$k = \bar{k}$.

$K$ function field over $k$. (f.g. extn of $k$)

Higher dimensional ($\mathrm{trdeg} \geq 2$).

$$\left[\begin{array}{l} M \quad \text{arbitrary field} \\ F. \end{array}\right]$$

$\mathrm{Hom}_{\mathbb{Z}}(M^\times, \mathbf{F}) = \mathcal{G}_M(F)$ $(M, \mathbf{F}$ arbitrary fields$)$

mult. additive

$M^\times = \{m \in M \mid m \neq 0\}$.

a t.v.s. / F. pointwise convergence

$$\mathcal{G}_M(F) \subseteq F^{M^\times}$$

Milnor K-theory of $M$:

$$K^M_*(M) = \frac{T_*(M^\times)}{\langle x \otimes y \mid x+y=1 \rangle}. \qquad \left(= H^{*,*}_{Mot}(M)\right)$$

~~Hover~~ Voevodsky - Bloch.

$$K^M_1(M) = M^\times.$$

one of

$\forall\ f, g \in \mathcal{A}_M(F)$ say $f, g$ are alternating if the following equiv. conditions are true.

- $\forall\ x, y \in M^\times$, $x+y=1$, $f(x)\cdot g(y) = f(y)\cdot g(x)$
- $\forall\ x, y \in M^\times$, $x \vee y = 0$ in $K_2(M)$ then $f(x)\cdot g(y) = f(y)\cdot g(x)$

Let $v$ be a val on $M$.

$$\mathcal{O}_v^\times = U_v \longrightarrow I_v = \{ f \in \mathfrak{g}_M \mid f|_{U_v} = 0 \}$$
$$= \mathrm{Hom}(M^\times/U_v, F).$$

$$1 + \mathfrak{m}_v = U_v^1.$$

$$\longrightarrow D_v = \{ f \in \mathfrak{g}_M \mid f|_{U_v^1} = 0 \}$$
$$= \mathrm{Hom}(M^\times/U_v^1, F).$$

$I_v \subset D_v \subset \mathfrak{g}_M$ closed subspaces.

**Fact:** if $f \in I_v$, $g \in D_v$ then $f, g$ are alternating

pf: Ultrametric inequality

Converse: let $f, g \in \mathcal{G}_M(F)$ sps. $f, g$ are alternating ($f(-1) = g(-1) = 0$).

Thm: if condition $(*)$ holds true, $f, g$ as above. Then $\exists\ v$ val. on $M$, s.t. $f, g \in D_v$, and $\langle f, g \rangle \cap I_v$ has codim $\leq 1$ in $\langle f, g \rangle$.

$(*)$: one of the ~~f~~ following holds

1) $F$ is a prime field.

2) $\operatorname{char} M > 0$, $\operatorname{char} F \neq \operatorname{char} M$. $\operatorname{char} F \neq 2$.

# Relationships w/ Cohomology

1) Gal cohomology if char $M \neq \ell$.

$$K_*^M(M) \longrightarrow H^*(M, \mathbb{Z}/\ell(*))$$

Kummer in degree 1, extended by taking cup products.

Thm (Rost-Voevodsky):

$$K_*^M(M) \otimes \mathbb{Z}/\ell \xrightarrow{\cong} H^*(M, \mathbb{Z}/\ell(*))$$

In this case: $(\mu_\ell \subset M)$

$x, y \in M^\times$ then TFAE:

1) $x \cup y = 0$ in $H^2 \cong \mathbb{Z}/\ell$.

2) $x \in N_{M(\sqrt[\ell]{y})|M}\left(M(\sqrt[\ell]{y})^\times\right)$

2) $K|k$ function field. $k \hookrightarrow \mathbb{C}$.

$X$ a model of $K|k$.

$$\rightsquigarrow \varinjlim_{\substack{U \subset X \\ \text{Zariski open}/k}} H^*(U, \mathbb{Z}) =: H^*(K|k, \mathbb{Z})$$

$$K^\times \xrightarrow{\kappa} H^1(K|k, \mathbb{Z}(1)) \rightsquigarrow K^M_*(K) \longrightarrow H^*(K|k, \mathbb{Z}(*))$$

<u>Fact:</u> TFAE: given $x, y \in K^\times$

1) $x \cup y = 0$ in $H^2(K|k, \mathbb{Z}(2))$

2) $x, y$ are alg. dependent $/ k$.

Results using thms on alt pairs.

Bogomolov's program

<u>Thm</u> (Pop, Bogomolov-Tschinkel): if $k = \overline{\mathbb{F}}_p$

$\mathrm{trdeg}(K|k) \geq 2$, then $H^*(K, \mathbb{Z}_\ell(*))$ determines

$K|k$ upto inseparable extensions.

(formulated in terms of quotients of $G_K$)

Thm (T.): if $k \hookrightarrow \mathbb{C}$, then

$$H^*(K|k, \mathbb{Z}(*)) \text{ determines } K|k$$

if one endows $H^1$ w/ MHS.

theorem (T.): $M$ arbitrary field abs. trdeg $\geq 5$

then $\mathbb{Q} \otimes K_*^M(M)$ determines $M$ upto insep. extns.

$G_m \rtimes G_a \curvearrowright \mathbb{A}^1$.

## Definability:

$$\left( \mathfrak{g}_M(\mathbf{F}) \times M^\times \longrightarrow \mathbf{F} \quad , \quad \text{alt. pairs} \right)$$
$$(f, x) \longmapsto f(x).$$

w/ this structure, $I_n \subset D_n \subset \mathfrak{g}_M(F)$

are unif. definable for "many" $n$.

if conditions (*) hold true.

if $K|k$, as above

among function fields of fixed trdg.

the collection $\{I_v \subset D_v\}$ w/ $v$ is a "$q$-div val" is unif. definable

w/ $(\mathcal{G}_M(F) \times M^\times \to F$, alt. pairs$)$

$q$-div. val: a val. $v$ on $K$ s.t.

1) $vK/vk \cong \mathbb{Z}$.
2) $\operatorname{trdeg}(K|k) = \operatorname{trdeg}(Kv|kv) + 1$.
3) $v$ is minimal w/ these conditions

Pop.

Question: if $F = \mathbb{Z}/\ell$,

$\left( \mathcal{G}_M(\mathbb{Z}/\ell) \times M^\times \to \mathbb{Z}/\ell, \quad \text{alt. pairs} \right)$

Can $\mathcal{G}_M(\mathbb{Z}/\ell)$ be replaced by the collection of its finite quotients (this is interpretable in $M$),

while retaining some sort of definability of InvDiv.

???