda}$ defining $\widehat{t}^d$. Equivalently, t^d/a_n is an n^{th} power in Λ_i and L. Thus we can write

$$t^d/a_n = x_n^n = y_n^n = z_n^n$$

for some $x_n \in \Lambda_1$, $y_n \in \Lambda_2$, and $z_n \in \mathrm{L}$. In particular, there exist n^{th} roots of unity ζ_n and ζ'_n in L, such that $x_n\zeta_n = z_n = y\zeta'_n$. Taking into account that the group of roots of unity in L is finite, say of order $m \geq 1$, it follows that

$$t^{dm}/a_n^m = (x^m)_n^n = (y^m)_n^n = (z^m)_n^n$$

where $x_n^m = y_n^m = z_n^m$ are equal, thus elements of Λ. ¿From this it follows that a_n^m are elements of Λ too. Thus finally, $t^{md} \pmod{\Lambda^{\mathrm{abs}\times}}$ is divisible in $\Lambda^\times/\Lambda^{\mathrm{abs}\times}$. □

Now the proof of the Theorem follows immediately from the Lemma above and the comments preceding it. □

For a fixed finite family $\mathcal{X}$ of classical fields we denote by $\mathbf{clos}^{\mathcal{X}}(K)$ the set of all classical closures of K which are contained in elements of $\mathbf{loc}^{\mathcal{X}}(K)$. It is clear that $\mathbf{clos}^{\mathcal{X}}(K)$ is closed by conjugation, and we have:

Theorem 2.5. *The space $\mathbf{clos}^{\mathcal{X}}(K)$ is τ^{et}-compact and the canonical mapping*

$$cl : \big(\mathbf{loc}^{\mathcal{X}}(K), \tau^{\mathrm{st}}\big) \to \big(\mathbf{clos}^{\mathcal{X}}(K), \tau^{\mathrm{et}}\big), \quad \mathrm{L} \mapsto \mathrm{L}_0.$$

is continuous.

Proof. It is sufficient to consider the case when $\mathcal{X}$ is the family of all classical subfields of some given classical field $\mathcal{K}$. But in this case $\mathbf{loc}^{\mathbb{K}}(K)$ is τ^{st}-closed by Proposition 2.1, and it does not meet its residue set. To proceed apply Appendix, 4.4. □

The orbit space of $(\mathbf{clos}^{\mathcal{X}}(K), \tau^{\mathrm{et}})$ with respect to the conjugation is denoted by $\mathbf{Spec}^{\mathcal{X}}(K)$ and is called the (classical) $\mathcal{X}$-spectrum of K. The correspondence

$$K \to \mathbf{clos}^{\mathcal{X}}(K) \to \mathbf{Spec}^{\mathcal{X}}(K)$$

has good functorial properties. Special cases of this definition but in a more general context can be found in [Be], [B–S], [C–R], [Ro], and others.

We now want to analyze more deeply the action of G_K on the space of all classical localities of K.

Lemma 2.6. *For every classical locality* L *of* K*, its normalizer in* G_K *is exactly the normalizer of* L *in the absolute Galois group of the unique closure* L_0 *contained in* K*. The canonical projection*

$$\pi_K : G_K \to G_{K^{\mathrm{abs}}}$$

maps the normalizer of any L *bijectively onto the normalizer of* $\mathrm{L}^{\mathrm{abs}}$ *in the absolute Galois group of* $\mathrm{L}_0^{\mathrm{abs}}$.

Proof. From the elementary theory of the classical fields, see [Pr1], [P–R], it follows that there exists a valuation (possibly trivial) w of K with the properties:

i) If w is non-trivial, then it is henselian and wK is divisible.
ii) Kw is dense in $\mathcal{K}$ and elementary equivalent to $\mathcal{K}$.

If w is trivial then the assertion of the Lemma is well known. Now suppose that w is non-trivial. Let $g \in G_K$ lie in the normalizer of L, i.e., $g(\mathrm{L}) = \mathrm{L}$. Denote by M the fixed field of g in K. Then $\mathrm{L}|M$ is a pro-cycle Galois extension. Hence by general valuation theory, it follows that M is henselian with respect to the restriction v of w to M. We show that the value group vM of v is divisible. Suppose the contrary. Let $t \in L$ and take any rational prime number q such that $v(t)$ is not divisible by q. Then for every natural number m there exist $a_m \in \mathrm{L}^{\mathrm{abs}}$ and $x_m \in \mathrm{L}$ with $t = a_m x^{q^m}$. Thus $u_m = t/a_m$ has the property: $v(t) = v(u_n) = q^m \cdot v(x_m)$ for all m. Therefore, for every m, the polynomial $X^{q^m} - u_m$ is irreducible over M and has a root x_m in L. Since $\mathrm{L}|M$ is Galois, it follows that all the roots of these polynomials are in L. Hence L contains all the q^m roots of unity. Contradiction! Since the residue field $Mv \subseteq \mathrm{L}w$ has characteristic 0, and vM is divisible, all the algebraic extensions of M are inert. Therefore, the normalizer of L in G_K is isomorphic to the normalizer of $\mathrm{L}w$ in G_{Mv}. To proceed, apply the case when w is trivial. □

Summarizing we have:

Theorem 2.7. *For any finite family* $\mathcal{X}$ *of classical fields, the space* $\mathbf{clos}^{\mathcal{X}}(K)$ *of all classical closures of* K *which are contained in elements of* $\mathbf{loc}^{\mathcal{X}}(K)$ *is* τ^{et}*-compact and consists only of self-normalizing elements.*

Proof. Apply Lemma above and Theorem 2.5. □

B) Pseudo classically closed fields

Definition. Let K be an arbitrary field. A simple classical family of localities of K is a τ^{st}-closed subset $\mathcal{K} = \mathcal{K}_{\mathbb{K}}$ of $\mathbf{loc}^{\mathbb{K}}(K)$ for some classical field $\mathbb{K}$. A classical family of localities of K is a finite union of simple classical families of localities of K, i.e., a τ^{st}-closed subset of some $\mathbf{loc}^{\mathcal{X}}(K)$.

Definition. Let K be an arbitrary field and $\mathcal{K} \subseteq \mathbf{loc}(K)$ a family of localities of K. We say that K is pseudo $\mathcal{K}$-closed if every absolutely irreducible affine variety $V|K$ has K-rational points, provided it has regular L-rational points for any $\mathrm{L} \in \mathcal{K}$.

We say that a field K is pseudo classically closed, if K is pseudo closed with respect to a classical family $\mathcal{K}$ of localities of K.

Let K be an arbitrary field. For an affine variety $V|K$, let $V(K)$ denote the set of K-rational points of V, and $V_r(K)$ the set of regular K-rational points of V. The next lemma shows that an absolutely irreducible affine variety over a pseudo closed field K which locally has regular rational points, has also regular K-rational points.

Lemma 2.8. *Let $\mathcal{K} \subseteq \mathbf{sf}(K)$ be a family of separable algebraic extensions of K. Then the following assertions are equivalent:*

(1) *For every absolutely irreducible affine variety $V|K$, the set $V(K)$ is non-empty, provided $V_r(\mathrm{L})$ is non-empty for all $\mathrm{L} \in \mathcal{K}$.*
(2) *For every absolutely irreducible affine variety $V|K$, the set $V_r(K)$ is non-empty, provided $V_r(\mathrm{L})$ is non-empty for all $\mathrm{L} \in \mathcal{K}$.*

Proof. This is a straightforward assertion. □

Before going further, we recall that a subspace $\mathcal{K} \subseteq \mathbf{sf}(K)$ is called a standard G_K-subspace, if it is quasi τ^{et}-compact, closed under conjugation by G_K, and $\mathcal{K} = \mathcal{K}_{\min}$.

Theorem 2.9 (Density & Uniqueness Theorem). *Let K be a pseudo $\mathcal{K}$-closed, with $\mathcal{K}$ some standard G_K-subspace of $\mathbf{loc}(K)$. Then K is dense in every $\mathrm{L} \in \mathcal{K}$, and moreover, $\mathcal{K} = \mathbf{lochens}(K)$. Therefore, $\mathcal{K}$ depends only on K. It holds:*

1) *Every $\mathrm{L} \in \mathcal{K}$ is self-normalizing.*
2) *The compositum of any tow distinct elements of $\mathcal{K}$ equals the separable closure of K.*

Proof. The main technical step is the following Lemma, comp. with [H–P]:

Lemma 2.10. *Let K be an arbitrary field and $\mathcal{K} \subseteq \mathbf{sf}(K)$ satisfying the equivalent conditions from above lemma. Let τ be any V-topology on K, and let K^τ denote the relative separable closure of K in the completion of K with respect to τ. Then there exists an element L_0 in the τ^{st}-closure $\overline{\mathcal{K}}$ of $\mathcal{K}$ in $\mathbf{sf}(K)$ which is K-embeddable in K^τ.*

Proof. Comp. with [H–P]:

Claim: *Let $f(X) \in K[X]$ be a separable polynomial. If $f(X)$ has zeros in every $\mathrm{L} \in \mathcal{K}$, then $f(X)$ has zeros in K^τ.*

Indeed, suppose that $f(X)$ has no zeros in K^τ. Let $c = f(b)$ be a fixed element in $f(K)$. Since $0 \notin f(K)$, it follows that $1 - c/f(K)$ is τ-bounded, hence there exist $d \in K$ and a neighborhood U of 0 with the properties:

i) $f(K^\tau) \cap U = \emptyset$.
ii) $d \cdot \big(1 - c/f(K^\tau)\big) \subseteq U$.

Set $h(X,Y) = f(Y)\big(1 - d^{-1}f(X)\big) - c$. The affine variety $V|K$ defined by $K[V] = K[X,Y]/(h)$ is absolutely irreducible, and $V_r(\mathrm{L})$ is non-empty for every $\mathrm{L} \in \mathcal{K}$. Indeed, if $a \in \mathrm{L}$ is any zero of $f(X)$, a direct verification shows that (a,b) is regular point of V. Hence, by our assumption on K, it follows that $V(K)$ is non-empty. Considering any point (x,y) in $V(K)$, we obtain $h(x,y)=0$, hence $f(y)=d\cdot\big(1-c/f(x)\big)$. This contradicts i), ii).

We now finish the proof of the Lemma. If no $\mathrm{L}\in\overline{\mathcal{K}}$ can be K-embedded in K^τ, then for every L, there exists some $a_{\mathrm{L}}\in\mathrm{L}$ such that the irreducible polynomial of a_{L} over K has no zeros in K^τ. For each L, denote by $\mathcal{U}_{\mathrm{L}}$ the set of all L' which contain a_K. Since L lies in $\mathcal{U}_{\mathrm{L}}$, it follows that $(\mathcal{U}_{\mathrm{L}})_{\mathrm{L}}$ is a τ^{et}-open covering of $\overline{\mathcal{K}}$. Let $(\mathcal{U}_{\mathrm{L}_k})_k$ be a finite sub-covering of it, and let $f_k(X)$ denote the irreducible polynomial of a_{L_k} over K. Then the lowest common multiple $f(X)=\mathrm{lcm}\big(f_k(X)\big)_k$ is a separable polynomial over K and has zeros in all $\mathrm{L}\in\mathcal{K}$, but not in K^τ. This is a contradiction. □

We now come to the proof of Theorem 2.9. Let v be a non-trivial valuation of K, and τ the V-topology on K defined by v. Suppose that the separable closure K^τ of K in the v-completion of K is not separably closed. By the above Lemma, there exists an element L' of $\mathcal{K}$ which is K-embeddable in K^τ. If τ' denotes the V-topology defined by the valuation v' of L', and one defines correspondingly L'^τ, then $\mathrm{L}'^\tau\subseteq\mathrm{L}'\subseteq K^\tau$. By [P–Z], it follows that $\tau=\tau'$ and $K^\tau=\mathrm{L}'^\tau$. Therefore, $\mathrm{L}'^\tau=\mathrm{L}'=K^\tau$, and so K^τ lies in $\mathcal{K}$.

The remaining assertions of the theorem now follow easily from the one proved and general valuation theory. □

Corollary 2.11. *Let K be a pseudo $\mathcal{K}$-closed field, with $\mathcal{K}$ a G_K-invariant classical family of localities of K. Then $\mathcal{K}_{\min}$ is the space of all classical closures of K, and so depends only on K.*

The next theorem shows how pseudo closed fields behave under algebraic extensions.

Theorem 2.12 (Overfield Theorem). *Let K be a pseudo $\mathcal{K}$-closed field with $\mathcal{K}$ a standard G_K-subspace of* $\mathbf{loc}(K)$*. Let $M|K$ be an algebraic extension, and let $\mathcal{M}$ denote the family of all non-separably closed localities Λ of M of the form $\Lambda=M\mathrm{L}$ $(\mathrm{L}\in\mathcal{K})$. Then $\mathcal{M}$ is a standard G_M-subspace of* $\mathbf{loc}(M)$*, and moreover, M is pseudo $\mathcal{M}$-closed.*

Proof. It is obvious that $\mathcal{M}$ is G_M-invariant; and using Theorem 2.9 combined with Appendix, 4.7, (3), it follows that $\mathcal{M}$ is standard.

We show that M is pseudo $\mathcal{M}$-closed. Let $V|M$ be an absolutely irreducible affine variety. Then there exists a finite sub-extension $K_1|K$ of $M|K$, and an absolutely irreducible affine variety $W|K_1$ such that $V=W\times_K M$. If $\mathcal{K}_1$ denotes the family of all non-separably closed localities $\mathrm{L}_1=\mathrm{L}K_1$ of K_1, then $\mathcal{M}$ is exactly the family of all non-separably closed localities of M of the form $\Lambda=\mathrm{L}_1M$. Obviously, $\mathcal{K}_1$ is quasi τ^{et}-compact and G_{K_1}-invariant, and by Heinemann–Prestel [H–P], we know that K_1 is pseudo $\mathcal{K}_1$-closed. Hence, we can assume that $K_1=K$. Now, let $V_r(\Lambda)\neq\emptyset$ for all $\Lambda\in\mathcal{M}$. Since for every Λ there exists $\mathrm{L}\in\mathcal{K}$ such that $\Lambda=\mathrm{L}M$, we have: For every L, there exists a finite extension $K_{\mathrm{L}}|K$ contained in L, and a finite extension $M_{\mathrm{L}}|K$ contained in M, such that V has regular M_KK_{L}-rational points. By construction, $(\mathcal{U}_{K_{\mathrm{L}}})_{\mathrm{L}}$ is a τ^{et}-open covering of $\mathcal{K}$. Therefore, there exist finitely many L_m $(m=1,\dots,n)$ such that: For every L, there exists m with $K_{\mathrm{L}_m}\subseteq\mathrm{L}$. Let M_0 be the compositum of the fields M_{L_m}. Then $M_0|K$ is finite, and M_0 is pseudo $\mathcal{M}_0$-closed, where $\mathcal{L}_0$ is the set of all non-separably closed localities of M_0 of the form $\Lambda_0=\mathrm{L}M_0$.

We now check that V has Λ_0-rational regular points for every $\Lambda_0\in\mathcal{M}_0$. Namely, for every $\Lambda_0=\mathrm{L}M_0$, there exists m such that $K_{\mathrm{L}_m}\subseteq\mathrm{L}$. Further, since M_{L_m} are contained in M_0, it follows that $K_{\mathrm{L}_m} M_{\mathrm{L}_m} \subseteq \Lambda_0$. Thus the $K_{\mathrm{L}_m} M_{\mathrm{L}_m}$-rational points of V are also Λ_0-rational points. Hence this last set is non-empty.

Therefore, V has M_0-rational points, and so, L-rational points. The proof is finished. □

Corollary 2.13. (1) *An algebraic extension of a PRC field is also PRC (comp.* [Pr2]*).*

(2) *An algebraic extension M of a PpC field K is PpC if and only if for any p-adic closure L of K, either $M \subset \mathrm{L}$ or $\mathrm{L}M$ is separably closed.*

(3) *An algebraic extension M of a pseudo classically closed field K is pseudo classically closed if and only if there exists a positive bound $c > 0$ such that for every classical closure L of K, either $\mathrm{L}M$ is separably closed or $[\mathrm{L}M : K] < c$. In particular, a finite extension of a pseudo classically closed field is itself pseudo classically closed.*

Corollary 2.14. *Let K be pseudo $\mathcal{K}$-closed. Let $M|K$ be an algebraic extension of K with pro-solvable absolute Galois group G_M. Suppose that $\mathrm{cd}_q G_M \geq 2$ for at least two rational prime numbers q. Then M is a locality of K, hence there is some $\mathrm{L} \in \mathcal{K}$ which is K-embeddable in M and moreover, this L is unique.*

In particular, if all $\mathrm{L} \in \mathcal{K}$ are endowed with maximal valuations v, and G_{L} is isomorphic to a classical Galois group $\mathbb{G} \subseteq G_{\mathbb{Q}_p}$, then L is p-adically closed.

Proof. Let $\mathcal{M}$ be the prolongation of $\mathcal{K}$ to M. Then M is pseudo $\mathcal{M}$-closed. In particular, its Brauer group $Br\ L$ satisfies a local-global principle with respect to $\mathcal{M}$, i.e., the canonical mapping

$$Br\, M \to \prod_{\Lambda} Br\, \Lambda$$

is injective. In particular, $\mathrm{cd}_q G_\Lambda \geq 2$ for at least two rational prime numbers q, and corresponding Λ 's. But then M is itself henselian, see [P1], (1.13). Since by Theorem 2.9, all Λ are henselisations of M, it follows that $\{M\} = \mathcal{M}$. In other words: $M = \mathrm{L}M$ for some $\mathrm{L} \in \mathcal{K}$. The uniqueness follows by Theorem 2.9.

For the second assertion we remark that the valuation of L is maximal (it prolonging the valuation of the unique K which is contained in L). Further, apply [P1], (E.9), and §2. □

C) Construction of pseudo closed fields

We will now solve the problem of constructing pseudo closed fields starting from some "initial conditions" as follows. Let K be an arbitrary field, and $\mathcal{K}$ a family of localities of K. Under supplementary conditions on $\mathcal{K}$ we will construct a field extension $L|K$ endowed with a family of localities $\mathcal{L}$ such that L is pseudo closed with respect to $\mathcal{L}$, and $\mathcal{L}$ is closely related to the family of localities $\mathcal{K}$. In the above context, in the case of a finite family $\mathcal{K}$, such extensions $L|K$ were constructed by Heinemann–Prestel [H–P], and other special cases have been dealt with by Ershov [E], Haran–Jarden [H–J1], [H–J2], and others. The idea we use can be found in van den Dries [vdD]. Therefore we will develop here the things only as far as to be in the position to make the machinery from loc.cit. work. We further will consider from the start that the fields we deal with are perfect fields.

For an arbitrary field extension $L|K$ let $\pi_{LK} : G_L \to G_K$ be the canonical projection, and $\rho_{LK} : \mathbf{sf}(L) \to \mathbf{sf}(K)$, $L' \mapsto K' := L' \cap K^{\mathrm{alg}}$ the canonical restriction map. In general, ρ_{LK} does not map localities of L to localities of K, as L could have valuations which are trivial on K. Nevertheless, if Λ is a maximal locality of L, then $\Lambda \cap K^{\mathrm{alg}}$ is a locality of K (unless K is algebraic over a finite field). This is the case for instance if Λ is a classical locality of Λ.

Definition. Let $\Lambda|\mathrm{L}$ be a field extension. We will say that Λ dominates L if Λ is henselian with respect to a non-trivial valuation v having the properties:

(i) v is trivial on L, and the residue field of v is $\Lambda v = \mathrm{L}$.

(ii) Λ has only inert extension.

In particular the canonical projection $\pi_{\Lambda\mathrm{L}} : G_\Lambda \to G_\mathrm{L}$ is an isomorphism, and $\mathbf{sg}(G)_\Lambda \to \mathbf{sg}(G)_\mathrm{L}$ and $\mathbf{sf}(\Lambda) \to \mathbf{sf}(\mathrm{L})$ are homeomorphisms.

We remark that if in the above context L is itself is henselian to some non-trivial valuation v_L, then setting $v_\Lambda = v \circ v_\mathrm{L}$, it follows that $(\mathrm{L}, v_\mathrm{L})$ is existentially closed in (Λ, v_Λ) as valued fields. In particular, if L is elementarily equivalent to some classical field $\mathbb{K}$, then every field Λ dominating L is also elementarily equivalent to $\mathbb{K}$.

Definition. Let K be a perfect field, and $\mathcal{K}$ a quasi τ^{et}-compact set of localities of K. A cover of $(K, \mathcal{K})$ is a field extension $L|K$ with L a perfect field together with a τ^{et}-compact family $\mathcal{L}$ of localities of L such that:

(i) ρ_{LK} maps $\mathcal{L}$ and its orbit space $\mathcal{L}^\wedge$ with respect to the conjugation bijectively onto $\mathcal{K}$, respectively $\mathcal{K}^\wedge$.

(ii) Every Λ dominates the corresponding $\mathrm{L} = \rho_{LK}(\Lambda) = \Lambda \cap K^{\mathrm{alg}}$.

(iii) For any $\Lambda, \Lambda_1 \in \mathcal{L}$ and $g \in \ker(\pi_{LK})$ one has: $\Lambda^g \Lambda_1 = L^{\mathrm{alg}}$.

We recall that a subspace $\mathcal{K} \subseteq \mathbf{sf}(K)$ is called special, if $\mathcal{K}^{G_K}$ is τ^{et}-compact and contains only self-normalizing over-fields of K.

Theorem 2.15. *Let K be a perfect field endowed with a special G_K-subspace of $\mathbf{sf}(K)$. Then there exist covers $({}^*K, {}^*\mathcal{K})$ for $(K, \mathcal{K})$ such that ${}^*K$ is pseudo ${}^*\mathcal{K}$-closed, and moreover, ${}^*\mathcal{K}$ is special.*

In particular, ${}^*\mathcal{K}$ consists of all minimal non-separably closed henselisations of ${}^*K$, and its orbit space ${}^*\mathcal{K}^\wedge$ is mapped by $\rho_{{}^*KK}$ homeomorphically onto $\mathcal{K}^\wedge$.

Proof. The key technical step in the proof is the following

Lemma 2.16. *In the context of the above Theorem, let $V \to K$ be an absolutely irreducible variety, and $K(V)$ its field of rational functions. Suppose that V has regular* L*-rational points for every $\mathrm{L} \in \mathcal{K}$. Then there exits a regular field extension $R|K$ containing $K(V)$, and a space $\mathcal{R}$ of localities of R such that $(R, \mathcal{R})$ is a cover for $(K, \mathcal{K})$. Moreover, $\mathcal{R}$ is special.*

Proof. For every $x_\mathrm{L} \in V_r(\mathrm{L})$, let $\mathcal{U}_\mathrm{L}$ be the τ^{et}-open subset of $\mathcal{K}$ consisting of all L' such that $x_\mathrm{L} \in V(\mathrm{L}')$. It is obvious that $\{\mathcal{U}_\mathrm{L}\}_\mathrm{L}$ is a τ^{et}-open covering of $\mathcal{K}$. Since $\mathcal{K}$ is τ^{et}-compact, there exists a finite Galois extension $N|K$ and finitely many L_l such that the covering defined by the basic τ^{et}-open subsets $\{\mathcal{U}_N(\mathrm{L}_l)\}_{\mathrm{L}_l}$ is a refinement of the given τ^{et}-open covering. In other words, for every L, there exists l such that $N_l := N \cap \mathrm{L}_l \subseteq \mathrm{L}$, and $V_r(N_l) \neq \emptyset$. By standard set topology arguments, we can suppose that the finite open covering $\{\mathcal{U}_N(\mathrm{L}_l)\}_l$ of $\mathcal{K}$ has a finite refinement $(\mathcal{K}_l)_l$ consisting of disjoint τ^{et}-open subsets of $\mathcal{K}$.

For every l, let $x \in V_r(N_l)$ be a regular N_l-rational point of V. Then $\mathcal{O}_{V,x}$ is a regular ring with residue field $\kappa(x) = N_l$. Therefore there exits a valuation w of $K(V)$ with value group $\cong \mathbb{Z}^d$, where $d = \dim(V)$, and residue field $\kappa(x) = N_l$. Let $K(V)^{\mathrm{h}}$ be a fixed henselisation of $K(V)$ with respect to w in a fixed algebraic closure of $K(V)$, and finally Ω a maximal purely ramified extension of $K(V)^{\mathrm{h}}$. For the sake of simplicity, we denote the canonical prolongation of w to Ω again by w.

Lemma 2.17. *Let R^0 be an arbitrary field, and $R^1 = R^0((X_i)_i)$ the rational function field in I variables over R^0, with I having cardinality $|I| \geq |R^0|$. Then for every algebraic extension $R'|R^0$, there exits a valuation v^1 of R^1 which is trivial on R^0 and has the properties: The residue field of v^1 is $R^1 v^1 = R'$, and the values $v^1(X_i)$ are rationally independent in the value group of v^1.*

In particular, if R^0 is perfect, one inductively constructs a purely transcendental extension $R^* = \cup_n R^n$ of R^0 of transcendence degree depending only on $|R^0|$ such that: If $R'|R^0$ is any given algebraic extension, then there exits a valuation v^* on R^* which is trivial on R^0 and satisfies:

1. *$R^* v^* = R'$, and the henselisation $R^{*\mathrm{h}}$ of (R^*, v^*) has only inert extensions.*
2. *$R^*|R^0$ has a transcendence basis T such that $v^*(t)$, $(t \in T)$, are rationally independent in the value group of v^*.*

Proof. The proof is quite straightforward, and therefore left to the reader. □

Coming back to the proof of Lemma 2.16: We apply the Lemma above with the hypothesis: $R^0 = K(V)$, and $R' = \Omega$. Let $(R^{*\mathrm{h}}, v^{*\mathrm{h}})$ a fixed henselisation of R with respect to v^*. By construction, $R^{*\mathrm{h}}$ has only inert extension, and therefore $\rho_{R^{*\mathrm{h}}\Omega} : \mathbf{sf}(R^{*\mathrm{h}}) \to \mathbf{sf}(\Omega)$ is a homeomorphism in both τ^{et} and τ^{st}. We further let denote $v = v^{*\mathrm{h}} \circ w$ the compositum of $v^{*\mathrm{h}}$ with the valuation w of Ω. Then by general valuation theory, $R^{*\mathrm{h}}$ is also a henselisation of R^* with respect to (the restriction of) v (to R^*), and further $R^{*\mathrm{h}} v = (R^{*\mathrm{h}} v^{*\mathrm{h}}) w = \Omega w = N_l$. In particular, $(R^{*\mathrm{h}}, v^{\mathrm{h}})$ has inert extension only, and $\rho_{R^{*\mathrm{h}} N_l} : \mathbf{sf}(R^{*\mathrm{h}}) \to \mathbf{sf}(N_l)$ is a homeomorphism in both τ^{et} and τ^{st}.

For every $\mathrm{L} \in \mathcal{K}_l$, let Π be its pre-image by $\rho_{R^{*\mathrm{h}} N_l}$ in $\mathbf{sf}(R^{*\mathrm{h}})$. If we endow Π with the unique prolongation v^*_Π of $v^{*\mathrm{h}}$ to Π, then Π dominates L. Further, we will endow every Π with the the unique prolongation v_Π of v^{h} to Π, and view it as a locality (Π, v_Π) of R and of $R^{*\mathrm{h}}$.

Let $\mathcal{R}_l$ be the set of all the localities (Π, v_Π) of the above form. Then $\rho_{R^{*\mathrm{h}} N_l}$ maps $\mathcal{R}_l$ homeomorphically onto $\mathcal{K}_l$ both in τ^{et} and τ^{st}.

Claim. *Every $\Pi \in \mathcal{R}_l$ is self-normalizing in G_{R^*}.*

Indeed, let H_Π be the normalizer of Π in G_{R^*}. Then, since the residue field $\Pi v_\Pi = \mathrm{L}$ is not separably closed, by the generalization of a Theorem of F. K. Schmidt [P1], (1.9), it follows that H_Π is contained in the decomposition group of v_Π. On the other hand, v_Π is simply the prolongation of v^{h} to Π, thus H_Π is mapped via $\pi_{R^{*\mathrm{h}} N_l}$ isomorphically onto the normalizer of $G_{\mathrm{L}} = \pi_{R^{*\mathrm{h}} N_l}(G_\Pi)$ in G_K. Since by hypothesis, L is self-normalizing in G_K, thus in G_{N_l} too, it follows that H_Π is mapped onto G_{L}. Thus $H_\Pi = G_\Pi$, i.e., Π is self-normalizing.

To finish the proof of Lemma 2.16, we first suppose that $\mathcal{K}$ is G_K invariant. Setting $G = G_K$ and $\mathcal{G} = \{G_{\mathrm{L}} \mid \mathrm{L} \in \mathcal{K}\}$, we apply Appendix, 4.10, B) with notations as there: For each k, let $Z_k \subset G_R$ be a pre-image of X_k by π_{RK}, and $\mathcal{K}_k = \{\mathrm{L} \mid G_{\mathrm{L}} \in \mathcal{G}_k\}$. For every $l = k$, one defines the valuation $v_l = v$ and $\rho_l = \rho_{R^{*\mathrm{h}} N_l}$ as above; and for the conjugates $g\mathcal{K}_l g^{-1}$, $(g \in X_k)$, of $\mathcal{K}_l$ one works with the conjugated valuations $v \circ h^{-1}$, where $h \in Z_k$ is the pre-image of $g \in X_k$. Then $(h\mathcal{R}_k h^{-1})_{k, h \in Y_k}$ are τ^{et}-compact disjoint subsets of $\mathbf{sf}(R^*)$; and obviously, their union $\mathcal{R}$ gives rise to a covering $(R^*, \mathcal{R}) \to (K, \mathcal{K})$.

For the general case, when $\mathcal{K}$ is not necessarily G_K invariant, we first make the construction for $\mathcal{K}' = \mathcal{K}^{G_K}$ instead of $\mathcal{K}$. If now $\mathcal{R}$ denotes the pre-image of $\mathcal{K}$ in $\mathcal{R}'$, then $\mathcal{R}$ is special. □

The next step in proving Theorem 2.15 is the following

Lemma 2.18. *In the context of Theorem 2.15 there exists a cover $(K^1, \mathcal{K}^1)$ of $(K, \mathcal{K})$ with the following property: Every absolutely irreducible variety $V \to K$ has a K^1-rational point, provided V has a regular* L*-point for every* $\mathrm{L} \in \mathcal{K}$. *Moreover, $\mathcal{K}^1$ is a special subset of* $\mathbf{sf}(K^1)$.

Proof. We use the idea from van den Dries, compare [vdD], p. 28. Let $(V_\lambda)_\lambda$ be the family of all absolutely irreducible varieties over K, the index set being well ordered. Setting $K_0 = K, \mathcal{K}_0 = \mathcal{K}$, using Lemma 2.16, one constructs inductively a chain of pairs $(K_\lambda, \mathcal{K}_\lambda)$ such that for $\mu \le \nu$, the resulting $(K_\nu, \mathcal{K}_\nu)$ is a cover for $(K_\mu, \mathcal{K}_\mu)$, and moreover, the following holds:

i) If λ is not a limit ordinal, then V_λ has a regular K_λ-rational point, provided V_λ has regular $\mathrm{L}_{\lambda-1}$-rational points for all $\mathrm{L}_{\lambda-1} \in \mathcal{K}_{\lambda-1}$

ii) If λ is a limit ordinal, then setting $L = \cup_{\mu<\lambda} K_\mu$, and $\mathcal{L} = \varprojlim \mathcal{K}_\mu$, the following holds: If V_λ has a regular Λ-point for all $\Lambda \in \mathcal{L}$, then V_λ has a K_λ-rational point.

Indeed, if λ is not a limit ordinal, then the existence/construction of $(K_\lambda, \mathcal{K}_\lambda)$ follows immediately from Lemma 2.16. If now λ is a limit ordinal, then it follows that the above $(L, \mathcal{L})$ is a cover for all $(K_\mu, \mathcal{K}_\mu)$. Then one uses again Lemma 2.16, with hypothesis $(L, \mathcal{L})$ in order to get $(K_\lambda, \mathcal{K}_\lambda)$.

Finally set $K^1 = \cup_\lambda K_\lambda$ and $\mathcal{K}^1 = \varprojlim \mathcal{K}_\lambda$. It is clear that $(K^1, \mathcal{K}^1)$ has the desired properties. □

We finally come to the proof of Theorem 2.15. We set $K^0 = K$ and $\mathcal{K}^0 = \mathcal{K}$. Using last Lemma above, we get a chain of covers $(K^n, \mathcal{K}^n)$ with the following properties:

j) $K^{n+1}|K^n$ is a regular field extension for every $n \ge 0$.

jj) If $V \to K^n$ is an absolutely irreducible variety which has regular L^n-rational points for all $\mathrm{L}^n \in \mathcal{K}^n$, then V has K^{n+1}-rational points.

It now follows without difficulty that K^* endowed with $\mathcal{K}^* = \varprojlim \mathcal{K}^n$ has the desired properties. □

As a corollary we have:

Theorem 2.19. *Let K be an arbitrary field, and $\mathcal{K}$ a τ^{et}-compact family of classical closures of K. Then there exists a regular field extension $K^*|K$ endowed with a τ^{et}-compact set $\mathcal{K}^*$ of classical localities of K, such that $(K^*, \mathcal{K}^*)$ is a cover for $(K, \mathcal{K})$, and K^* is pseudo $\mathcal{K}^*$-closed.*

In particular, this applies in the case $\mathcal{K}$ is the set of all classical closures of K which are contained in the elements of some classical family $\mathbf{clos}^{\mathcal{X}}(K)$ *of localities of K.*

Proof. Clear from Theorem 1.5 and the Theorem 2.15 above. □

3. Classically projective groups as absolute Galois groups

A) The absolute Galois group of a pseudo closed field

Lemma 3.1. *Let $N|M$ be an arbitrary Galois extension, and X a finite set on which $G :=$* $\mathrm{Gal}(N|M)$ *acts. We let $N[X]$ be the polynomial ring in X variables on which G acts by $g(ax) = g(a)gx$ for $a \in N$ and $x \in X$. Then the fixed ring of G in $N[X]$ is a polynomial ring $M[T]$ over M, and moreover, $N[T] = N[X]$.*

Proof. Let $\{X_k\}_k$ be the orbits of X with respect to the action of G. If the assertion of Lemma is true for every $N[X_k]$, then it is true for $N[X]$. Hence it is sufficient to consider the case $r = 1$, i.e., G acts transitively on X. Fixing $x \in X$, we let G_x be its stabilizer, and N_x the fixed field of G_x in N. Let a be a primitive element of $N_x|M$. Remark that for $0 \leq k \leq n-1$, the elements, $a^k x$ have the same stabilizers, namely $G_{a^k x} = G_x = G_a$. Let σ_m be a system of representatives for G/G_x. For each k as above, let $t_k = \sum \sigma_m(a^k x)$ be the relative trace of $a^k x$. Denote by T the set of all t_k. We claim that this T does the job. Indeed, by their definition, all t_k are G invariant. Next, the determinant of the system of equations defining $\sigma_m(x)$ in relation to t_k is the Vandermonde one defined by $\big(\sigma_m(a)\big)_m$, hence it does not vanish. Therefore, $\sigma_m(x)$ are linear forms in t_k with coefficients in N, hence $N[T] = N[X]$. □

The above construction will be applied in the following way:

3.2. Let K be an arbitrary field endowed with a family $\mathcal{K}$ of separable over-fields and set $\mathcal{G} = \{G_{\mathrm{L}} \mid \mathrm{L} \in \mathcal{K}\}$. Let $(\gamma_K : G_K \to A, \alpha : B \to A, \mathcal{B})$ be any finite embedding problem for $(G_K, \mathcal{G})$. Thus for every $\Gamma \in \mathcal{G}$, there exits a local splitting $\beta_\Gamma : \Gamma \to \Delta \subset B$ such that $(\gamma_K)|_\Gamma = \alpha\, \beta_\Gamma$. Moreover, let us suppose that the conditions 1), 2) from Fact 1.1 are satisfied. We denote by N the fixed field of $D_K = \ker(\gamma_K)$ in K^{sep}, and identify $\gamma_K : G_K \to G_K/D_K$ with the canonical restriction map $pr_N : G_K \to G = \mathrm{Gal}\,(N|M)$. Further, let $X = |B|$ be the underlying set of B viewed as B-space by the multiplication from the left. Set $R = N(X)$ the rational function field on X variables over N. We identify B with a subgroup of $\mathrm{Aut}(R)$ by setting $h(ax) = a(h)(a)hx$ for all $h \in B$, $a \in N$ and $x \in X$. Let $S = R^B$ denote the fixed field of B in R. Then $R|S$ is a Galois extension with Galois group $\mathrm{Gal}\,(R|S) = B$; $S|K$ is a regular extension; and the canonical projection

$$pr_{RN} : B = \mathrm{Gal}\,(R|S) \to \mathrm{Gal}\,(N|K) = G_K/D_K$$

is exactly α. Further, for a fixed $\Gamma \in \mathcal{G}$, let N_Γ be the fixed field of Γ in N, and $B_\Gamma = \beta_\Gamma(\Gamma)$ a pre-image in $\mathcal{B}$ of $\gamma_K(\Gamma) = \Gamma/D_K$. Then Γ acts on $X = |B|$ by $g(x) := \beta_\Gamma(g)\,x$ for $g \in \Gamma$, $x \in |B|$. Moreover, since $(\gamma_K)|_\Gamma = \alpha\beta_\Gamma$ we get $g(a) = \beta_G(g)(a)$ for all $a \in N$ and $g \in \Gamma$. By Lemma above we get: The fixed ring of Γ in $N[X]$ with respect to the action of B is a polynomial ring $N_\Gamma[T]$ and moreover, $N[X] = N[T]$. Therefore, the fixed field $S_\Gamma = N_\Gamma(T)$ of Γ in R is a rational function field over N_Γ and furthermore, $R = S_\Gamma N$.

Theorem 3.3. *Let K be a pseudo $\mathcal{K}$-closed with $\mathcal{K} \subseteq \mathbf{loc}(K)$ quasi τ^{et}-compact, and set $\mathcal{G} = \{G_{\mathrm{L}} \mid \mathrm{L} \in \mathcal{K}\}$. Then G_K is $\mathcal{G}$-projective.*

Proof. We show that any special finite embedding problem as mentioned above $\mathrm{EP} = (\gamma_K, \alpha, \mathcal{B})$ for $(G_K, \mathcal{G})$ has solutions. Namely, in the above construction, for any $\Gamma \in \mathcal{G}$, we have $S \subseteq S_\Gamma \subseteq R$. We let V be any absolutely irreducible affine variety over K with $S = K[V]$. Correspondingly, let V_Γ be an absolutely irreducible affine variety over N_Γ with $N_\Gamma[T] = N_\Gamma[V_\Gamma]$. By the lemma above, V_Γ is isomorphic to the $(\dim V)$-dimensional affine space over N_Γ. Let L be the locality of K with $\Gamma = G_{\mathrm{L}}$. Then the inclusion $\mathrm{L}[V] \subseteq \mathrm{L}[V_\Gamma]$, gives on the geometry side a projection $pr_\Gamma : V_\Gamma(\mathrm{L}) \to V(\mathrm{L})$. Taking into account that $V_\Gamma(\mathrm{L})$ is Zariski dense, it follows that its image in $V(\mathrm{L})$ is Zariski dense too. In particular, $V_r(\mathrm{L})$ is non-empty. Since K is pseudo $\mathcal{K}$-closed, it follows that V also has regular K-rational points. Using any regular K-rational points x of V, one gets a valuation v on $S = K(V)$ such that its valuation ring $\mathcal{O}_v$ dominates the local ring $\mathcal{O}_{V,x}$ of x, and its residue field $Sv = \kappa(x) = K$.

Let S^h be any henselisation of (S, v) and Ω a maximal, totally ramified extention of S^h. Then the canonical projection $\pi_S : G_S \to G_K$ induces by restriction an isomorphism $\pi : G_\Omega \to G_K$. Therefore, denoting by $pr : G_S \to \mathrm{Gal}(R|S) = B$, the canonical projection we get a group homomorphism

$$\beta : G_K \xrightarrow{\pi^{-1}} G_\Omega \hookrightarrow G_S \xrightarrow{pr} \mathrm{Gal}(R|S) = B.$$

We claim that β is a solution of the embedding problem EP. Namely, we have $\alpha\,\beta = \alpha \circ pr \circ \pi^{-1}$. On the other hand, α is by construction exactly the canonical projection $B = \mathrm{Gal}(R|S) \to \mathrm{Gal}(N|K)$ by construction. Thus $\alpha \circ pr$ is the canonical projection $G_S \to \mathrm{Gal}(R|S) \to \mathrm{Gal}(N|K)$. Hence, we get: $\alpha\beta\pi^{-1} = \gamma_K\pi^{-1}$. □

B) Strongly relatively projective groups as absolute Galois groups

We start with the following:

Definition. Let G be an arbitrary profinite group and $\mathcal{G}$ a set of closed subgroups of G. A Galois approximation for $(G, \mathcal{G})$ is a pair $(K, \mathcal{K})$ consisting of a field K endowed with a subspace $\mathcal{K} \subseteq \mathbf{sf}(K)$ together with a surjective group homomorphism

$$\kappa : G \longrightarrow G_K$$

of G onto the absolute Galois group G_K of K which has the properties:

i) κ is injective on each $\Gamma \in \mathcal{G}$.

ii) The fixed field K_Γ of $\kappa(\Gamma)$ in K^{sep} lies in $\mathcal{K}$.

We say that a Galois approximation κ is classical if K_Γ is a classical locality of K for every classical subgroup $\Gamma \in \mathcal{G}$.[1]

If $\mu : G \to G_M$ is another Galois approximation of $(G, \mathcal{G})$, we say that μ dominates κ, written $\kappa \prec \mu$, if the following holds:

j) $K \subseteq M$ and $\kappa = \pi\,\mu$, where π is the canonical projection $G_M \to G_K$.

jj) K_Γ is dominated by Λ_Γ for all $\Gamma \in \mathcal{G}$.

It is obvious that $\prec$ is an ordering on any family of Galois approximations for $(G, \mathcal{G})$. Furthermore, if $\kappa \prec \mu$, then $\ker(\mu) \subseteq \ker(\kappa)$.

Theorem 3.4. *Let G be strongly $\mathcal{G}$-projective with $\mathcal{G}$ τ^{et}-compact. Suppose that there exists a Galois approximation*

$$\kappa : G \to G_K$$

for $(G, \mathcal{G})$ with K a perfect field. Then there exists $\lambda : G \to G_L$ a Galois approximation for $(G, \mathcal{G})$ such that $\kappa \prec \lambda$, and λ is an isomorphism, and L is pseudo $\mathcal{L} = \{\Lambda_\Gamma \mid \Gamma \in \mathcal{G}\}$-closed. In particular, $L|K$ is regular.

Proof. We can suppose that $\mathcal{G}$ is G-invariant. An obvious argument using Zorn Lemma shows that there exist Galois approximations

$$\mu : G \longrightarrow G_M$$

for $(G, \mathcal{G})$ such that $\kappa \prec \mu$ and any other Galois approximation μ_1 for $(G, \mathcal{G})$ satisfies: If $\mu \prec \mu_1$, then $\ker(\mu) = \ker(\mu_1)$. We now show that any such μ is actually an isomorphism.

[1]It was conjectured that the condition ii) is always satisfied, i.e. if G_K is a classical Galois group, then K is a classical field. In the case $G_K \cong G_{\mathbb{R}}$ it is the famous Artin-Schreier Theorem. This conjecture is now *completely proved* by Neukirch, Pop, Koenigsmann, Efrat.

Equivalently, it is sufficient to show that $H = \ker(\mu)$ is the trivial group. *Mutatis mutandis,* we can suppose that κ is maximal in the above sense.

Let D be an arbitrary open subgroup of G. Using the construction from 3.2, we shall construct a Galois approximation μ for $(G, \mathcal{G})$ with $\ker(\mu) \subseteq D$. Let namely consider the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\kappa} & G_K \\ \downarrow{\scriptstyle \gamma} & & {\scriptstyle \gamma_K}\downarrow \\ G/D & \xrightarrow{\alpha} & G_K/D_K \end{array}$$

where $D_K = \kappa(D)$ and the mappings are the canonical projections. Let N be the fixed field of D_K and for $\Gamma \in \mathcal{G}$ denote by N_Γ the fixed field of $\gamma_K\kappa(\Gamma) = \alpha\gamma(\Gamma)$. Since κ is injective on all $\Gamma \in \mathcal{G}$, and $\mathcal{G}$ is τ^{et}-compact, after replacing D by a sufficiently small open normal subgroup of G, we can suppose that α is injective on all B in $\gamma(\mathcal{G})$. Then $\big(\gamma_K, \alpha, \gamma(\mathcal{G})\big)$ is a special embedding problem for $\big(G_K, \kappa(\mathcal{G})\big)$. Hence, we can apply the construction 3.2 and obtain:

For every $\Gamma \in \mathcal{G}$, the fixed field S_Γ of $\kappa(\Gamma)$ in $R = N(X)$, is a rational function field $S_\Gamma = N_\Gamma(T)$ over N_Γ. Let v be the valuation

$$v : S_\Gamma \to \mathbb{Z}^T$$

with $\mathbb{Z}^T$ ordered lexicographically, and $v(t) =$ (the t^{th} element of the canonical basis of $\mathbb{Z}^T$) for any $t \in T$. Then the residue field of v is $S_\Gamma v = M_\Gamma$. Denote by S^h some henselisation of (S_Γ, v) (together with a prolongation of v, which we again denote by v). Finally let Ω_Γ be a fixed maximally purely ramified extension of S^h (together with a prolongation of v, which we again denote by v). Thus, the canonical projection

$$\pi_\Gamma : G_{\Omega_\Gamma} \longrightarrow G_{M_\Gamma}$$

is an isomorphism. Now consider any $\Gamma_1 \in \mathcal{G}$ with $\Gamma_1/D \subseteq \Gamma/D$ and so, $\alpha\gamma(\Gamma_1) \subseteq \alpha\gamma(\Gamma)$. Then $M_\Gamma \subseteq M_{\Gamma_1}$, hence we can consider the pre-image Δ_1 of $\kappa(\Gamma_1)$ in G_{Ω_Γ}. Let Σ_1 be the fixed field of Δ_1 in the separable closure of Ω_Γ endowed with the unique prolongation v_{Σ_1} of v to Σ_1. By construction, it follows that $\varphi_{SK}(\Sigma_1) = K_1$ and moreover, Σ_1 dominates K_1. If now $\mathcal{S}_1$ is the family of all Σ_1 as above, when Γ varies in $\mathcal{G}$ we show that there exist Galois approximations $\mu : (G, \mathcal{G}) \to (S, \mathcal{S})$ with $\mathcal{S} \subseteq \mathrm{con}(\mathcal{S}_1)$, $\kappa \prec \mu$ and $\ker(\mu) \subseteq D$.

For the open normal subgroup D of G let namely

$$\pi = \pi_D : (F_D, \mathcal{F}) \longrightarrow (G, \mathcal{G})$$

be a "a canonical cover" for $(G, \mathcal{G})$ associated with D as in the Appendix, 4.10, C). Further, we let g_S be a pre-image of $\kappa(g)$ in G_S (such pre-images do exist, because $S|K$ is regular, thus $G_S \to G_K$ is surjective). From the universal property of the usual profinite product, it follows that there exists a unique homomorphism $\varphi : F_D \to G_S$ such that $\varphi\,|_{D_k} = \pi_{\Gamma_k}^{-1} \circ \kappa\,|_{D_k}$, and $\varphi(|x|) = g_S$. Obviously, ϕ maps $\mathcal{F}$ into $\mathrm{con}(\mathcal{S}_1)$ and makes the following diagram commutative:

$$\begin{array}{ccccc} F_D & \xrightarrow{\phi} & G_S & \xrightarrow{\mathrm{pr_R}} & \mathrm{Gal}\,(R|S) = B \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi_S}\downarrow & & {\scriptstyle \alpha}\downarrow \\ G & \xrightarrow{\kappa} & G_S & \xrightarrow{\mathrm{pr_N}} & \mathrm{Gal}\,(N|K) = A \end{array}$$

On the other hand, since π is a cover for $(G, \mathcal{G})$, there exists a right inverse σ of π with $\sigma(\mathcal{G}) \subseteq \mathcal{F}^F$. Therefore, $\mu = \phi \circ \sigma$ is a Galois approximation for $(G, \mathcal{G})$ with $\kappa \prec \mu$ and $\ker(\mu) \subseteq D$. The maximality of κ implies: $\ker(\kappa) = \ker(\mu)$. Hence, $\ker(\mu) \subseteq D$. Since D was arbitrary, it follows that $\ker(\kappa) = \{1\}$.

Now let us consider an arbitrary Galois approximation $\mu : G \to G_M$ satisfying $\kappa \prec \mu$, and $\ker(\mu) = \{1\}$. Let us denote $\mathcal{G}_M = \kappa(\mathcal{G})$, and $\mathcal{M} = \{\Lambda_\Gamma \mid \Gamma \in \mathcal{G}\}$. Then $(M, \mathcal{M})$ satisfies the hypothesis of Theorem 2.12, and moreover, $\mathcal{M}$ is a special G_M-subspace of $\mathbf{sf}(M)$. Let $(M^*, \mathcal{M}^*)$ be as in Theorem 3.2, and set $\mathcal{G}_{M^*} = \{G_{\Lambda^*} \mid \Lambda^* \in \mathcal{M}^*\}$. Then the canonical projection $G_{M^*} \to G_M$ defines in fact a cover in the sense of Section 1, B)

$$\pi_{M^* M} : (G_{M^*}, \mathcal{G}_{M^*}) \to (G_M, \mathcal{G}_M)$$

for $(G_M, \mathcal{G}_M) = \mu(G, \mathcal{G})$. Since G_M is strongly $\mathcal{G}_M$-projective, we can apply Theorem 1.10, and get a right inverse $\sigma^* : G_M \to G_{M^*}$ of π^* such that $\sigma^*(\mathcal{G}_M) \subseteq \mathcal{G}_{M^*}$, and for all $\Delta^* \in \mathrm{con}(\mathcal{G}_{M^*})$ either $\Delta^* \subseteq \sigma^*(G_M)$, or $\Delta^* \cap \sigma^*(\mathcal{G}_M) = \{1\}$. In field theoretical terms this means: Let L, Λ^* be the fixed fields of $\sigma^*(G_M)$, respectively Δ^* in the algebraic closure of M^*. Then either Λ^* contains L or their compositum $\Lambda^* L$ is algebraically closed. Applying Theorem 2.12, it follows that L is pseudo closed with respect to $\mathcal{L} = \sigma^*(\mathcal{M})$. Finally, $\lambda = \sigma^* \mu$ is a Galois approximation for $(G, \mathcal{G})$ with $\kappa \prec \lambda$. □

Without proof we give the following corollary:

Theorem 3.5. *Let G be strongly $\mathcal{G}$-projective with $\mathcal{G} = \mathcal{G}_{\max}$, and τ^s-compact, and suppose that all $\Gamma \in \mathcal{G}$ are isomorphic to the decomposition group Z_v of a valuation v of some field. If Z_v has finite corank, then $(G, \mathcal{G})$ has Galois approximations. Hence, it can be realized as absolute Galois group of a pseudo closed field.*

We conclude this section by the main result of the paper:

Theorem 3.6. *A profinite group G is classically projective if and only if it is isomorphic to the absolute Galois group G_K of a pseudo classically closed field K.*

Proof. In one direction, one applies Theorem 3.3 together with Theorem 1.2. In the other direction, apply Theorem 1.2 combined with Theorem 3.4, and Theorem 1.5 (which gives one Galois approximations for classically relatively projective groups). □

4. Appendix: Boolean and quasi-boolean G-spaces

Let X be an arbitrary topological space. For any $x \in X$, and any subspace $Y \subseteq X$ we define:

1) $\mathrm{env}(x)$ the set of all $x' \in X$ which have x as specialization, i.e., $x \in \overline{\{x'\}}$.
2) $\mathrm{env}(Y) = \cup_{y \in Y} \mathrm{env}(y)$, the envelope of Y.
3) $\mathbf{gen}(Y)$, the residue set of Y, i.e., the set of all $y \in Y$ having more than a closed point as specialization in Y.

It is clear that any neighborhood U of x contains $\mathrm{env}(x)$, and any neighborhood U of Y contains $\mathrm{env}(Y)$. Further, it is obvious that $\mathrm{env}\big(\mathbf{gen}(Y)\big) = \mathbf{gen}(Y)$.

Let X be a finite set. Giving a T_1-topology on X is the same as giving an ordering $\leq$ on X. Indeed, if the topology is given, then for x, x' we define: $x' \leq x$ if and only if $x' \in \mathrm{env}(x)$. In particular, the sets $\mathrm{env}(x)$ $(x \in X)$ constitute a basis of the given topology. Conversely, given an ordering $\leq$ on X, the family of all sets $\mathrm{env}(x)$ defined as above is a basis of a T_1-topology on X.

A mapping $f : X \to Y$ between two finite T_1-topological spaces is continuous if and only if it is order preserving.

Definition. We say that a topological space is quasi-boolean if it is a projective limit of finite T_1 topological spaces. For technical reasons we usually denote a quasi-boolean topology by τ^{et}. We say that a topological space is boolean if it is projective limit of discrete finite spaces.

Equivalently, a quasi-boolean topological space is a T_1 quasi-compact topological space which has a topology basis consisting of quasi-compact sets. A boolean space is a compact topological space which has a topology basis consisting of compact sets.

The category of all boolean spaces can be viewed as a completion of the category of all quasi-boolean spaces in the following sense: For every quasi-boolean topological space (X, τ^{et}) there exists on X a unique smallest Hausdorff topology τ^{st} containing τ^{et} such that $\tau^{\mathrm{et}} \mapsto \tau^{\mathrm{st}}$ has functorial behavior, i.e.:

Any τ^{et}-continuous map $f : (X, \tau^{\mathrm{et}}) \to (Y, \tau^{\mathrm{et}})$ is also τ^{st}-continuous.

In particular, if $(X, \tau^{\mathrm{et}}) = \varprojlim (X_i, \tau^{\mathrm{et}})$ with X_i finite and T_1, then τ^{st} is the projective limit of the discrete topology on X_i.

Directly from the definition it follows that any τ^{st}-closed subset $Y \subseteq X$ of a quasi-boolean topological space X endowed with the τ^{et}-subspace topology, is also a quasi-boolean topological space. Moreover, if $(X, \tau^{\mathrm{et}}) = \varprojlim (X_i, \tau^{\mathrm{et}})$ with X_i finite and T_1, then $(Y, \tau^{\mathrm{et}}) = \varprojlim (Y_i, \tau^{\mathrm{et}})$ with $Y_i \subseteq X_i$ the image of Y in X_i by the canonical projection $\varphi_i : X \to X_i$.

Therefore, the next assertions on a quasi-boolean topological space (X, τ^{et}) also hold for any τ^{st}-closed subspace of it, but endowed with the τ^{et}-induced topology.

Lemma 4.1.
(1) *For every $x \in X$ there exist minimal elements x_0 and maximal elements x^0 of X with the property $x_0 \leq x \leq x^0$. We denote by $X_{\max}$ and $X_{\min}$ the set of all maximal, respectively minimal elements of X.*
(2) *Let x_1, x_2 be arbitrary elements of X. Then there exists $x \in X_{\min}$ with $x \leq x_1, x_2$ if and only if any τ^{et}-neighborhoods U_1, U_2 of x_1, x_2 have non-empty intersection.*

Lemma 4.2. *Let $Y \subseteq X$ be an arbitrary subset. Then the following assertions are equivalent:*
(1) *Y is quasi τ^{et}-compact.*
(2) $\mathrm{env}(Y)$ *is quasi τ^{et}-compact.*
(3) $\mathrm{env}(Y)$ *is intersection of τ^{et}-open quasi-compact sets.*
(4) *Y has a basis of τ^{et}-open quasi-compact neighborhoods.*

As an immediate corollary we have

Lemma 4.3. *For an arbitrary subspace $Y \subseteq X$ the following assertions are equivalent:*
(1) *Y is quasi τ^{et}-compact.*
(2) $\mathrm{env}(Y)$ *is τ^{st}-compact.*

In particular, if Y is quasi τ^{et}-compact, then $Y_{\max}$ is defined, and $\mathrm{env}(Y) = \mathrm{env}(Y_{\max})$.

Definition. We say that a subspace $Y \subseteq X$ is standard if it is quasi τ^{et}-compact and $Y = Y_{\max}$.

Lemma 4.4. *Let $Y \subseteq X$ be an arbitrary subspace and denote by $\overline{Y}$ its τ^{st}-closure. Then the following assertions are equivalent:*

(1) *Y is τ^{et}-compact.*

(2) *$Y = (\overline{Y})_{\max}$ and $\overline{Y} \cap \mathbf{gen}(Y) = \emptyset$*

Thus, if Y is τ^{et}-compact, then for any $y \in \overline{Y}$ there exists a unique $y_0 \in Y$ with $y \leq y_0$.

Proof. 1) $\Rightarrow$ 2). By **(4.3)** it follows that $\overline{Y} \subseteq \text{env}(Y)$. Take $y', y'' \in Y$ and $y \in X$ with $y \leq y', y''$. Then for any τ^{et}-open neighborhoods U' and U'' of y' respectively y'' in X it follows that $U' \cap U''$ is also a τ^{et}-neighborhood of y. In particular, if y lies in $\overline{Y}$ then $U' \cap U'' \cap Y \neq \oslash$. Therefore, $U' \cap Y$ and $U'' \cap Y$ have non-empty intersection, hence $y' = y''$.

2) $\Rightarrow$ 1) By **(4.3)** it follows that $Y = (\overline{Y})_{\max}$ is quasiτ^{et}-compact. We show that it is also Hausdorff. Let y', y'' be points of Y such that any τ^{et}-open quasi-compact neighborhoods U' and U'' of them in Y have non-empty intersection. Now by **(4.3)** it follows that $\overline{Y} \cap \text{env}(U') \cap \text{env}(U'')$ is a τ^{st}-closed non-empty set for every U' and U''. Therefore, the intersection of all the sets of this form is also non-empty. On the other hand, this intersection equals $Z = \overline{Y} \cap \text{env}(y') \cap \text{env}(y'')$. Now for every $y \in Z$ one has $y \leq y', y''$. Therefore, $y' = y''$. ☐

Taking into account that any τ^{et}-closed subset of X is also τ^{st}-closed, by **(4.4)** one has the following

Corollary 4.5. *Let $Y \subseteq X$ be a standard subspace of X. Then for every τ^{st}-closed subset D of X which does not meet $\mathbf{gen}(\overline{Y})$ the set $D \cap Y$ is τ^{et}-compact.*

Definition. Let (X, τ^{et}) be a quasi-boolean topological space. We say that X is separating if $\mathbf{gen}(X)$ is τ^{st}-closed.

Lemma 4.6. *Let X be a quasi-boolean topological space. The following assertions are equivalent:*

(1) *X is separating.*

(2) *Any $y \in X_{\max}$ has a τ^{st}-neighborhood U with $U \cap \mathbf{gen}(X) = \emptyset$.*

(3) *Any $y \in X_{\max}$ has a basis of τ^{st}-clopen neighborhoods U which do not meet $\mathbf{gen}(X)$.*

Examples of quasi-boolean topological spaces: $\mathbf{sg}(G)$ and $\mathbf{sf}(K)$.

Let X be an arbitrary boolean space and let $\mathbf{exp}X$ denote the family of all τ^{st}-closed subsets of X. Let $X = \varprojlim(X_i, \varphi_{ji})$ be a realization of X as projective limit of finite T_1 spaces and $\varphi_i : X \to X_i$ the canonical projections. Then $\mathbf{exp}X = \varprojlim \ \mathbf{exp}X_i$.

Let us endow all $\mathbf{exp}X_i$ with the ordering defined by the inclusion. The corresponding T_1-topology on X_i is called the étale topology, denoted τ^{et}. It is obvious that all φ_{ji} respect these orderings, or equivalently, they are τ^{et}-continuous. We define τ^{et} on $\mathbf{exp}X$ as being the projective limit of the étale topology on X_i. A basis of quasi-compact τ^{et}-open neighborhoods of any $Y \in \mathbf{exp}X$ is

$$\mathcal{U}_i(Y) = \{Y_i \in \mathbf{exp}X \mid \varphi_i(Y_1) \subseteq \varphi_i(Y)\}, \forall i$$

One shows without difficulty that τ^{et} does not depend on the concrete realizations $X = \varprojlim(X_i, \varphi_{ji})$ of X as projective limit of finite T_1 spaces.

The strict topology constructed from τ^{et} is denoted as usually by τ^{st}. We remark that a basis of compact τ^{st}-open neighborhoods of any $Y \in \mathbf{exp}X$ is

$$\mathcal{V}_i(Y) = \{Y_i \in \mathbf{exp}X \mid \varphi_i(Y_i) = \varphi_i(Y)\}, \forall i$$

Further, τ^{st} does not depend on the concrete realizations $X = \varprojlim(X_i, \varphi_{ji})$ of X as projective limit of finite T_1 spaces.

Finally, we remark that

$$X \to (\mathbf{exp}X, \tau^{\text{et}}), \qquad X \to (\mathbf{exp}X, \tau^{\text{st}})$$

are covariant functors from the category of all boolean spaces into the category of all quasi-boolean spaces, respectively boolean spaces. Namely, for every given continuous mapping $\varphi : X \to Y$ of boolean spaces one defines

$$\mathbf{exp}\varphi : \mathbf{exp}X \to \mathbf{exp}Y \quad \text{by} \quad (\mathbf{exp}\varphi)(X') = \varphi(X')$$

and these mappings are τ^{et} and τ^{st}-continuous.

An interesting situation arises when X is a profinite group G viewed as boolean space. To emphasize that we forget about the group structure of G we shall write $X = |G|$. $|G|$ has a canonical realization as a projective limit of finite discrete spaces, namely that defined by

$$(*) \qquad\qquad G = \varprojlim\, (G/D', \gamma_{D''D'})$$

where D' runs over all open normal subgroups of G and $\gamma_{D''D'}$ is the canonical projection $G/D'' \to G/D'$ for $D'' \subseteq D'$.

A prominent subspace of $\mathbf{exp}|G|$ is the space of all closed subgroups of G, which we denote by $\mathbf{sg}(G)$ (Remark: we now write G to emphasize that we consider the projective group and not only its topological structure). $\mathbf{sg}(G)$ is τ^{st}-closed in $\mathbf{exp}|G|$, hence $(*)$ also defines the topologies τ^{et} and τ^{st} on $\mathbf{sg}(G)$.

We remark that for every $\Gamma \in \mathbf{sg}(G)$, the family

$$\mathcal{U}_D(\Gamma) = \{\Gamma_1 \in \mathbf{sg}(G) \mid D\Gamma_1 \subseteq D\Gamma\}, \quad (D \subset G \text{ open})$$

is a basis of quasi-compact τ^{et}-open neighborhoods of Γ. Further

$$\mathcal{V}_D(\Gamma) = \{\Gamma_1 \in \mathbf{sg}(G) \mid D\Gamma_1 = D\Gamma\} \quad (D \subset G \text{ open})$$

is a basis of compact τ^{st}-open neighborhoods of $\Gamma \in \mathbf{sg}(G)$.

It is clear that $\mathbf{sg}(\,)$ defines a co-variant functor from the category of all profinite groups to the category of quasi-boolean spaces, i.e.,

For any morphism of profinite groups $\varphi : F \to G$ the mapping

$$\mathbf{sg}(\varphi) : \mathbf{sg}(F) \to \mathbf{sg}(G), \quad \mathbf{sg}(\varphi)(\Gamma) = \varphi(\Gamma)$$

has functorial properties. We shall usually denote $\mathbf{sg}(\varphi)$ simply by φ.

Now suppose that G acts continuously and faithfully on a discrete field L. Then by definition, the orbit Gx of any element $x \in L$ is finite (or equivalently, the stabilizer $G_x \subseteq G$ of any element $x \in L$ is an open subgroup of G). Therefore, L is a Galois extension of the fixed field $L_0 = L^G$, and G is canonically isomorphic to the Galois group $\text{Gal}\,(L|L_0))$ of $L|L_0$.

Definition. 1) In the above context, we denote the space of all subextensions of $L|L_0$ by $\mathbf{sf}(L|L_0)$. Via the canonical isomorphism of $\mathrm{Gal}\,(L|L_0)$ with G, we endow $\mathbf{sf}(L|L_0)$ with the topologies τ^{et} and τ^{st}.

2) For an arbitrary field K, we denote by $\mathbf{sf}(K)$ the set of all algebraic extensions of K in some separable closure K^{sep} of K, endowed with the topologies τ^{et} and τ^{st} as above.

We remark that for any $\Lambda \in \mathbf{sf}(L|L_0)$, the family

$$\mathcal{U}_M(\Lambda) = \{\Lambda_1 \in \mathbf{sf}(L|L_0) \mid M \cap \Lambda \subseteq M \cap \Lambda_1\} \quad M|L_0 \subseteq L|L_0 \text{ finite}$$

is a basis of quasi-compact τ^{et}-open neighborhoods of Λ. Further,

$$\mathcal{V}_M(\Lambda) = \{\Lambda_1 \in \mathbf{sf}(L|L_0) \mid M \cap \Lambda = M \cap \Lambda\} \quad M|L_0 \subseteq L|L_0 \text{ finite}$$

is a basis of compact τ^{st}-open neighborhoods of $\Lambda \in \mathbf{sf}(L|L_0)$.

Proposition 4.7. *The following holds:*

(1) *For any injective morphism of profinite groups $\iota : H \to G$, the induced mapping*

$$\iota : \mathbf{sg}(H) \to \mathbf{sg}(G) \quad \text{defined by} \quad \Delta \mapsto \iota(\Delta)$$

is a τ^{et} and τ^{st}-immersion.

(2) *For every surjective morphism of profinite groups $\varphi : F \to G$ the induced mapping*

$$\varphi : \mathbf{sg}(F) \to \mathbf{sg}(G) \quad \text{defined by} \quad \Phi \mapsto \varphi(\Phi)$$

is a τ^{et}- and τ^{st}-projection.

(3) *If $G_1 \subseteq G$ is a subgroup of G then the mapping*

$$\pi : \mathbf{sg}(G) \to \mathbf{sg}(G_1) \quad \text{defined by} \quad \Gamma \mapsto \Gamma \cap G_1$$

is τ^{et}-continuous.

Corresponding assertions holds for $\mathbf{sf}(L|L_0)$.

Proposition 4.8. *Let G be an arbitrary profinite group acting continuously on a boolean space T.*

A) *For $x \in T$, let $G_{x'}$ be an open subgroup of G containing G_x for some $x' \in Gx$ and further, $U_{x'}$ an open neighborhood of $G_{x'}x'$. Then there exists a finite family $\{x_k\}_k$ of elements x', and clopen disjoint G_{x_k}-invariant neighborhoods $U_k \subseteq U_{x_k}$ of x_k with the properties:*
   (1) *$\{GU_k\}_k$ is a disjoint clopen covering of T.*
   (2) *For all m, $x \in U_m$ and $g \in G$, if $gx \in \cup_k U_k$, then $g \in G_m$ and hence $gx \in U_m$.*

B) *Let H be an open normal subgroup of G such that all G_x are conjugate to subgroups of H. Then there exists a boolean H-subspace $U \subseteq T$ with the properties:*
   (1) *$GU = T$*
   (2) *If $gx = y$ for some $g \in G$ and $x, y \in U$, then $g \in H$.*
   In particular, the canonical projection $T/H \to T/G$ has continuous sections.

Proof. The proof of A) follows applying the following Lemma. □

Lemma 4.9. *Let T be a boolean topological G-space. For some $x \in T$ let H be an arbitrary open subgroup of G containing G_x. Then any neighborhood of Hx contains an H-invariant clopen neighborhood U of Hx which has the property:*

$$g_1U \cap g_2U \neq \emptyset \Leftrightarrow g_1U = g_2U \Leftrightarrow g_1H = g_2H, \ \forall g \in G$$

Proof. Fix an open neighborhood U'' of x containing Gx. For each clopen neighborhood U' of x set $B'_{U'} = \{(g, x') \in G \times U' \mid gx' \notin U''\}$. By the continuity of the action of G on T, it follows that $B'_{U'}$ is closed. On the other hand, since $Gx \subseteq U''$ it follows that $\cap_{U'} B'_{U'}$ is empty. Thus, there exists U' with $GU' \subseteq U''$.

Working with H instead of G it follows that any neighborhood U'' of Hx contains HU' for some clopen neighborhood U' of x. We now claim: There exist such U' that $U = HU$ has the properties we want. Indeed, let $g_1H \neq g_2H$. Since $G_x \subseteq H$, we deduce $g_1Hx \cap g_2Hx = \emptyset$. Therefore, there exists a clopen neighborhood U' of x such that $g_1HU \cap g_2HU' = \emptyset$. Taking into account that there exist only finitely many classes gH, we deduce the existence of U' with the desired properties.

To prove B) we use once again Lemma 4.9. Namely, for any x take $x' \in Gx$ with $G_{x'} \subseteq H$. Further, set $G_{x'} = H$ and $U_{x'}$ an arbitrary neighborhood of Hx'.

With the notations from A) we then have: $U = \cup_k U_k$ is a boolean H-space and if $gx = y$ for some $g \in G$ and $x, y \in U$, then $g \in H$. □

Let G be an arbitrary profinite group acting continuously on a boolean space X. Then G acts in a canonical way on $\mathbf{exp}(X)$, and this action is τ^{et} and τ^{st} continuous.

In particular, we can consider $X = |G|$ endowed with the inner conjugation

$$G \times |G| \to |G|, \quad (g, |x|) \mapsto |gxg^{-1}|$$

which is a τ^{et} and τ^{st} continuous left action of G on $|G|$. In this way we make $\mathbf{sg}(G)$ into a τ^{et} and τ^{st} G-space. We shall always consider $\mathbf{sg}(G)$ endowed with this action of G.

4.10. Conventions and Notations

1) For $x \in |G|$, $\mathcal{G} \subseteq \mathbf{sg}(G)$ and $g \in G$ we sometimes write x^g instead of gxg^{-1}, and also $\mathcal{G}^g$ instead of $g\mathcal{G}g^{-1}$.
2) For G' subgroup of G and $\mathcal{G} \subseteq \mathbf{sg}(G)$, we denote by $\mathcal{G}^{G'}$ the least G'-invariant subset of $\mathbf{sg}(G)$ containing $\mathcal{G}$.
3) For $\mathcal{G} \subseteq \mathbf{sg}(G)$, we denote $\text{con}(\mathcal{G}) = \text{env}(\mathcal{G}^G) = \big(\text{env}(\mathcal{G})\big)^G$.

Next, we fix some notations and facts which will be through out the paper.

Let G be an arbitrary profinite group.

A) A subset $\mathcal{G} \subseteq \mathbf{sg}(G)$ is called special if it is τ^{et}-compact and consists only of self-normalizing elements, i.e., it $\Gamma^g = \Gamma$ for some $\Gamma \in \mathcal{G}$ and $g \in G$, then $g \in \Gamma$.
B) Let $\mathcal{G} \subseteq \mathbf{sg}(G)$ be a special G-subspace and D a fixed open normal subgroup of G. By **(4.9)** and the definition of $\mathcal{G}$, there exist finitely many $\Gamma_k \in \mathcal{G}$, τ^{et}-compact disjoint subsets $\mathcal{D}_k \subseteq \mathcal{V}_D(\Gamma_k)$ of $\mathcal{G}$ such that setting $D_k = D\Gamma_k$ the following holds:
   (1) $\mathcal{D}_k \subseteq \mathbf{sg}(D_k)$ is D_k-invariant.
   (2) If $\mathcal{D}_k \cap \mathcal{D}_\iota^g \neq \emptyset$ for some k, l and $g \in G$, then $k = 1$ and $g \in D_k$.
   (3) For each k let X_k be a set of representatives for G/D_k. Then the finite family $(\mathcal{G}_k^g)_{k, g \in X_k}$ represents a disjoint covering of $\mathcal{G}$.

C) With the notations from B) we construct a "**canonical cover**" associated to D in the following way: For every $x \in G/D$, let $g \in G$ be a pre-image of it. Further, for each k, let $C_k = \iota_k(D_k)$ be an isomorphic copy of D_k, and set $\mathcal{C}_k = \iota_k(\mathcal{D}_k)$ the family $\mathcal{D}_k$ viewed in $\mathbf{sg}(C_k)$. Now consider the following usual profinite free product together with the canonical projection

$$\pi = \pi_D : F_D = F_{|G/D|} * *_k C_k \to G$$

where $F_{|G/D|}$ is the profinite free group on the set $|G/D|$, and π is defined by $\pi(|g|) = g$ and $\pi|_{C_k} = \iota_k^{-1}$. Further, for every k let Y_k be a pre-image of X_k in F_D. Then $h\mathcal{C}_k h^{-1}$ $(k; h \in Y_k)$ are τ^{et}-compact disjoint subsets of $\mathbf{sg}(F_D)$, and we let $\mathcal{F}$ be their union. It is clear that π maps $\mathcal{F}$ τ^{et}-homeomorphically onto $\mathcal{G}$. Moreover, using the result of Herfort–Ribes [H-R], and taking into account that all $\Gamma \in \mathcal{G}$ are self-normalizing it follows that all $\Phi \in \mathcal{F}$ are self-normalizing. Hence, $\mathcal{F}$ is a special subspace of $\mathbf{sg}(F_D)$.

- We say that $\pi : (F_D, \mathcal{F}) \to (G, \mathcal{G})$ is a "**canonical cover**" associated to D.

The same holds for $\mathbf{sf}(\,)$.

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