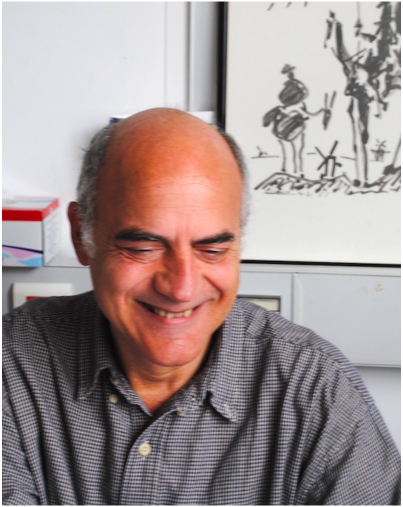
# Remembering Thanases Pheidas

Héctor Pastén

Pontificia Universidad Católica de Chile

Definability, Decidability and Computability over Arithmetically Significant Fields

# Thanases Pheidas



Thanases was one of my academic grandfathers:

$$\text{T. Pheidas} \to \text{X. Vidaux} \to \text{H. Pasten}$$

I first met him when I was an undergraduate student in Concepcion, Chile, in 2007-2008. There he also met my wife Natalia Garcia-Fritz and my friend Javier Utreras.

Thanases visited Xavier Vidaux in Chile many times, greatly contributing to the development of the logic and number theory in my country.

Always generous with the younger generations, he shared his ideas and wisdom, making a profound impact on the mathematical community in Chile.

# Thanases scientific work

Thanases had very wide interests. Mathematically, his research and ideas focused in the following topics (among several others):

- Hilbert's tenth problem for function fields, rings of integers, number fields, and rings and fields of analytic functions
- Definability in arithmetic
- Fragments of arithmetic
- Algebraic differential equations (Grothendieck–Katz conjecture)

I'll mention a couple of highlights from his research on the topics I am more familiar with.

# Rings of integers

His first contribution on $H10(O_K)$ was the following:

**Theorem (1988)**

*Let $K$ be a number field with exactly 2 complex (non-real) conjugate embeddings. Then $H10(O_K)$ is undecidable.*

This was proved simultaneously by Pheidas, Shlapentokh, and Videla. This was one of the last advances made before a long hiatus of 2 decades.

# Function fields

In his breakthrough *Inventiones* paper from 1991 he proved:

**Theorem**

*Let $k$ be a finite field of odd characteristic. Then $H10(k(z))$ is undecidable.*

Here he introduced the method

$$\text{Defining a valuation} + \text{Defining } p^s\text{-powers} \to \text{H10}$$

which is by now standard. Many people extended his ideas. Most recently, there is the work of Eisentraeger and Shlapentokh on very general cases of positive characteristic function fields.

**Trivia:** The paper was first rejected from *Crelle* because of a typo.

# Analytic functions

One of Thanases's favorite problems was $H10$ for the ring of complex entire functions in one variable. He made substantial contributions in this topic, showing undecidability in the following cases:

- $p$-adic entire functions in one variable (1994, with Lipshitz)
- complex entire functions in at least 2 variables (2017, with Vidaux).
- complex exponential polynomials (2020, with Chompitaki, Garcia-Fritz, P., and Vidaux)

# Power series

Thanases had very general works on H10 for power series over enriched languages. Just to mention one (1986):

**Theorem**

*Let $A$ be an integral domain of characteristic $p > 0$. Let $R = A[[z]]$ and let $P = \{1, z, z^2, z^3, ...\} \subseteq R$. The positive existential theory of $R$ over the language $\{0, 1, z, +, \times, =, P\}$ is undecidable.*

This should be contrasted with classical **decidability** results of Ax, Kochen, Ershov, and Macintyre for power series over $\mathbb{C}$ or $\mathbb{Q}_p$ (say).

# Büchi's problem

Another of Thanases favorite problems was Büchi's problem:

**Problem (Büchi's problem)**

*Show that there is a uniform $M$ with the following property:*
*If a sequence of $M$ integer squares has second differences $2, 2, \ldots, 2$, then it is in fact a sequence of consecutive integer squares.*

This has direct connections with definability in arithmetic. His contributions in this context were numerous, solving analogues of the question over many different structures (mainly in joint work with Vidaux).

An unexpected application was joint work with Vidaux and myself achieving a uniform definition of $p^s$-powers across function fields of a fixed genus $g$ and positive characteristic $p$, as $p$ varies.

# Hilbert's tenth problem for $\mathbb{Q}$

Thanases had several striking strategies to attack $H10(\mathbb{Q})$. Although he did not publish much on this subject, he always generously shared his ideas with other people and explained them in different survey articles.

His last attack on this problem (to appear soon) was joint work with Garcia-Fritz, P., and Vidaux, initiated at the MSRI DDC meeting in 2022. Let me explain the context.

# Hilbert's tenth problem for $\mathbb{Q}$

The **height** of a rational number $q = a/b$ with $\gcd(a,b) = 1$ is

$$H(q) = \max\{|a|, |b|\}.$$

There is a similar notion for $\mathbf{q} \in \mathbb{Q}^n$.

Quite often, problems in Diophantine geometry are not just about the existence of rational points in varieties, but they also include conditions on heights. For instance:

**Conjecture (A case of effective Mordell)**

*Let $d \geq 5$. There is $M = M(d)$ such that for every $q \in \mathbb{Q} - \{-1, 0, 1\}$, all rational solutions of $y^2 = x^d + q$ satisfy*

$$H(x,y) \leq H(q)^M.$$

Even for $d = 5$ this is open.

# Hilbert's tenth problem for $\mathbb{Q}$

For $m, n$ define the **height comparison predicate**

$$C_{m,n}(x_1, ..., x_m; y_1, ..., y_n) : \quad H(\mathbf{x}) \leq H(\mathbf{y}).$$

Let $\mathcal{L}_{Height} = \{0, 1, +, \times, =\} \cup \{C_{m,n} : m, n \geq 1\}$.

Pheidas's last contribution on $H10(\mathbb{Q})$ is:

**Theorem (GF-P-Pheidas-V; up to checking again all the details!)**

*The Diophantine problem of $\mathbb{Q}$ with height comparisons is undecidable. More precisely: The positive existential theory of $\mathbb{Q}$ over the language $\mathcal{L}_{Height}$ is undecidable.*

We prove a similar result over $\mathbb{C}(z)$.

# Thanases explaining his ideas to us at Caffe Strada, Berkeley, 2022.

This is the exact moment Thanases was explaining to us a possible approach to overcome some technical difficulties in the result over $\mathbb{C}(z)$.

Thanks for your attention.
And thanks to the organizers for dedicating this amazing workshop to the memory of Thanases Pheidas.