MARKOV CHAINS IN A FIELD OF TRAPS

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ABSTRACT:
We consider a Markov chain on a countable state space, on which is placed a random field of traps, and ask whether the chain gets trapped almost surely. We show that the quenched problem (when the traps are fixed) is equivalent to the annealed problem (when the traps are updated each unit of time) and give a criterion for almost sure trapping versus positive probability of non-trapping. The hypotheses on the Markov chain are minimal, and in particular, our results encompass the results of den Hollander, Menshikov and Volkov (1995).

Keywords: Markov Chain, Greens function, traps, random traps, killing, annealed, quenched.

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1 Introduction and statement of results

Let $\mathcal{S}$ be a countable space and let $p : \mathcal{S}^2 \to [0,1]$ be a set of transition probabilities on $\mathcal{S}$, i.e., $\sum_y p(x,y) = 1$ for all $x \in \mathcal{S}$. Let $\mathbf{P}_x$ denote any probability measure such that the random sequence $X_0, X_1, X_2, \ldots$ is a Markov chain with transition probabilities $\{p(x,y)\}$, starting from $X_0 = x$. Given a function $q : \mathcal{S} \to [0,1)$, representing a set of trapping probabilities, we define two new Markov chains as follows.

Quenched problem. "site x is a trap with probability $q(x)$ forever". Let $T \subseteq \mathcal{S}$ be the random set of traps, i.e., $\mathbf{P}_x(\{x_1, \ldots, x_n\} \subset T) = \prod_{i=1}^n q(x_i)$ for all finite subsets of $\mathcal{S}$, and T is independent of $X_0, X_1, \ldots$. We are interested in the quantities

$$\pi(x) = \mathbf{P}_x(X_n \in T \text{ for some } n \geq 0). \tag{1.1}$$

We say that the quenched field is *trapping* or *non-trapping*, according to whether $\pi(x) = 1$ for all x, or whether $\pi(x) < 1$ for some x.

Annealed problem. "The state of site x is a trap with probability $q(x)$, but each unit of time the state of site x is chosen afresh with this probability". This gives us an IID sequence of trap sets $\{T_n : n \geq 0\}$, each distributed as T in the quenched problem. We let

$$\tilde{\pi}(x) = \mathbf{P}_x(X_n \in T_n \text{ for some } n \geq 0). \tag{1.2}$$

We say that the annealed field is trapping or non-trapping, according to whether $\tilde{\pi}(x) = 1$ for all x, or whether $\tilde{\pi}(x) < 1$ for some x.

Our first result is the equivalence of these two problems. Let

$$g(x,y) = \sum_{n=0}^{\infty} \mathbf{P}_x(X_n = y)$$

denote the Greens function.

Theorem 1.1 *Assume there is a constant K such that*

$$g(x,x) \leq K \text{ for all } x \in \mathcal{S}. \tag{1.3}$$

Then the annealed field is trapping if and only if the quenched field is trapping.

Remarks:

1. The necessity of an assumption such as (1.3) is illustrated by Example 2 below.

2. These two problems were considered by den Hollander, Menshikov and Volkov (1995) in the special case of mean-zero, finite-variance random walk on $\mathbf{Z}^d$, $d \geq 3$. They showed (in their Section 5) that the quenched and annealed problems were equivalent under an additional technical assumption. The present theorem thus removes their technical hypothesis, as well as showing that the particular setting of $\mathbf{Z}^d$ is irrelevant.

3. It is elementary (see the next paragraph) that $\pi(x) \leq \tilde{\pi}(x)$ for all x. Thus only one direction of Theorem 1.1 is interesting.

Let $\tau(x) = \inf\{k : X_k = x\}$ denote the first hitting time of the point x and define the two quantities

$$R_n = \sum_{i=0}^{n} -\log(1 - q(X_i))\mathbf{1}_{\tau(X_i)=i} \; ; \tag{1.4}$$

$$\tilde{R}_n = \sum_{i=0}^{n} -\log(1 - q(X_i)).$$

It is easy to see that the probability of no trapping in the annealed chain up to time n is given by

$$\mathbf{P}_x(X_i \notin T_i \text{ for all } i \leq n \,|\, X_0, \ldots, X_n) = \exp(-\tilde{R}_n).$$

Similarly, since the conditional probability of $X_n \notin T$ given $X_0, X_1, \ldots$ and given $\{X_i \notin T : i < n\}$, is equal to $1 - q(X_n)\mathbf{1}_{\tau(X_n)=n}$, the probability of no trapping in the quenched chain to time n may be computed as

$$\mathbf{P}_x(X_i \notin T \text{ for all } i \leq n \,|\, X_0, \ldots, X_n) = \exp(-R_n).$$

Thus

$$\begin{aligned} \pi(x) &= \mathbf{E}[1 - \exp(-R_\infty)] \text{ and} \\ \tilde{\pi}(x) &= E[1 - \exp(-\tilde{R}_\infty)]. \end{aligned} \tag{1.5}$$

From equations (1.4) and (1.5), it is evident that $R_n \le \tilde{R}_n$, and hence that $\tilde{\pi}(x) \ge \pi(x)$ for all x.

We turn now to the determination of a criterion for trapping. We consider only the annealed case, which of course solves the quenched case as well under condition (1.3). By (1.5), the problem reduces to the determination of when $\tilde{R}_\infty = \infty$ with probability 1. Define

$$\begin{aligned} S_n &= \sum_{i=0}^{n} q(X_i) \mathbf{1}_{\tau(X_i)=i} \\ \tilde{S}_n &= \sum_{i=0}^{n} q(X_i) \end{aligned}$$

and observe that $\tilde{R}_\infty = \infty$ if and only if $\tilde{S} = \infty$, and that $R_\infty = \infty$ if and only if $S_\infty = \infty$. Thus an obvious necessary condition for trapping in the anealed case is that for all x_0,

$$\mathbf{E}\tilde{S}_\infty = \sum_{x \in \mathcal{S}} g(x_0, x) q(x) = \infty. \tag{1.6}$$

Later, we will discuss some conditions under which this is sufficient as well, but the next example shows that (1.6) is not always sufficient for trapping.

Definition 1 *Say that a subset $A \subseteq \mathcal{S}$ is* transient for x *if* $\mathbf{P}_x(X_n \in A$ *for some* $n) < 1$. *Say that A is* transient *if it is transient for some x. We remark that when all states communicate in A^c, then A must be transient for all x or for no x; with the present definition, our results make sense for more general chains.*

Example 1: Let A be any transient set satisfying $\sum_{x \in A} g(x_0, x) = \infty$. An example of such a set A for simple random walk in $\mathbf{Z}^3$ is the set

$$\{x : ||x - (0, 0, 2^n)|| \le 2^{n/2} \text{ for some } n \ge 1\}.$$

Let $q(x) = c \in (0,1)$ for $x \in A$ and 0 for $x \notin A$. Then clearly $\tilde{\pi}(x_0) < 1$, while $\sum_{x \in \mathcal{S}} g(0,x)q(x) = \infty$.

Our second result is that this is the only way that (1.6) can fail to imply trapping.

Theorem 1.2 *Suppose that the annealed field is non-trapping, i.e., for some x_0, $\tilde{\pi}(x_0) < 1$. Then there exists a subset $A \subseteq \mathcal{S}$ such that*

$$(i) \qquad A \text{ is transient for } x_0, \qquad\qquad \textit{and}$$

$$(ii) \qquad \sum_{x \in \mathcal{S} \setminus A} g(x_0, x) q(x) < \infty .$$

These two theorems are proved in Section 2. The final section discusses conditions under which the condition $\sum_{x \in \mathcal{S}} g(x_0,x)q(x) = \infty$ is necessary and sufficient for trapping. These cases include the case of a simple random walk on $\mathbf{Z}^d$, $d \geq 3$, and a spherically symmetric function $q(x) = q(||x||)$ discussed in den Hollander, Menshikov and Volkov (1995).

2 Proofs of the two theorems

Proof of Theorem 1.1: By (1.5), it suffices to show that the event $\{S_\infty < \infty = \tilde{S}_\infty\}$ has probability zero. Recall that $\tau(x)$ is defined to be the first hitting time of x, and let $\mathcal{F}_{\tau(x)}$ be the σ-field generated by the Markov chain up to time $\tau(x)$. Observe that for any x and any event $G \in \mathcal{F}_{\tau(x)}$,

$$\mathbf{E}_{x_0} \mathbf{1}_G \mathbf{1}_{\tau(x) < \infty} \left(\sum_{i=0}^{\infty} \mathbf{1}_{X_i = x} \right) = g(x,x) \mathbf{P}_{x_0} (G \cap \{\tau(x) < \infty\}) . \tag{2.1}$$

Given $M \geq 0$, define a time $\tau_M = \inf\{k : S_k \geq M\}$ and define

$$\tilde{S}^{(M)} = \sum_{i=0}^{\infty} q(X_i) \mathbf{1}_{\tau(X_i) \leq \tau_M} .$$

From the definitions, it is evident that $\tilde{S}^{(M)} = \tilde{S}_\infty$ whenever $\tau_M = \infty$. A second useful fact, also immediate from the definitions, is that the event $\{\tau(x) \leq \tau_M\}$ is the same as the intersection of events

$$\{\tau(x) < \infty\} \cap \{S_{\tau(x)-1} < M\},$$

from which also we see it is in $\mathcal{F}_{\tau(x)}$. From these two facts we get

$$\begin{aligned} \mathbf{E}_{x_0} \tilde{S}^{(M)} &= \sum_{i=0}^{\infty} \sum_{x \in \mathcal{S}} q(x) \mathbf{E}_{x_0} \mathbf{1}_{X_i = x} \mathbf{1}_{\tau(x) \leq \tau_M} \\ &= \sum_{x \in \mathcal{S}} q(x) \mathbf{E} \sum_{i=0}^{\infty} \mathbf{1}_{X_i = x} \mathbf{1}_{\tau(x) < \infty} \mathbf{1}_{S_{\tau(x)-1} < M} . \end{aligned}$$

Applying (2.1) for each x shows that this is equal to

$$\sum_{x \in \mathcal{S}} q(x) g(x,x) \mathbf{P}_{x_0} (\tau(x) < \infty, S_{\tau(x)-1} < M) \, ,$$

which is bounded above by

$$\sum_{x \in \mathcal{S}} K q(x) \mathbf{P}_{x_0} (\tau(x) \leq \tau_M) = K \mathbf{E}_{x_0} S_{\tau_M} \, .$$

Since $S_{\tau_M} < M + 1$, this shows that for any M, $\tilde{S}^{(M)}$ has finite expectation and is hence almost surely finite. Since $\tilde{S}^{(M)} = \tilde{S}_\infty$ whenever $\tau_M = \infty$, we see that $\tilde{S}_\infty < \infty$ whenever $\tau_M = \infty$ for some M, which happens exactly when $S_\infty < \infty$. $\square$

Before proving Theorem 1.2, we give an example to show that the hypothesis of a bounded Greens function is necessary.

Example 2: Let $\mathcal{S} = \{2, 3, 4 \ldots\}$ with transition probabilities $p(n, n+1) = 1/n$, $p(n,n) = 1 - 1/n$ and $p(n,k) = 0$ for $k \neq n, n+1$. The quenched field traps the chain at n with probability $q(n)$, conditional on its not being trapped at any $k < n$, hence the quenched field is trapping if and only if $\sum_n q(n) = \infty$. The annealed field traps the chain at n with probability $nq/(nq + 1 - q)$, conditional on its not being trapped at any $k < n$, hence the annealed field is trapping if and only if $\sum_n nq(n) = \infty$, which is the Greens function criterion.

One can replace this example if desired by an example where the Markov chain is a simple random walk. Let G be a binary tree, rooted at a vertex 0, to which is appended at each vertex v in generation $n > 0$ a single chain of vertices of length n. Setting $q(x) = 1/n$ if x is the end of one of these chains for some n, and $q(x) = 0$ otherwise, makes the annealed field trapping and the quenched field non-trapping for simple random walk.

We now give the proof of Theorem 1.2. The idea is that $\tilde{\pi}(X_n)$ is a martingale if one stops at the value 1 upon being trapped, and therefore that $\tilde{\pi}(X_n)$ is a $\mathbf{P}_x$-supermartingale. The excess of a positive superharmonic function ($\tilde{\pi}$) is integrable against the Greens function; this excess is usually close to q, and in fact the set where the excess of $\tilde{\pi}$ is not near q must be a transient set. We begin by stating two facts whose proofs are immediate.

Lemma 2.1 *For all x, under no assumptions on the Markov chain,*

$$\tilde{\pi}(x) = q(x) + (1 - q(x))\mathbf{E}_x \tilde{\pi}(X_1).$$

□

Proof of Theorem 1.2: Let $B_a = \{x : \tilde{\pi}(x) \geq a\}$. Observe that if

$$x \notin B_{1/16} \;\Rightarrow\; P_x(X_1 \in B_{1/4}) < \frac{1}{4}. \tag{2.2}$$

For any a, the set B_a is the union of a finite set and a set transient for x_0. To see this, note that $\tilde{\pi}(X_n) \to 0$ almost surely on the event of nontrapping, and hence that $\mathbf{P}_{x_0}(\tilde{\pi}(X_n) \to 0) > 0$; choosing N so that $\mathbf{P}_{x_0}(\tilde{\pi}(X_n) < a \forall n \geq N) > 0$, and letting $F = \{x_1, x_2, \ldots, x_N\}$ such that

$$\mathbf{P}_{x_0}(\tilde{\pi}(X_n) < a \forall n \geq N \,|\, X_i = x_i \text{ for } i = 1, \ldots N) > 0,$$

gives a set $B_a \setminus F$ that is transient for x_0. We now show that $\sum_{x \in \mathcal{S} \setminus B_{1/16}} g(x_0, x) q(x) < \infty$, which proves the theorem with $A = B_{1/16} \setminus F$. By (2.2), for any $x \notin B_{1/16}$,

$$\mathbf{E}_x \tilde{\pi}(X_1) \leq \frac{1}{4} \mathbf{P}_x(X_1 \notin B_{1/4}) + \mathbf{P}_x(X_1 \in B_{1/4}) \leq \frac{1}{2}.$$

Applying Lemma 2.1 we see that for any x,

$$\begin{aligned} \mathbf{E}_x \tilde{\pi}(X_1) &= \tilde{\pi}(x) - q(x)(1 - \mathbf{E}_x \tilde{\pi}(X_1)) \\ &\leq \tilde{\pi}(x) - \frac{1}{2} q(x) \mathbf{1}_{x \notin B_{1/16}} \,. \end{aligned}$$

Iterating this, using the Markov property, gives

$$0 \leq \mathbf{E}_x \tilde{\pi}(X_n) \leq \tilde{\pi}(x) - \sum_{i=0}^{n-1} \frac{1}{2} \mathbf{E}_x q(X_i) \mathbf{1}_{X_i \notin B_{1/16}}$$

and hence

$$\sum_{i=0}^{\infty} \mathbf{E}_{x_0} q(x_i) \mathbf{1}_{X_i \notin B_{1/16}} \leq 2\tilde{\pi}(x_0) < \infty.$$

But

$$\sum_{i=0}^{\infty} \mathbf{E}_{x_0} q(x_i) \mathbf{1}_{X_i \notin B_{1/16}} = \sum_{x \in \mathcal{S} \setminus B_{1/16}} g(x_0, x) q(x) \,,$$

so this concludes the proof. □.

3 Chains with well behaved Greens function geometry

We have not yet imposed any geometry on $\mathcal{S}$. In order to formulate regularity conditions under which (1.6) is equivalent to trapping, the geometry inherent in the Greens function must be reasonable, and must be compatible with the geometry imposed by q. Assume throughout this section that the Greens function bound (1.3) holds.

Definition 2 *For $x_0 \in \mathcal{S}$, $L \leq K$ and $\alpha \in (0,1)$, define the Greens function annulus $H_\alpha(L, x_0)$ to be the set*

$$\{x \in \mathcal{S} : L \geq g(x_0, x) \geq \alpha L\} \,.$$

Say that the Markov chain has reasonable annuli *if for some $\alpha \in (0,1)$ and for every $L \leq K$, $x_0 \in \mathcal{S}$ and $A \subseteq \mathcal{S}$ transient with respect to x_0, the annulus $H_\alpha(L, x_0)$ has finite cardinality and*

$$\limsup_{L \to 0} \frac{|H_\alpha(L, x_0) \cap A|}{|H_\alpha(L, x_0)|} < 1. \tag{3.1}$$

Most nice chains satisfy this definition. For example, consider a simple random walk in $\mathbf{Z}^d$, $d \geq 3$. Annuli for this chain are spherical shells, and any set A taking up more than a fraction β of such a shell is hit with probability at least $\beta + o(1)$. Thus the probability of hitting A infinitely often is at least the limsup in (3.1), which is less than 1 for a transient set (actually 0, by tail triviality). For another example, take the Markov chain on an infinite rooted binary tree which always walks away from the root, choosing either of the two children with equal probability. This chain is far from irreducible, yet when α is large enough so that the annuli are nonempty, it clearly satisfies (3.1).

Theorem 3.1 *Suppose the Markov chain on $\mathcal{S}$ with transitions $p(x, y)$ has reasonable annuli, and suppose that, for some $C, C' > 1$, the function $q : \mathcal{S} \to [0, 1)$ satisfies the following regularity condition:*

$$\frac{1}{C} g(x_0, x) \leq g(x_0, y) \leq C g(x_0, x) \Rightarrow \frac{1}{C'} q(x) \leq q(y) \leq C' q(x). \tag{3.2}$$

Then $\tilde{\pi}(x_0) = 1$ if and only if $\sum_x g(x_0, x) q(x) = \infty$.

Proof: If $\tilde{\pi}(x_0) < 1$, then Theorem 1.2 gives a set A, transient for x_0, with

$$\sum_{x \in \mathcal{S} \setminus A} g(x_0, x) q(x) < \infty \, .$$

By the assumption of reasonable annuli, the transience of A and condition (3.2),

$$\limsup_{n \to \infty} \frac{\sum_{x \in H_\alpha(L\alpha^n, x_0) \cap A} g(x_0, x) q(x)}{\sum_{x \in H_\alpha(L\alpha^n, x_0) \setminus A} g(x_0, x) q(x)} < \infty \, ,$$

and hence $\sum_x g(x_0, x) q(x) < \infty$. □

Remarks:

4. In the case of a simple random walk on $\mathbf{Z}^d$, we find that condition (3.2) is equivalent to the existence of a $C < 1$ for which

$$\sup_x \sup_{y : C^{-1}|x| \leq |y| \leq C|x|} \frac{q(x)}{q(y)} < \infty.$$

This is a more natural formulation than the equivalent condition (4.1) in den Hollander, Menshikov and Volkov (1995), namely that

$$\sup_x \sup_{y:|y|\leq\sqrt{d}|x|} \frac{q(x)}{q(y)} < \infty.$$

Hence our Theorem 3.1 generalizes their integrability test at the end of their Section 4.1.

5. Condition (3.1) may appear cumbersome but it is more natural than it looks. First, it is a condition on the Markov chain alone, so that Theorem 3.1 identifies a class of chains such that the $\sum g(x_0, x)q(x) = \infty$ is a sharp criterion for all functions q that are "spherically symmetric up to constant factor" as defined by (3.2). Secondly, it is nearly sharp, meaning that if

$$\lim_{n\to\infty} \frac{|H_\alpha(L\alpha^n, x_0) \cap A|}{|H_\alpha(L\alpha^n, x_0)|} = 1,$$

then one can always choose a sequence $\{a_n\}$ such that setting $q(x) = a_n$ on $H_\alpha(L\alpha^n, x_0)$ gives a non-trapping field for which $\sum g(x_0, x)q(x) = \infty$.

References

[1] den Hollander, F., Menshikov, M. and Volkov, S. (1995). Two problems about random walks in a random field of traps. *Markov Proc. Relat. Fields* **1** 185 - 202.