Hilbert's 10th Problem for complete discretely valued fields

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Motivation

In research surrounding definability, decidability and computability, most often *global* objects are considered: number fields, function fields, their rings of (S-)integers

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Algebraic number theory/arithmetic geometry/commutative algebra tell us: Often helpful to first understand *local* objects, which tend to be simpler

$\mathbb{Q}_p$ instead of $\mathbb{Q}$, $\mathbb{Z}_p$ instead of $\mathbb{Z}$, $F(\!(t)\!)$ instead of $F(t)$, $F[\![t]\!]$ instead of $F[t]$

Objects of study

We consider complete discretely valued fields K; that is, K is fraction field of a PID $\mathcal{O}$ with a unique maximal ideal $\mathfrak{m} \neq 0$ (discrete valuation ring), and $\mathcal{O} \cong \varprojlim_n \mathcal{O}/\mathfrak{m}^n$ (completeness)
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If $\mathrm{char}(K) = \mathrm{char}(\kappa)$, all examples are of the form $\mathcal{O} = \kappa[\![t]\!]$, $K = \kappa(\!(t)\!)$, and (parameterfree) H10 is well-understood (Anscombe–Fehm 2016)

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Thus assume *mixed characteristic*: $\mathrm{char}(K) = 0$, $\mathrm{char}(\kappa) = p > 0$
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(Model-theoretically correct generality: henselian finitely ramified valued fields)

Questions

Let K, $\mathcal{O}$ as above

Question (Hilbert 10 over K)

Is there an algorithm to decide which polynomials in $\mathbb{Z}[X_1, \ldots]$ have zeroes in K? Equivalently, is $\mathrm{Th}_\exists(K)$ decidable? More precisely, how hard is $\mathrm{Th}_\exists(K)$ to decide?

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In both cases, want to reduce questions to properties of the residue field.

Pathologies

Theorem (D. 2022)

There exists K CDVF with $\mathrm{Th}_\exists(K)$ undecidable, even though $\mathrm{Th}_\exists(\kappa)$ decidable.

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Theorem (D. 2022, after K. Thanagopal 2019)

Let p prime. There exists κ of characteristic p and λ/κ separable quadratic with $\mathrm{Th}_\exists(\kappa)$ *decidable,* $\mathrm{Th}_\exists(\lambda)$ *undecidable.*

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Then construct K CDVF with residue field κ which encodes ($\exists$-$\emptyset$-interprets) λ

Previous work

Let K CDVF of characteristic 0, residue characteristic $p > 0$, $\mathcal{O}$, κ as above

Theorem (Ershov 1966; Anscombe–Jahnke 2022)

Assume p generates the maximal ideal of $\mathcal{O}$ ("K is unramified"). Then $\mathrm{Th}(K)$ is axiomatised by fixing $\mathrm{Th}(\kappa)$. In particular, $\mathrm{Th}(K)$ is decidable if and only if $\mathrm{Th}(\kappa)$ is decidable.

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Theorem (Basarab 1978)

$\mathrm{Th}(K)$ is axiomatised by the data of the $\mathrm{Th}(\mathcal{O}/p^n)$ for all $n > 0$. In particular: $\mathrm{Th}(K)$ is decidable if and only if the $\mathrm{Th}(\mathcal{O}/p^n)$ are uniformly *decidable.*

New results (Anscombe–D.–Jahnke 2023)

Goal: Find invariants in terms of the residue field for $\mathrm{Th}_{\exists}(K)$, $\mathrm{Th}(K)$, without assuming unramified or κ perfect

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Let K, $\mathcal{O}$, κ as above, $e \in \mathbb{N}$ the *initial ramification index*, i.e. $p\mathcal{O} = \mathfrak{m}^e$

Thus $e = 1$ in $\mathbb{Q}_p$, since $p\mathbb{Z}_p$ is the maximal ideal of $\mathbb{Z}_p$; $e = 2$ in $\mathbb{Q}_3(\sqrt{6})$, since the maximal ideal of $\mathcal{O} = \mathbb{Z}_3[\sqrt{6}]$ is $\sqrt{6}\mathbb{Z}_3$, and $3\mathbb{Z}_3[\sqrt{6}] = \sqrt{6}^2\mathbb{Z}_3[\sqrt{6}]$

We solve the problem above for fixed p and e.

New results (Anscombe–D.–Jahnke 2023)

Theorem

There is $m = m(p, e) \geq 0$ *such that, for a certain* $\Omega \subseteq \kappa^m$*:*

- $\mathrm{Th}_{\exists}(K)$ *is as hard as (1-equivalent to)* $\mathrm{Th}_{\exists+}(\kappa, \Omega)$
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$\mathrm{Th}_{\exists^+}(\kappa, \Omega)$ encodes for which $f_1, \ldots, f_k \in \mathbb{F}_p[X_1, \ldots, X_n, Y_1, \ldots, Y_{ml}]$ we have

$$\exists x_1, \ldots, x_n \in \kappa \, \exists (y_1, \ldots, y_m), \ldots, (y_{ml-m+1}, \ldots, y_{ml}) \in \Omega : \bigwedge_i f_i(\underline{x}, \underline{y}) = 0$$

$\mathrm{Th}(\kappa, \Omega)$ also allows universal quantification, and asserting that certain tuples do not lie in Ω

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The set $\Omega \subseteq \kappa^m$ is itself defined in a natural way: there exists $\Xi \subseteq \mathcal{O}^m$, $\exists$-$\emptyset$-definable (parameterfreely diophantine) in K such that Ω is the reduction of Ξ; and the definition of Ξ only depends on p, e (not K)

Model-theoretic formulation

The computability-theoretic results are corollaries of axiomatisability results:

Theorem (A–D–J)

Let K, L both CVDFs with residue fields κ, λ of characteristic p, with the same initial ramification index e.

- $\mathrm{Th}_{\exists}(K) = \mathrm{Th}_{\exists}(L) \iff \mathrm{Th}_{\exists^+}(\kappa, \Omega_K) = \mathrm{Th}_{\exists^+}(\lambda, \Omega_L)$
- $\mathrm{Th}(K) = \mathrm{Th}(L) \iff \mathrm{Th}(\kappa, \Omega_K) = \mathrm{Th}(\lambda, \Omega_L)$

Various bonus statements: Ω_K precisely describes the *structure induced* by K on κ, and (κ, Ω_K) is *stably embedded* in K

Commentary

- The reduction of $\mathrm{Th}_{\exists}(K)$ to $\mathrm{Th}_{\exists^+}(\kappa, \Omega)$ is purely theoretical
- Even the arity $m = m(p, e)$ of Ω grows extremely quickly with e
- Further results concerning Hilbert 10/full theories with parameters (quantifier elimination) would be desirable