An Ax-Kochen/Ershov principle for deeply ramified fields

jt work with Konstantinos Kartas

§ Introduction

Notation: (K,v) valued field, vK value group, Kv residue field.

$\mathcal{L}_{ring} = \{0,1,+,\cdot\}$ $\quad \mathcal{L}_{val} = \mathcal{L}_{ring} \cup \{\mathcal{O}\}$

$\mathcal{L}_{oag} = \{0,+,<\}$

Thm (Ax-Kochen/Ershov, '65):
Let (K,v) and (L,w) be two henselian fields of residue characteristic 0. Then

$$\underbrace{(K,v) \equiv (L,w)}_{\text{in } \mathcal{L}_{val}} \iff \underbrace{Kv \equiv Lw}_{\text{in } \mathcal{L}_{ring}} \text{ and } \underbrace{vK \equiv wL}_{\text{in } \mathcal{L}_{oag}}$$

and, if $(K,v) \subseteq (L,w)$, we have

$$(K,v) \preccurlyeq (L,w) \iff Kv \preccurlyeq Lw \text{ and } vK \preccurlyeq wL \qquad (\star)$$

This has been generalized in several ways; e.g. replace "residue characteristic 0" by
- unramified with perfect residue field (AK/E)
- unramified with imperfect residue field (Anscombe-J.)
- alg. maximal Kaplansky (Ziegler/Ershov)
- tame of pos. char (F.V. Kuhlmann)

& $(\star)$ for tame mixed characteristic. (FVK)

Problem: $(\star)$ fails when (K,v) admits a proper immediate algebraic extension (L,w)
as $Kv \preccurlyeq Lw$ and $vK \preccurlyeq wL$ but $(K,v) \not\preccurlyeq (L,w)$.
e.g. for $(K,v) = (\mathbb{F}_p((t))^{perf}, v_t)$, $((\mathbb{F}_p(t)^h)^{perf}, v_t)$

We understand $\mathbb{Q}_p$, but not $\mathbb{F}_p((t))$ or $\mathbb{F}_p^{alg}((t))$.

TODAY: Replace Kv by an infinitesimal thickening.
$\rightsquigarrow$ gain new understanding of $\mathbb{F}_p((t))^{perf}$, $(\mathbb{F}_p(t)^h)^{perf}$

§ A new AK/E type phenomenom

Open problems:
- $(\mathbb{F}_p(t)^{hens}, v_t) \preccurlyeq (\mathbb{F}_p((t)), v_t)$?
- $(\mathbb{F}_p^{alg}(t)^{hens}, v_t) \preccurlyeq (\mathbb{F}_p^{alg}((t)), v_t)$?
- $(\mathbb{F}_p((t)), v_t) \preccurlyeq (\underbrace{\mathbb{F}_p((t))((s^{\mathbb{Q}}))}_{v_t}, v_t)$?

Theorem (J.-Kartas, 2023)
Let $(K,v) \subseteq (L,w)$ be two henselian fields of residue characteristic p s.th. $\mathcal{O}_v/(p)$ and $\mathcal{O}_w/(p)$ are semiperfect (i.e., Frob : $\mathcal{O}/(p) \to \mathcal{O}/(p)$ is surj.)
Suppose there is $\varpi \in \mathfrak{m}_v$ s.th. $\mathcal{O}_v[1/\varpi]$ and $\mathcal{O}_w[1/\varpi]$ are algebraically maximal. Then

$$(K,v) \preccurlyeq (L,w) \iff \mathcal{O}_v/\varpi \preccurlyeq \mathcal{O}_w/\varpi \text{ and } (vK, v\varpi) \preccurlyeq (wK, w\varpi)$$

Corollary: • $(\mathbb{F}_p(t)^{hens,perf}, v_t) \preccurlyeq (\mathbb{F}_p((t))^{perf}, v_t)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \wedge$

$\qquad\qquad\qquad\qquad\qquad (\underbrace{\mathbb{F}_p((t))^{perf}((s^{\mathbb{Q}}))}_{v_t}, v_t)$

Why? $\mathcal{O}/(t) = k[t^{1/p^\infty}]/(t)$ in each case

and $(\mathbb{F}_p((t))^{perf}, v_t) \preccurlyeq (\widehat{\mathbb{F}_p((t))^{perf}}, v_t)$

• $(\mathbb{F}_p^{alg}(t)^{hens,perf}, v_t) \preccurlyeq (\mathbb{F}_p^{alg}((t))^{perf}, v_t)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \wedge$

$\qquad\qquad\qquad\qquad\qquad (\mathbb{F}_p^{alg}\underbrace{((t))^{perf}((s^{\mathbb{Q}}))}_{v_t}, v_t)$

Note: • The class of perfect henselian fields of pos. residue characteristic $p>0$ s.th. $\mathcal{O}_v/(p)$ is semiperfect & with a distinguished element ϖ s.th. $\mathcal{O}_v[1/\varpi]$ is alg. maximal is an elementary class in $\mathcal{L}_{val}(t)$.

$\rightsquigarrow$ suffices to check $\mathcal{O}_v[1/\omega]$ is alg. max. in some model

• If vK and wL are regularly dense, the value group condition can be omitted.

• The theorem applies in particular to any perfectoid fields $(K,v) \subseteq (L,w)$, where ϖ is a pseudouniformizer in K.
It also applies to all deeply ram. fields of mixed char, choosing $\varpi = p$.
In pos. char., the existence of ϖ is not guaranteed.

• In positive char, we also get a version for $\equiv$.

Theorem (J. - Kartas, 2023)

Let $(K,v) \subseteq (L,w)$ be two henselian fields of residue characteristic p s.th. $\mathcal{O}_v/(p)$ and $\mathcal{O}_w/(p)$ are **semiperfect** (i.e., $\mathrm{Frob}: \mathcal{O}/(p) \to \mathcal{O}/(p)$ is surj.) Suppose there is $\varpi \in \mathfrak{m}_v$ s.th. $\mathcal{O}_v[1/\varpi]$ and $\mathcal{O}_w[1/\varpi]$ are **algebraically maximal**. Then

$$(K,v) \preccurlyeq (L,w) \iff \mathcal{O}_v/\varpi \preccurlyeq \mathcal{O}_w/\varpi \text{ and } (vK, v\varpi) \preccurlyeq (wL, w\varpi)$$

Proof sketch: "$\Rightarrow$" clear

"$\Leftarrow$" Let (K,v), (L,w), ϖ be as in the assumption, by above wma (K,v,ϖ) and (L,w,ϖ) $\aleph_1$-sat. [assume $\mathrm{char}(K) = \mathrm{char}(L) = p$]

We consider convex subgroups of vK

Δ_0 = min convex subgroup containing ϖ

Δ_ϖ = max convex subgroup not containing ϖ

and obtain a decomposition of v:

$$K \xrightarrow{vK/\Delta_0} K_0 \xrightarrow{\Delta_0/\Delta_\varpi} K_\varpi \xrightarrow{\Delta_\varpi} Kv \qquad \mathcal{O}_{v_0} = \mathcal{O}_v[1/\varpi]$$

perfect; tame by assumption; tame by sat.

$$K_\varpi = \mathrm{Frac}((\mathcal{O}_v/\varpi)_{red})$$

as $\mathfrak{m}_w = \{x \in \mathcal{O}_v : \exists n\ x^n \in (\varpi)\}$

$\Rightarrow \mathcal{O}_{\bar v} = \mathcal{O}_v/\mathfrak{m}_w = \mathcal{O}_v/\sqrt{(\varpi)} = (\mathcal{O}_v/\varpi)_{red}$

$$\mathrm{Frac}((\mathcal{O}_v/\varpi)_{red}) \preccurlyeq \mathrm{Frac}((\mathcal{O}_w/\varpi)_{red})$$

by $\mathcal{O}_v/\varpi \preccurlyeq \mathcal{O}_w/\varpi$ (not obvious)

$$\mathrm{Frac}((\mathcal{O}_w/\varpi)_{red}) = L_\varpi$$

$$L \xrightarrow{wL/\Delta_0^L} L_0 \xrightarrow{\Delta_0^L/\Delta_\varpi^L} L_\varpi \xrightarrow{\Delta_\varpi^L} Lw$$

tame; tame

By $(vK, v\varpi) \preccurlyeq (wL, w\varpi)$, get $vK/\Delta_\varpi \preccurlyeq wL/\Delta_\varpi^L$

FVK $\Rightarrow$ $(K, v_\varpi) \preccurlyeq (L, w_\varpi)$
for v_ϖ, w_ϖ =

by $\mathcal{O}_v/\varpi \preccurlyeq \mathcal{O}_w/\varpi$, get $(K_\varpi, \bar{v}) \preccurlyeq (L_\varpi, \bar{w})$
as $\mathcal{O}_{\bar{v}} = (\mathcal{O}_v/\varpi)_{red}$.

[...] $\Rightarrow$ $(K, v) \preccurlyeq (L, w)$ "⧄"

stable emb. of
res. fields in
tame fields

§ Background facts & definitions

§ Tame fields

Definition: A valued field (K,v) with $\mathrm{char}(Kv) = p > 0$ is called **tame** if
- (K,v) is algebraically maximal
- vK is p-divisibel
- Kv is perfect.

Thm (F.V. Kuhlmann, '16)
Let (K,v) and (L,w) be two tame henselian fields of residue characteristic $p>0$. Then if $(K,v) \subseteq (L,w)$

$$(K,v) \preccurlyeq (L,w) \iff Kv \preccurlyeq Lw \text{ and } vK \preccurlyeq wL$$

If $\mathrm{char}(K) = \mathrm{char}(L) = p$, we also have

$$(K,v) \equiv (L,w) \iff Kv \equiv Lw \text{ and } vK \equiv wL.$$

§ Perfectoid fields

Def: A valued field (K,v) of residue char. $p>0$ is called **perfectoid**, if
(1.) vK is archimedean but not discrete
(2.) (K,v) is complete
(3.) $\mathrm{Frob}: \mathcal{O}/p \to \mathcal{O}/p,\ x \mapsto x^p$ is surjective.

Tilting construction (Scholze, based on Fontaine):

(K,v) perfectoid, $\mathrm{char}(K, Kv) = (0,p)$

$$\mathcal{O}^\flat = \quad \cdots \xrightarrow{\text{Frob}} \mathcal{O}/p \xrightarrow{\text{Frob}} \mathcal{O}/p, \quad K^\flat = \mathrm{Frac}(\mathcal{O}^\flat)$$

$\Rightarrow$ $(K^\flat, v^\flat)$ is called the **tilt** of (K,v), and there is $t \in \mathfrak{m}_{v^\flat}$ with
$$\mathcal{O}/p \cong \mathcal{O}^\flat/t$$

$$\rightsquigarrow Kv \cong K^\flat v^\flat \text{ and } vK \cong v^\flat K^\flat.$$

Quintessence: "K and $K^\flat$ are similar."
$\rightsquigarrow$ tilting allows to transfer arithmetic properties between K and $K^\flat$ and vice versa

Corollary: (K,v), (L,w) perfectoid. Then

$$(K,v) \preceq (L,w) \Leftrightarrow (K^\flat, v^\flat) \preceq (L^\flat, w^\flat)$$

and

$$(K,v) \equiv (L,w) \Rightarrow (K^\flat, v^\flat) \equiv (L^\flat, w^\flat)$$

§ Deeply ramified fields

Def: A valued field (K,v) of residue characteristic $p>0$ is called **deeply ramified** if

- $\mathrm{Frob}: \hat{\mathcal{O}}_v/p \to \hat{\mathcal{O}}_v/p$ is surjective and
- Δ_2/Δ_1 is non-discrete, for any convex subgroups $\Delta_1 \lneq \Delta_2 \leq vK$

Remark:
- If $\mathrm{char}(K) = p > 0$, then (K,v) is deeply ramified iff $(\hat{K},\hat{v})$ is perfect.
- If $\mathrm{char}(K) = 0$, then
$$\hat{\mathcal{O}}_{\hat{v}}/p = \mathcal{O}_v/p$$
- any perfectoid field is deeply ramified.

δ version for $\equiv$

Theorem (J.-Kartas): Let (K,v), (L,w) be two perfect henselian fields over $(\mathbb{F}_p(t), v_t)$ s.th. both $\mathcal{O}_v[1/t]$ and $\mathcal{O}_w[1/t]$ are alg. max.

Suppose that $\mathcal{O}_v/(t) \equiv \mathcal{O}_w/(t)$ in $\mathcal{L}_{\text{ring}}$ and $(vK, vt) \equiv (wL, wt)$ in $\mathcal{L}_{\text{oag}}(vt)$.

Then $(K,v) \equiv (L,w)$.

Note: "perfect" is a crucial assumption!
(have counterexample with
$K = \mathbb{F}_p((t))^{\text{perf}}$
L an imperfect field with
$(\hat{L}, v_t) = (\widehat{\mathbb{F}_p((t))^{\text{perf}}}, v_t)$
$\Rightarrow K \not\equiv L$!
)

... also version for $\preceq_\exists$ ...