On the size of Heyting Semi-Lattices and Equationally Linear Heyting Algebras *

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The theory of **Heyting Semi-Lattices**, HSLs for short, is obtained by adding to the theory of meet-semi-lattices-with-top a binary operation whose values are denoted $x \to y$ characterized by

$$x \ \leq \ y \to z \quad \text{iff} \quad x \wedge y \ \leq \ z$$

The easiest equational treatment uses the symmetric version, the double arrow rather than the single. The binary operation $x \leftrightarrow y$ is characterized by

$$x \ \leq \ y \leftrightarrow z \quad \text{iff} \quad x \wedge y \ = \ x \wedge z$$

Each may be defined in terms of the other:

$$x \to y \quad = \quad x \leftrightarrow (x \wedge y)$$

$$x \leftrightarrow y \quad = \quad (x \to y) \wedge (y \to x)$$

The $\leftrightarrow$ operation may be equationally characterized by

$$x \leftrightarrow 1 \quad = \quad x \quad = \quad 1 \leftrightarrow x$$

$$x \leftrightarrow x \quad = \quad 1$$

$$x \wedge (y \leftrightarrow z) = x \wedge ((x \wedge y) \leftrightarrow (x \wedge z))$$ [1]

Any finite distributive lattice has a unique Heyting structure. Every finite Heyting semi-lattice has, of course, a unique lattice structure and it is necessarily distributive.

* This ms was born in 2002. The first appendix was added in 2005, the first footnote in 2013, the subscorings in 2015 and the addendum in 2017.

[1] See 2$^{\text{nd}}$ appendix for subscorings. First x and y meet $x \leftrightarrow y$ in the same way: $x \wedge (x \leftrightarrow y) = x \wedge ((x \wedge x) \leftrightarrow (x \wedge y)) = x \wedge (x \leftrightarrow (x \wedge y)) = x \wedge ((x \wedge 1) \leftrightarrow (x \wedge y)) = x \wedge (1 \leftrightarrow y) = x \wedge y$ and similarly $y \wedge (x \leftrightarrow y) = x \wedge y$. Hence for any $z \leq x \leftrightarrow y$ we have $z \wedge x = z \wedge (x \leftrightarrow y) \wedge y = z \wedge x \wedge y$ and similarly $z \wedge y = z \wedge (x \leftrightarrow y) \wedge y = z \wedge x \wedge y$. Finally, if $z \wedge x = z \wedge y$ then $z \wedge (x \leftrightarrow y) = z \wedge ((z \wedge x) \leftrightarrow (z \wedge y)) = z \wedge 1 = z$, that is, $z \leq x \leftrightarrow y$.

Finitely generated HSL*s are finite.*

The proof takes a while. First, say that an HSL is **local** if there's an element $m < 1$ such that for all $x < 1$ it is the case that $x \ \leq \ m$. (The local HSLs are the "subdirectly irreducibles" of the subject for those who know what that ugly phrase means.) The eponymous example is obtained by starting with the HSL of open sets of a space, choosing a point, identifying two open sets iff they agree when restricted to some neighborhood of the point. We'll call a local quotient a **localization**.

Every HSL *is embedded into the product of its localizations.*

For two elements a and b we need to find a localization in which a and b remain distinct. Define $m = a \leftrightarrow b$. It suffices to find a localization in which m remains different from 1. Reduce by a maximal congruence that does just that. In the resulting quotient HSL m has the property that for any non-trivial congruence it is the case that $m \equiv 1$. Let k be an element strictly less than 1. Define a congruence by $x \equiv y$ iff $x \wedge k \ = \ y \wedge k$. (The third defining equation for $\leftrightarrow$ makes it easy to see that this a congruence.) Since $\equiv$ is a non-trivial congruence ($k \equiv 1$) it must be the case that $m \equiv 1$, that is $m \wedge k \ = \ k$. In other words, $k \leq m$.

Assume now that we know that all HSLs generated by n elements are finite. Let A_n be the number of elements in the free HSL on n generators.

A local HSL *on* $n+1$ *generators has at most* $1+A_n$ *elements.*

If m is removed the complement is a sub-algebra, because in a local HSL $x \to y = m$ only if $y = m$ (and $x = 1$) and $x \wedge y = m$ only if either x or y is m. One of the $n+1$ generators must be m. The complement is therefore generated by n of its elements.

Hence for an HSL on $n+1$ generators there are only a finite number of isomorphism types of localizations. From the finiteness of the generating set we know that there are only finitely many maps into any finite HSL. We may conclude that there are only finitely many localizations. Which is quite enough to force the HSL on $n+1$ generators to be finite.

(If the join operation and a bottom are added to the structure, that is, if we consider not HSLs but full-fledged Heyting algebras, then the free HA on one generator is infinite. For a quick proof, partially order the positive integers by taking $x \ll y$ iff $x+1 < y$. The HA of downdeals is generated by the one-element downdeal $\{1\}$, which is quite enough for the infinitude of the free HA. Actually, up to isomorphism, there is only one infinite HA on one generator, this one. Indeed, any pair of one-generator HAs of the same cardinality are isomorphic.)

The very first sequence, A000001, in Sloane's Encyclopedia Of Integer Sequences (`http://oeis.org/`) counts the number of isomorphism types of groups. Both the HSL and HA analog are the same as his A006982. That is for X equal to "distributive lattice", "Heyting algebra", or "Heyting semi-lattice" the number of isomorphism types of X of order n is given by the sequence 1, 1, 1, 1, 2, 3, 5, 8, 15, 26, 47, 82, 151, 269, 494, 891, 1639, 2978, 5483, 10006, 18428, 33749, 62162, 114083, 210189, 386292, 711811, 1309475, 2413144, 4442221, 8186962, 15077454, 27789108, 51193086, 94357143, 173859936, 320462062, 590555664, 1088548290, 2006193418, 3697997558, 6815841849, 12563729268, 23157428823, 42686759863, 78682454720, 145038561665, 267348052028, 492815778109, ...

It's easy to see that the free HSL on one generator has 2 elements. We'll discover below that the free HSL on two generators has 18 elements and is isomorphic the the product $2 \times 3 \times 3$ where the numbers name totally ordered sets.

Define an **equationally linear Heyting semi-lattice**, or ELHSL, to be an HSL that satisfies the further equation:

$$((x \to y) \to z) \wedge ((y \to x) \to z)) \quad = \quad z$$

Any totally ordered set with a top element is a ELHSL. And any *local* ELHSL is linear, that is, totally ordered (specialize to the case that $z = m$). Hence an HSL is equationally linear iff it can be embedded in a product of linear HSLs.

For a finite set, G, of variables let $F(G)$ denote the free ELHSL on G. $F(\emptyset) = \{1\}$. *Theorem:*

$$F(G) \quad \cong \prod_{S \subset G,\, S \neq G} F(S)_\perp$$

where we use the standard domain-theoretical notation for what is there called the "lifting": $F(S)_\perp$ is the result of adjoining a new bottom element to the poset $F(S)$.

Let $f(n)$ be the order of $F(n)$. $f(0) = 1$. We may read off the following recursion formula for $n > 0$:

$$f(n) = \prod_{r=0}^{n-1} \left[(1 + f(r))^{\binom{n}{r}} \right]$$

The first few terms are: 1, 2, 18, 370386, 14359142810110969797351100018504 2. It isn't hard to see that there is a constant, C, such that $f(n)$ is asymptotically equal to

$$C^{\frac{n!}{(log 2)^n}}.$$

C is approximately equal to 2.03827.

For the proofs, note first that for any element k in an HSL, H, the downdeal $\{x \mid x \leq k\}$ has a natural HSL structure and we'll denote that HSL as H/k. The inclusion map is not a homomorphism but the map from H to H/k that sends x to $x \wedge k$ is. That is, H/k may be viewed as a quotient structure of H. (Indeed, if H is finite or, more generally, satisfies the descending chain condition, then every quotient structure so arises.) The product decompositions of H are of the form $H/k_1 \times H/k_2 \times \cdots$ where the sequence $\{k_1, k_2, \ldots\}$ is a "partition of unity", that is, a sequence of elements in H above the bottom element whose join is 1 and whose pairwise meets are each the bottom element. In the finite case there is a unique largest such. That is, any finite HSL uniquely factors as a product of "prime" algebras. (All of this is easy in the distributive lattice context. It continues to work in the HA context. And even in the HSL context.)

Given a finite set, G, let 0 denote the element $\bigwedge_{a \in G} a$ in $F(G)$ (0 is the bottom element). For any element x let $\neg x$ denote $x \to 0$. For each subset $S \subset G$ define

$$t_S \quad = \quad (\bigwedge_{a \in G \setminus S} \neg a) \wedge (\bigwedge_{a \in S} \neg\neg a)$$

The family $\{t_S\}_{S \subset G,\, S \neq G}$ is a partition of unity. To see that, it suffices to show in any local ELHSL generated by G, that there is a unique $S \subset G$ such that t_S names the top element and for all other proper subsets of G, t_S names the bottom element.

When we regard $F(G)/t_S$ as a finitely presented ELHSL (where $t_S = 1$ is the unique relation) it is easily seen to be isomorphic to $F(S)_\perp$.

It is easy to verify that any local HSL on two generators is totally ordered, hence any HSL on two generators is linear. Necessarily the free HSL on two generators is the same as the free ELHSL on two generators.

In any local ELHSL

$$((x \to y) \to y) \wedge ((y \to x) \to x)$$

can be verified to satisfy the equations of a join operation (because, of course, it can be verified to be the join operation), hence continues to satisfy those equations in any ELHSL. If we add a bottom constant then a ELHSL is a full-fledged Heyting algebra and we could as well call them linear HAs. (The definition of linear HAs can be made a bit simpler: they are those HAs that satisfy the equation $(x \to y) \vee (y \to x) \ = 1$.) All of which allows us to prove that the free linear Heyting algebra on G is constructible as:

$$\prod_{S \subseteq G} F(S)_\perp$$

and that the order of the free linear HA on n generators is:

$$(f(n) + 1) f(n)$$

The first few values are 2, 6, 342, 137186159382. (The same asymptotic expression works as for $f(n)$ with C approximately 4.1545.)

(There is something of a converse of the fact that join is definable in ELHSLs: if the join operation is definable in a variety of HSLs then necessarily it is a variety of equationally linear HSLs. See Appendix. There aren't many such varieties, by the way. Each is determined by the sizes of its local members, that is, each is determined by the smallest n, if any, for which the equation $1 \ = \ \bigvee_{i=1}^{n} (x_i \to x_{i+1})$ is satisfied.)

The number of isomorphism types of ELHSLs of order n already appears in Sloane as A050318, albeit shifted by 1, credited to Christian G. Bower whose description of the sequence there is:

> Number of ways to write n as an **mterm**, where an mterm is an unordered sum which is either 2, or 1 + an unordered product of mterms. Formula: Shifts left under transform T where Ta has Dirichlet g.f.:
>
> $$\prod_{n=1}^{\infty} (1/(1 - 1/n^s)^{a(n)})$$

I would rewrite that as

> Number of ways to write n as an **mterm**, where an mterm is the sum 1 + an unordered product (possibly empty) of mterms.

The first few entries are: 1, 1, 1, 2, 2, 3, 3, 5, 6, 8, 8, 12, 12, 15, 17, 23, 23, 31, 31, 41, 44, 52, 52, 69, 72, 84, 90, 108, 108, 135, 135, 161, 169, 192, 198, 246, 246, 277, 289, 342, 342, 404, 404, 464, 491, 543, 543, 644, 650, 734, 757, 853, 853, 978, 994, 1123, 1154, 1262, 1262, ...

I'll use the convention that 2 names the mterm 1+1 (where the second 1 is the empty product), 3 names 1+2, 4 names 1+3, etc. The first number that may be written as an mterm in more that one way is five, to wit, $5 = 1 + 4$ and $1 + (2\times 2)$. There remain only two ways for six, to wit, $6 = 1 + 5$ and $1 + (1 + (2\times 2))$. For seven there are three ways: $7 = 1+6$, $1+(1+(1+(2\times 2)))$ and $1+(2\times 3)$. For thirteen there are twelve ways (using a further simplification of notation): $13 = 1 + 12$, $9+(2\times 2)$, $7+(2\times 3)$, $6+(2\times 4)$, $5+(2\times 2\times 2)$, $4 + (3\times 3)$, $3 + (2\times 5)$, $3 + (2\times(1 + (2\times 2))))$, $1+(2\times(2+(2\times 2)))$, $1+(2\times 2\times 3)$, $1+(3\times 4)$, $1 + (2\times 6)$.

The connection with ELHSLs is this: Every prime ELHSL is the lifting of an ELHSL of one fewer element, which ELHSL is, as already pointed out, uniquely an unordered product of prime ELHSLs. The argument is as follows. If 0 is a prime element of the given ELHSL, that is, if 0 is not the meet of two larger elements then—since we are in a finite lattice—there is an element just on top of 0 that is below all other elements above 0, hence the ELHSL is a lifting of an ELHSL of one fewer element. If 0 is not prime we will argue that the ELHSL is not prime. It suffices to find a non-trivial partition of unity. Let $a > 0$ and $b > 0$ be such that $a \wedge b = 0$. Then $\neg a \wedge \neg\neg a = 1$ (it suffices to verify in any totally ordered HSL that $(x \to y) \wedge ((x \to y) \to y) = 1)$. $\neg a > 0$ since $\neg a \geq b$. $\neg\neg a > 0$ since $\neg\neg a \geq a$. Hence A050318 counts the number of prime ELHSLs. But the number of prime ELHSLs of order n is equal to the number of all ELHSLs of order $n-1$ since each is the lifting of a unique ELHSL of one fewer element.

As with A006982, we may describe A050318 as counting the number of isomorphism classes of equationally linear Heyting algebras (any finite HSL is automatically an HA). But unlike the case with A006982, there is no happy translation into distributive lattices. Perhaps the least unhappy: A050318 shifted by 1 is the number of isomorphism types of distributive lattices in which any element below a coprime is itself a coprime. Which, of course, is the same as the number of distributive lattices in which any element above a prime is itself a prime.

Finally, Bower's sequence A050365, can be interpreted to count the number of **anti-symmetric** ELHSLs (and ELHAs), to wit, those for which all automorphisms are one. His description (emphasis added):

> Ways to write n as an identity mterm, where an identity mterm is an unordered sum which is either 2, or 1 + an unordered product of *distinct* identity mterms. Formula: Shifts left under transform T where Ta has Dirichlet g.f.:
>
> $$\prod_{n=1}^{\infty}(1 + 1/n^s)^{a(n)}$$

The first few terms are 1, 1, 1, 1, 1, 2, 2, 3, 3, 4, 4, 6, 6, 8, 9, 11, 11, 15, 15, 19, 21, 25, 25, 33, 33, 39, 42, 50, 50, 63, 63, 74, 78, 89, 91, 110, 110, 125, 131, 152, 152, 181, 181, 206, 217, 242, 242, 285, 286, 322, 333, 372, 372, 428, 432, 486, 501, 551, 551, 636, 636, 699, 724, 799 ...

There is yet another interpretation. Recall that in formal set theory the rank of a set is defined, recursively, as the first ordinal larger than the rank of any of its elements. For a set of finite rank we may define its **Bower-rank** as the first ordinal larger than the *product* of the Bower-ranks of its elements. Then A050365 counts the number of sets of given Bower-rank. If we extend these definitions to "mull-sets" then A050318, besides counting mterms, ELHSLs and ELHAs, counts the number of multi-sets of given Bower-rank.

Appendix 1

The following are equivalent conditions on a variety of HSL*s:*

a: All members are ELHSL*s;*

b: All local members are totally ordered;

c: All members with a bottom are Heyting algebras.

We have already proved that a and b are equivalent and that they imply c. We need only show that c implies b. We'll attack the contrapositive: if a variety of HSLs has a local member that is not totally ordered then it has a member with a finite subset of elements whose set of upper bounds does not have a bottom.

If there's a local member that is not totally ordered we may cut down to the finite subalgebra generated by m and a pair of incomparable elements. We may then chose a smallest such example. We'll call it A.

Let m' be the join of all the elements less than m. If $m' < m$ then A/m [2] is still local and still not totally ordered. Hence m is the join of the smaller elements.

Let P be the countable cartesian power of A. We will treat its elements as sequences of A-elements. We carve out a particular subalgebra as the (disjoint) union of two subsets C and B. C is the set of all constant sequences with constant value less than m. B is the set of all sequences that are equal to 1 except for a finite number of ms. Note that every element of C is less than every element of B.

Clearly both C and B are closed under the meet operation, and the last remark makes it clear that the meet of a C-element and a B-element is a C-element.

Clearly $C \cup \{1\}$ and B are each closed under the arrow operation. Clearly if $c \in C$ and $b \in B$ then $c \to b = 1$ and $b \to c = c$. Hence $C \cup B$ is a subalgebra. It lies in the variety under discussion. B is the set of upper bounds of the finite set C. It does not have a least element.

[2] At the end of page 2 it's defined as the set $\{x \in A \mid x \leq m\}$ with the induced HSL structure.

Appendix 2

$$
\begin{array}{ccc}
x \wedge (x \leftrightarrow y) & & y \wedge (x \leftrightarrow y) \\
= & & = \\
x \wedge ((x \wedge x) \leftrightarrow (x \wedge y)) & & y \wedge ((y \wedge x) \leftrightarrow (y \wedge y)) \\
= & & = \\
x \wedge (x \leftrightarrow (x \wedge y)) & & y \wedge ((y \wedge x) \leftrightarrow y) \\
= & & = \\
x \wedge ((x \wedge 1) \leftrightarrow (x \wedge y)) & & y \wedge ((y \wedge x) \leftrightarrow (y \wedge 1)) \\
= & & = \\
x \wedge (1 \leftrightarrow y) & & y \wedge (x \leftrightarrow 1) \\
= & & = \\
x \wedge y & = & y \wedge x
\end{array}
$$

$$z \leq x \leftrightarrow y$$

$$
\begin{array}{ccc}
z \wedge x & & z \wedge y \\
= & & = \\
z \wedge (x \leftrightarrow y) \wedge y & & z \wedge (x \leftrightarrow y) \wedge x \\
= & & = \\
z \wedge x \wedge y & = & z \wedge y \wedge x
\end{array}
$$

$$z \wedge x = z \wedge y$$

$$
\begin{array}{c}
z \wedge (x \leftrightarrow y) \\
= \\
z \wedge ((z \wedge x) \leftrightarrow (z \wedge y)) \\
= \\
z \wedge ((z \wedge x) \leftrightarrow (z \wedge x)) \\
= \\
z \wedge 1 \\
= \\
z
\end{array}
$$

Addendum

The theory of HSLs is an example of an equational theory with a unique maximal consistent extension,[3] and only one equation is needed to achieve that extension:

$$(x \to z) \wedge ((x \to y) \to z) \;=\; z$$

Its consistency is established by checking its correctness on a two-element HSL.[4] Since any non-trivial HSL contains a two-element subalgebra (to wit, $\{x, 1\}$ for any $x < 1$) any equation is consistent with the theory of HSLs iff it holds for a two-element HSL.

We establish that this one new equation suffices for all consistent equations by establishing that any HSL that satisfies it can be embedded in a cartesian power of the two-element HSL. We know that any HSL is embedded in the product of its localizations, hence we finish by showing that any local HSL with more than two elements fails the new equation. So let y be less than m, take $x = z = m$ to obtain $(x \to z) \wedge ((x \to y) \to z) = (m \to m) \wedge ((m \to y) \to m) = 1 \wedge (y \to m) = 1 \wedge 1 \neq z$.

Finally, any HSL with a bottom that satisfies this equation is a Boolean algebra.[5]

Following the lead of footnote [17] in `www.math.upenn.edu/~pjf/amplifications.pdf` wherein abelian group theory without zero is named—in imitation of Jacobson's *rngs*—the theory of *grups* we can call such algebras *bolean*.

Better known is its upside-down version. Consider the join-semilattice of finite subsets (of an infinite set). Add the relative-complement operation, $A \setminus B$, and the dual of $(x \to z) \wedge ((x \to y) \to z) \;=\; z$ to obtain $(C \setminus A) \cup (C \setminus (B \setminus A)) = C$. If this doesn't look familiar, use AB to denote $A \setminus (A \setminus B)$ and $A+B$ to denote $(A \setminus B) \cup (B \setminus A)$. It's not called a *boolan algebra*. It's called a *Boolean rng*.

Available at `http://www.math.upenn.edu/~pjf/Heyting.pdf`

[3] See footnote [35] in `http://www.math.upenn.edu/~pjf/analysis.pdf`

[4] The equation easily holds when $z = 1$ and $(x \to 0) \wedge ((x \to y) \to 0) = 0$ easily holds when either $x = 1$ or $y = 1$. And that leaves only $(0 \to 0) \wedge ((0 \to 0) \to 0) = 1 \wedge (1 \to 0) = 0$.

[5] It's clearly an ELHSL hence has a join operation. (Indeed, $x \vee y$ is even more easily defined just as $(x \to y) \to y$.) Add a bottom 0 and use $x \to 0$ for the complement.