A Phase Transition in Random Coin Tossing

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Suppose that a coin with bias θ is tossed at renewal times of a renewal process, and a fair coin is tossed at all other times. Let μ_θ be the distribution of the observed sequence of coin tosses, and let u_n denote the chance of a renewal at time n. Harris and Keane in [10] showed that if $\sum_{n=1}^{\infty} u_n^2 = \infty$, then μ_θ and μ_0 are singular, while if $\sum_{n=1}^{\infty} u_n^2 < \infty$ and θ is small enough, then μ_θ is absolutely continuous with respect to μ_0. They conjectured that absolute continuity should not depend on θ, but only on the square-summability of $\{u_n\}$. We show that in fact the power law governing the decay of $\{u_n\}$ is crucial, and for some renewal sequences $\{u_n\}$, there is a *phase transition* at a critical parameter $\theta_c \in (0, 1)$: for $|\theta| < \theta_c$ the measures μ_θ and μ_0 are mutually absolutely continuous, but for $|\theta| > \theta_c$, they are singular. Moreover, for these sequences, the unknown bias can be recovered from the observations if it is large enough. When $\sum_{n=1}^{\infty} u_n^2 = \infty$, the unknown bias can always be recovered.

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1. Introduction. A *coin toss* with *bias* θ is a $\{-1,1\}$-valued random variable with mean θ, and a *fair coin* is a coin toss with mean zero. Kakutani's dichotomy for independent sequences reduces, in the case of coin tosses, to the following:

THEOREM A ([13]) *Let μ_0 be the distribution of i.i.d. fair coin tosses on $\{-1,1\}^{\mathbf{N}}$, and let ν_θ be the distribution of independent coin tosses with biases $\{\theta_n\}_{n=1}^{\infty}$.*

(i) If $\sum_{n=1}^{\infty} \theta_n^2 = \infty$ then $\nu_\theta \perp \mu_0$, where $\nu \perp \mu$ means that the measures ν and μ are mutually singular.

(ii) If $\sum_{n=1}^{\infty} \theta_n^2 < \infty$, then $\nu_\theta \ll \mu_0$ and $\mu_0 \ll \nu_\theta$, where $\nu \ll \mu$ means that ν is absolutely continuous with respect to μ.

For a proof of Theorem A see, for example, Theorem 4.3.5 of [7].

Harris and Keane [10] extended Theorem A(i) to sequences with a specific type of dependence. Let $\{\Gamma_n\}$ be a (hidden) recurrent Markov chain with initial state o, called the *origin*. Suppose that whenever $\Gamma_n = o$, an independent coin with bias $\theta \geq 0$ is tossed, while at all other times an independent fair coin is tossed. Write $X = (X_1, X_2, \ldots)$ for the record of coin tosses, and let μ_θ be the distribution of X. Let $\Delta_n = \mathbf{1}_{\{\Gamma_n = o\}}$ and denote by

$$u_n = \mathbf{P}[\Gamma_n = o] = \mathbf{P}[\Delta_n = 1]$$

the probability of a return of the chain to the origin at time n. The random variables $\{\Delta_n\}$ form a *renewal process*, and their joint distribution is determined by the corresponding *renewal sequence* $\{u_n\}$; see the next section. Harris and Keane established the following theorem.

THEOREM B ([10])

(i) If $\sum_{n=1}^{\infty} u_n^2 = \infty$*, then* $\mu_\theta \perp \mu_0$*.*

(ii) If $\sum_{n=1}^{\infty} u_n^2 = \| u \|^2 < \infty$ *and* $\theta < \| u \|^{-1}$*, then* $\mu_\theta \ll \mu_0$*.*

Harris and Keane conjectured that singularity of the two laws μ_θ and μ_0 should not depend on θ, but only on the return probabilities $\{u_n\}$. In particular, they asked whether the condition $\sum_{k=0}^{\infty} u_k^2 < \infty$ implies that $\mu_\theta \ll \mu_0$, analogously to the independent case treated in Theorem A. We answer this negatively in Sections 4 and 5, where the following is proved.

Notation: Write $a_n \asymp b_n$ to mean that there exist positive finite constants C_1, C_2 so that $C_1 \leq a_n/b_n \leq C_2$ for all $n \geq 1$.

Theorem 1.1. *Let* $1/2 < \gamma < 1$*. Suppose that the return probabilities* $\{u_n\}$ *satisfy* $u_n \asymp n^{-\gamma}$ *and* $\max\{u_i \,:\, i \geq 1\} > 2^{\gamma-1}$*.*

(i) If $\theta > \frac{2^\gamma}{\max\{u_i \,:\, i \geq 1\}} - 1$*, then* $\mu_\theta \perp \mu_0$*.*

(ii) The bias θ *can be a.s. reconstructed from the coin tosses* $\{X_n\}$*, provided* θ *is large enough. More precisely, we exhibit a measurable function* g *so that, for all* $\theta > \frac{2^\gamma}{\max\{i \,:\, i \geq 1\}} - 1$*, we have* $\theta = g(X)$ μ_θ*-almost surely.*

Part (i) is proved, in a stronger form, in Proposition 4.1, and (ii) is contained in Theorem 5.1 in Section 5, where g is defined.

In Section 4 we provide examples of random walks having return probabilities satisfying the hypotheses of Theorem 1.1. We provide other examples of Markov chains in this category in Section 8.

For this class of examples, Theorem B(ii) and Theorem 1.1(i) imply that there is a **phase transition** in θ: there is a critical $\theta_c \in (0, 1)$ so that for $\theta < \theta_c$, the measures μ_θ and μ_0 are equivalent, while for $\theta > \theta_c$, μ_θ and μ_0 are mutually singular. See Section 3 for details. Consequently, there are cases of absolute continuity, where altering the underlying Markov chain by introducing delays can produce singularity.

Most of our current knowledge on the critical parameter

$$\theta_c \stackrel{\text{def}}{=} \sup\{\theta : \mu_\theta \ll \mu_0\}$$

is summarized in the following table. Choose r such that $u_r = \max\{u_i : i \geq 1\}$, and let $\theta_s = (\sum_{n=1}^{\infty} u_n^2)^{-1/2} \wedge 1$. (The arguments of Harris and Keane [10] imply that θ_s is the critical parameter for μ_θ to have a square-integrable density with respect to μ_0.)

asymptotics of u_n	**critical parameters**
$u_n \asymp n^{-1/2}$	$0 = \theta_s = \theta_c$
$u_n \asymp n^{-\gamma}$, $\frac{1}{2} < \gamma < 1$	$0 < \theta_s \leq \theta_c \leq u_r^{-1} 2^\gamma - 1$
$u_n = O(n^{-1})$	$0 < \theta_s \leq \theta_c = 1$

There are renewal sequences corresponding to the last row for which $0 < \theta_s < \theta_c = 1$; see Theorem 1.4 and the remark following it.

Theorem 1.1(ii) shows that for certain chains satisfying $\sum_{n=0}^{\infty} u_n^2 < \infty$, for θ large enough, the bias θ of the coin can be reconstructed from the observations X. Harris and Keane described how this can be done for all θ in the case where Γ is the simple random walk on the integers, and asked whether it is possible whenever $\sum_n u_n^2 = \infty$. In Section 6 we answer affirmatively, and prove the following theorem:

THEOREM 1.2. *If $\sum_n u_n^2 = \infty$, then there is a measurable function h so that $\theta = h(X)$ μ_θ-a.s. for all θ.*

In fact, h is a limit of linear estimators (see the proof given in Section 6). Theorem 1.2 is extended in Theorem 6.1.

There are examples of renewal sequences with $\sum_k u_k^2 < \infty$ which do not exhibit a phase transition:

THEOREM 1.3. *If the return probabilities* $\{u_n\}$ *satisfy* $u_n = O(n^{-1})$, *then* $\mu_\theta \ll \mu_0$ *for all* $0 \leq \theta \leq 1$.

For example, the return probabilities of (even a delayed) random walk on $\mathbf{Z}^2$ have $u_k \asymp k^{-1}$.

Remark: The significance of this result is that the asymptotic conditions on $\{u_n\}$ still holds if the underlying Markov chain is altered to increase the transition probability from the origin to itself.

This result is proved in Section 9. It is much easier to prove that μ_θ and μ_0 are always mutually absolutely continuous in the case where the Markov chain is "almost transient", for example if $u_k \asymp (k \log k)^{-1}$. We include the argument for this case as a warm-up to Theorem 1.3. In particular, we prove the following theorem:

THEOREM 1.4. *If the return probabilities* $\{u_n\}$ *satisfy* $u_k = O(k^{-1})$, *and obey the condition*

$$\sum_{k=0}^{n} u_k = o\left(\frac{\log n}{\log \log n}\right),$$

then $\mu_\theta \ll \mu_0$ *for all* $0 \leq \theta \leq 1$.

Theorem 1.4 is extended in Theorem 8.2 in Section 8, and Proposition 8.5 provides examples of Markov chains satisfying the hypotheses. Then Theorem 1.3 is proved in Section 9

Write $J = \sum_{n=0}^{\infty} \mathbf{1}_{\{\Gamma_n = o\}} \mathbf{1}_{\{\Gamma'_n = o\}}$, where Γ and Γ' are two independent copies of the underlying Markov chain. The key to the proof by Harris and Keane of Theorem B(ii) is the implication

$$\mathbf{E}[(1+\theta^2)^J] < \infty \;\Rightarrow\; \mu_\theta \ll \mu_0 \,.$$

To prove Theorem 1.3 and Theorem 1.4 we refine this and show that

$$\mathbf{E}[(1+\theta^2)^J \mid \Gamma] < \infty \;\Rightarrow\; \mu_\theta \ll \mu_0 \,.$$

The model discussed here can be generalized by substituting real-valued random variables for the coin tosses. We consider the model where observations are generated with distribution α at times when the chain is away from o, and a distribution η is used when the chain visits o.

Similar problems of "random walks on scenery" were considered by Benjamini and Kesten in [3] and by Howard in [11, 12]. Vertices of a graph are assigned colors, and a viewer, provided only with the sequence of colors visited by a random walk on the graph, is asked to distinguish (or reconstruct) the coloring of the graph.

The rest of this paper is organized as follows. In Section 2, we provide definitions and introduce notation. In Section 3, we prove a useful general zero-one law, to show that singularity and absolute continuity of the measures are the only possibilities. In Section 4, Theorem 1.1(i) is proved, while Theorem 1.1(ii) is established in Section 5. We prove a more general version of Theorem 1.2 in Section 6. In Section 7, we prove a criterion for absolute continuity, which is used to prove Theorem 1.4 in Section 8 and Theorem 1.3 in Section 9. A connection to long-range percolation and some unsolved problems are described in Section 10.

2. Definitions. Let $\Upsilon = \{0,1\}^\infty$ be the space of binary sequences. Denote by Δ_n the n^{th} coordinate projection from Υ. Endow Υ with the σ-field $\mathcal{H}$ generated by $\{\Delta\}_{n\geq 0}$ and let $\mathbf{P}$ be a renewal measure on $(\Upsilon, \mathcal{H})$, that is, a measure obeying

$$\mathbf{P}[\Delta_0 = 1, \Delta_{n(1)} = 1, \ldots, \Delta_{n(m)} = 1] = \prod_{i=1}^{m} u_{n(i)-n(i-1)}, \tag{2.1}$$

where $u_n \stackrel{\text{def}}{=} \mathbf{P}[\Delta_n = 1]$. We let $\{T_k\}_{k=1}^\infty$ denote the *inter-arrival times* of the renewal process: If $S_n = \inf\{m > S_{n-1} : \Delta_m = 1\}$ is the time of the n^{th} renewal, then $T_n = S_n - S_{n-1}$. The condition (2.1) implies that $T_1, T_2, \ldots$ is an i.i.d. sequence. We will use f_n to denote $\mathbf{P}[T_1 = n]$.

In the introduction we defined u_n as the probability for a Markov chain Γ to return to its initial state at time n. If $\Delta_n = \mathbf{1}_{\{\Gamma_n = o\}}$, then the Markov property guarantees that (2.1) is satisfied. Conversely, any renewal process Δ can be realized as the indicator of return times of a Markov chain to its initial state. (Take, for example, the chain whose value at epoch n is the time until the next renewal, and consider returns to 0.) Thus we can move freely between these points of view. For background on renewal theory, see [8] or [15].

Suppose that α, η are two probabilities on $\mathbf{R}$ which are mutually absolutely continuous, that is, they share the same null sets. In the coin tossing case discussed in the Introduction, these measures are supported on $\{-1, 1\}$. Given a renewal process, independently generate observations according to η at renewal times, and according to α at all other times. We describe the distribution of these observations for various choices of η.

Let $\mathbf{R}^\infty$ denote the space of real sequences, endowed with the σ-field $\mathcal{G}$ generated by coordinate projections. Write η^∞ for the product probability on $(\mathbf{R}^\infty, \mathcal{G})$

with marginal η. Let $\mathbf{Q}_\eta$ be the measure $\alpha^\infty \times \eta^\infty \times \mathbf{P}$ on $(\mathbf{R}^\infty \times \mathbf{R}^\infty \times \Upsilon, \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{H})$. In the case where η is the coin tossing measure with bias θ, write $\mathbf{Q}_\theta$ for $\mathbf{Q}_\eta$. The random variables Y_n, Z_n are defined by $Y_n(y, z, \delta) = y_n, \;\; Z_n(y, z, \delta) = z_n$. Finally, the random variables X_n are defined by

$$X_n = (1 - \Delta_n)Y_n + \Delta_n Z_n.$$

The distribution of $X = \{X_n\}$ on $\mathbf{R}^\infty$ under $\mathbf{Q}_\eta$ will be denoted μ_η.

The natural questions in this setting are: if β and π are two mutually absolutely continuous measures on $\mathbf{R}$, under what conditions is $\mu_\beta \perp \mu_\pi$? Under what conditions is $\mu_\beta \ll \mu_\pi$? When can η be reconstructed from the observations $\{X_n\}$ generated under μ_η? Partial answers are provided in Proposition 4.1, Theorem 1.1, Theorem 5.1, Theorem 6.1, and Theorem 8.2.

3. A Zero-One Law and Monotonicity. We use the notation established in the previous section. Let $\mathcal{G}_n$ be the σ-field on $\mathbf{R}^\infty$ generated by the first n coordinates. If μ_β and μ_π are both restricted to $\mathcal{G}_n$, then they are mutually absolutely continuous, and we can define the Radon-Nikodym derivative $\rho_n = \frac{d\mu_\pi}{d\mu_\beta}|_{\mathcal{G}_n}$. Write ρ for $\liminf_{n\to\infty} \rho_n$; the Lebesgue Decomposition Theorem (see Theorem 4.3.3 in [7]) implies that for any $A \in \mathcal{G}$,

$$\mu_\pi[A] = \int_A \rho d\mu_\beta + \mu_\pi^{sing}(A) = \int_A \rho d\mu_\beta + \mu_\pi[\{\rho = \infty\} \cap A], \tag{3.2}$$

where $\mu_\pi^{sing} \perp \mu_\beta$. Thus to prove that $\mu_\pi \ll \mu_\beta$, it is enough to show that

$$1 = \mu_\pi[x : \rho(x) < \infty] = \mathbf{Q}_\pi[\rho(X) < \infty]. \tag{3.3}$$

For any process Γ, let $\Theta_n\Gamma = (\Gamma_n, \Gamma_{n+1}, \ldots)$, and let $\mathcal{T}(\Gamma) = \bigcap_{n=1}^\infty \sigma(\Theta_n\Gamma)$ be the tail σ-field.

Lemma 3.1 Zero–One Law. *The tail σ-field $\mathcal{T}(Y, Z, \Delta)$, and hence $\mathcal{T}(X)$, is $\mathbf{Q}_\eta$-trivial. That is, $A \in \mathcal{T}(Y, Z, \Delta)$ implies $\mathbf{Q}_\eta(A) \in \{0, 1\}$.*

Proof. By the Kolmogorov Zero-One Law, $\mathcal{T}(Y)$ and $\mathcal{T}(Z)$ are trivial. The inter-arrival times $\{T_n\}$ form an i.i.d. sequence, and clearly $\mathcal{T}(\Delta) \subset \mathcal{E}(T_1, T_2, \ldots)$, where $\mathcal{E}$ is the exchangeable σ-field. The Hewitt-Savage Zero-One law implies that $\mathcal{E}$, and hence $\mathcal{T}(\Delta)$, is trivial.

Let f be a bounded $\mathcal{T}(Y, Z, \Delta)$-measurable function on $\mathbf{R}^\infty \times \mathbf{R}^\infty \times \Upsilon$ which can be written as

$$f(y, z, \delta) = f_1(y) f_2(z) f_3(\delta)\,. \tag{3.4}$$

By independence of Y,Z, and Δ, and triviality of $\mathcal{T}(Y)$,$\mathcal{T}(Z)$, and $\mathcal{T}(\Delta)$, it follows that

$$\mathbf{E}[f_1(Y) f_2(Z) f_3(\Delta)] = \mathbf{E} f_1(Y) \mathbf{E} f_2(Z) \mathbf{E} f_3(\Delta) = f_1(Y) f_2(Z) f_3(\Delta) \ a.s.$$

Consequently, for all functions of the form (3.4),

$$\mathbf{E} f(Y, Z, \Delta) = f(Y, Z, \Delta) \ a.s. \tag{3.5}$$

The set of bounded functions of the form (3.4) is closed under multiplication, includes the indicator functions of rectangles $A \times B \times C$ for $A, B \in \mathcal{H}$ and $C \in \mathcal{G}$, and these rectangles generate the σ-field $\mathcal{G} \times \mathcal{G} \times \mathcal{H}$. Since the collection of bounded functions satisfying (3.5) form a monotone vector space, a Monotone Class Theorem implies that all bounded $\mathcal{G} \times \mathcal{G} \times \mathcal{H}$-measurable functions obey (3.5). We conclude that $\mathcal{T}(Y, Z, \Delta)$ is trivial. □

Proposition 3.2. *Either μ_π and μ_β are mutually absolutely continuous, or $\mu_\pi \perp \mu_\beta$.*

PROOF. Suppose that $\mu_\pi \not\ll \mu_\beta$. From (3.2), it must be that $\rho < \infty$ with positive μ_π probability. Because the event $\{\rho < \infty\}$ is in $\mathcal{T}$, Lemma 3.1 implies $\rho < \infty$ μ_π-almost surely. Using (3.2) again, we have that $\mu_\pi \ll \mu_\beta$. The same argument with the roles of β and π reversed, yields that $\mu_\beta \ll \mu_\pi$ also. □

We return to the special case of coin tossing here, and justify our remarks in the introduction that for certain sequences $\{u_n\}$, there is a phase transition. In particular, we need the following monotonicity result.

PROPOSITION 3.3. *Let $\theta_1 < \theta_2$. If $\mu_{\theta_1} \perp \mu_0$, then $\mu_{\theta_2} \perp \mu_0$.*

PROOF. Couple together the processes X for all θ: At each epoch n, generate a variable V_n, uniformly distributed on $[0, 1)$. If Δ is a renewal process independent of $\{V_n\}$, define X^θ by

$$X_n^\theta = \begin{cases} +1 \text{ if } V_n \leq \frac{1+\theta\Delta_n}{2} \\ -1 \text{ if } V_n > \frac{1+\theta\Delta_n}{2} \end{cases} \tag{3.6}$$

Then $X^{\theta_1} \leq X^{\theta_2}$ for $\theta_1 < \theta_2$, and X^θ has law μ_θ for all $\theta \in [0, 1]$. Thus μ_{θ_2} stochastically dominates μ_{θ_1}.

Suppose now that $\mu_{\theta_1} \perp \mu_0$. Then (3.2) implies that

$$\mu_{\theta_1}[\rho_{\theta_1} = \infty] = 1 \text{ and } \mu_0[\rho_{\theta_1} = 0] = 1. \tag{3.7}$$

Because the functions

$$\rho_n(x) = \int_\Upsilon \prod_{k=0}^n (1 + \theta x_k \Delta_k) d\mathbf{P}(\Delta)$$

are increasing in x, it follows that ρ is an increasing function and the event $\{\rho = \infty\}$ is an increasing event. Because μ_{θ_2} stochastically dominates μ_{θ_1}, we have

$$\mu_{\theta_2}[\rho_{\theta_1} = \infty] = 1. \tag{3.8}$$

Putting together (3.8) and the second part of (3.7) shows that we have decomposed $\mathbf{R}^\infty$ into the two disjoint sets $\{\rho_{\theta_1} = 0\}$ and $\{\rho_{\theta_1} = \infty\}$ which satisfy

$$\mu_0[\rho_{\theta_1} = 0] = 1 \quad \text{and} \quad \mu_{\theta_2}[\rho_{\theta_1} = \infty] = 1\,.$$

In other words, $\mu_{\theta_2} \perp \mu_0$. □

Consequently, it makes sense to define for a given renewal sequence $\{u_n\}$ the *critical bias* θ_c by

$$\theta_c \stackrel{\text{def}}{=} \sup\{\theta \le 1 : \mu_\theta \ll \mu_0\}.$$

We say there is a *phase transition* if $0 < \theta_c < 1$. The results of Harris and Keane say $\sum u_n^2 = \infty$ implies $\theta_c = 1$ and there is no phase transition. In Section 4, we provide examples of $\{u_n\}$ with $\sum_n u_n^2 < \infty$ having a phase transition. In Section 8, we provide examples with $\sum_n u_n^2 < \infty$ without a phase transition.

4. Existence of Phase Transition. In this section, we confine our attention to the coin tossing situation discussed in the Introduction. In this case, α and β are both the probability on $\{-1, 1\}$ with zero mean, and π is the probability with mean θ (the θ-biased coin). The distributions μ_β and μ_π are denoted by μ_0 and μ_θ respectively. Let $U_n \stackrel{\text{def}}{=} \sum_{k=0}^n u_k$.

Proposition 4.1. *Let $\{u_n\}$ be a renewal sequence with*

$$\sum_{k=0}^n u_k \;=\; U_n \;\asymp\; n^{1-\gamma}\ell(n)\,,$$

for $\frac{1}{2} < \gamma < 1$ and ℓ a slowly varying function. If

$$(1+\theta)\max\{u_i \;:\; i \ge 1\} \;>\; 2^\gamma\,,$$

then $\mu_\theta \perp \mu_0$.

Remark. The conditions on θ specified in the statement above are not vacuous. That is, there are examples where the lower bound on θ is less than 1. There are random walks with return times obeying $u_n \asymp n^{-\gamma}$, as shown in Theorem 4.3. By introducing delays at the origin, u_1 can be made to be close to 1, so that $2u_1 > 2^\gamma$.

PROOF. Let $\mathbf{E}$ denote expectation with respect to the renewal measure $\mathbf{P}$ and let $\mathbf{E}_\theta$ denote expectation with respect to $\mathbf{Q}_\theta$. Let $u_r = \max\{u_i \ : \ i \geq 1\}$ and assume for now that $r = 1$. Let $b = \frac{1}{2}(1+\theta)$ and $k(n) = \lfloor (1+\epsilon)\log_2 n \rfloor$, where ϵ is small enough that $(1+\epsilon)(-\log_2 u_1 b) < 1-\gamma$. Define A_j^n as the event that at all times $i \in [jk(n), (j+1)k(n))$ there are renewals and the coin lands "heads", i.e.,

$$A_j^n \stackrel{\text{def}}{=} \bigcap_{\ell=0}^{k(n)-1} \{\Delta_{jk(n)+\ell} = 1 \ \text{ and } X_{jk(n)+\ell} = 1\}.$$

Let $D_n \stackrel{\text{def}}{=} \sum_{j=1}^{n/k(n)} \mathbf{1}_{A_j^n}$, and

$$c(n) \ \stackrel{\text{def}}{=} \ \mathbf{Q}_\theta[A_j^n | \Delta_{jk(n)} = 1] \ = \ b(u_1 b)^{k(n)-1} = u_1^{-1}(u_1 b)^{k(n)} \, .$$

Note that we have defined things so that $c(n) \asymp n^{-p}$, where $p < 1-\gamma$. Then

$$\mathbf{E}_\theta D_n = \sum_{j=1}^{n/k(n)} u_{jk(n)} b(u_1 b)^{k(n)-1} = c(n) \sum_{j=1}^{n/k(n)} u_{jk(n)}. \tag{4.9}$$

We need the following simple lemma:

LEMMA 4.2. *For all $r \geq 0$,*

$$u_r + u_{r+k} + \cdots + u_{r+mk} \leq u_0 + u_k + \cdots + u_{mk} \, . \tag{4.10}$$

PROOF. Recall that $u_0 = 1$. Let $\tau^* = \inf\{j \geq 0 : \Delta_{r+jk} = 1\}$. Then

$$\begin{aligned} \mathbf{E}[\sum_{j=0}^{m} \Delta_{jk+r} | \tau^*] &= (1 + u_k + \cdots + u_{(m-\tau^*)k}) \mathbf{1}_{\{\tau^* \leq m\}} \\ &\leq u_0 + u_k + \ldots + u_{mk} \, . \end{aligned}$$

Taking expectation proves the lemma. □

By this lemma,

$$\sum_{j=0}^{n/k(n)} u_{jk(n)} \geq \frac{1}{k(n)} \sum_{j=0}^{n} u_j = \frac{U_n}{k(n)}\,, \tag{4.11}$$

and thus

$$\sum_{j=1}^{n/k(n)} u_{jk(n)} \geq \frac{U_n}{k(n)} - 1 \asymp \frac{U_n}{k(n)} \asymp n^{1-\gamma}\frac{\ell(n)}{k(n)}\,. \tag{4.12}$$

Combining (4.9) and (4.12), we find that

$$\mathbf{E}_\theta D_n \geq C_1 n^{-p} n^{1-\gamma}\frac{\ell(n)}{k(n)} = C_1 n^{1-\gamma-p}\frac{\ell(n)}{k(n)}\,.$$

Since $1-\gamma-p>0$, it follows that $\mathbf{E}_\theta D_n \to \infty$.

Also,

$$\mathbf{E}_\theta D_n^2 = \sum_{i=1}^{n/k(n)} \mathbf{Q}_\theta[A_i^n] + 2\sum_{i=1}^{n/k(n)} \sum_{j=i+1}^{n/k(n)} \mathbf{Q}_\theta[A_j^n|A_i^n]\mathbf{Q}_\theta[A_i^n]$$

$$= \mathbf{E}_\theta D_n + 2\sum_{i=1}^{n/k(n)} \sum_{j=i+1}^{n/k(n)} c(n)u_{k(n)(j-i-1)+1}c(n)u_{k(n)i}$$

$$\leq \mathbf{E}_\theta D_n + 2c(n)^2 \sum_{i=1}^{n/k(n)} u_{k(n)i} \sum_{j=0}^{n/k(n)} u_{k(n)j+1}$$

$$\leq \mathbf{E}_\theta D_n + 2c(n)^2 \sum_{i=1}^{n/k(n)} u_{k(n)i} \sum_{j=0}^{n/k(n)} u_{k(n)j} \tag{4.13}$$

$$\leq \mathbf{E}_\theta D_n + 2c(n)^2 \sum_{i=1}^{n/k(n)} u_{k(n)i} \sum_{j=1}^{n/k(n)} u_{k(n)j} + 2u_0c(n)\mathbf{E}_\theta D_n \tag{4.14}$$

$$\leq \mathbf{E}_\theta D_n + 2c(n)^2 \left(\sum_{i=1}^{n/k(n)} u_{k(n)i}\right)^2 + 2u_0c(n)\mathbf{E}_\theta D_n$$

$$\leq C(\mathbf{E}_\theta D_n)^2 \tag{4.15}$$

(4.13) follows from Lemma 4.2, and the last term in (4.14) comes from the contributions when $j=0$.

If A_n is the event that there is a run of length $k(n)$ after epoch $k(n)$ and before n, then (4.15) and the second moment inequality yield

$$\mathbf{Q}_\theta[A_n] \geq \mathbf{Q}_\theta[D_n > 0] \geq \frac{(\mathbf{E}_\theta D_n)^2}{\mathbf{E}_\theta D_n^2} \geq \frac{1}{C} > 0.$$

Finally, we have

$$\mathbf{Q}_\theta[\limsup A_n] \geq \limsup \mathbf{Q}_\theta[A_n] > 0,$$

and by the Zero-One Law (Lemma 3.1) we have that $\mathbf{Q}_\theta[\limsup A_n] = 1$. A theorem of Erdős and Rényi (see, for example, Theorem 7.1 in [21]) states that under the measure μ_0, $L_n / \log_2 n \to 1$, where L_n is the length of the longest run before epoch n. But under the measure μ_θ, we have just seen that we are guaranteed to, infinitely often, see a run of length $(1+\epsilon)\log_2 n$ before time n.

If $u_1 \neq \max\{u_i \ : \ i \geq 1\}$, consider the renewal process $\{\Delta_{nr}\}_{n=0}^\infty$ and the sequence $\{X_{nr}\}_{n=0}^\infty$, where $u_r = \max\{u_i \ : \ i \geq 1\}$. Apply the proceeding argument to this subsequence to distinguish between μ_θ and μ_0. □

Proposition 4.3. *There exists a renewal measure* $\mathbf{P}$ *with* $u_n \sim Cn^{-\gamma}$ *for* $1/2 < \gamma < 1$.

Proof. For a distribution function F to be in the domain of attraction of a stable law, only the asymptotic behavior of the tails $F(t), 1 - F(-t)$ is relevant (see, for example, Theorem 8.3.1 in [4]). Thus if the symmetric stable law with exponent $1/\gamma$ is discretized so that it is supported on $\mathbf{Z}$, then the modified law F is in the domain of attraction of this stable law. Then if Γ is the random walk with increments distributed according to F, Gnedenko's Local Limit Theorem (see Theorem 8.4.1. of [4]) implies that

$$\lim_{n\to\infty} |n^\gamma \mathbf{P}[\Gamma_n = 0] - g(0)| = 0,$$

where g is the density of the stable law. Thus if $\Delta_n \stackrel{\text{def}}{=} \mathbf{1}_{\{\Gamma_n=0\}}$, then $\{\Delta_n\}$ form a renewal sequence with $u_n \sim Cn^{-\gamma}$. □

For a sequence to satisfy the hypotheses of Proposition 4.1 and 1.1, we also need that $\max\{u_i \,:\, i \geq 1\} > 2^{\gamma-1}$. By introducing a delay at the origin for the random walk Γ in Proposition 4.3, u_1 can be made arbitrarily close to 1. Thus there do exist Markov chains which have $0 < \theta_c < 1$.

An example of a Markov chain with $U_n \asymp n^{1/4}$ will be constructed by another method in Section 8.

5. Determining the bias θ. In this section we refine the results of the previous section and give conditions that allow reconstruction of the bias from the observations.

For $a \geq 1$, let

$$\Lambda^*(a) \stackrel{\text{def}}{=} \lim_{m\to\infty} \frac{-\log_2 \mathbf{P}[T_1 + \cdots + T_m \leq ma]}{m}\,. \tag{5.16}$$

($\Lambda^*(a) = \infty$ for $a < 1$ (since each $T_i \geq 1$), hence we restrict attention to when $a \geq 1$.)

Because $\mathbf{E}T_i = \infty$, Cramér's Theorem (see, e.g., [6]) implies that $\Lambda^*(a) > 0$ for all a. Since $\lim_{a\uparrow\infty} \mathbf{P}[T_1 \leq a] = 1$, it follows that $\lim_{a\uparrow\infty} \Lambda^*(a) = 0$. Also, $\Lambda^*(1) = -\log_2 u_1$.

It is convenient to reparameterize so that we keep track of $\varphi \stackrel{\text{def}}{=} \log_2(1+\theta)$ instead of θ itself. Let

$$\widehat{\psi}(\varphi,\xi) \stackrel{\text{def}}{=} \xi \cdot (\varphi - \Lambda^*(\xi^{-1})) \text{ and } \psi(\varphi) \stackrel{\text{def}}{=} \sup_{0<\xi\leq 1} \widehat{\psi}(\varphi,\xi). \tag{5.17}$$

Observe that $\lim_{\xi\to 0} \widehat{\psi}(\varphi,\xi) = 0$. For $\epsilon > 0$ small enough so that $\Lambda^*(\epsilon^{-1}) < \frac{\varphi}{2}$,

$$\widehat{\psi}(\varphi,\epsilon) > \epsilon(\varphi - \frac{\varphi}{2}) = \epsilon\frac{\varphi}{2} > 0\,.$$

Hence, The maximum of $\widehat{\psi}(\varphi, \cdot)$ over $(0, 1]$ is attained, so we can define

$$\xi_0(\varphi) \stackrel{\text{def}}{=} \inf\{0 < \xi \le 1 \ : \ \widehat{\psi}(\varphi, \xi) = \psi(\varphi)\} \, .$$

We show now that $\widehat{\psi}(\varphi, \xi_0) > \widehat{\psi}(\varphi, 1)$, a fact which we will use later (see the remarks following Theorem 5.1). Let $\ell = \min\{n > 1 : f_n > 0\}$, and note that $f_1 = u_1$. If in the interval $[0, k(1 + \lfloor \epsilon\ell \rfloor)\,]$ there are $k - \lfloor \epsilon k \rfloor$ inter-renewal times of length 1 and $\lfloor \epsilon k \rfloor$ inter-renewal times of length ℓ, then in particular there are at least k renewals. Consequently,

$$\mathbf{P}[T_1 + \cdots + T_k \le k(1 + \epsilon\ell)] \ \ge \ \binom{k}{\lfloor \epsilon k \rfloor} f_1^{k - \lfloor \epsilon k \rfloor} f_\ell^{\lfloor \epsilon k \rfloor} \tag{5.18}$$

Taking logs, normalizing by k, and then letting $k \to \infty$ yields

$$\begin{aligned} -\Lambda^*(1 + \epsilon\ell) &= \lim_{k \to \infty} k^{-1} \log_2 \mathbf{P}[T_1 + \cdots + T_k \le k(1 + \epsilon\ell)] \\ &\ge h_2(\epsilon) + \log_2 f_1 + \epsilon \log_2(f_\ell / f_1) \, , \end{aligned}$$

where $h_2(\epsilon) = \epsilon \log_2 \epsilon^{-1} + (1 - \epsilon) \log_2 (1 - \epsilon)^{-1}$. Therefore

$$\psi(\varphi, \frac{1}{1 + \epsilon\ell}) - \psi(\varphi, 1) = \frac{1}{1 + \epsilon\ell}\varphi - \frac{1}{1 + \epsilon\ell}\Lambda^*(1 + \epsilon\ell) - \varphi - \log_2 f_1 \tag{5.19}$$

$$\ge \frac{1}{1 + \epsilon\ell} \left\{ -\epsilon(\ell\varphi + \log_2(f_\ell / f_1)) + h_2(\epsilon) \right\} \tag{5.20}$$

Thus for ϵ bounded above, the left-hand side of (5.19) is bounded below by $C_1(h_2(\epsilon) - C_2\epsilon)$. Since the derivative of h_2 tends to infinity near 0, there is a positive ϵ where the difference is strictly positive. Thus, the maximum of $\widehat{\psi}(\varphi, \cdot)$ is *not* attained at $\xi = 1$.

Finally, ψ is strictly increasing: let $\varphi < \varphi'$, and observe that

$$\psi(\varphi') = \widehat{\psi}(\varphi', \xi_0(\varphi')) \ge \widehat{\psi}(\varphi', \xi_0(\varphi)) > \widehat{\psi}(\varphi, \xi_0(\varphi)) = \psi(\varphi).$$

THEOREM 5.1. *Recall that*

$$\mathbf{P}[X_k = 1 \mid \Delta_k = 1] = 2^{-1}(1 + \theta) = 2^{\varphi - 1}, \textit{ for } \varphi \stackrel{\text{def}}{=} \log_2(1 + \theta) \, .$$

Let

$$R_n = \sup\{m : X_{n+1} = \cdots = X_{m+n} = 1\} \quad \textit{and} \quad \widehat{R}(X) = \limsup_n R_n(\log_2 n)^{-1} .$$

Suppose that $\frac{1}{2} < \gamma < 1$ *and* ℓ *is a slowly varying function. If* $U_n \asymp n^{1-\gamma}\ell(n)$*, then*

$$\widehat{R}(X) = \frac{1-\gamma}{1-\psi(\varphi)} \bigvee 1,$$

where ψ *is the strictly monotone function defined in (5.17).*

In particular, for $\varphi > \psi^{-1}(\gamma)$ *(equivalently,* $\theta \geq 2^{\psi^{-1}(\gamma)} - 1$*), we can recover* φ *(and hence* θ*) from* X*:*

$$\varphi \;=\; \psi^{-1}\left(1 - \frac{1-\gamma}{\widehat{R}(X)}\right) .$$

Remark. Suppose $u_1 = \max\{u_i \;:\; i \geq 1\}$. Since $\psi(\varphi) > \widehat{\psi}(\varphi, 1)$, (see the comments before the statement of Theorem 5.1) we have that

$$\psi(\varphi) \;>\; \varphi + \log_2 u_1 . \tag{5.21}$$

Substituting $\psi^{-1}(\gamma)$ for φ in (5.21) yields

$$\psi^{-1}(\gamma) \;<\; \gamma - \log_2 u_1 .$$

Thus

$$2^{\psi^{-1}(\gamma)} - 1 \;<\; 2^{\gamma - \log_2 u_1} - 1. \tag{5.22}$$

The right-hand side of (5.22) is the upper bound on θ_c obtained in Proposition 4.1, while the left-hand side is the upper bound given by Theorem 5.1. Thus this section strictly improves the results achieved in the previous section.

PROOF. Let $\zeta = (1-\gamma)/(1-\psi(\varphi))$. We begin by proving that $\widehat{R}(X) \leq \zeta \vee 1$, or equivalently, that

$$\forall c > \zeta \vee 1,\; \mathbf{Q}_\theta[R_n \geq c \log_2 n \; i.\; o.\;] = 0. \tag{5.23}$$

Fix $c > \zeta \vee 1$. If $k(n,c) = k(n) \stackrel{\text{def}}{=} \lfloor c \log_2 n \rfloor$, then it is enough to show that

$$\mathbf{Q}_\theta[\limsup_n \{X_{n+1} = \cdots = X_{n+k(n)} = 1\}] = 0. \tag{5.24}$$

Let E_n be the event $\{X_{n+1} = \cdots = X_{n+k(n)} = 1\}$, and define

$$F_n \stackrel{\text{def}}{=} \inf\{m > 0 : \Delta_{n+m} = 1\}$$

as the waiting time at n until the next renewal (the residual lifetime at n). We have

$$\mathbf{Q}_\theta[E_n] \leq \mathbf{Q}_\theta[E_n \mid F_n > k(n)] + \sum_{m=1}^{k(n)} \mathbf{Q}_\theta[E_n \mid F_n = m]\mathbf{Q}_\theta[F_n = m]. \tag{5.25}$$

Notice that

$$\{F_n = m\} = \{\Delta_{n+1} = \cdots = \Delta_{n+m-1} = 0,\ \Delta_{n+m} = 1\},$$

and consequently we have

$$\begin{aligned} &\mathbf{Q}_\theta[E_n \mid F_n = m,\ \Delta_{n+m+1}, \ldots, \Delta_{n+k(n)}] \\ &= \ 2^{-k(n)}(1+\theta)^{1+\Delta_{n+m+1}+\cdots+\Delta_{n+k(n)}} \end{aligned} \quad . \tag{5.26}$$

Taking expectations over $(\Delta_{n+m+1}, \ldots, \Delta_{n+k(n)})$ in (5.26) gives that

$$\begin{aligned} \mathbf{Q}_\theta[E_n \mid F_n = m] &= 2^{-k(n)}\mathbf{E}[(1+\theta)^{1+\Delta_{n+m+1}+\cdots+\Delta_{n+k(n)}} \mid \Delta_{n+m} = 1] \\ &= 2^{-k(n)}\mathbf{E}[(1+\theta)^{1+\Delta_1+\cdots+\Delta_{k(n)-m}}] \end{aligned} \tag{5.27}$$

The equality in (5.27) follows from the renewal property, and clearly the right-hand side of (5.27) is maximized when $m = 1$. Therefore the right-hand side of (5.25) is bounded above by

$$2^{-k(n)} + (U_{n+k(n)} - U_n)\mathbf{Q}_\theta[E_n \mid \Delta_{n+1} = 1]. \tag{5.28}$$

We now examine the probability $\mathbf{Q}_\theta[E_n \mid \Delta_{n+1} = 1]$ appearing on the right-hand side of (5.28). Let $N[i,j] \stackrel{\text{def}}{=} \sum_{k=i}^{j} \Delta_k$ be the number of renewals appearing between times i and j. In the following, $N = N[n+1, n+k(n)]$. We have

$$\begin{aligned} \mathbf{Q}_\theta[E_n \mid \Delta_{n+1} = 1] &= 2^{-k(n)}\mathbf{E}[(1+\theta)^N \mid \Delta_{n+1} = 1] \\ &= \mathbf{E}[2^{k(n)(-1+\varphi N/k(n))} \mid \Delta_{n+1} = 1]\,. \end{aligned} \tag{5.29}$$

By conditioning on the possible values of N, (5.29) is bounded by

$$\sum_{m=1}^{k(n)} 2^{k(n)(-1+\varphi m/k(n))}\mathbf{P}[T_1 + \cdots T_m \le k(n)]\,. \tag{5.30}$$

By the superadditivity of $\log \mathbf{P}[T_1 + \cdots + T_m \le ma]$, the probabilities in the sum in (5.30) are bounded above by $2^{-m\Lambda^*(k(n)/m)}$. Consequently, (5.30) is dominated by

$$\begin{aligned} \sum_{m=1}^{k(n)} 2^{k(n)(-1+m/k(n)(\varphi-\Lambda^*(k(n)/m)))} &\le \sum_{m=1}^{k(n)} 2^{k(n)(-1+\widehat{\psi}(\varphi,m/k(n)))} \\ &\le k(n)2^{k(n)(\psi(\varphi)-1)} \end{aligned}$$

Hence, returning to (5.28),

$$\begin{aligned} \mathbf{Q}_\theta[E_n] &\le 2^{-k(n)} + (U_{n+k(n)} - U_n)k(n)2^{-k(n)(1-\psi(\varphi))} \\ &\le 2n^{-c} + 2k(n)(U_{n+k(n)} - U_n)n^{-c(1-\psi(\varphi))}. \end{aligned} \tag{5.31}$$

Let $q = c(1-\psi(\varphi))$, and since $c > \zeta \vee 1$, we have that $q + \gamma > 1$. Letting $m(n) = n + k(n)$, since $m(n) \ge n$, we have

$$\begin{aligned} \sum_{n=1}^{L} k(n)U_{n+k(n)}n^{-q} &\le \sum_{n=1}^{L} k(m(n))U_{m(n)}(m(n) - k(n))^{-q} \\ &\le \sum_{n=1}^{L} k(m(n))U_{m(n)}(m(n) - k(m(n)))^{-q} \\ &\le \sum_{m=1}^{L+k(L)} k(m)U_m(m - k(m))^{-q}\,. \end{aligned} \tag{5.32}$$

Then, using (5.32), it follows that

$$\sum_{n=1}^{L} k(n)(U_{n+k(n)} - U_n)n^{-q} \le \sum_{n=1}^{L} k(n)U_n \left((n-k(n))^{-q} - n^{-q}\right)$$
$$+ \sum_{n=L+1}^{L+k(L)} k(n)U_n(n-k(n))^{-q}\,. \tag{5.33}$$

Since $a^{-q} - b^{-q} \le C(b-a)a^{-1-q}$, and $U_n \le Cn^{1-\gamma}$, the right-hand side of (5.33) is bounded above by

$$\begin{aligned} &C_1 \textstyle\sum_{n=1}^{L} k(n)n^{1-\gamma}k(n)(n-k(n))^{-q-1} \\ &+ C_2 k(L)k(L+k(L))(L+k(L))^{1-\gamma}(L-k(L))^{-q} \end{aligned}\,. \tag{5.34}$$

We have that (5.34), and hence (5.33), is bounded above by

$$C_3 \sum_{n=1}^{L} k(n)^2 n^{-(q+\gamma)} + o(1)\,. \tag{5.35}$$

Since $q + \gamma > 1$, (5.35) is bounded as $L \to \infty$. We conclude that (5.31) is summable. Applying the Borel-Cantelli lemma establishes (5.24).

We now prove the lower bound, $\widehat{R}(X) \ge \zeta \vee 1$.

It is convenient to couple together monotonically the processes X^θ for different θ. See (3.6) in the proof of Proposition 3.3 for the construction of the coupling, and let $\{V_i\}$ be the i.i.d. uniform random variables used in the construction.

First, using the coupling, we have that $\widehat{R}(X^\theta) \ge \widehat{R}(X^0) = 1$. Hence,

$$\mu_\theta[x : \widehat{R}(x) \ge 1] \;=\; 1\,.$$

It is enough to show that if $c < \zeta$, then

$$\mathbf{Q}_\theta[R_n \ge k(c,n)\ i.\ o.\ ] = 1\,.$$

Fix φ, and write ξ_0 for $\xi_0(\varphi)$.

Let $\tau_i = \tau_i^n$ be the time of the $\lfloor \xi_0 k(n) \rfloor^{th}$ renewal after time $ik(n) - 1$. The event G_i^n of a *good run* in the block $I_i^n = [ik(n), (i+1)k(n) - 1] \cap \mathbf{Z}^+$ occurs when

1. there is a renewal at time $ik(n)$: $\Delta_{ik(n)} = 1$,
2. there are at least $\xi_0 k(n)$ renewals in I_i: $\tau_i \leq (i+1)k(n) - 1$,
3. until time τ_i, all observations are "heads": $X_j = 1$ for $ik(n) \leq j \leq \tau_i$,
4. $V_j \leq 1/2$ for $\tau_i < j \leq (i+1)k(n) - 1$.

The importance of the coupling and the last condition is that a good run in I_i implies an observed run ($X_j = 1 \, \forall \, j \in I_i$).

Let $N_i = N[I_i]$. The probability of G_i^n is given by

$$\mathbf{Q}_\theta[G_i^n] = 2^{-k(n)}(1+\theta)^{\xi_0 k(n)} p_i u_{ik(n)}, \tag{5.36}$$

where $p_i \stackrel{\text{def}}{=} \mathbf{P}[N_i \geq \xi_0 k(n) \mid \Delta_{ik(n)} = 1]$ is the probability of at least $\xi_0 k(n)$ renewals in the interval I_i, given that there is a renewal at $ik(n)$. Note that $p_i \equiv p_1$ for all i, by the renewal property.

Following the proof of Proposition 4.1, we define $D_n = \sum_{j=1}^{n/k(n)} \mathbf{1}_{G_j^n}$, and compute the first and second moments of D_n. Using (5.36) gives

$$\mathbf{E}_\theta[D_n] = 2^{-k(n)}(1+\theta)^{\xi_0 k(n)} p_1 \sum_{j=1}^{n/k(n)} u_{jk(n)}. \tag{5.37}$$

Since $c < \zeta = \frac{1-\gamma}{1-\psi(\varphi)}$, we also have for some $\epsilon > 0$ that

$$c < \frac{1-\gamma}{1+\epsilon\xi_0 - \psi(\varphi)}\,. \tag{5.38}$$

By definition of Λ^*, we can bound below the probability p_1: For n sufficiently large,

$$p_1 = \mathbf{P}[N_1 \geq \xi_0 k(n) \mid \Delta_{k(n)} = 1]$$

$$= \mathbf{P}[T_1 + \cdots + T_{\xi_0 k(n)} \le k(n)]$$

$$\ge 2^{-\xi_0 k(n)(\Lambda^*(\xi_0^{-1})+\epsilon)}\,, \tag{5.39}$$

where $\epsilon > 0$ is arbitrary. Thus, plugging (5.39) into (5.37) shows that for n sufficiently large,

$$\mathbf{E}_\theta[D_n] \ge 2^{-k(n)}(1+\theta)^{\xi_0 k(n)} 2^{-\xi_0 k(n)(\Lambda^*(\xi_0^{-1})+\epsilon)} \sum_{j=1}^{n/k(n)} u_{jk(n)}$$

$$= 2^{k(n)(-1-\epsilon\xi_0+\varphi\xi_0-\xi_0\Lambda^*(\xi_0^{-1}))} \sum_{j=1}^{n/k(n)} u_{jk(n)}$$

$$\ge 2^{-1} n^{-q} \sum_{j=1}^{n/k(n)} u_{jk(n)}\,, \tag{5.40}$$

where $q = (1 + \epsilon\xi_0 - \psi(\varphi))c$. By (5.38), $1 - \gamma - q > 0$. Using (4.12), $\sum_{j=1}^{n/k(n)} u_{jk(n)} \asymp \frac{\ell(n)}{k(n)} n^{1-\gamma}$, in (5.40), gives that for n large enough,

$$\mathbf{E}_\theta[D_n] \ge C_3 \frac{\ell(n)}{k(n)} n^{1-\gamma-q} \overset{n\to\infty}{\longrightarrow} \infty.$$

We turn now to the second moment, which we show is bounded by a multiple of the square of the first moment.

$$\mathbf{E}_\theta[D_n^2] = 2 \sum_{i=1}^{n/k(n)} \sum_{j=i+1}^{n/k(n)} \mathbf{Q}_\theta[G_i^n \cap G_j^n] + \mathbf{E}_\theta[D_n]. \tag{5.41}$$

We compute the probabilities appearing in the sum by first conditioning on the renewal process:

$$\mathbf{Q}_\theta[G_i^n \cap G_j^n \mid \Delta] = \left(2^{-k(n)}(1+\theta)^{\xi_0 k(n)}\right)^2 \mathbf{1}_{\{N_i \ge \xi_0 k(n),\ \Delta_{ik(n)}=1\}}$$

$$\times\, \mathbf{1}_{\{N_j \ge \xi_0 k(n),\ \Delta_{jk(n)}=1\}}\,. \tag{5.42}$$

Taking expectations of (5.42), if $d(n) = 2^{-k(n)}(1+\theta)^{\xi_0 k(n)}$, then

$$\mathbf{Q}_\theta[G_i^n \cap G_j^n] = d(n)^2 \mathbf{P}[N_j \ge \xi_0 k(n), \Delta_{jk(n)} = 1 \text{ and } N_i \ge \xi_0 k(n), \Delta_{ik(n)} = 1]$$

$$= d(n)^2 p_1 \mathbf{P}[\Delta_{jk(n)} = 1 \mid N_i \ge \xi_0 k(n), \Delta_{ik(n)} = 1] p_1 u_{ik(n)}$$

$$= d(n)^2 p_1^2 u_{ik(n)} \mathbf{P}[\Delta_{jk(n)} = 1 \mid N_i \geq \xi_0 k(n), \Delta_{ik(n)} = 1] \tag{5.43}$$

Summing (5.43) over $i < j$ shows that $\sum_{i=1}^{n/k(n)} \sum_{j=i+1}^{n/k(n)} \mathbf{Q}_\theta[G_i^n \cap G_j^n]$ equals

$$d(n)^2 p_1^2 \sum_{i=1}^{n/k(n)} u_{ik(n)} \sum_{j=i+1}^{n/k(n)} \mathbf{P}[\Delta_{jk(n)} = 1 \mid N_i \geq \xi_0 k(n), \Delta_{ik(n)} = 1]\,. \tag{5.44}$$

Let $\sigma = (i+1)k(n) - \tau_i$. For $m = n/k(n)$, write

$$\sum_{j=i+1}^{m} \mathbf{P}[\Delta_{jk(n)} = 1 \mid N_i \geq \xi_0 k(n), \Delta_{ik(n)} = 1, \sigma] \tag{5.45}$$

as

$$\mathbf{E}[\sum_{j=i+1}^{m} \Delta_{jk(n)} \mid \tau_i < (i+1)k(n)\,,\, \sigma]\,. \tag{5.46}$$

Then observe that (5.46) is bounded above by

$$u_\sigma + u_{\sigma+k(n)} + \cdots + u_{\sigma+mk(m)}\,. \tag{5.47}$$

We can apply Lemma 4.2 to bound (5.47) above by $\sum_{j=0}^{m} u_{jk(n)}$. To summarize,

$$\sum_{j=i+1}^{m} \mathbf{P}[\Delta_{jk(n)} = 1 \mid N_i \geq \xi_0 k(n), \Delta_{ik(n)} = 1, \sigma] \;\leq\; \sum_{j=0}^{m} u_{jk(n)}\,. \tag{5.48}$$

Taking expectation over σ in (5.48), and then plugging into (5.44) shows that

$$\begin{aligned}\sum_{i=1}^{n/k(n)} \sum_{j=i+1}^{n/k(n)} \mathbf{Q}_\theta[G_i^n \cap G_j^n] &\leq d(n)^2 p_1^2 \sum_{i=1}^{n/k(n)} u_{ik(n)} \sum_{j=0}^{n/k(n)} u_{jk(n)} \\ &\leq (\mathbf{E}_\theta D_n)^2 + u_0 \mathbf{E}_\theta D_n\,,\end{aligned} \tag{5.49}$$

where we have used the expression (5.37) for $\mathbf{E}_\theta D_n$. Finally, using (5.49) in (5.41) yields that

$$\mathbf{E}_\theta[D_n^2] \leq C_3(\mathbf{E}_\theta[D_n])^2.$$

Now, we have, as in the proof of Proposition 4.1, that

$$\mathbf{Q}_\theta[\limsup\{D_n > 0\}] \geq \limsup_{n\to\infty} \mathbf{Q}_\theta[D_n > 0] \geq C_3^{-1} > 0.$$

Using Lemma 3.1 shows that $\mathbf{Q}_\theta[\limsup\{D_n > 0\}] = 1$. That is, the events $\bigcup_{i=1}^{n/k(n)} G_i^n$ happen infinitely often. But since a good run is also an observed run, also the events

$$\{\exists j,\, 1 \leq j \leq n/k(n) \text{ with } R_{jk(n)} \geq k(n)\}$$

happen infinitely often. But, if $R_{jk(n)} \geq k(n)$, then certainly $R_{jk(n)} \geq k(jk(n))$. Thus, in fact the events

$$\{\exists j \geq k(n) \text{ with } R_j \geq k(j)\}$$

happen infinitely often. That is

$$\mathbf{Q}_\theta[R_n \geq k(c,n)\, i.\ o.\ ] = 1.$$

We conclude that $\zeta \leq \widehat{R}(X)$. □

6. Linear estimators work when u_n are not square-summable. Before stating and proving a generalization of Theorem 1.2, we indicate how a weak form of that theorem may be derived by rather soft considerations; these motivated the more concrete arguments in our proof of Theorem 6.1 below. In the setting of Theorem 1.2, let

$$\mathcal{T}_n \stackrel{\text{def}}{=} \frac{\sum_{i=1}^n u_i X_i}{\sum_{i=1}^n u_i^2}\,.$$

It is not hard to verify that $\mathbf{E}_\theta \mathcal{T}_n = \theta$ and $\sup_n \mathtt{Var}_\theta(\mathcal{T}_n) < \infty$. Since $\{\mathcal{T}_n\}$ is a bounded sequence in $L^2(\mu_\theta)$, it has an L^2-weakly convergent subsequence. Because the limit $\mathcal{T}$ of this subsequence must be a tail function, $\mathcal{T} = \theta$ a.s. Finally, standard results of functional analysis imply that there exists a sequence of convex combinations of the estimators $\mathcal{T}_n$ that tends to θ in $L^2(\mu_\theta)$ and a.s.

The disadvantage of this approach is that the convergent subsequence and the convex combinations used may depend on θ; thus the argument sketched above only works for fixed θ. The proof of Theorem 6.1 below provides an explicit sequence of estimators not depending on θ.

We return to the general setting described in Section 2. A collection Ψ of bounded Borel functions on $\mathbf{R}$ is called a *determining class* if $\mu = \nu$ whenever $\int_{\mathbf{R}} \psi d\mu = \int_{\mathbf{R}} \psi d\nu$ for all $\psi \in \Psi$.

The following theorem generalizes Theorem 1.2.

THEOREM 6.1. *If $\sum_{k=0}^{\infty} u_k^2 = \infty$, then for any bounded Borel function ψ, there exists a sequence of functions $h_N : \mathbf{R}^N \to \mathbf{R}$ with the following property:*

for any probability measure η on $\mathbf{R}$, we have

$$h_N(X_1, \ldots, X_N) \;\to\; \int \psi d\eta \quad \textit{a.s. with respect to } \mu_\eta\,.$$

Thus the assumptions of the theorem imply that for any countable determining class Ψ of bounded Borel functions on $\mathbf{R}$, a.s. all the integrals $\{\int \psi d\eta\}_{\psi \in \Psi}$ can be computed from the observations X, and hence a.s. the measure η can be reconstructed from the observations.

PROOF. Fix $\psi \in \Psi$, and assume for now that $\alpha(\psi) = \int_{\mathbf{R}} \psi d\alpha = 0$. Without loss of generality, assume that $\| \psi \|_\infty \le 1$. Define

$$w(n) = w_n \stackrel{\text{def}}{=} \sum_{i=0}^{n} u_i^2, \text{ and } w(m,n) \stackrel{\text{def}}{=} \sum_{i=m+1}^{n} u_i^2.$$

For each pair $m_i < n_i$, let

$$L_i = L_i(\psi) = \frac{1}{w(m_i, n_i)} \sum_{j=m_i+1}^{n_i} u_j \psi(X_j).$$

Let $\{\epsilon_j\}$ be any sequence of positive numbers. We will inductively define $\{m_i\}, \{n_i\}$ with $m_i < n_i$, so that

$$(6.50) \qquad w(m_i, n_i) \ge w(m_i) \text{ for all } i, \text{ and } \mathtt{Cov}(L_i, L_j) \le \epsilon_i \text{ for all } j > i\,.$$

We now show how to define (m_{i+1}, n_{i+1}), given n_i, so that (6.50) is satisfied. Observe that

$$\begin{aligned}\texttt{Cov}(L_i, L_\ell) &= \frac{\sum_{k=m_i+1}^{n_i} \sum_{s=m_\ell+1}^{n_\ell} u_k u_s \eta(\psi)^2 (u_k u_{s-k} - u_k u_s)}{w(m_i, n_i) w(m_\ell, n_\ell)} \\ &= \frac{\eta(\psi)^2}{w(m_i, n_i) w(m_\ell, n_\ell)} \sum_{k=m_i+1}^{n_i} u_k^2 \left(\sum_{s=m_\ell+1}^{n_\ell} u_s u_{s-k} - u_s^2 \right). \end{aligned} \tag{6.51}$$

Fix k, and write m, n for m_ℓ, n_ℓ respectively. We claim that

$$\sum_{m+1}^{n} u_s u_{s-k} - u_s^2 \ \leq \ k\,. \tag{6.52}$$

Assume that $\sum_{m+1}^{n} u_s u_{s-k} - u_s^2 > 0$; if not (6.52) is trivial. Applying the inequality $a - b \leq (a^2 - b^2)/b$, valid for $b \leq a$, yields

$$\sum_{s=m+1}^{n} u_s u_{s-k} - u_s^2 \leq \frac{(\sum_{s=m+1}^{n} u_s u_{s-k})^2 - w(m,n)^2}{w(m,n)}. \tag{6.53}$$

Then applying Cauchy-Schwarz to the right-hand side of (6.53) bounds it by

$$\begin{aligned}\frac{w(m,n) w(m-k, n-k) - w(m,n)^2}{w(m,n)} &\leq w(m-k, n) - w(m,n) \\ &= w(m-k, m) \\ &\leq k\,,\end{aligned}$$

establishing (6.52). Using the bound (6.52) in (6.51), and recalling that $|\psi| \leq 1$, yields

$$\texttt{Cov}(L_i, L_\ell) \leq \frac{1}{w(m_i, n_i) w(m_\ell, n_\ell)} \sum_{k=m_i+1}^{n_i} u_k^2 k \leq \frac{n_i}{w(m_\ell, n_\ell)}\ . \tag{6.54}$$

Pick m_{i+1} large enough so that

$$w(m_{i+1}) \geq \frac{n_i}{\epsilon_i}\ , \tag{6.55}$$

and let $n_{i+1} \stackrel{\text{def}}{=} \inf\{t : w(m_{i+1}, t) \geq w(m_{i+1})\}$. Then for any $\ell \geq i+1$, since $w(m_\ell, n_\ell) \geq w(m_\ell) \geq w(m_{i+1})$, (6.55) and (6.54) yield that $\texttt{Cov}(L_i, L_\ell) \leq \epsilon_i$.

Observe that $\mathbf{E}[L_i] = \eta(\psi)$, and

$$\mathbf{E}\left[\left(\sum_{j=m_i+1}^{n_i} u_j\psi(X_j)\right)^2\right] = 2\eta(\psi)^2 \sum_{j=m_i+1}^{n_i}\sum_{k=j+1}^{n_i} u_j u_k u_j u_{k-j} + \sum_{j=m_i+1}^{n_i} \mathbf{E}[\psi(X_j)^2]\, u_j^2$$

$$\leq \|\psi\|_\infty^2 \left\{2\sum_{j=m_i+1}^{n_i} u_j^2 \sum_{k=j+1}^{n_i} u_k u_{k-j} + w(m_i,n_i)\right\}. \tag{6.56}$$

Fix i, let $m = m_i, n = n_i$. For j fixed, using Cauchy-Schwarz yields

$$\sum_{k=j+1}^{n} u_k u_{k-j} \leq \sqrt{w(j,n)w_{n-j}} \leq w_n . \tag{6.57}$$

Plugging (6.57) into (6.56), and recalling that $\|\psi\|_\infty < 1$, gives that

$$\mathbf{E}\left[\left(\sum_{j=m_i+1}^{n_i} u_j\psi(X_j)\right)^2\right] \leq 2w_{n_i}^2 + w_{n_i} . \tag{6.58}$$

Thus,

$$\mathbf{E}[L_i^2] \leq \frac{2w_{n_i}^2 + w_{n_i}}{w_{n_i}^2/4} = 8 + \frac{4}{w_{n_i}} \leq B .$$

Choosing, for example, $\epsilon_i = i^{-3}$, one can apply the strong law for weakly correlated random variables (see Theorem A in section 37 of [19]), to get that

$$G_n(\psi) \stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=1}^{n} L_i(\psi) \to \eta(\psi) \ a.\ s. \tag{6.59}$$

For general ψ, define $H_n(\psi) = G_n(\psi - \alpha(\psi)) + \alpha(\psi)$. From (6.59), it follows that

$$H_n(\psi) \to \eta(\psi - \alpha(\psi)) + \alpha(\psi) = \eta(\psi)\,. \tag{6.60}$$

To finish the proof, define $h_N(X_1,\ldots,X_N) \stackrel{\text{def}}{=} H_{k(N)}(\psi)$, where $k(N)$ is the largest integer k such that $n_k \leq N$. □

7. Quenched Large Deviations Criterion. Recall that $\rho_n = \frac{d\mu_\eta}{d\mu_\alpha}_{|\mathcal{G}_n}$, the density of the measure μ_η restricted to $\mathcal{G}_n$ with respect to the measure μ_α restricted to $\mathcal{G}_n$.

We make the additional assumption that

$$r \; = \; \int_{\mathbf{R}} \left(\frac{d\eta}{d\alpha}\right)^2 d\alpha \; = \; \int_{\mathbf{R}} \frac{d\eta}{d\alpha} d\eta \; < \; \infty. \tag{7.61}$$

For two binary sequences δ, δ', define $J(\delta, \delta') = |\{n : \delta_n = \delta'_n = 1\}|$, the number of joint renewals.

Lemma 7.1. *If* $\mathbf{E}[r^{J(\Delta,\Delta')} \mid \Delta] < \infty$, *then* $\mu_\eta \ll \mu_\alpha$.

Proof. Let $x(y,z,\delta)_n = z_n\delta_n + y_n(1-\delta_n)$. We have

$$\mathbf{E}_{\mathbf{Q}_\eta}[\rho_n(X) \mid \Delta = \delta] \; = \int\limits_{\mathbf{R}^\infty}\int\limits_{\mathbf{R}^\infty} \rho_n(x(y,z,\delta)) d\alpha^\infty(y) d\eta^\infty(z)\,, \tag{7.62}$$

and expanding ρ_n shows that (7.62) equals

$$\int\limits_{\mathbf{R}^\infty}\int\limits_{\mathbf{R}^\infty}\int\limits_{\Upsilon} \prod_{i=1}^n \left[\frac{d\eta}{d\alpha}(x(y,z,\delta)_i)\delta'_i + 1 - \delta'_i\right] d\mathbf{P}(\delta') d\alpha^\infty(y) d\eta^\infty(z)\,. \tag{7.63}$$

Using Fubini's Theorem and the independence of coordinates under product measure, (7.63) is equal to

$$\int_\Upsilon \prod_{i=1}^n \int_{\mathbf{R}}\int_{\mathbf{R}} \left[\frac{d\eta}{d\alpha}(x(y,z,\delta)_i)\delta'_i + 1 - \delta'_i\right] d\alpha(y) d\eta(z) d\mathbf{P}(\delta'). \tag{7.64}$$

If

$$I \stackrel{\text{def}}{=} \int_{\mathbf{R}}\int_{\mathbf{R}} \left[\frac{d\eta}{d\alpha}(x(y,z,\delta)_i)\delta'_i + 1 - \delta'_i\right] d\alpha(y) d\eta(z)\,,$$

then we have that

$$I \; = \; \begin{cases} 1 & \text{if } \delta'_i = 0 \\ \int \frac{d\eta}{d\alpha}(y) d\alpha(y) = 1 & \text{if } \delta'_i = 1, \delta_i = 0 \\ \int \frac{d\eta}{d\alpha}(z) d\eta(z) = r & \text{if } \delta' = 1, \delta_i = 1 \end{cases} \tag{7.65}$$

Plugging (7.65) into (7.64), we get that

$$\begin{aligned}\mathbf{E}_{\mathbf{Q}_\eta}[\rho_n(X) \mid \Delta = \delta] &= \int_\Upsilon \prod_{i=1}^n r^{\delta_i \delta_i'} d\mathbf{P}(\delta') \\ &\le \int_\Upsilon r^{J(\delta,\delta')} d\mathbf{P}(\delta') \\ &= \mathbf{E}[r^{J(\Delta,\Delta')} \mid \Delta = \delta]\,.\end{aligned}$$

Applying Fatou's Lemma, we infer that $\mathbf{E}\mathbf{Q}_\eta[\rho(X) \mid \Delta] < \infty$, whence

$$\mathbf{Q}_\eta[\rho(X) < \infty] \;=\; 1\,.$$

The Lebesgue Decomposition (3.2) implies that $\mu_\eta \ll \mu_\alpha$. □

8. Absence of Phase Transition in Almost Transient Case In this section, we apply the quenched moment generating function criterion established in the previous section.

Let $N[m,n]$ be the number of renewals in the interval $[m,n]$, and write $N_m = N[0,m]$. Let $U_m = U(m) = \mathbf{E}N_m = \sum_{k=0}^m u_k$.

Lemma 8.1. *For any integer $A \ge 1$, we have $\mathbf{P}[N_m \ge AeU_m] \le e^{-A}$.*

Proof. For $A = 1$, the inequality follows from Markov's inequality. Assume it holds for $A-1$. On the event E that $N_m \ge (A-1)eU_m$, define τ as the time of the $\lceil (A-1)eU_m \rceil^{th}$ renewal. Then

$$\mathbf{P}[N_m \ge AeU_m \mid E] \le \mathbf{P}[N[\tau,m] \ge eU_m|E] \le \mathbf{P}[N_m \ge eU_m] \le e^{-1}.$$

Consequently,

$$\mathbf{P}[N_m \ge AeU_m] \le \mathbf{P}[N_m \ge AeU_m|E]e^{-(A-1)} \le e^{-A}.$$

□

Theorem 8.2. *Suppose that the renewal probabilities* $\{u_n\}$ *satisfy*

$$U(e^k) \;=\; o(k/\log k)\,,$$

and also $u_k \le C_2 k^{-1}$*. If* $\eta \ll \alpha$ *and* $\frac{d\eta}{d\alpha} \in L^2(\alpha)$*, then* $\mu_\eta \ll \mu_\alpha$*.*

Proof. In this proof the probability space will always be Υ^2, endowed with the product measure $\mathbf{P}^2$, where $\mathbf{P}$ is the renewal probability measure. Let

$$J[m,n] \;=\; |\{n \le k \le m : \Delta_k = \Delta'_k = 1\}|$$

be the number of *joint renewals* in the interval $[m,n]$.

First we show that

$$\forall\, C, \quad T_1 + \cdots + T_k \ge e^{Ck} \quad \text{eventually.} \tag{8.66}$$

Observe that

$$\mathbf{P}[T_1 + \cdots + T_k \le e^{Ck}] = \mathbf{P}[N(e^{Ck}) \ge k] \le \exp\left(-\frac{k}{eU(e^{Ck})}\right)\,. \tag{8.67}$$

Our assumption guarantees that $k/eU(e^{Ck}) \ge 2\log k$ eventually, and hence the right-hand side of (8.67) is summable. Consequently, for almost all Δ, there is an integer $M = M(\Delta)$ such that $\sum_{j=1}^k T_j > e^{Ck}$ for all $k > M(\Delta)$. Equivalently, $N[0, \exp(Ck)] < k$ when $k > M$. To use Lemma 7.1, it suffices to show that

$$\sum_n s^n \mathbf{P}[J[0,n] \ge n \mid \Delta] < \infty \;\text{ a.s., for all real } s.$$

We have

$$\sum_n s^n \mathbf{P}[J[0,n] \ge n \mid \Delta] \;\le\; C_2(\Delta) + \sum_{n=M}^{\infty} s^n \mathbf{P}[J(e^{Cn}, \infty) \ge 1 \mid \Delta] \tag{8.68}$$

Observe that

$$\mathbf{E}[J(e^{Cn}, \infty)] = \sum_{k=\exp(Cn)}^{\infty} u_k^2 \le C_3 e^{-Cn}, \tag{8.69}$$

since we have assumed that $u_n \leq C_2 n^{-1}$. Thus the expectation of the sum on the right in (8.68), for C large enough, is finite. Thus the sum is finite Δ-almost surely, so the conditions of Lemma 7.1 are satisfied. We conclude that $\mu_\eta \ll \mu_\alpha$. □

We now discuss examples of Markov chains which satisfy the hypothesis of Theorem 8.2.

LEMMA 8.3. *Given two Markov chains with transition matrices P, P' on state spaces $\mathcal{X}$ and $\mathcal{Y}$ with distinguished states x_0, y_0 respectively, construct a new chain $\Phi = (X, Y)$ on $\mathcal{X} \times \mathcal{Y}$ with transition matrix*

$$Q((x_1, y_1), (x_2, y_2)) = \begin{cases} P(x_1, x_2)P'(y_1, y_2) & \text{if } y_1 = 0 \\ P'(y_1, y_2) & \text{if } y_1 \neq 0,\ x_1 = x_2 \end{cases}.$$

Let $A(s) = \sum_{n=1}^{\infty} f_n s^n$ be the moment generating function for the distribution of the time of first return to x_0 for the chain with transitions P, and let $B(s)$ be the corresponding generating function but for the chain P' and state y_0. Then the generating function for the distribution of the time of the first return of Φ to (x_0, y_0) is the composition $A \circ B$.

PROOF. Let $S_1, S_2, \ldots$ be the times of successive visits of Φ to $\mathcal{X} \times \{y_o\}$, and $T_k = S_k - S_{k-1}$. Observe that Y is a Markov chain with transition matrix P', so $\{T_k\}$ has the distribution of return times to y_0 for the chain P'.

Let $\tau = \inf\{n \geq 1 : X_{S_n} = x_0\}$. Note that $\{X_{S_n}\}_{n=0}^{\infty}$ is a Markov chain with transition matrix P, independent of $\{T_n\}$. Hence τ is independent of $\{T_n\}$, and

$$T = T_1 + \cdots + T_\tau$$

is the time of the first return of Φ to (x_0, y_0). A standard calculation (see, for example, XII.1 in [8]) yields that the generating function $\mathbf{E}s^T$ is $A \circ B$. □

Let F, U be the moment generating functions for the sequences $\{f_n\}$ and $\{u_n\}$ respectively. Define $L : (0, \infty) \to (1, \infty)$ by $L(y) = 1 - \frac{1}{y}$, and note that $F = L \circ U$. Denote $W(y) = U \circ L(y) = L^{-1} \circ F \circ L$. When $F_3 = F_1 \circ F_2$, it follows that $W_3 = W_1 \circ W_2$.

We make use of the following Tauberian theorem, found as Theorem 2.4.3 in [16].

Proposition 8.4. *Let $\{a_n\}$ be a sequence of non-negative reals, $A(s) = \sum_{n=0}^{\infty} a_n s^n$ its generating function, $W(y) \stackrel{\text{def}}{=} A(1 - y^{-1})$, $\alpha \geq 0$ a constant, and ℓ a slowly varying function. The following are equivalent:*

(i) $A(s) \asymp (1-s)^{-\alpha} \ell((1-s)^{-1})$ for $s < 1$ near 1.

(ii) $W(y) \asymp y^{\alpha} \ell(y)$ for large y.

(iii) $A_n = \sum_{k=0}^{n} a_k \asymp n^{\alpha} \ell(n)$.

We now show that there are Markov chains with no phase transition.

Proposition 8.5. *There is a Markov chain that satisfies $U_n \asymp \log\log n$, and $u_n \leq C n^{-1}$.*

Proof. For simple random walk on $\mathbf{Z}^2$, we have

$$U(s) \asymp \sum_{n=1}^{\infty} n^{-1} s^{2n} = -\log(1 - s^2)\,.$$

Thus, $W(y) \asymp \log y$. Consequently, $W \circ W(y) \asymp \log\log(y)$ corresponds to the chain in Lemma 8.3 with both P and P' the transition matrices for simple random walk on $\mathbf{Z}^2$. Proposition 8.4 implies that $U_n \asymp \log\log n$. Finally,

$$u_n \leq \mathbf{P}[X_n = 0] \leq C n^{-1}\,,$$

since X is a simple random walk on $\mathbf{Z}^2$. □

In conjunction with Theorem 8.2, this establishes Theorem 1.4.

Lemma 8.3 can be applied to construct Markov chains obeying the hypotheses of Proposition 4.1 and Theorem 5.1. Take as the chains X and Y the simple random walk on $\mathbf{Z}$. The moment generating function $U_{[1]}$ for the return probabilities u_n of the simple random walk is given by $U_{[1]}(s) = (1-s^2)^{-1/2}$ (see XIII.4 in [8]). Then $W_{[1]}(y) = U_{[1]} \circ L(y) = (\frac{y}{2-y^{-1}})^{1/2}$ satisfies $W_{[1]}(y) \sim (y/2)^{1/2}$ as $y \to \infty$. Hence $W(y) = W_{[1]} \circ W_{[1]}(y) \asymp y^{1/4}$, and by Proposition 8.4, $U_n \asymp n^{1/4}$.

The last example is closely related to the work of Gerl in [9]. He considered certain "lexicographic spanning trees" $\mathcal{T}_d$ in $\mathbf{Z}^d$, where the path from the origin to a lattice point $(x_1, \ldots, x_d)$ consists of at most d straight line segments, going through the points $(x_1, \ldots, x_k, 0, \ldots, 0)$ for $k = 1, \ldots, d$ in order. Gerl showed that for $d \geq 2$, the return probabilities of simple random walk on $\mathcal{T}_d$ satisfy $u_{2n} \asymp n^{2^{-d}-1}$; after introducing delays, this provides further examples of Markov chains with a phase transition ($0 < \theta_c < 1$).

9. Absence of Phase Transition in $\mathbf{Z}^2$. The results in [10] (as summarized in Theorem B of Section 1) show that for the random walk on $\mathbf{Z}^2$ which chooses at each unit of time a nearest neighbor with equal probability, the measures μ_θ and μ_0 are mutually absolutely continuous for all θ. The argument does not extend to Markov chains which are small perturbations of this walk. For example, if the walk is allowed to remain at its current position with some probability, the asymptotic behavior of $\{u_n\}$ is not altered, but Theorem B does not resolve whether $\mu_\theta \ll \mu_0$ always. In this section, we show that for any Markov chain whose return probabilities $\{u_n\}$ have the same asymptotic behavior as the simple random walk on $\mathbf{Z}^2$, that is, $u_n \asymp n^{-1}$, the measures μ_θ and μ_0 are always mutually absolutely continuous.

Recall that T is the time of the first renewal, and $T_1, T_2, \ldots$ are i.i.d. copies of T. Also, $S_n = \sum_{j=1}^n T_j$ denotes the time of the n^{th} renewal. Recall from before that Δ_n is the indicator of a renewal at time n, hence

$$\{S_n = k \text{ for some } n \geq 1\} \ = \ \{\Delta_k = 1\} \, .$$

Let S'_n and T'_n denote the renewal times and inter-renewal times of another independent renewal process. Recall that J is the total number of simultaneous renewals: $J = \sum_{k=0}^\infty \Delta_k \Delta'_k$. If $\mathcal{S}_k$ is the sigma-field generated by $\{T_j : 1 \leq j \leq k\}$, then define

$$q_n \ = \ \mathbf{P}[J \geq n \mid \Delta] \ = \ \mathbf{P}[\, |\{(i,j) : S_i = S'_j\}| \ \geq n \mid \mathcal{S}_\infty] \, . \tag{9.70}$$

In this section, we prove the following:

Theorem 9.1. *When $u_n = O(n^{-1})$, the sequence $\{q_n\}$ defined in (9.70) decays faster than exponentially almost surely, that is,*

$$n^{-1} \log q_n \to -\infty \text{ almost surely.}$$

Consequently, the quenched large deviations criterion Lemma 7.1 implies that if $\eta \ll \alpha$ and $\frac{d\eta}{d\alpha} \in L^2(\alpha)$, then $\mu_\eta \ll \mu_\alpha$.

The proof relies on the specific renewal process via the tails of the inter-return times. Specifically, we use that the renewal probabilities u_n are at most c/n and that

$$\mathbf{P}[\log T \geq t] \geq c t^{-1}. \tag{9.71}$$

(9.71) can be deduced, for example, from Proposition 8.4 and the relation

$$U(s) = [1 - F(s)]^{-1},$$

where $U(s)$ and $F(s)$ are the generating functions for the sequences $\{u_n\}$ and $\{\mathbf{P}[T = n]\}$ respectively. From this it follows that if $\omega(n)$ is any function going to infinity, then

$$\mathbf{P}[S_{n \log n\omega^2(n)} \leq e^{n\omega(n)}] \leq (1 - c/(n\omega(n)))^{n \log n\omega^2(n)} \leq n^{-c\omega(n)}.$$

This is summable, so by Borel-Cantelli, if we define $m = m(n) := n \log n\omega^2(n)$, then

$$n^{-1} \log S_{m(n)} \to \infty \tag{9.72}$$

almost surely.

Proof of Theorem 9.1. Define the random variables

$$J^m \stackrel{\text{def}}{=} |\{(i,j) \ : \ i \geq m, j > 1 \text{ and } S_i = S'_j \}|$$

and let $Q_m \stackrel{\text{def}}{=} \mathbf{P}[J^m \geq 1 \mid \mathcal{S}_\infty]$.

In what follows, $m(n)$ will be as defined above, and we will often write simply m for $m(n)$. Let

$$r_n \stackrel{\text{def}}{=} \mathbf{P}[\,|\{(i,j) \ : \ i < m(n) \text{ and } S_i = S'_j\}| \ \geq n \ \mid \ \mathcal{S}_\infty].$$

We have

$$q_n = \mathbf{P}[J \geq n \text{ and } J^{m(n)} \geq 1 \mid \mathcal{S}_\infty] + \mathbf{P}[J \geq n \text{ and } J^{m(n)} = 0 \mid \mathcal{S}_\infty]$$

$$\leq Q_{m(n)} + r_n . \tag{9.73}$$

Write $Q^*_m \stackrel{\text{def}}{=} \mathbf{E}[Q_m \mid \mathcal{S}_m] \ = \ \mathbf{P}[J^{m(n)} \geq 1 \mid \mathcal{S}_m]$. Then

$$Q^*_m \ \leq \ \mathbf{E}[J^{m(n)} \mid \mathcal{S}_m] \ \leq \ \sum_{k=1}^{\infty} u_k u_{k+S_m} \ \leq \ \sum_{k=1}^{\infty} \frac{c}{k} \frac{c}{k+S_m} \ \leq \ c \frac{\log S_m}{S_m} \,.$$

By (9.72), we see that $n^{-1}\log Q^*_{m(n)} \to -\infty$ almost surely.

Since $Q^*_m = \mathbf{E}[Q_m|\mathcal{S}_m]$, we see that $\mathbf{P}[Q_m \geq 2^m Q^*_m] \leq 2^{-m}$, hence

$$Q_m \geq 2^m Q^*_m \text{ finitely often}$$

and it follows that $n^{-1}\log Q_{m(n)} \to -\infty$. It therefore suffices by (9.73) to show that

$$\frac{\log r_n}{n} \to -\infty \text{ almost surely.} \tag{9.74}$$

Let $[m(n)] \stackrel{\text{def}}{=} \{1, 2, \ldots, m(n)\}$. We can bound r_n above by

$$\sum_{\substack{A\subset[m(n)]\\|A|=n}} \mathbf{P}[\forall i \in A, \exists j \geq 1 \text{ so that } S'_j = S_i \mid \mathcal{S}_\infty\,] \;\leq\; \binom{m}{n} R_n\,, \tag{9.75}$$

where

$$R_n \stackrel{\text{def}}{=} \max_{\substack{A\subset[m(n)]\\|A|=n}} \mathbf{P}[\forall i \in A, \exists j \geq 1 \text{ so that } S'_j = S_i \;\mid\; \mathcal{S}_\infty\,]\,.$$

We can conclude that

$$\log r_n \;\leq\; \log\binom{m}{n} + \log R_n\,. \tag{9.76}$$

Notice that $\binom{m}{n} = e^{O(n\log\log n)}$ when $\omega(n)$ is no more than polylog n; for convenience, we assume throughout that $\omega^2(n) = o(\log n)$. Hence, if we can show that

$$\frac{\log R_n}{n\log\log n} \to -\infty \text{ almost surely,} \tag{9.77}$$

then by (9.76), it must be that (9.74) holds.

For any n-element set $A \subset [m(n)]$, we use the following notation:

- $A = \{x_1 < x_2 < \cdots < x_n\}$, and $m' \stackrel{\text{def}}{=} x_m$.
- For any $k \leq m'$, let $I(k)$ be the set of indices i such that $\{T_i\}_{i\in I(k)}$ are the k largest inter-renewal times among $\{T_i\}_{i\leq m'}$.

- For $i \le n$, let $M(A,i) \stackrel{\text{def}}{=} \max\{T_j : x_{i-1}+1 \le j \le x_i\}$.

We have

$$\mathbf{P}[\forall X_i \in A, \exists j \ge 1 \text{ so that } S'_j = S_{x_i} \mid \mathcal{S}_\infty] = \prod_{i=1}^{n} u_{S_{x_i}-S_{x_{i-1}}} ,$$

where $S_0 \stackrel{\text{def}}{=} 0$. Recalling that $u_n \le c/n$, we may bound the right–hand side above by

$$\prod_{i=1}^{n} \frac{c}{S_{x_i} - S_{x_{i-1}}} = \prod_{i=1}^{n} \frac{c}{\sum_{j=x_{i-1}+1}^{x_i} T_j} \le R(A) \stackrel{\text{def}}{=} \prod_{i=1}^{n} \frac{c}{M(A,i)} . \tag{9.78}$$

To summarize, we have

$$R_n \le \max_{\substack{A\subset[m(n)]\\|A|=n}} R(A) = \max_{\substack{A\subset[m(n)]\\|A|=n}} \prod_{i=1}^{n} \frac{c}{M(A,i)} . \tag{9.79}$$

To see where this is going, compute what happens when $A = [n]$. From the tail behavior of T, we know that

$$\liminf_{n\to\infty} \frac{\log R([n])}{n \log n} > 0 .$$

To establish (9.77), we need something like this for R_n instead of $R([n])$.

In what follows, $k_0(n) \stackrel{\text{def}}{=} 10(\log n\omega(n))^2$.

LEMMA 9.2. *Almost surely, there is some (random) N so that if $n > N$, then for all n-element sets $A \subseteq [m]$, providing k satisfies $m' \ge k > k_0(n)$, at least $kn/(6m' \log\log n)$ values of i satisfy $M(A,i) \in \{T_i : i \in I(k)\}$.*

Assuming this for the moment, we finish the proof of the theorem. The following summation by parts principle will be needed.

LEMMA 9.3. *Let $E(k)$ be the k largest values in a given finite set E of positive real numbers. Suppose another set E' contains at least ϵk members of $E(k)$ for* *every $k_0 < k \le |E|$. Then*

$$\sum_{e \in E'} e \;\ge\; \epsilon \sum_{e \in E \setminus E(k_0)} e\,.$$

PROOF OF LEMMA 9.3. Let $E = \{e_j, j = 1, ..., N\}$ in decreasing order and let $e_{N+1} = 0$ for convenience. Write

$$f(j) \stackrel{\text{def}}{=} \mathbf{1}\{e_j \in E'\}\,, \text{ and let } F(k) = f(1) + ... + f(k)\,.$$

Then

$$\begin{aligned}
\sum_{j=1}^{N} f(j) e_j = \sum_{j=1}^{N} (F(j) - F(j-1)) e_j \;&=\; \sum_{k=1}^{N} F(k)(e_k - e_{k+1}) \\
\ge \sum_{k=k_0+1}^{N} F(k)(e_k - e_{k+1}) \;&\ge\; \sum_{k=k_0+1}^{N} \epsilon k (e_k - e_{k+1}) \\
= \epsilon \left\{ (k_0+1) e_{k_0+1} + \sum_{k=k_0+2}^{N} e_k \right\} \;&\ge\; \epsilon \sum_{k=k_0+1}^{N} e_k \\
= \epsilon \sum_{e \in e \setminus E(k_0)} e &
\end{aligned}$$

This proves the lemma. □

LEMMA 9.4. *Write $\{T_i\}_{i=1}^m$ in decreasing order:*

$$T_{(1)} \ge T_{(2)} \ge \cdots \ge T_{(m)}\,.$$

Then

$$\liminf_{m \to \infty} \frac{1}{m \log m} \sum_{i=k_0(n)+1}^{m} \log T_{(i)} > 0\,.$$

PROOF. It suffices to prove this lemma in the case where $u_n \asymp n^{-1}$, because in the case where $u_n \le cn^{-1}$, the random variables T_i stochastically dominate those in the first case.

Let $Y_i \stackrel{\text{def}}{=} \log T_i$; then Y_i are i.i.d. random variables with tails obeying

$$\mathbf{P}[Y_i \geq t] \asymp t^{-1}\,.$$

Write $Y_{(i)}$ for the i^{th} largest among $\{Y_i\}_{i=1}^n$. From [5], it can be seen that

$$\lim_{n\to\infty} \frac{1}{n\log n}\left(\sum_{i=2}^{k_0(n)} Y_{(i)} - n\log\log n\right) = 0\,. \tag{9.80}$$

From Theorem 1 of [20], we can deduce that

$$\liminf_{n\to\infty} \frac{1}{n\log n}\sum_{i=2}^{n} Y_{(i)} > 0\,. \tag{9.81}$$

Combining (9.80) and (9.81) yields

$$\liminf_{n\to\infty} \frac{1}{n\log n}\sum_{i=k_0(n)+1}^{n} Y_{(i)} > 0\,.$$

□

Recall that

$$R(A) = \prod_{i=1}^{n} cM(A,i)^{-1}\,.$$

From Lemma 9.2 we see that almost surely there exists an N so that, for all $n > N$ and $k_0(n) < k \leq m'$ the set $\{M(A,i) \;:\; 1 \leq i \leq n\}$ includes at least $kn/(6m'\log\log n)$ of the k greatest values of $\{T_j\}_{j=1}^{m'}$ Therefore by Lemma 9.3 (applied to the logs of the denominators), we see that for $n > N$ and all $A \subset [m(n)]$,

$$-\log R(A) \geq \frac{(n/m')\sum_{i=k_0(n)+1}^{m'} \log(T_{(i)}/c)}{(6\log\log n)}\,.$$

Since $(m'\log m')^{-1}\sum_{i=k_0(n)+1}^{m'} \log(T_i/c)$ has a nonzero liminf by Lemma 9.4, we see that $\log\log n \frac{\log R_n}{n\log n}$ is not going to zero, from which follow (9.77) and the theorem. □

It remains to prove Lemma 9.2. Define the event $G_{n,m'}$ to be the event

> for all n-element sets $A \subset [m(n)]$ with maximal element m', and k obeying $m' \geq k > k_0(n)$, at least $kn/(6m' \log\log n)$ values of i satisfy
>
> $$M(A,i) \in \{T_i \; : \; i \in I(k)\} \, .$$

Then define $G_n \stackrel{\text{def}}{=} \cap_{m'=n}^{m(n)} G_{n,m'}$. The conclusion of Lemma 9.2 is that

$$\mathbf{P}[G_n \text{ eventually}] \; = \; 1 \, . \tag{9.82}$$

If we can show that

$$\mathbf{P}[G_{n,m'}^c] \; \leq \; n^{-3} \, , \tag{9.83}$$

then by summing over $m' \in [n, n(m)]$, we can conclude that $\mathbf{P}[G_n^c] \leq n^{-2}$, and hence by Borel-Cantelli, that (9.82) holds.

We prove (9.83) for $m' = m$, the argument for other values of m being identical. The values $T_1, T_2, \ldots$ are exchangeable, so the set $I(k)$ is a uniform random k-element subset of $[m]$ and we may restate (9.83) (with $m' = m$):

Let

$$I(k) \; = \; \{r_1 < r_2 < \cdots < r_k\}$$

be a uniform k-subset of $[m(n)]$; then the event $G_{n,m}$ has the same probability as the event $\widetilde{G}_{n,m}$, defined as

> for all n-element sets $A = \{x_1 < \cdots < x_n = m\} \subseteq [m]$ and k satisfying $m \geq k > k_0(n)$, at least $kn/(6m \log\log n)$ of the intervals $[x_{i-1}+1, x_i]$ contain an element of $I(k)$.

Equivalently, $\widetilde{G}_{n,m}$ is the event that for all n-element sets $A = \{x_1 < \cdots < x_n = m\} \subseteq [m]$ and k satisfying $k > k_0(n)$, at least $kn/(6m \log\log n)$ of the intervals $[r_i, r_{i+1} - 1], 1 \le i \le k$ contains an element of A.

Finally, $\widetilde{G}_{n,m}$ can be rewritten again as the event

for k obeying $m \ge k > k_0(n)$, no $kn/(6m \log\log n) - 1$ of the intervals $[r_i, r_{i+1} - 1]$ together contain n points.

Proving the inequality (9.83) is then the same as proving that

$$\mathbf{P}[\widetilde{G}_{n,m}] \ \ge \ 1 - n^{-3} \,. \tag{9.84}$$

For $0 \le j \le k$ let D_j denote $r_{j+1} - r_j$ where $r_0 := 0$ and $r_{k+1} \overset{\text{def}}{=} m + 1$. For any $B \subseteq [k]$, let $W(B)$ denote the sum $\sum_{j \in B} D_j$. Then define the events $\widetilde{G}_{n,m,k}$ to be

For all sets $B \subset [k]$ with $|B| < kn/(m \log\log n)$, we have $W(B) < n$.

We have that $\widetilde{G}_{n,m} = \cap_{k=k_0(n)+1}^{m} \widetilde{G}_{n,m,k}$.

Set $\epsilon = n/m = (\log n\omega^2(n))^{-1}$, and set $\delta = \epsilon/(6 \log\log n)$, so that

$$\delta \log \frac{1}{\delta} = \frac{\epsilon}{6} \frac{\log(1/\epsilon)}{\log\log n} \le \frac{\epsilon}{5}$$

for sufficiently large n. We now need to use the following lemma:

Lemma 9.5. *Let $p(k, m, \epsilon, \delta)$ denote the probability that there is some set B of cardinality at most δk such that $W(B) \ge \epsilon m$. Then for ϵ sufficiently small and $\delta \log(1/\delta) \le \epsilon/5$,*

$$p(k, m, \epsilon, \delta) \le e^{-k\epsilon/2}.$$

The proof of this will be provided later.

Now applying Lemma 9.5, we have that for fixed k so that $m \geq k > k_0(n)$,

$$\mathbf{P}[\widetilde{G}_{n,m,k}] \;\geq\; 1 - n^{-5}\,,$$

since $\frac{k\epsilon}{2} \geq n^{-5}$. Summing over k gives that

$$\mathbf{P}[\widetilde{G}_{n,m}] \;\geq\; 1 - n^{-3}\,.$$

To prove Lemma 9.5, two more lemmas are required.

LEMMA 9.6. *Let $B \subseteq [k]$ and $W := \sum_{j \in B} D_j$. Then for $0 < \lambda < 1$,*

$$\mathbf{E}e^{\lambda k W/m} \leq \left(\frac{1}{1-\lambda}\right)^{|B|}. \tag{9.85}$$

PROOF. The collection $\{D_j : 0 \leq j \leq k\}$ is exchangeable and is stochastically increasing in m. It follows that the conditional joint distribution of any subset of these given the others is stochastically decreasing in the values conditioned on, and hence that for any $B \subseteq [k]$, and $\lambda > 0$,

$$\mathbf{E}\exp\left(\sum_{j \in B} D_j\right) \leq \prod_{j \in B} \mathbf{E}\exp(D_j) = (\mathbf{E}\exp(D_0))^{|B|}\,. \tag{9.86}$$

The distribution of D_0 is explicitly described by

$$\mathbf{P}(D_0 \geq j) \;=\; (1 - \frac{j}{m}) \cdots (1 - \frac{j}{m-k+1})\,.$$

Thus

$$\mathbf{P}(D_0 \geq j) \leq \left(1 - \frac{j}{m}\right)^k \leq e^{-kj/m}.$$

In other words, kD_0/m is stochastically dominated by an exponential of mean 1, leading to $\mathbf{E}e^{\lambda k D_0/m} \leq 1/(1-\lambda)$. Thus by (9.86), $\mathbf{E}\exp(\lambda k W/m) \leq (1-\lambda)^{-|B|}$, proving the lemma. □

LEMMA 9.7. *Let $|B| = j$ and let $W = \sum_{j \in B} D_j$ as in the previous lemma. Then*

$$\mathbf{P}(\frac{W}{m} \geq \frac{t}{k}) \leq e^{-t} \left(\frac{et}{j}\right)^j. \tag{9.87}$$

PROOF. Use Markov's inequality

$$\mathbf{P}(\frac{W}{m} \geq \frac{t}{k}) \leq \frac{\mathbf{E}e^{\lambda k W/m}}{e^{\lambda t}}.$$

Set $\lambda = 1 - j/t$ and use the previous lemma to get

$$\begin{aligned}\mathbf{P}(\frac{W}{m} \geq \frac{t}{k}) &\leq (1-\lambda)^{-j} e^{-\lambda t} \\ &= \left(\frac{t}{j}\right)^j e^{j-t},\end{aligned}$$

proving the lemma. □

PROOF OF LEMMA 9.5. We can assume without loss of generality that $j := \delta k$ is an integer and that $n := \epsilon m$ is an integer. By exchangeability, $p(k, m, \epsilon, \delta)$ is at most $\binom{k+1}{j}$ times the probability that $W(B)/m \geq \epsilon$ for any particular B of cardinality j. Setting $t = k\epsilon$ and plugging in the result of Lemma 9.7 then gives

$$\begin{aligned}p(k, m, \epsilon, \delta) &\leq \binom{k+1}{j} \left(\frac{\epsilon k}{j}\right)^j e^{j - \epsilon k} \\ &= \binom{k}{\delta k} \left(\frac{\epsilon}{\delta}\right)^{\delta k} e^{(\delta - \epsilon)k}.\end{aligned}$$

The inequality $\binom{a}{b} \leq (a/b)^b (a/(a-b))^{a-b}$ holds for all integers $a \geq b \geq 0$ (with $0^0 := 1$) and leads to the right-hand side of the previous equation being bounded above by

$$\left(\frac{1}{\delta}\right)^{\delta k} \left(\frac{1}{1-\delta}\right)^{(1-\delta)k} \left(\frac{\epsilon}{\delta}\right)^{\delta k} e^{(\delta-\epsilon)k}.$$

Hence $p(k,m,\epsilon,\delta) \le e^{kr(\epsilon,\delta)}$ where

$$r(\epsilon,\delta) = \delta(\log\epsilon - 2\log\delta + \log(1-\delta)) - \log(1-\delta) + \delta - \epsilon.$$

Since $\log\epsilon$ and $\log(1-\delta)$ are negative, we have

$$r(\epsilon,\delta) \le 2\delta\log(1/\delta) - \epsilon + \delta + \log(1/(1-\delta)).$$

For sufficiently small ϵ, hence small δ, we have $\delta + \log(1/(1-\delta)) < (1/2)\delta\log(1/\delta)$, hence

$$r(\epsilon,\delta) < (5/2)\delta\log(1/\delta) - \epsilon \le \epsilon/2 - \epsilon = -\frac{\epsilon}{2},$$

by the choice of δ. This finishes the proof. □

10. Concluding Remarks. A Markov chain Γ with state-space $\mathcal{X}$ and transition kernel P is *transitive* if, for each pair of states $x, y \in \mathcal{X}$, there is an invertible mapping $\Phi : \mathcal{X} \to \mathcal{X}$ so that $\Phi(x) = y$, and $P(y, \Phi(z)) = P(x, z)$ for all $z \in \mathcal{X}$. Random walks, for example, are transitive Markov chains. When the underlying Markov chain Γ is transitive, our model has an equivalent percolation description. Indeed, given the sample path $\{\Gamma_n\}$, connect two vertices $m, \ell \in \mathbf{Z}^+$ iff

$$\Gamma_m = \Gamma_\ell\,, \text{ but } \Gamma_j \ne \Gamma_m \text{ for } m < j < \ell.$$

A coin is chosen for each cluster (connected component), and labels are generated at each $x \in \mathbf{Z}^+$ by flipping this coin. The coin used for vertices in the cluster of the origin is θ-biased, while the coin used in all other clusters is fair. The bonds are hidden from an observer, who must decide which coin was used for the cluster of the origin. For certain Γ (e.g., for the random walks considered in Section 4), there is a phase transition: for θ sufficiently small, it cannot be determined which coin was used for the cluster of the origin, while for θ large enough, the viewer can distinguish. This is an example of a 1-dimensional, long-range, dependent percolation model which exhibits a phase transition. Another 1-dimensional models that exhibits a phase transition was studied by Aizenman, Chayes, Chayes, and Newman in [2].

In Sections 4 and 8 we construct renewal processes whose renewal probabilities $\{u_n\}$ have prescribed asymptotics. T. Liggett in [18] proves a theorem containing the following as a special case:

THEOREM C. *If* $u(0) = 1$ *and* $u(k-1)u(k+1) \geq u^2(k)$ *for* $k \geq 1$*, then* $\{u_k\}$ *is a renewal sequence.*

This was originally proven by Kaluza in [14]. See also Theorem 5.3.2 of [1] for a proof. Theorem C can be used to prove that the renewal sequences discussed in this paper exist.

A generalized version of this random coin tossing model, when the underlying graph is $\mathbf{Z}$, is studied in [17]. Each vertex $z \in \mathbf{Z}$ is assigned a coin with bias $\theta(z)$. At each move of a random walk on $\mathbf{Z}$, the coin attached to the walk's position is tossed. In [17], it is shown that if $|\{z : \theta(z) \neq 0\}|$ is finite, then the biases $\theta(z)$ can be recovered up to a symmetry of $\mathbf{Z}$.

Some unsolved problems. Recall that Δ and Δ' denote two independent and identically distributed renewal processes, and $u_n = \mathbf{P}[\Delta_n = 1]$. The distribution of the sequence of coin tosses, when a coin with bias θ is used at renewal times, is denoted by μ_θ.

1. Is the quenched moment generating function criterion in Lemma 7.1 sharp? That is, does $\mathbf{E}[r^{\sum_{n=0}^{\infty} \Delta_n \Delta'_n} \mid \Delta] = \infty$ for some $r < 1 + \theta^2$ imply that $\mu_\theta \perp \mu_0$?

2. Does $\mu_{\theta_1} \perp \mu_0$ imply that $\mu_{\theta_1} \perp \mu_{\theta_2}$ for all $\theta_2 \neq \theta_1$?

3. For renewal sequences exhibiting a phase transition at a critical parameter θ_c, is $\mu_{\theta_c} \perp \mu_0$?

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