Unsolved Problems Concerning Random Walks on Trees

BY RUSSELL LYONS, ROBIN PEMANTLE, AND YUVAL PERES

Abstract. We state some unsolved problems and describe relevant examples concerning random walks on trees. Most of the problems involve the behavior of random walks with drift: e.g., is the speed on Galton-Watson trees monotonic in the drift parameter? These random walks have been used in Monte-Carlo algorithms for sampling from the vertices of a tree; in general, their behavior reflects the size and regularity of the underlying tree. Random walks are related to conductance. The distribution function for the conductance of Galton-Watson trees satisfies an interesting functional equation; is this distribution function absolutely continuous?

§1. Introduction.

To explore the structure of irregular trees, we consider nearest-neighbor random walks on them. The behavior of simple random walk gives some information about the structure, but more can be gleaned by considering the one-parameter family of random walks RW_λ described below. That is, the behavior of such random walks on spherically symmetric trees is easy to analyze and quite regular. The results we describe and questions we pose concern the similarity of other trees, such as family trees of Galton-Watson processes, to spherically symmetric trees.

By a **tree**, we mean an undirected, connected, locally finite graph without cycles; one distinguished vertex is called the **root**. For any vertex v, the number of edges on the unique simple path between v and the root is called the **level** of v and denoted $|v|$. The vertices at level $|v|+1$ that are adjacent to v are called the **children** of v.

For $\lambda \geq 0$, the **λ-biased random walk** on a rooted tree T, denoted RW_λ, is the time-homogeneous Markov chain $\{X_n\}$ on the vertices of T such that if u is a vertex with k children $v_1, \ldots, v_k$ and a parent u_*, then $\mathbf{P}[X_{n+1} = v_i | X_n = u] = 1/(k+\lambda)$ for

1991 Mathematics Subject Classification. Primary 60J80, 60J15.
Key words and phrases. Galton-Watson, random walk, speed, rate of escape.
Partially supported by an Alfred P. Sloan Foundation Research Fellowship (Pemantle), by NSF Grants DMS-9306954 (Lyons), DMS-9300191 (Pemantle), and DMS-9404391 (Peres), and by a Presidential Faculty Fellowship (Pemantle). The authors are grateful to the IMA at the University of Minnesota for its hospitality.

$i = 1, \ldots, k$ and $\mathbf{P}[X_{n+1} = u_* | X_n = u] = \lambda/(k + \lambda)$; from the root all transitions to its children are equally likely, and we fix the initial state X_0 to be the root.

If T is a regular tree where every vertex has m children, then it is clear that RW_λ is transient for $\lambda < m$ and recurrent for $\lambda \geq m$; moreover, in the former case, the law of large numbers implies that the walk escapes from the root at a positive "speed" (formally defined in the next section). Generalizing this, Lyons (1990) showed that the critical parameter for transience of RW_λ on a general tree is exactly the exponential of the Hausdorff dimension of the tree boundary (defined in the next paragraph). However, determining on which trees RW_λ has positive speed is more subtle and is one of the subjects of this article.

An infinite self-avoiding path from the root of a tree T is called a **ray**. The space of rays is called the **boundary** of T and denoted ∂T. This space has a natural metric, where the distance between two rays is e^{-n} if they have exactly n edges in common. (As usual, this metric then yields a notion of Hausdorff dimension $\dim_H$ for sets and measures on ∂T.) It is convenient to use this metric also between rays and vertices by identifying each vertex v with the self-avoiding path from the root to v. In the resulting metric, any transient nearest-neighbor Markov chain on the vertices of a tree T must converge to a ray of T. The hitting measure on the boundary is then called **harmonic measure** for the chain.

Galton-Watson trees (family trees of Galton-Watson processes with mean number of offspring $m > 1$) are very close to regular trees in many respects (see, e.g., Pemantle and Peres (1995)), yet exhibit persistent random irregularities which are detected even by the simple random walk (see below). For these trees, the critical parameter is the mean m and the critical process RW_m is recurrent (Lyons (1992), Theorem 4.2). For simplicity, we shall consider below only the case in which the probability of no children, p_0, is zero.

The set of self-avoiding walks on a lattice has a natural tree structure; Berretti and Sokal (1985) suggested that biased random walks on this tree can be used to obtain almost uniform samples from the set of self-avoiding walks of a given length. Refinements of this idea are in Lawler and Sokal (1988), Sinclair and Jerrum (1989) and Randall (1994). These papers are primarily concerned with recurrent walks, while we will discuss the transient case; the behavior of RW_λ near the critical parameter is of special interest from both perspectives. The value of the critical parameter, which is the growth rate of the number of self-avoiding walks, is not explicitly known.

Conductance of trees from their roots to infinity are intimately related to properties of random walks. In particular, in Section 4, we discuss the functional equation for the conductance of Galton-Watson trees.

§2. The Speed of Biased Random Walks.

Given any random process X_n with values in the set of vertices of a tree, its **speed** is defined to be

$$\liminf_{n\to\infty} \frac{|X_n|}{n}$$

We denote the speed of RW_λ on T by $\mathsf{speed}(\lambda, T)$. In general, this is a random variable, but in all the explicit examples discussed below, the limit exists and is almost surely constant. If the tree is evident from the context, we omit it from the notation.

Let T be the family tree of a Galton-Watson process with offspring distribution $\{p_k\}_{k\geq 0}$ and mean $m = \sum kp_k > 1$. Assume that $p_0 = 0$. In Lyons, Pemantle and Peres (1994b), it is shown that $\mathsf{speed}(\lambda, T)$ is a.s. a constant which depends only on λ and the offspring distribution, and this constant is positive for all $0 \leq \lambda < m$.

Question 2.1 *Is the speed of* RW_λ *on Galton-Watson trees monotonic nonincreasing in the parameter* λ*?*

Of course, this is true in the deterministic case where $p_k = 1$ for some k. In general, however, no proof, nor indeed much evidence, is known. At first sight, it seems that $\mathsf{speed}(\lambda, T)$ should be nonincreasing in λ for *any* tree T, but this is wrong, as the following two examples attest. In fact, these examples show that the speed is not monotonic on multitype Galton-Watson trees.

Example 2.1. Binary tree with pipes: Let T be a binary tree to every vertex of which is joined a unary tree, which we refer to as a pipe; see Figure 2.1. This is also a deterministic 2-type Galton-Watson tree, in which a particle of type 1 has one child of type 1 and a particle of type 2 has a child of type 1 and two children of type 2. Simple random walk on T spends an infinite expected time in each pipe which it enters, whence its speed is zero. Yet simple random walk on T is transient: this can be seen either by restricting one's attention to the times when the walk is not on a pipe; or by using the fact that the walk is reversible and hence, by Rayleigh's principle (see, e.g., Doyle and Snell (1984)), transience on the subgraph of the binary tree implies transience on the whole graph T. On the other hand, for $\lambda > 1$, the expected time RW_λ spends on each pipe it visits is finite, whence its speed is positive provided RW_λ has positive speed on a binary tree, i.e., $\lambda < 2$. Indeed, it is easy to calculate that for $1 \leq \lambda \leq 2$, the speed is

$$\frac{(2-\lambda)(\lambda-1)}{\lambda^2+3\lambda-3}\,, \tag{2.1}$$

which is unimodal in λ, maximized at $\lambda = (5+\sqrt{7})/6$, where the speed is $(4-\sqrt{7})/(14+\sqrt{7})$.

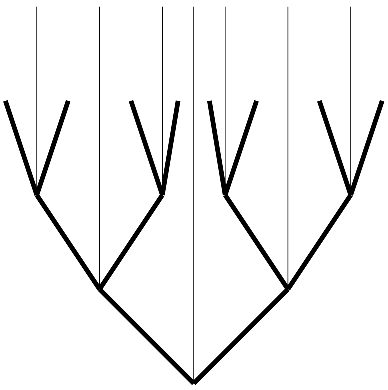

Figure 2.1.

EXAMPLE 2.2. A COVERING TREE: Let T be the tree described in Figure 2.2. Then T is the universal covering tree of the graph G shown in Figure 2.3. The tree T may be viewed as a deterministic, irreducible, multi-type Galton-Watson tree with 33 types. We claim that the speed of simple random walk on T is less than the speed of $\mathsf{RW}_{4/3}$. This is seen as follows.

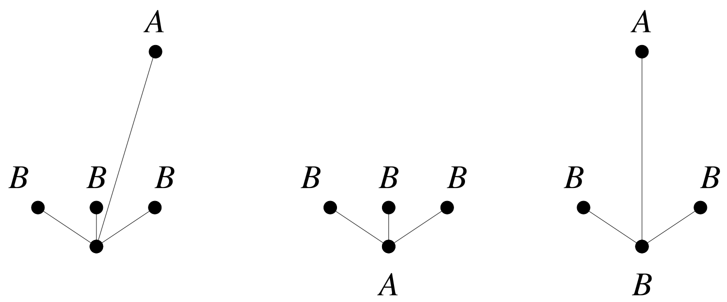

Figure 2.2. Three trees are shown, one with root of type A and one with root of type B. In all cases, the long edges are of length 31. The full tree T is obtained by adding a copy of the tree of type A to the leaf of type A in the first tree and a copy of the tree of type B to each of the leaves of type B of the first tree, and repeating *ad infinitum*.

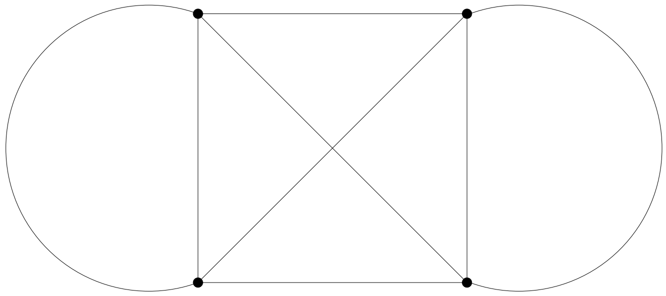

Figure 2.3. The curved edges each have 30 vertices of degree 2 that are not drawn.

Let V be the number of vertices in G and E the number of edges. The path of simple random walk on T projects to the path of simple random walk on G. When the walk is at a vertex of degree d in T, its distance from the root has expected increment $(d-2)/d$. The stationary distribution for simple random walk on G is $\{\deg(v)/(2E)\,;\ v \in G\}$, whence the speed of simple random walk on T is

$$\sum_{v\in G} \frac{\deg(v)}{2E}\frac{\deg(v)-2}{\deg(v)} = \frac{E-V}{E}\,. \tag{2.2}$$

In the present case, this turns out to be $1/17$.

By (2.1), the speed of $\mathsf{RW}_{4/3}$ on the binary tree with pipes is $2/25$, which is larger than $1/17$. It is easy to see that the speed of $\mathsf{RW}_{4/3}$ on T is even greater than the speed of $\mathsf{RW}_{4/3}$ on the binary tree with pipes.

EXAMPLE 2.3. THE FIBONACCI TREE: We present a calculation of speed on the Fibonacci tree, a special case of a deterministic 2-type Galton-Watson tree, but more complicated than the binary tree with pipes. Namely, a type 1 particle has one child of type 2; while a type 2 particle has two children, one of type 1 and one of type 2. For $0 \le \lambda < (\sqrt{5}+1)/2$, let $C_i(\lambda)$ denote the probability that a walk started at a particle of type i will never visit its parent. Then

$$C_1(\lambda) = \left\{1 + \frac{\lambda}{C_2(\lambda)}\right\}^{-1} \qquad \text{and} \qquad C_2(\lambda) = \left\{1 + \frac{\lambda}{C_1(\lambda)+C_2(\lambda)}\right\}^{-1}, \tag{2.3}$$

whence

$$C_1(\lambda) = \frac{\sqrt{\lambda+1}-\lambda}{\sqrt{\lambda+1}} \qquad \text{and} \qquad C_2(\lambda) = \sqrt{\lambda+1}-\lambda\,.$$

Let $p_i(\lambda)$ be the probability that a walk started at a vertex of type i never visits its parent nor returns to its starting point. Then

$$p_1(\lambda) = \frac{1}{\lambda+1}C_2(\lambda) = \frac{\sqrt{\lambda+1}-\lambda}{\lambda+1}\,,$$

$$p_2(\lambda) = \frac{1}{\lambda+2}(C_1(\lambda)+C_2(\lambda)) = \frac{1+(1-\lambda)\sqrt{\lambda+1}}{\sqrt{\lambda+1}(2+\lambda)}\,.$$

Let $\pi_i(\lambda)$ be the limiting frequency of visits to vertices of type i. These exist by a standard "regeneration" argument: Each time the walking particle reaches a type 1 vertex v it has never visited before, it has a fixed chance $p_1(\lambda)$ of continuing to the child of v and never returning to v. The portions of the random walk between these occurrences are i.i.d.; their length has finite mean, so the strong law of large numbers yields an almost surely constant asymptotic frequency of visits to each type of vertex. We have the following equations:

$$\pi_1(\lambda) + \pi_2(\lambda) = 1\,,$$

$$\mathsf{speed}(\lambda) = \pi_1(\lambda)p_1(\lambda) + \pi_2(\lambda)p_2(\lambda)\,,$$

$$\mathsf{speed}(\lambda) = \pi_1(\lambda)\frac{1-\lambda}{1+\lambda} + \pi_2(\lambda)\frac{2-\lambda}{2+\lambda}\,.$$

The last equation comes from the fact that the distance from the root has expected increment $(i-\lambda)/(i+\lambda)$ when the walk is at a vertex of type i, via the strong law for martingale differences (see Feller (1970), Section VII.9).

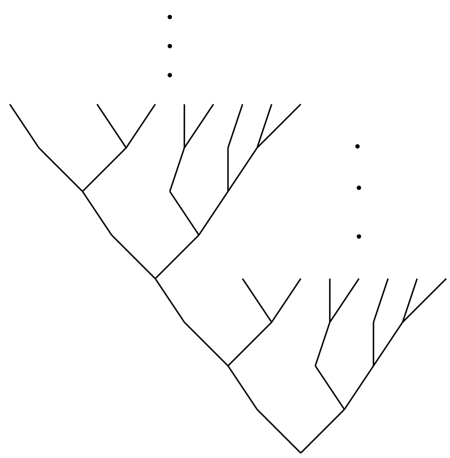

Figure 2.4. Part of the Fibonacci tree.

Solving these equations, we obtain

$$\pi_1(\lambda) = \frac{\sqrt{\lambda+1}}{2+\lambda+\sqrt{\lambda+1}}\,,$$

$$\pi_2(\lambda) = \frac{\lambda+2}{2+\lambda+\sqrt{\lambda+1}}\,,$$

$$\mathsf{speed}(\lambda) = \frac{(\sqrt{\lambda+1}+2)(\sqrt{\lambda+1}-\lambda)}{\sqrt{\lambda+1}(2+\lambda+\sqrt{\lambda+1})}\,.$$

It is easy to verify from this formula that $\mathsf{speed}(\lambda)$ *is* strictly monotonic for $0 \le \lambda < (\sqrt{5}+1)/2$. Note that since $\mathsf{speed}(1) = 1/(3+\sqrt{2})$ is irrational, T is *not* the covering tree of any finite undirected graph (see (2.2)), even if the tree were infinitely extended in the "negative" direction so that the root had degree 3. Also, harmonic measure for RW_λ is a

Markov measure with transitions governed by the $C_i(\lambda)$, whence different values of λ give mutually singular harmonic measures.

There is a natural one-to-one correspondence between unit flows on a tree T and Borel probability measures on ∂T, where the flow into a vertex v equals the measure of the set of rays through v; additivity of the measure is equivalent to the Kirchhoff equations holding for the flow. We use this correspondence without further comment below.

EXAMPLE 2.4. THE REPEATED FILTERING METHOD: We now describe a general method that, given any pair $0 < \lambda_1 < \lambda_2$ (even when both are less than 1), produces a tree for which the speed of RW_{λ_1} is less than the speed of RW_{λ_2}.

Let T_f be a tree such that RW_{λ_2} is transient and the harmonic measures $\theta(\lambda)$ of RW_λ are singular for λ_1 and λ_2. Such a tree was given in Example 2.3 for $\lambda_1 < \lambda_2 < (1+\sqrt{5})/2$; a simple modification of T_f works for larger values of λ_2. Also, almost every tree produced by a Galton-Watson process with mean larger than λ_2 has this property, as shown in Lyons, Pemantle and Peres (1996). Thus, given $\epsilon > 0$, we may choose N sufficiently large so that there are complementary subsets A_1, A_2 of level N of T_f with $\theta(\lambda_i)(A_i) > 1-\epsilon$. There is also an M sufficiently large so that if B_i denotes the vertices at level M that are descendants of those in A_i, then the chance that the first visit of RW_{λ_i} to level M is in B_i is at least $1-\epsilon$. Choose $K_1 > 2\lambda_2$ and $K_2 > K_1\lambda_2/\lambda_1$. (The first inequality ensures that the expected number of returns to the root of RW_{λ_2} on the K_1-ary tree is at most 2, while the second inequality makes the speed of RW_{λ_2} on the K_2-ary tree greater than the speed of RW_{λ_1} on the K_1-ary tree.) Now build a tree T' by taking the first M levels of T_f and adding $\lceil \lambda_2/\epsilon \rceil$ copies of the regular K_i-ary tree to each vertex in B_i. Then the harmonic measure of A_i for RW_{λ_i} on T' is more than $1-3\epsilon$ provided M is large enough. Truncate T' after n levels, add to each leaf a copy of the first n levels of T', and so on *ad infinitum*, thus constructing a tree T. If n is sufficiently large, then T clearly has the following property: For $i = 1, 2$, on almost all trajectories of RW_{λ_i}, at least proportion $1-4\epsilon$ of the vertices have K_i children. Thus the speed of RW_{λ_2} on T is greater than the speed of RW_{λ_1} on T if ϵ is small enough.

The next two questions involve smoothness of and estimates for $\mathsf{speed}(\lambda, T)$ on Galton-Watson trees.

Question 2.2: *Is* $\mathsf{speed}(\lambda, T)$ *a real-analytic function of* $\lambda \in (0, m)$ *for Galton-Watson trees* T*?*

In certain examples of random walks on matrix groups (Ruelle 1979, Peres 1992), the speed, measured as a top Lyapunov exponent, is known to depend analytically on parameters even though it cannot be calculated explicitly; the techniques of those papers (the implicit function theorem and polynomial approximation) may be applicable to Question 2.2.

Question 2.3: *Let* $0 < \lambda < m$. *Is* $\mathsf{speed}(\lambda, T) \le (m-\lambda)/(m+\lambda)$ *a.s. for Galton-Watson trees of mean offspring* $m > 1$*?*

The upper bound here is the speed on an m-ary tree. This inequality is true for $\lambda = 1$ by the explicit formula in Lyons, Pemantle, and Peres (1995). A more ambitious question in this direction, which is unresolved even for $\lambda = 1$, is in Question 3.5. Examples due to R. Kenyon (personal communication to Y. Peres, Nov. 1994) show that* the following strengthening of the inequality in Question 2.3 is not valid for general trees, even for those produced by multitype Galton-Watson processes: $\mathsf{speed}(\lambda, T) \le \mathsf{speed}(\lambda, T^*)$ if T^* is a spherically symmetric tree with the same level sizes as T.

Most results on Galton-Watson processes have natural extensions to the multitype case. We have not found such an extension for the speed formula of simple random walk.

Question 2.4: *Is there an explicit formula for the speed of simple random walk on a (supercritical, irreducible) multitype Galton-Watson tree?*

§3. Dimension of Harmonic Measures and Approximately Uniform Sampling.

Fix a nondegenerate offspring distribution with a finite mean $m = \mathbf{E}[Z] > 1$ and $p_0 = 0$. A natural measure on the boundary of a Galton-Watson tree T is the weak limit as $n \to \infty$ of the uniform measure on the nth level of T. (In general, the a.s. existence of this weak limit is a consequence of the Seneta-Heyde theorem.) Call this measure "limit uniform measure" and denote it by UNIF_T.

Under the assumption that $\mathbf{E}[Z \log^2 Z]$ is finite, it was shown by Hawkes (1981) that the boundary of the Galton-Watson tree has Hausdorff dimension $\log m$ a.s. and, moreover, the limit uniform flow UNIF_T has dimension $\log m$ almost surely. The moment assumption on the offspring distribution was weakened to $\mathbf{E}[Z \log Z] < \infty$ in Lyons, Pemantle and Peres (1995).

Question 3.1: *What is the dimension of* UNIF_T *when* $\mathbf{E}[Z \log Z] = \infty$*?*

More generally, consider a measurable function M on trees that assigns to each infinite tree T a measure $M(T)$ on its boundary. The descendant tree of a child of the root can be

* In the published version of this article, we mistakenly said that the examples show that the inequality in Question 2.3 is not valid for general trees. In fact, B. Virág has a 1998 preprint establishing this inequality, as well as that in Question 3.5, for general trees.

naturally identified with a subset of ∂T. Call M a **consistent flow rule** if, given such a descendant tree T' of positive $M(T)$-measure, the measure obtained by conditioning $M(T)$ to T' is precisely $M(T')$.

Examples of consistent flow rules are the **equally splitting** flow (the flow entering a vertex v is split equally among its children), the limit uniform flow, and harmonic measure for RW_λ (for any $0 < \lambda < m$). (In fact, the harmonic measure for RW_0 is the equally splitting flow.)

Question 3.2: *Is it true, as we conjecture, that for every consistent flow rule* $M \neq$ UNIF *a.s., the Hausdorff dimension satisfies* $\dim_H(M(T)) < \log m$ *a.s.?*

This conjecture was also made by V. Kaimanovich (personal communication, Cornell, April 1993). Under a further hypothesis, a version of this is proved in Lyons, Pemantle and Peres (1995). In particular, it is shown there that the dimension of harmonic measure for simple random walk on the boundary of T is a.s. less than $\log m$; this is extended to RW_λ in Lyons, Pemantle and Peres (1996).

Denote the **Hausdorff dimension of harmonic measure** for RW_λ on the tree T by $\dim(\lambda, T)$. Given an offspring distribution for Galton-Watson trees, this is a.s. constant in T.

Question 3.3: *Is* $\dim(1, T)$ *for a Galton-Watson tree* T *a.s. greater than* $\mathbf{E}[\log Z]$*?*

F. Ledrappier (personal communication, Cornell, April 1993) asked us this question since the dimension of the equally splitting flow on T equals $\mathbf{E}[\log Z]$. A more general question is:

Question 3.4: *For* $0 \leq \lambda < m$, *is* $\dim(\lambda, T)$ *for a Galton-Watson tree* T *monotonic nondecreasing in the parameter* λ*? Is it strictly increasing?*

Let T be any tree. By the result of Lyons (1990) quoted earlier, RW_λ is recurrent on the subtree of T corresponding to any closed subset of ∂T of dimension less than $\log \lambda$. That is, $\dim(\lambda, T) \geq \log^+ \lambda$ on any tree T for all $\lambda < \dim_H \partial T$. From Lyons (1994), Corollary 4.3, it follows that this inequality is strict when T is the covering tree of a (directed or undirected) graph. However, the repeated filtering method, described in the previous section, shows that there are trees T for which $\dim(\lambda, T)$ is *not* monotonic in λ.

It was shown in Lyons, Pemantle and Peres (1995) that for almost all nondegenerate Galton-Watson trees T, there exists a subtree T' such that $\dim_H(\partial T') = \dim(1, T)$ and such that the trajectory of simple random walk on T is confined to T' with overwhelming probability. The proof is quite robust and extends to RW_λ and many other trees (see Lyons, Pemantle and Peres (1996)). This motivates a sharpening of Question 2.3:

Question 3.5: *For $0 < \lambda < m$, is*

$$\mathsf{speed}(\lambda, T) \le \frac{\exp(\dim(\lambda, T)) - \lambda}{\exp(\dim(\lambda, T)) + \lambda} \tag{3.1}$$

a.s. on Galton-Watson trees, T?

This is unresolved even for simple random walk.* However, we can show that $\dim(\lambda, T)$ is monotonic and (3.1) holds for many trees. This is illustrated by the Fibonacci tree:

Example 2.3 on the Fibonacci tree continued: In this case, the dimension of harmonic measure is simply the entropy of the two-state Markov chain in which state 1 leads deterministically to state 2 and the transition probabilities from state 2 to state $i \in \{1, 2\}$ are proportional to $C_i(\lambda)$ given in (2.3). (This follows from the well-known relation between Hölder exponents and Hausdorff dimension and the Shannon-McMillan-Breiman Theorem; see, e.g., Billingsley (1965).) Thus, we find that

$$\dim(\lambda, T) = \frac{1 + \sqrt{\lambda + 1}}{2 + \sqrt{\lambda + 1}} \log(1 + \sqrt{\lambda + 1}) - \frac{\sqrt{\lambda + 1}}{(2 + \sqrt{\lambda + 1})} \log \sqrt{\lambda + 1}\,.$$

This is easily shown to be monotonic for $0 \le \lambda < (\sqrt{5} + 1)/2$ and quite close to constant, going from $2(\log 2)/3 \approx 0.46$ up to $\log\bigl((\sqrt{5} + 1)/2\bigr) \approx 0.48$. Figure 3.1 shows that the inequality (3.1) holds for the Fibonacci tree.

The next question is the only vague one in this note; it is included because it is of wide interest and some of the approaches proposed so far involve RW_λ. See Sinclair and Jerrum (1989) for motivation and background. Note that the gap $\dim \partial T - \dim(\lambda, T)$ measures how far from uniform is the last visit to the nth level of T as $n \to \infty$.

Question 3.6: *What is the most efficient procedure to sample (approximately) uniformly from the leaves of a (non-regular) finite tree?*

* As noted in a previous footnote, B. Virág has a 1998 preprint establishing this inequality for general trees.

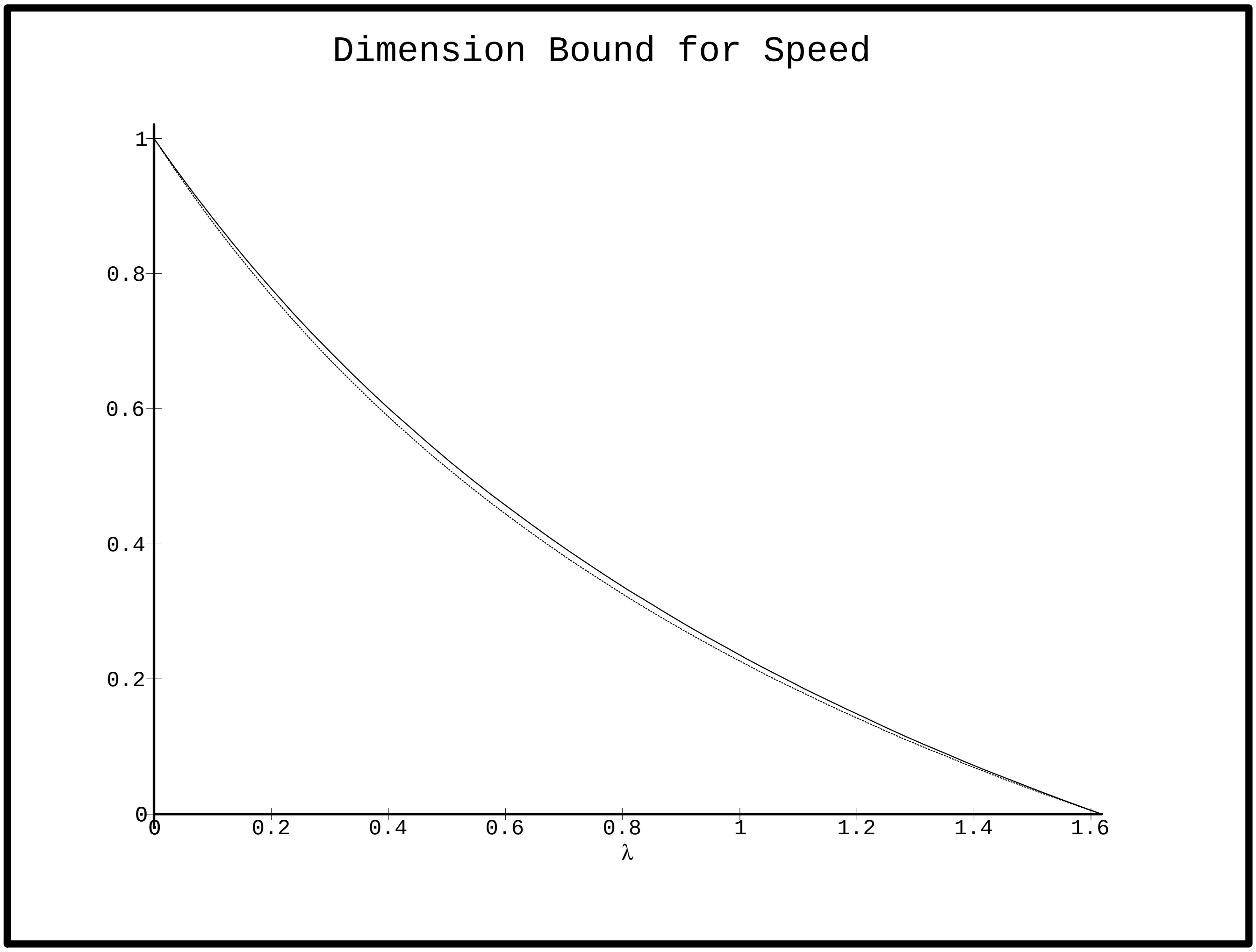

Figure 3.1. The dimension bound on the speed of the Fibonacci tree, Equation (3.1).

§4. A Functional Equation for the Conductance of a Galton-Watson Tree.

Given a tree T, form a new tree T_Δ by joining the root of T to a new vertex, Δ. The probability, $\gamma(T)$, that simple random walk on T_Δ started at the root of T will never visit Δ is easily expressed via $\mathcal{C}(T)$, the effective conductance of T (from its root to infinity) when the edges of T have unit conductance:

$$\gamma(T) = \frac{\mathcal{C}(T)}{1 + \mathcal{C}(T)} \,.$$

The conductance is also a key tool in analyzing other properties of simple random walk on T, such as its Hausdorff dimension (see Lyons, Pemantle and Peres (1995)). Now

$$\gamma(T) = \frac{\sum_{|x|=1} \gamma(T(x))}{1 + \sum_{|x|=1} \gamma(T(x))} \,. \tag{4.1}$$

Thus, if $\{p_k\}$ is the offspring distribution of a Galton-Watson process, then the recursive structure of Galton-Watson trees gives that the c.d.f. F_γ of γ satisfies

$$F(s) = \begin{cases} \sum_k p_k F^{*k}\left(\frac{s}{1-s}\right), & \text{if } s \in (0,1); \\ 0, & \text{if } s \le 0; \\ 1, & \text{if } s \ge 1\ . \end{cases} \tag{4.2}$$

Of course, the c.d.f. of $\mathcal{C}(T)$ is $F_\gamma(x/(1+x))$ for $x \ge 0$. We shall assume that $p_0 = 0$.

THEOREM 4.1. *The functional equation (4.2) has exactly two solutions, F_γ and the Heaviside function $\mathbf{1}_{[0,\infty)}$. Define the operator on c.d.f.'s*

$$\mathcal{K} : F \mapsto \sum_k p_k F^{*k}\left(\frac{s}{1-s}\right) \qquad (s \in (0,1))\,.$$

For any initial c.d.f. F with $F(0) = 0$ and $F(1) = 1$ other than the Heaviside function, we have weak convergence under iteration to F_γ:

$$\lim_{n\to\infty} \mathcal{K}^n(F) = F_\gamma\,.$$

In order to prove this, we require the following lemma.

LEMMA 4.2. *Let T be a tree without leaves on which simple random walk is transient and let $T^{(k)}$ be the finite subtree of the first k generations. From $T^{(k)}$, form the finite tree $T[k,R]$ by adding to each leaf an edge of resistance R. Then*

$$\mathcal{R}(T[k,R]) \le \mathcal{R}(T) + o(R/k)$$

uniformly in $R \ge 0$ as $k \to \infty$, where $\mathcal{R}$ denotes effective resistance from the root to the boundary of a tree and the edges of T have unit resistance.

Proof. Let I be the unit current flow on T. (This is the flow corresponding to the harmonic measure for simple random walk (Lyons 1990, §4).) Then $\mathcal{R}(T)$ equals the energy of I:

$$\mathcal{R}(T) = \sum_{|x|\ge 1} I(x)^2 < \infty\,. \tag{4.3}$$

Since

$$I(x)^2 = \left(\sum I(y)\right)^2 \ge \sum I(y)^2\,,$$

where the summations extend over the children y of x, it follows that $\sum_{|x|=k} I(x)^2$ decreases in k, whence from (4.3), that

$$\sum_{|x|=k} I(x)^2 = o\left(\frac{1}{k}\right)\,.$$

Since $\mathcal{R}(T[k,R])$ is the minimum energy of unit flows on $T[k,R]$, we have

$$\mathcal{R}(T[k,R]) \le \sum_{0<|x|\le k} I(x)^2 + \sum_{|x|=k} I(x)^2 R \le \mathcal{R}(T) + o\left(\frac{R}{k}\right). \blacksquare$$

Proof of Theorem 4.1. Let $\mathbf{GW}$ denote the law of the family trees of our Galton-Watson process. The fact that there are only the two solutions mentioned of the functional equation (4.2) is a consequence of the convergence of $\mathcal{K}^n$. To establish this convergence, let $R_F(x)$ be a collection of i.i.d. random variables indexed by the vertices $x \in T$ and independent of $\mathbf{GW}$ so that $1/(R_F(x)+1)$ has c.d.f. F. Write ν_T^F for the law of $\langle R_F(x)\rangle_{x\in T}$. For a tree T, let $T[n,R_F]$ be as described in Lemma 4.2. Thus, $\mathcal{K}^n(F)$ is the $\nu_T^F \times \mathbf{GW}$-distribution function of the conductance of the tree $T_\Delta[n,R_F]$, i.e., the c.d.f. of $\gamma(T[n,R_F])$. In particular (and as can be seen directly), if $F_1 \le F_2$, then $\mathcal{K}^n(F_1) \le \mathcal{K}^n(F_2)$. We shall compare F to two other distributions: to $F_\infty := \mathbf{1}_{[1,\infty)}$, so that $\mathcal{K}^n F_\infty$ is the distribution of $\gamma(T[n,0])$; and to $F_\epsilon := (1-\epsilon)\mathbf{1}_{[0,\infty)} + \epsilon \mathbf{1}_{[\epsilon/(1+\epsilon),\infty)}$. Thus, $F \ge F_\infty$ and, provided $F \ne \mathbf{1}_{[0,\infty)}$, for all sufficiently small $\epsilon > 0$, we have $F \le F_\epsilon$. By definition, $\gamma(T) = \lim_{n\to\infty} \gamma(T[n,0])$, whence $F_\gamma = \lim_{n\to\infty} \mathcal{K}^n(F_\infty)$ weakly. Therefore, all weak limit points of $\{\mathcal{K}^n F\}$ are bounded below by F_γ.

On the other hand, let T^n be the nth level of T and let $\overline{T}^{(n)}$ be the finite subtree of T consisting of those vertices one of whose descendants $x \in T^n$ has $R_\epsilon(x) < \infty$, where we write $R_\epsilon := R_{F_\epsilon}$ for simplicity; by definition, if $R_\epsilon(x) < \infty$, then $R_\epsilon(x) = 1/\epsilon$. Also write $\nu_T^\epsilon := \nu_T^{F_\epsilon}$. We may assume that ϵ is so small that $p_1 + \epsilon < 1$. Let $\delta > 0$ be so small that $m^\delta (p_1+\epsilon)^{(1-\delta)} < 1$. Let n_0 be sufficiently large that for $n \ge n_0$, we have

$$\mathcal{R}(T[k_n, j_n + 1/\epsilon]) \le \mathcal{R}(T) + \epsilon\delta^2 \frac{j_n + 1/\epsilon}{k_n}, \tag{4.4}$$

where $k_n := \lceil \delta n \rceil$ and $j_n := n - k_n$; such an n_0 exists by Lemma 4.2. We have

$$\begin{aligned}\nu_T^\epsilon\left(T^{(k_n)} \not\subseteq \overline{T}^{(n)}\right) &\le \sum_{|x|=k_n} \nu_T^\epsilon(x \notin \overline{T}^{(n)}) \\ &= \sum_{|x|=k_n} (1-\epsilon)^{|T(x)\cap T^n|}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{E}\Big[\nu_T^\epsilon\left(T^{(k_n)} \not\subseteq \overline{T}^{(n)}\right)\Big] &\le \mathbf{E}\left[\mathbf{E}\Big[\sum_{|x|=k_n} (1-\epsilon)^{|T(x)\cap T^n|} \;\Big|\; T^{k_n}\Big]\right] \\ &= \mathbf{E}\left[|T^{k_n}| \cdot f_{j_n}(1-\epsilon)\right] = m^{k_n} f_{j_n}(1-\epsilon).\end{aligned}$$

Here, f_j denotes the j-fold composition of f with itself. Since

$$f_j(1-\epsilon) = o((p_1+\epsilon)^j)$$

as $j \to \infty$, we have that for all large n,

$$\mathbf{E}\left[\nu_T^\epsilon\left(T^{(k_n)} \not\subseteq \overline{T}^{(n)}\right)\right] < \delta$$

by choice of δ. Now when n is at least n_0 and $T^{(k_n)} \subseteq \overline{T}^{(n)}$, we have

$$\begin{aligned}\gamma(T[n,R_\epsilon])^{-1} = 1+\mathcal{R}(T[n,R_\epsilon]) = 1+\mathcal{R}(\overline{T}^{(n)}[n,1/\epsilon]) \le 1+\mathcal{R}(T[k_n, j_n+1/\epsilon])\\ \le 1+\mathcal{R}(T)+\epsilon\delta^2\frac{j_n+1/\epsilon}{k_n} \le \gamma(T)^{-1}+\left(\epsilon+\frac{1}{n}\right)\delta\end{aligned}$$

by (4.4). Hence for all large n,

$$\mathcal{K}^n F_\epsilon(s) \le F_\gamma(s+\delta)+\delta\,.$$

Since δ can be chosen arbitrarily small, this shows that all weak limit points of $\mathcal{K}^n F_\epsilon$, hence of $\mathcal{K}^n F$, are bounded above by F_γ. ∎

Question 4.1: *Is F_γ absolutely continuous? In other words, does the effective conductance of a Galton-Watson tree have an absolutely continuous distribution?*

We believe so: Theorem 4.1 provides a method of calculating F_γ. Using it, we have calculated some examples numerically. From the graphs (Figures 4.1 and 4.2), it appears that the derivative of the distribution function is well behaved.

Note how these figures reflect the stochastic self-similarity of the Galton-Watson trees. Consider, for example, Figure 4.1. Roughly speaking, the peaks represent the number of generations with no branching. For example, note that the full binary tree has conductance 1, whence its γ value is 1/2. Thus, the tree with one child of the root followed by the full binary tree has conductance 0.5 and γ value 1/3. The wide peak at the right of Figure 4.1 is thus due entirely to those trees which begin with two children of the root; the nth peak to the left of it is due to n generations without branching.

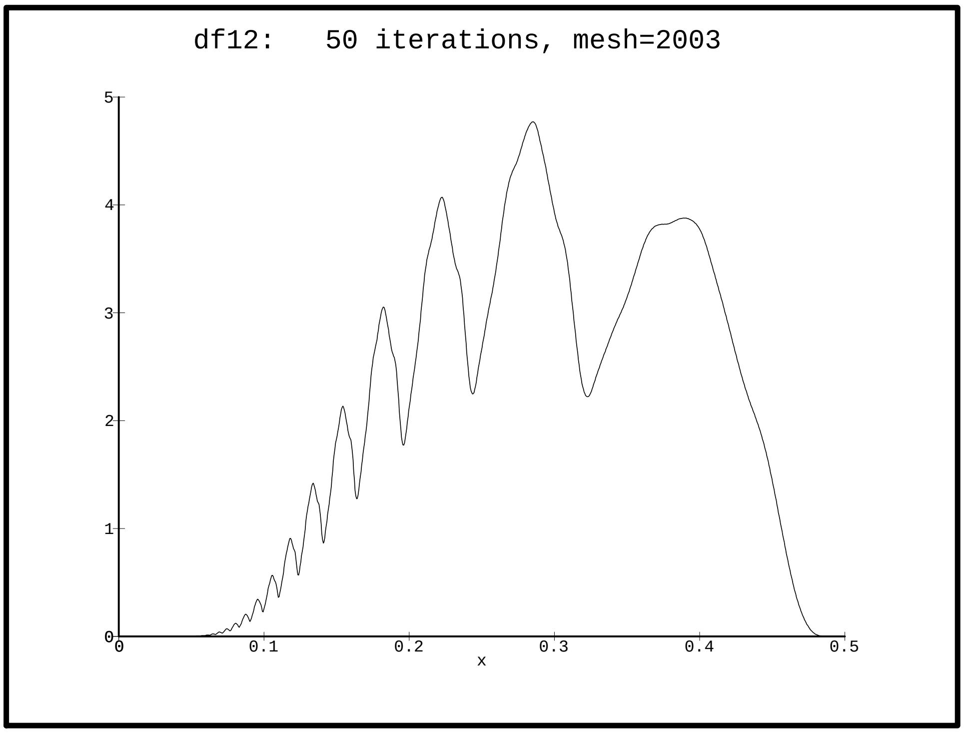

Figure 4.1. The apparent **GW**-density of $\gamma(T)$ for $f(s) = (s + s^2)/2$.

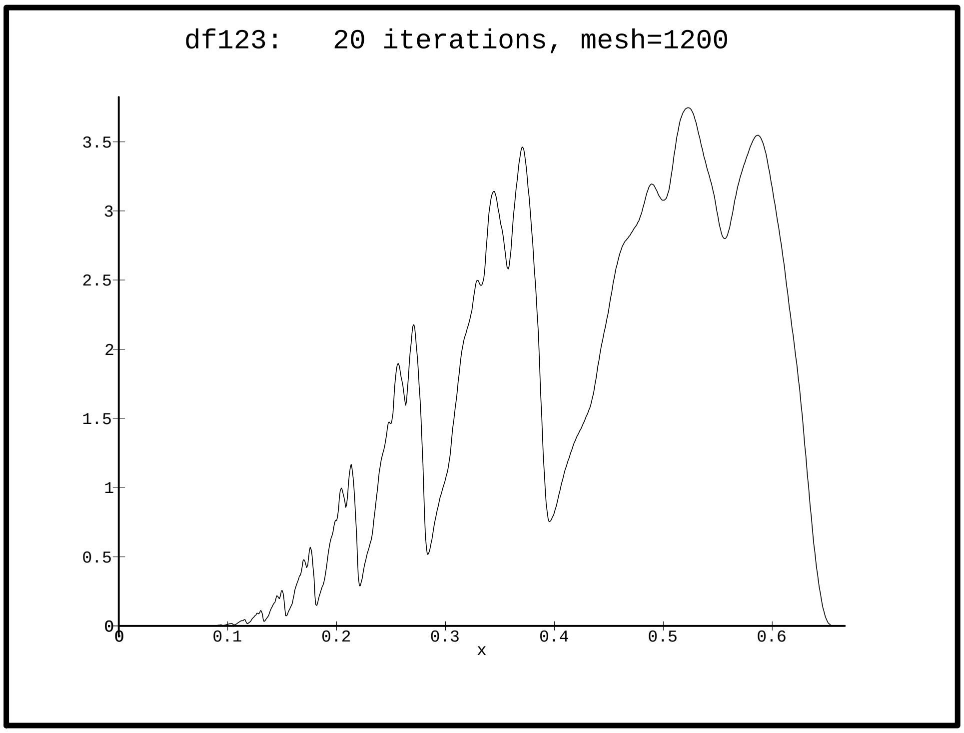

Figure 4.2. The apparent **GW**-density of $\gamma(T)$ for $f(s) = (s + s^2 + s^3)/3$.

REFERENCES

BERRETTI, A. and SOKAL, A. D. (1985). New Monte Carlo method for the self-avoiding walk. *J. Stat. Physics* **40**, 483–531.

BILLINGSLEY, P. (1965). *Ergodic Theory and Information.* Wiley, New York.

FELLER, W. (1970). *An Introduction to Probability Theory and Its Applications.* Volume II, 2nd ed. Wiley, New York.

HAWKES, J. (1981). Trees generated by a simple branching process, *J. London Math. Soc.* **24**, 373–384.

KESTEN, H. (1986). Subdiffusive behavior of random walk on a random cluster, *Ann. Inst. Henri Poincaré Probab. Statist.* **22**, 425–487.

LAWLER, G. F. and SOKAL, A. D. (1988). Bounds on the L^2 spectrum for Markov chains and Markov processes: A generalization of Cheeger's inequality. *Trans. Amer. Math. Soc.* **309**, 557–589.

LYONS, R. (1990). Random walks and percolation on trees, *Ann. Probab.* **18**, 931–958.

LYONS, R. (1992). Random walks, capacity, and percolation on trees, *Ann. Probab.* **20**, 2043–2088.

LYONS, R. (1994). Equivalence of boundary measures on covering trees of finite graphs, *Ergodic Theory Dynam. Systems* **14**, 575–597.

LYONS, R., PEMANTLE, R. and PERES, Y. (1995). Ergodic theory on Galton-Watson trees: speed of random walk and dimension of harmonic measure. *Ergodic Theory Dynamical Systems* **15**, 593–619.

LYONS, R., PEMANTLE, R. and PERES, Y. (1996). Biased random walks on Galton-Watson trees, *Probab. Theory Relat. Fields* **106**, 249–264.

PEMANTLE, R. and PERES, Y. (1995). Galton-Watson trees with the same mean have the same polar sets, *Ann. Probab.* **23**, 1102–1124.

PERES, Y. (1992). Domains of analytic continuation for the top Lyapunov exponent, *Ann. Inst. Henri Poincaré Probab. Statist.* **28**, 131–148.

RANDALL, D. (1994). *Counting in Lattices: Combinatorial Problems from Statistical Mechanics.* Ph. D. thesis, University of California, Berkeley.

RUELLE, D. (1979). Analyticity properties of the characteristic exponents of random matrix products, *Adv. Math.* **32**, 68–80.

SINCLAIR, A. J. and JERRUM, M. R. (1989). Approximate counting, uniform generation and rapidly mixing Markov chains, *Information and Computation* **82**, 93–133.

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, IN 47405-5701
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WI 53706
DEPARTMENT OF STATISTICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720-3860