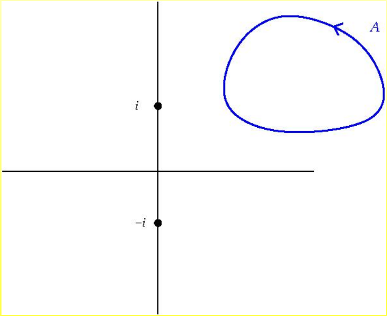
Residues are about integrals over closed curves in the complex plane. Let's take as an example, the integral

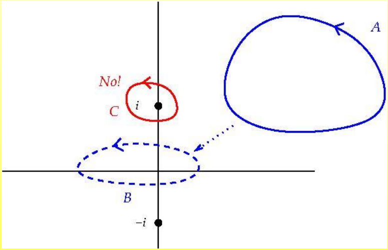
$$\oint_A \frac{1}{1+z^2}\,dz\,.$$




Let $f(z)$ denote the integrand, that is, $f(z) := \dfrac{1}{1+z^2}$. The function $f$ has singularities at $\pm i$. That's where the denominator is zero, hence the function is undefined there.

We say that the singularities at $\pm i$ are poles. A pole is an isolated singularity at a point $a$ that goes away if you multiply by some power of $z-a$. For example, if I multiply by $z-i$, the function becomes $(z-i)f(z) = 1/(z+i)$ which no longer has a pole at $i$.

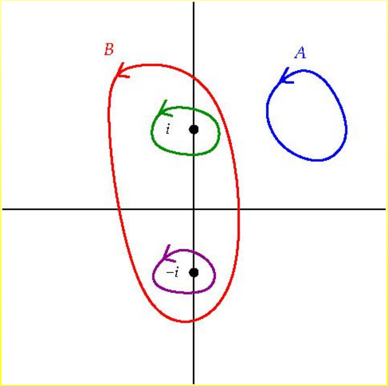
A theorem due to Cauchy (not the big one, namely Cauchy's integral formula) says that you can move the contours without changing the integral. There are some necessary hypotheses to make this work.




1. You have to deform the contour without crossing any singularities. In the picture, contour $A$ can be deformed to $B$ but not to $C$. It can't enclose the singularity $i$ if it didn't already.

2. This whole thing only works if the thing you're integrating is an *analytic function*. More on that in a minute.

This means that no matter what contour $\gamma$ you integrate over, there are only a few possible values for $\oint_\gamma f(z)\,dz$. At every singularity $y$ there is a fixed value for the integral that holds for every contour that encloses $y$ and only $y$. We call this value $2\pi\mathrm{i}\,\mathrm{Res}(\mathrm{f};\mathrm{y})$.




In the figure, the green integral has value $2\pi\mathrm{i}\ \mathrm{Res}(\mathrm{f};\mathrm{i})$ and the purple one has the value $2\pi\mathrm{i}\,\mathrm{Res}(\mathrm{f};-\mathrm{i})$.

The integral over red contour will be the sum of these. The integral over the blue contour will be zero.

## Theorem 1 (Cauchy integral formula)

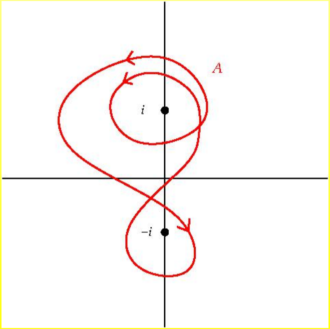
*Suppose* $f$ *is analytic, defined everywhere except for poles at* $z_1, \dots z_n$*. Then*

$$\oint_\gamma f(z)\, dz = (2\pi i) \sum_{k=1}^{n} n_k \operatorname{Res}(f; z_k)$$

*where* $n_k$ *is the* winding number *telling how many times* $\gamma$ *winds counterclockwise around* $z_k$*.*

Example:
Cauchy's integral formula tells us that

$$\oint_A \frac{1}{1+z^2}\, dz = 2 \operatorname{Res}(f; i) - \operatorname{Res}(f; -i)\,.$$



# Analytic functions

One way to think of an analytic function on a domain in the complex plane is that it's a function that looks like one of your favorite real functions – a polynomial, an exponential, a trig function, etc. – except that it has been extended to the complex numbers.

Another definition of an analytic function is that it has a convergent power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. If you were wondering how something like $e^x$ gets extended to the complex numbers, that's it: use the power series. A $\mathrm{meromorphic}$ function, means a function that's analytic except at some poles. It can have a more general looking power series: near each pole $z_k$, of order $m_k$ (poles have orders),

$$f(z) = \sum_{n=-m_k}^{\infty} a_n (z - z_k)^n .$$

# What is this thing we call "Residue"?

One way to think of / define a residue is as a power series coefficient: $\mathrm{Res}(f; z_k)$ is the coefficient of $(z - a_k)^{-1}$ in the power series expansion. For example, if $f(z) = g(z)/(z - a)$ where $g$ is analytic everywhere (no poles), then $\mathrm{Res}(f; a) = g(a)$. To see this, recall that $g$ has a power series near $a$ with no negative powers, so $f$ has a pole of order 1 and a power series in powers $(z - a)^m$, $m \geq -1$. Near $a$, we have

$$\begin{aligned} g(z) &= a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots \\ f(z) &= a_0(z-a)^{-1} + a_1 + a_2(z-a)^1 + \cdots \end{aligned}$$

so $\mathrm{Res}(f; a) = a_0 = g(a)$.

Example: take $f(z) = 1/(1 + z^2)$, $a = i$ and $g(z) = 1/(z + i)$. Then $\mathrm{Res}(1/(1 + z^2); i) = g(i) = 1/(2i)$. Cauchy's integral formula gives

$$\oint_A f(z)\,dz = 2\pi i\,\mathrm{Res}(f; i) = (2\pi i)/(2i) = \pi\,.$$

# Another way to think of residues

This will show you why this definition gives you Cauchy's integral formula. Let's say we are only interested in one contour: the unit circle, $\gamma$, oriented clockwise. Also, we are only interested in functions with a single pole at the origin of order 1. Parametrize the unit circle by $\mathrm{z} := \mathrm{e}^{\mathrm{i}\theta}$. Changing variables,

$$\oint_\gamma \mathrm{f}(\mathrm{z})\,\mathrm{dz} = \int_{\theta=0}^{2\pi} \mathrm{f}\left(\mathrm{e}^{\mathrm{i}\theta}\right) \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta\,.$$

On the unit circle, the functions $\{\mathrm{e}^{\mathrm{in}\theta} : \mathrm{n} \in \mathbb{Z}\}$ are a basis, that is, for some coefficients $\{\mathrm{a_n}\}$,

$$\mathrm{f}\left(\mathrm{e}^{\mathrm{i}\theta}\right) = \sum_{\mathrm{n}=-\infty}^{\infty} \mathrm{a_n}\mathrm{e}^{\mathrm{in}\theta}$$

Remember how to extract $\mathrm{a_n}$? Multiply by $\mathrm{e}^{-\mathrm{in}\theta}$ and integrate.

Therefore, integrating over $\theta \in [0, 2\pi]$,

$$\int_{\theta=0}^{2\pi} \mathrm{f}\left(\mathrm{e}^{\mathrm{i}\theta}\right) \mathrm{i}\mathrm{e}^{\mathrm{i}\theta} \mathrm{d}\theta = \sum_{\mathrm{n}=-\infty}^{\infty} \mathrm{i}\mathrm{a}_{\mathrm{n}} \int_{\theta=0}^{2\pi} \mathrm{e}^{(\mathrm{n}+1)\mathrm{i}\theta}$$
$$= 2\pi \mathrm{i}\, \mathrm{a}_{-1}$$

because $\int_0^{2\pi} \mathrm{e}^{\mathrm{i}\mathrm{m}\theta}\, \mathrm{d}\theta$ is zero when $\mathrm{m}$ is a nonzero integer and 1 when $\mathrm{m} = 0$.

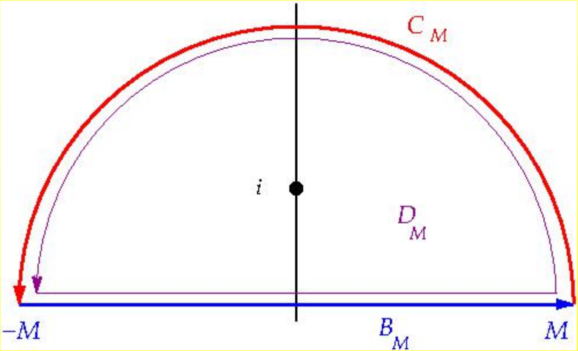
In other words we have proved:

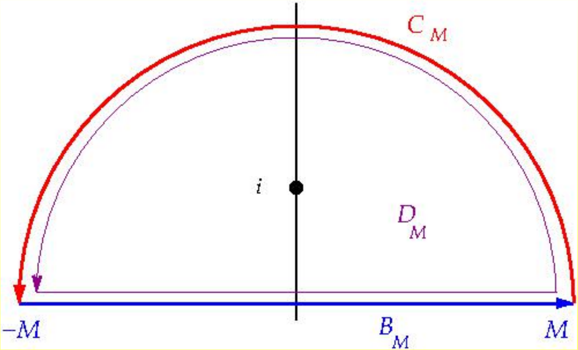
$$\oint_{\gamma} \mathrm{f}(\mathrm{z})\, \mathrm{d}\mathrm{z} = 2\pi \mathrm{i}\, \mathrm{a}_{-1}\,.$$

The integral of a function analytic except at the poles over a closed contour can be given by evaluating the residue $\mathrm{a}_{-1}$ at each pole $\mathrm{z}_{\mathrm{k}}$, and summing $\mathrm{n}_{\mathrm{k}}(2\pi\mathrm{i})\mathrm{a}_{-1}(\mathrm{z}_{\mathrm{k}})$ where $\mathrm{n}_{\mathrm{k}}$ is the winding number of $\gamma$ around $\mathrm{z}_{\mathrm{k}}$.

# Using residues to compute real integrals

Let's compute $L := \int_{-\infty}^{\infty} \frac{1}{1+z^2}\, dz$. Let $B_M$ be the blue contour, a line segment $[-M, M]$. Let $C_M$ be the red contour, a semicircle from $M$ to $iM$ to $-M$. Let $D_M = B_M + C_M$ be the purple closed loop, drawn just inside its true position so you can see it distinct from $B_M$ and $C_M$.






1. Observe that $\mathrm{L} = \lim_{\mathrm{M}\to\infty} \mathrm{L_M}$, where $\mathrm{L_M} = \int_{\mathrm{B_m}} \mathrm{f(z)\,dz}$.

2. Observe that $\int_{\mathrm{C_M}} \mathrm{f(z)\,dz} \to 0$ because $|1/(1+\mathrm{z}^2)| \leq 1/(\mathrm{M}^2 - 1)$, and integrating over a contour of length $\pi\mathrm{M}$ gives at most $\pi\mathrm{M}/(\mathrm{M}^2 - 1) \to 0$.

3. Therefore $\mathrm{L} = \lim_{\mathrm{M}\to\infty} \oint_{\mathrm{D_M}} \mathrm{f(z)\,dz} = (2\pi\mathrm{i})\mathrm{Res(f,i)} = \pi$.

J. Bak and D. Newman.
*Complex Analysis*.
undergraduate Texts in Mathematics. Springer-Verlag, Berlin, 1982.

John B. Conway.
*Functions of One Complex Variable*.
Springer-Verlag, New York, second edition, 1978.

Peter Henrici.
*Applied and Computational Complex Analysis. Vol. 1*.
John Wiley & Sons Inc., New York, 1988.
Power series—integration—conformal mapping—location of zeros, Reprint of the 1974 original, A Wiley-Interscience Publication.

R. Pemantle and M. Wilson.
*Analytic Combinatorics in Several Variables*, volume 140 of *Cambridge Studies in Advanced Mathematics*.
Cambridge University Press, New York, 2013.