# FRG Workshop (online), June 21-22, 2023

## — In memory of Thanases C. Pheidas —

## FRG: Definability, Decidability and Computability over Arithmetically Significant Fields

This workshop aims at presenting and exploring new results concerning first-order theories in (enriched versions of) the language of rings over algebraic extensions of function fields over arithmetically significant fields and/or Henselian (discretely) valued fields.

### Organizers:


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## PROGRAM (US East Coast Int′l Time; Central Europe +6h)

### June 21, 2023

11:00 Caleb Springer

**Title:** *On decidability, definability, and rings of integers in algebraic extensions of the rational numbers*

**Abstract:** When considering all algebraic extensions of the rational numbers, which fields have (un)decidable rings of integers? The two extreme cases have been long-established. Specifically, Julia Robinson showed that rings of integers in number fields are undecidable, while Rumely proved that the ring of all algebraic integers is decidable. More recently, a preprint of Mazur, Rubin and Shlapentokh established the undecidability of rings of integers in so-called "non-big" fields, thereby generalizing many recent results in the literature. In this talk, we turn our attention to fields which can be considered "big" or "large", in either a technical or non-technical sense. For example, we exhibit an existential formula such that, if $K$ is a totally real field which contains "enough" elements and $L$ is any totally imaginary quadratic extension of $K$, then the fixed formula defines the ring of integers of $K$ within the ring of integers of $L$. The techniques deployed here exploit the unit groups of non-maximal orders. Consequently, the undecidability follows from an appeal to so-called JR-numbers for totally real fields, in the sense of Vidaux and Videla.

### 12:00 Hector Pasten

**Title:** *A diophantine definition of* $\mathbb{Q}$ *in* $\mathbb{Q}(z)$

**Abstract:** It is an old open question dating back to the sixties to determine whether $\mathbb{Q}$ is Diophantine in $\mathbb{Q}(z)$. In the 2022 MSRI meeting "Definability, decidability, and computability in number theory" Thanases Pheidas raised this question again, and during the same MSRI meeting Natalia Garcia-Fritz and I proved a positive answer if one assumes two standard conjectures on elliptic surfaces. In this lecture I will try to give an overview of the proof.

### 12:45 Franziska Jahnke

**Title:** *An Ax-Kochen/Ershov principle for deeply ramified fields*

**Abstract:** Deeply ramified fields are a generalization of perfectoid fields and were introduced by Gabber and Ramero. As deeply ramified fields may admit immediate extensions, there is no hope for a classical Ax-Kochen/Ershov Theorem, and hence they have eluded model-theoretic machinery so far. In this talk, we present an AKE Theorem for certain perfect deeply ramified fields with a distiguished element $t$ down to the pointed value group and "thickened" residue field. In particular, our Theorem applies to any perfectoid field, choosing a (pseudo)uniformizer for $t$. As a consequence, we obtain that the perfect hull of the henselization of $\mathbb{F}_p(t)$ is an elementary substructure of the perfect hull of $\mathbb{F}_p((t))$. The results are joint work with Konstantinos Kartas.

### 13:30 Philip Dittmann

**Title:** *Hilbert's* $10^{th}$ *Problem for complete discretely valued fields*

**Abstract:** Many of the known undecidability results (and conjectures) for Hilbert's 10th Problem concern number fields and function fields - these are related to the global fields of algebraic number theory. In this talk I will take a closer look at complete discretely valued fields, i.e. a generalization of local fields. Although we think of these as simpler, I pointed out last year that there are examples of such fields such that Hilbert's 10th Problem is undecidable, even though Hilbert's 10th Problem is decidable over the residue field. I will discuss recent joint work with Sylvy Anscombe and Franziska Jahnke, in which we completely analyze the situation (also for full first-order theories) by adding a natural predicate on the residue field.

## June 22, 2023

### 11:00 Adam Topaz

**Title:** *Arithmetic, geometry and Galois theory of geometric function fields*

**Abstract:** In this talk, I will discuss a few results and questions related to the arithmetic and geometry of function fields over algebraically closed fields, arising from anabelian Geometry and Galois theory. I will focus primarily on topics related to *valuations* of such fields, while highlighting relationships with the themes of the workshop.

### 12:00 Natalia Garcia-Fritz

**Title:** *Hilbert's* $10^{th}$ *Problem for lacunary entire functions*

**Abstract:** One of the main open cases of Hilbert's 10th Problem is that of complex entire functions in one variable. The problem has been solved negatively for complex polynomials

(Denef) and for complex exponential polynomials (Chompitaki, GF, Pasten, Pheidas, Vidaux). In this talk I will explain a new contribution to this problem: The case of finite order entire functions with lacunary power series at zero. The proof is based on the theory of holonomic functions. This is joint work with Hector Pasten.

12:45 Sylvy Anscombe

**Title:** *Transfer of decidability for existential theories of (valued) fields*

**Abstract:** In previous work with Fehm we found that the existential theory of an equicharacteristic henselian valued field is axiomatized using the existential theory of its residue field. From this we deduced a transfer of decidability: for a complete theory $T$ of residue fields, the existential consequences of $T$ are decidable if and only if the existential consequences of the theory $H(T)$ are decidable, where $H(T)$ is 'equicharacteristic, henselian, and residue field models $T$'. In more recent work with Dittmann and Fehm we considered a similar problem in which $H(T)$ is expanded to a theory that distinguishes a uniformizer, using an additional constant symbol. In this case Denef and Schoutens gave a transfer of existential decidability conditional on Resolution of Singularities. We isolated a consequence of Resolution and prove that it implies a similar transfer of existential decidability.

13:30 Arno Fehm

**Title:** *Universal-existential theories of fields*

**Abstract:** I will discuss axiomatizability and decidability of certain universal-existential fragments of theories of fields, in particular function fields and Laurent series fields. Some of these fragments are directly related to Hilbert's 10th Problem over these fields. This is joint work with Sylvy Anscombe.

14:15 Hector Pasten: *In memory of* ***Thanases Pheidas***

## Participants (preliminary):

### Senior/junior researchers:

Irfan Alam
Sylvy Anscombe
Matthias Aschenbrenner
Francesca Balestrieri
Wesley Calvert
Gabriel Conant
Nicolas Daans
Philip Dittmann
Lou van den Dries
Kirsten Eisentraeger
Arno Fehm
Natalia Garcia Fritz
Valentina Harizanov
Franziska Jahnke
Marianthe Malliaris

Barry Mazur
Travis Morrison
Jennifer Park
Hector Pasten
Anand Pillay
Bjorn Poonen
Florian Pop
Karl Rubin
Thomas Scanlon
Andre Scedrov
Alexandra Shlapentokh
Caroline Terry
Adam Topaz
Henry Towsner
Meng-Che Turbo Ho
Xavier Vidaux
Carlos Videla

**Grad students:**

Eben Bleisdel
Zhaodong Cai
Krishan Canzius
Hunter Handley
Henry Klatt
Andrew Kwon
Ian Lewis
Juan Pablo De Rasis
Gabriela Pinto
Souparna Purohit
Keshav Srinivasan
The Hoang The Gia
Zana Tran
Philip White