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Random Walks on the Lamplighter Group

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Abstract. Kaimanovich and Vershik (1983) described certain finitely generated groups of exponential growth such that simple random walk on their Cayley graph escapes from the identity at a sublinear rate, or equivalently, all bounded harmonic functions on the Cayley graph are constant. Here we focus on a key example, called G_1 by Kaimanovich and Vershik, and show that *inward-biased* random walks on G_1 move *outward* faster than simple random walk. Indeed, they escape from the identity at a linear rate provided that the bias parameter is smaller than the growth rate of G_1. These walks can be viewed as random walks interacting with a dynamical environment on $\mathbb{Z}$. The proof uses potential theory to analyze a stationary environment as seen from the moving particle.

§1. Introduction.

The study of random walks and harmonic functions on finitely generated groups has a long history. For a random walk supported by a finite generating set, Avez (1974) showed that a necessary condition for the existence of non-constant bounded harmonic functions is the positivity of a quantity called the entropy of the random walk. Kaimanovich and Vershik (1983) and Derriennic (1980) showed that this condition is sufficient as well and extended it to more general random walks. A more natural quantity for probabilists is the speed (or rate of escape) of the random walk. By **speed** of a random walk $\langle X_n \rangle$ on the Cayley graph of a group, we mean $\lim_{n\to\infty} |X_n|/n$ (if the limit exists), where $|X_n|$ denotes the distance of X_n from the identity element. It follows from the result of Avez that the existence of non-constant bounded harmonic functions implies positive speed. Varopoulos (1985) established that for finitary symmetric random walks, this is an equivalence. In particular, Varopoulos showed that the speed is zero on any group of subexponential growth. It is perhaps surprising that the converse is false; a particularly

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illuminating example of a Cayley graph of exponential growth on which simple random walk has zero entropy (and zero speed) was described by Kaimanovich and Vershik (1983). Random walks on this Cayley graph with bias toward the identity element will be the focus of the present article. These are not group-invariant random walks, but they do capture the growth rate of the group: Lyons (1995) showed that such biased random walks on any Cayley graph are transient for all values of the bias less than the exponential growth rate of the group. Here, we prove the surprising result that this inward biasing can change the speed from zero to a positive number.

More precisely, for $\lambda > 0$, define the **λ-biased random walk** RW_λ on a connected locally finite graph with a distinguished vertex Θ as the time-homogeneous Markov chain $\langle X_n\,;\ n \ge 0\rangle$ with the following transition probabilities. The **distance** $|v|$ from a vertex v to Θ is the number of edges on a shortest path joining the two vertices. Suppose that v is a vertex of the graph. Let $v_1, \ldots, v_k$ ($k \ge 1$ unless $v = \Theta$) be the neighbors of v at distance $|v| - 1$ from Θ and $u_1, \ldots, u_j$ ($j \ge 0$) be the other neighbors of v. Then the transition probabilities are $p(v,\, v_i) = \lambda/(k\lambda + j)$ for $i = 1, \ldots, k$ and $p(v,\, u_i) = 1/(k\lambda + j)$ for $i = 1, \ldots, j$. This is a reversible Markov chain with edge weights (or conductances) λ^{-n} on edges at distance n from Θ. Note that simple random walk, when all neighbors are equally likely, is the particular case $\lambda = 1$. In the special case of a Cayley graph, we take Θ to be the identity element. Define the **growth rate** of a finitely generated group G, denoted $\operatorname{gr} G$, to be the limit as $n \to \infty$ of the nth root of the number of vertices in its Cayley graph at distance n from Θ. The result of Lyons (1995) mentioned above is that RW_λ is transient for $\lambda < \operatorname{gr} G$ and recurrent for $\lambda > \operatorname{gr} G$. This may not be surprising if we think of the Cayley graph of G as being something like spherically symmetric; after all, it is vertex transitive. However, this point of view is not well justified. For example, there are amenable groups of exponential growth; thus, the balls in these groups do not form Følner sets. For one such group, called G_1 by Kaimanovich and Vershik (1983, Section 6.1), we show that the speed of RW_λ is positive for $1 < \lambda < \operatorname{gr} G_1 = \varphi := (1 + \sqrt{5})/2$, although it vanishes for $\lambda = 1$. This demonstrates how far this Cayley graph is from being spherically symmetric since on any spherically symmetric graph, the speed of RW_λ is monotone decreasing (when it exists).

The group G_1, also known as the lamplighter group, is defined as a semidirect product $G_1 := \mathbb{Z} \ltimes \sum_{x \in \mathbb{Z}} \mathbb{Z}_2$ of $\mathbb{Z}$ with the direct sum of copies of $\mathbb{Z}_2$ indexed by $\mathbb{Z}$; for $m, m' \in \mathbb{Z}$ and $\eta, \eta' \in \sum_{x \in \mathbb{Z}} \mathbb{Z}_2$, the group operation is

$$(m, \eta)(m', \eta') := (m + m', \eta \oplus \mathcal{S}^{-m} \eta')\,,$$

where $\mathcal{S}$ is the left shift, $\mathcal{S}(\eta)(j) := \eta(j + 1)$, and $\oplus$ is componentwise addition modulo 2.

We call an element $\eta \in \sum_{m\in\mathbb{Z}} \mathbb{Z}_2$ a **configuration** and call $\eta(k)$ the **bit** at k. We identify $\mathbb{Z}_2$ with $\{0,1\}$. The first component of an element $x = (m,\eta) \in G_1$ is called the **position of the marker in the state** x, denoted $M(x)$. Generators of G_1 are $(1,\mathbf{0})$, $(-1,\mathbf{0})$, and $(0,\mathbf{1}_0)$. The reason for the name of this group is that we may think of a streetlamp at each integer with the configuration η representing which lights are on, namely, those where $\eta = 1$. We also may imagine a lamplighter at the position of the marker. The first two generators of G_1 correspond to the lamplighter taking a step either to the right or to the left (leaving the lights unchanged); the third generator corresponds to flipping the light at the position of the lamplighter. See Figure 1.1.

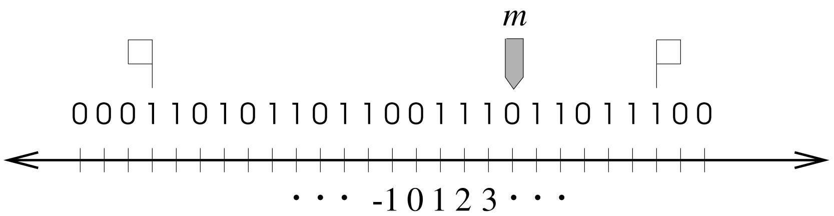

Figure 1.1. A typical element of G_1; the flags are defined in (1.1).

Simple random walk on G_1 thus corresponds to the marker moving according to simple random walk on $\mathbb{Z}$ with delays one-third of the time when it changes the bit at its location. However, RW_λ is quite different in that the configuration influences the transition probabilities of the marker as a random walk on $\mathbb{Z}$. Namely, for $\lambda > 1$, there is a tendency for the walk to return to the initial state Θ, which means that the marker has a greater tendency to change bits to 0 than to 1. In order to do so, rather than head for the origin, the marker heads for the bit equal to 1 that is on the same side of the origin as the marker and is most extreme since this is a shortest path back to Θ.

THEOREM 1.1. *Whenever $1 < \lambda < \varphi$, the speed of RW_λ on G_1 is a.s. a strictly positive constant. In fact, a lower bound for the speed is given by*

$$\lim_{n\to\infty} \frac{|X_n|}{n} \geq \lim_{n\to\infty} \frac{|M(X_n)|}{n} \geq \frac{(\lambda-1)(\sqrt{\lambda+1}-\lambda)}{3\lambda(2+\lambda)(1+\lambda-\sqrt{\lambda+1})} > 0 \qquad a.s.$$

As a corollary, we see that for RW_λ with $1 < \lambda < \varphi$, there are non-constant bounded harmonic functions on G_1, e.g., the function whose value at x is the probability that the bit at the origin is eventually 0 given that the random walk starts at x.

A crucial element in the proof of Theorem 1.1 is the 1-dimensionality of the underlying space $\mathbb{Z}$. This allows an easy determination of the shortest paths from Θ to any element of G_1, so that the transition probabilities for RW_λ admit simple expressions in terms of the configuration; see below. In contrast, the higher-dimensional analogues of G_1 require the

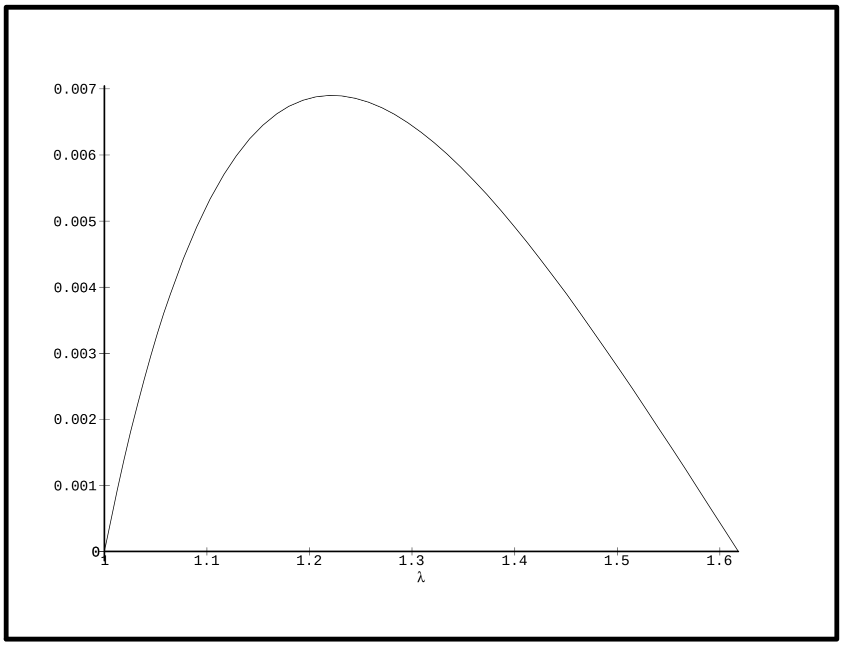

Figure 1.2. The lower bound for speed in Theorem 1.1.

solution of a traveling-salesman problem to determine the transition probabilities of RW_λ and it is unknown whether the speed is still positive.

Coming back to the 1-dimensional situation, consider outward-biased random walks on G_1, i.e., RW_λ for $0 < \lambda < 1$. Surprisingly, these escape from the identity even more slowly than simple random walk:

PROPOSITION 1.2. *Whenever $0 < \lambda < 1$, the speed of RW_λ on G_1 is a.s. 0. In fact,*

$$\liminf_{n\to\infty} \frac{|X_n|}{\log n} > 0 \quad \textit{and} \quad \limsup_{n\to\infty} \frac{|X_n|}{\log n} < \infty \quad \textit{a.s.}$$

This is much easier to prove than Theorem 1.1; see Section 4. Also, it follows from standard shift-coupling techniques that the only bounded harmonic functions are the constants when $0 < \lambda < 1$.

To make explicit the distances and transition probabilities on G_1, define

$$\mathsf{flag}_R := \sup\{k \ge 0\,;\ \eta(k) = 1\} \quad \text{and} \quad \mathsf{flag}_L := \inf\{k \le 0\,;\ \eta(k) = 1\}\,. \tag{1.1}$$

We call these the **right** and **left flags**; note that when $\eta(k) = 0$ for all $k \ge 0$, we have $\mathsf{flag}_R = -\infty$ and similarly for flag_L. When $m \ge 0$, we have

$$|(m, \eta)| = 2|\mathsf{flag}_L \wedge 0| + m + 2(\mathsf{flag}_R - m)^+ + \sum_{k\in\mathbb{Z}} \eta(k) \tag{1.2}$$

and similarly when $m < 0$. The transition probabilities are as follows for the case $m \geq 0$; the case $m < 0$ is symmetric. First, we have $p(\Theta,\, x) = 1/3$ for all the generators x of G_1. If $(m, \eta) \neq \Theta$ and $\mathsf{flag}_R < m$, then $p\big((m,\eta),\,(m-1,\eta)\big) = \lambda/(\lambda+2)$ and

$$p\big((m,\eta),\,(m+1,\eta)\big) = p\big((m,\eta),\,(m,\eta \oplus \mathbf{1}_m)\big) = 1/(\lambda+2)\,.$$

If $\mathsf{flag}_R = m$, then $p\big((m,\eta),\,(m,\eta \oplus \mathbf{1}_m)\big) = \lambda/(\lambda+2)$ and the other transition probabilities are $1/(\lambda+2)$. Finally, if $m < \mathsf{flag}_R$ and $\eta(m) = 0$, then $p\big((m,\eta),\,(m+1,\eta)\big) = \lambda/(\lambda+2)$ and the other transition probabilities are $1/(\lambda+2)$, while if $\eta(m) = 1$, then

$$p\big((m,\eta),\,(m+1,\eta)\big) = p\big((m,\eta),\,(m,\eta \oplus \mathbf{1}_m)\big) = \lambda/(2\lambda+1)$$

and $p\big((m,\eta),\,(m-1,\eta)\big) = 1/(2\lambda+1)$.

Thus, RW_λ on G_1 can be studied directly as a random walk interacting with a dynamical environment on $\mathbb{Z}$, without any reference to the group structure. However, the only proof we know that establishes positivity of speed for all $\lambda \in (1, \varphi)$ uses explicitly the structure of the Cayley graph. Simpler comparison arguments are available to show positivity of speed for λ in the smaller range $(1, 1.5)$.

§2. The Fibonacci Tree and Semitightness.

In this section, we begin our analysis of the dynamics of RW_λ. We first obtain a lower bound on the escape probability from Θ by using a subtree of the Cayley graph. This is then given a "stationary" version. Next, we prove that the marker is unlikely to be much closer to the origin than are the flags and, finally, that when the marker is at a flag, the expected time until a flag separates the marker from the origin is finite.

We first define a subgraph of G_1 which is a tree rooted at Θ. The vertices consist of states $x = (m, \eta)$ for which $m \geq \mathsf{flag}_R(x)$ and $\eta(k) = 0$ for all $k < 0$. Each vertex (m, η) with $\mathsf{flag}_R(m, \eta) = m$ has the single child $(m+1, \eta)$ and each vertex (m, η) with $\mathsf{flag}_R(m, \eta) < m$ has the two children $(m+1, \eta)$ and $(m, \eta \oplus \mathbf{1}_m)$. This is called the Fibonacci tree (see Figure 2.1). Since the number of vertices at distance n from the root of the Fibonacci tree is asymptotically a constant times φ^n, this shows that $\operatorname{gr} G_1 \geq \varphi$. From this, it is not hard to see that an upper bound for the number of vertices at distance n from Θ in the Cayley graph of G_1 is a constant times $\sum_{k \leq n} \varphi^k$, which, again, is just asymptotically a constant times φ^n. Hence, $\operatorname{gr} G_1 = \varphi$.

Transience of RW_λ implies that $\limsup |M(X_n)| = \infty$, but it does not immediately imply that $\lim |M(X_n)|$ exists. In fact, this limit does exist when $1 < \lambda < \varphi$, so that the

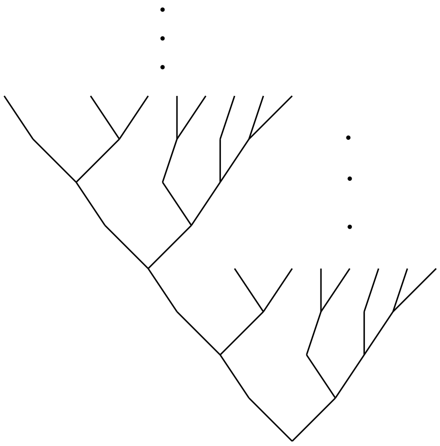

Figure 2.1.

marker tends either to ∞ or to $-\infty$. These cases are clearly symmetric and it is convenient to deal with them separately by removing a half line. Thus, define the subset $G_1^+ \subseteq G_1$ to be the set of (m, η) such that $m \geq 0$ and $\eta(j) = 0$ for all $j < 0$. Observe that for any $x \in G_1^+$, the shortest paths connecting x to Θ in G_1 are contained in G_1^+. Thus RW_λ on G_1^+ has the same transitions as RW_λ on G_1 except that transitions where the marker moves to -1 are suppressed. We use the method of Lyons (1995) to show that RW_λ is transient on G_1^+.

LEMMA 2.1. *Assume that $1 < \lambda < \varphi$. The probability that RW_λ on G_1^+ started from Θ never returns to Θ is at least $h(\lambda)$, where*

$$h(\lambda) := \frac{\sqrt{\lambda+1} - \lambda}{2(1 + \lambda - \sqrt{\lambda+1})} > 0\,.$$

In particular, RW_λ is transient on G_1^+.

Proof. Observe that the Fibonacci tree is also a subgraph of G_1^+. Let $C(\lambda)$ denote the effective conductance of the Fibonacci tree from the root to infinity. Observe that if v in generation n has two children, then the effective conductance from v to infinity in the descendant subtree rooted at v is $\lambda^{-n} C(\lambda)$. Since the edges incident to Θ have conductance 1, the obvious recursions of the Fibonacci tree and the usual series-parallel laws give the equation

$$C(\lambda) = \frac{1}{1 + \lambda/C(\lambda)} + \frac{1}{1 + \lambda + \lambda^2/C(\lambda)}\,.$$

Since the Fibonacci tree is subperiodic, $C(\lambda) > 0$ for $\lambda < \varphi$ by Lyons (1990), whence the unique positive solution is

$$C(\lambda) = \frac{\sqrt{\lambda+1} - \lambda}{1 + \lambda - \sqrt{\lambda+1}}\,. \tag{2.1}$$

By Rayleigh's Monotonicity Principle (see Doyle and Snell (1984)), the effective conductance from Θ to infinity in G_1^+ is at least the conductance on any subgraph, hence at least $C(\lambda)$. The escape probability of RW_λ on G_1^+ from Θ, i.e., the probability that the random walk never returns to Θ, is at least $C(\lambda)$ divided by the total conductance incident to Θ (see again Doyle and Snell); the latter is 2, which proves the lemma. ∎

We are interested in looking at the configuration from the viewpoint of the marker in order to find a stationary measure and thus be able to use ergodic theory. Now, after a long time, RW_λ on G_1^+ will be far right of the root and many bits will be 1. In order to allow (in the limit) infinitely many 1's to the left of the marker, define the space $\Gamma \supseteq G_1 \supseteq G_1^+$ to be the set of $(m, \eta) \in \mathbb{Z} \times \{0,1\}^{\mathbb{Z}}$ such that $\sum_{j>0} \eta(j) < \infty$. Let $\mathsf{RW}_\lambda^{(-\infty)}$ denote the Markov chain on Γ obtained by ignoring the left flag and not assigning any special status to the origin of $\mathbb{Z}$. More precisely, define $\mathsf{flag} := \sup\{k \in \mathbb{Z}\,;\ \eta(k) = 1\}$. The transitions give relative weights 1 for the marker moving away from the flag, λ for the marker moving toward the flag, and $\lambda^{\eta(m)}$ for flipping the bit at the marker.

Lemma 2.2. *For $k \in \mathbb{Z}$, let*

$$A_k := \{x \in \Gamma\,;\ \mathsf{flag}(x) < M(x) \le k\}\,.$$

Suppose that $\langle Y_0, Y_1, \ldots\rangle$ is the Markov chain $\mathsf{RW}_\lambda^{(-\infty)}$ started from any initial state with $\mathsf{flag}(Y_0) < M(Y_0)$. Then

$$\mathbf{P}[Y_n \notin A_{M(Y_0)} \textit{ for all } n > 0] \ge \frac{2}{2+\lambda} h(\lambda)\,.$$

Proof. Without loss of generality, we may assume that $M(Y_0) = 0$. Define the map $Q : \Gamma \to G_1^+$ by

$$Q(m, \eta) := (\max\{m, 0\}, \eta \mathbf{1}_{[0,\infty)})\,.$$

Observe that $Q(Y_n)$ will sometimes remain constant, but that the (possibly finite) sequence of successive changes of state will have the distribution of RW_λ.

It is evident that if $Y_n \in A_0$, then $Q(Y_n) = Q(Y_0)$. Therefore, the event $\{Y_n \notin A_0 \text{ for all } n > 0\}$ contains the event $\{Q(Y_n) \ne Q(Y_0) \text{ for all } n > 0\}$. Conditional on $\{Q(Y_1) \ne Q(Y_0)\}$, the probability of $\{Q(Y_n) \ne Q(Y_0) \text{ for all } n > 0\}$ is at least $h(\lambda)$ by Lemma 2.1. Thus

$$\begin{aligned}\mathbf{P}[Y_n \notin A_0 \text{ for all } n > 0] &\ge h(\lambda)\mathbf{P}[Q(Y_1) \ne Q(Y_0)] \\ &= \frac{2}{2+\lambda} h(\lambda)\,. \quad ∎\end{aligned}$$

We now establish the semitightness property that the marker is unlikely to be far to the left of the flag. Let $\mathcal{F}_n := \sigma(Y_0, \ldots, Y_n)$.

Lemma 2.3. *Let $\langle Y_n \rangle$ be the Markov chain $\mathsf{RW}_\lambda^{(-\infty)}$ and $D_n := \mathsf{flag}(Y_n) - M(Y_n)$. If $D_0 = 0$, then for any $n, k \geq 1$ and $1 < \lambda < \varphi$,*

$$\mathbf{P}[D_n = k] \leq \lambda^{-(k-1)/2} \frac{1 + 2\lambda}{(\sqrt{\lambda} - 1)^2}\,.$$

Proof. For any $r \geq 0$, let $\tau_r := \min\{n \geq r\,;\; D_n \leq 0\}$. Set $\beta := (1 + 2\lambda)/(\lambda + 2\sqrt{\lambda}) > 1$ and define

$$V_n := (\sqrt{\lambda})^{D_n} \beta^n\,.$$

We show that $\{V_{n \wedge \tau_r}\,;\; n \geq r\}$ is a supermartingale. Observe that $|D_{n+1} - D_n| \leq 1$ as long as $D_n > 0$. Let $p_+^{(n)}$ and $p_-^{(n)}$ denote the conditional probabilities of D_n respectively increasing by one and decreasing by one conditional on $\mathcal{F}_n$. Then for $r \leq n < \tau_r$,

$$\begin{aligned}\mathbf{E}[V_{(n+1)\wedge\tau_r} \mid \mathcal{F}_n] = \mathbf{E}[V_{n+1} \mid \mathcal{F}_n] &= \beta V_n \left[1 + (\sqrt{\lambda} - 1)p_+^{(n)} + (\frac{1}{\sqrt{\lambda}} - 1)p_-^{(n)}\right] \\ &\leq \beta V_n \left(1 + \frac{\sqrt{\lambda} - 1 + (\frac{1}{\sqrt{\lambda}} - 1)\lambda}{1 + 2\lambda}\right) = V_n = V_{n\wedge\tau_r}\end{aligned}$$

since the numerator is negative and the only possibilities for the vector $(p_+^{(n)}, p_-^{(n)})$ are $(1 + 2\lambda)^{-1}(1, \lambda)$ or $(2 + \lambda)^{-1}(1, \lambda)$.

It follows that on the event $D_r = 1$ and for $n \geq r$,

$$\mathbf{E}[V_{n\wedge\tau_r} \mid \mathcal{F}_r] \leq V_r = \sqrt{\lambda}\beta^r$$

and therefore by Markov's inequality,

$$\mathbf{P}[D_n = k, \tau_r \geq n \mid \mathcal{F}_r] \leq \mathbf{P}[V_{n\wedge\tau_r} = \lambda^{k/2}\beta^n \mid \mathcal{F}_r] \leq \lambda^{-(k-1)/2}\beta^{-(n-r)} \tag{2.2}$$

on $\{D_r = 1\}$. Now decomposing the event $\{D_n = k\}$ according to the last r such that $D_r = 1$, we get

$$\mathbf{P}[D_n = k] = \sum_{r=1}^{n} \mathbf{P}\big[D_n = k,\, \max\{j \leq n\,;\; D_j = 1\} = r\big]\,.$$

The rth summand is at most the right-hand side of (2.2), and summing over r yields the desired bound since $1/(1 - \beta^{-1}) = (1 + 2\lambda)/(\sqrt{\lambda} - 1)^2$. ∎

Finally, we show that when the marker is at the flag, the time until the marker is to the right of the flag has finite mean.

LEMMA 2.4. *Let* $\langle Y_n \rangle$ *be the Markov chain* $\mathsf{RW}_\lambda^{(-\infty)}$ *with* $M(Y_0) = \mathsf{flag}(Y_0)$. *If* τ *is the first time that* $M(Y_\tau) > \mathsf{flag}(Y_\tau)$, *then*

$$\mathbf{E}[\tau] \le \frac{1+2\lambda}{\lambda - 1}\,.$$

Proof. Set $\widehat{D}_n := \max\bigl(\mathsf{flag}(Y_n) - M(Y_n), -1\bigr)$. Note that for $n < \tau$, we have

$$\mathbf{E}[\widehat{D}_{n+1} \mid \mathcal{F}_n] \le \widehat{D}_n - \frac{\lambda-1}{1+2\lambda}\,.$$

Let

$$W_n := \widehat{D}_n + \frac{\lambda-1}{1+2\lambda} n\,.$$

Then $\{W_{n\wedge\tau}\,;\ n \ge 0\}$ is a supermartingale. By the optional stopping theorem (see Durrett (1991), Theorem 7.6), we get that

$$0 \ge \mathbf{E}[W_\tau] = \mathbf{E}[\widehat{D}_\tau] + \frac{\lambda-1}{1+2\lambda}\mathbf{E}[\tau] = -1 + \frac{\lambda-1}{1+2\lambda}\mathbf{E}[\tau]\,. \quad \blacksquare$$

§3. Proof of the theorem.

We are now in a position to look at the configuration from the viewpoint of the marker. Recall that $\mathcal{S}$ denotes the left shift operator on $\{0,1\}^{\mathbb{Z}}$. Let $\Gamma^* := \{\eta\,;\ \sum_{j>0} \eta(j) < \infty\}$ and $\mathsf{flag}^*(\eta) := \sup\{k \in \mathbb{Z}\,;\ \eta(k) = 1\}$. Define $\mathcal{S}^* : \Gamma \to \Gamma^*$ by

$$\mathcal{S}^*(m, \eta) := \mathcal{S}^m(\eta)$$

and set $\xi_n := \mathcal{S}^*(Y_n)$. Define

$$\Delta_n := M(Y_n) - M(Y_{n-1}) \in \{-1, 0, 1\}$$

for $n \ge 1$ and $\Delta_0 := 0$, say. Observe that $\langle (\xi_n, \Delta_n)\,;\ n \ge 0 \rangle$ is a Markov chain.

The transition probability kernel for $\langle (\xi_n, \Delta_n) \rangle$ is denoted $K_\lambda^*(\cdot,\cdot)$ and may be described as follows. When $\xi_n(0) = 0$,

$$\Delta_{n+1} = \begin{cases} \operatorname{sign}(\mathsf{flag}^*(\xi_n)) & \text{with probability } \lambda/(2+\lambda), \\ -\operatorname{sign}(\mathsf{flag}^*(\xi_n)) & \text{with probability } 1/(2+\lambda), \\ 0 & \text{with probability } 1/(2+\lambda). \end{cases}$$

When $\xi_n(0) = 1$ and $\mathsf{flag}^*(\xi_n) = 0$,

$$\Delta_{n+1} = \begin{cases} 1 & \text{with probability } 1/(2+\lambda), \\ -1 & \text{with probability } 1/(2+\lambda), \\ 0 & \text{with probability } \lambda/(2+\lambda)\ , \end{cases}$$

and when $\xi_n(0) = 1$ but $\mathsf{flag}^*(\xi_n) > 0$,

$$\Delta_{n+1} = \begin{cases} 1 & \text{with probability } \lambda/(1+2\lambda), \\ -1 & \text{with probability } 1/(1+2\lambda), \\ 0 & \text{with probability } \lambda/(1+2\lambda)\,. \end{cases}$$

Finally,

$$\xi_{n+1} = \begin{cases} \mathcal{S}^{\Delta_{n+1}}\xi_n & \text{if } \Delta_{n+1} = \pm 1, \\ \xi_n \oplus \mathbf{1}_0 & \text{if } \Delta_{n+1} = 0. \end{cases} \tag{3.1}$$

Note that $\mathcal{S}^*$ takes the set $\{\mathsf{flag}(x) < M(x)\}$ to the set $\{\mathsf{flag}^* < 0\}$. Suppose that $\mathsf{flag}^*(\xi_0) < 0$. Let $n(k)$ be the kth return time of the sequence $\langle \xi_j \rangle$ to $\{\mathsf{flag}^* < 0\}$ and

$$Z_k := \sum_{j=1}^{n(k)} \Delta_j = M(Y_{n(k)}) - M(Y_0)\,.$$

If $Z_k \le 0$, then $Y_{n(k)} \in A_{M(Y_0)}$. It follows from Lemma 2.2 that from any initial state with $\mathsf{flag}^* < 0$,

$$\mathbf{P}[Z_k > 0 \text{ for all } k > 0] \ge \frac{2h(\lambda)}{2+\lambda}\,. \tag{3.2}$$

Now assume that $\mathsf{flag}^*(\xi_0) = 0$. Equip $\Gamma^* \times \{-1, 0, 1\}$ with the metric

$$d\Big((\eta, \delta), (\eta', \delta')\Big) := |\delta - \delta'| + \sum_{j=-\infty}^{\infty} 2^j |\eta(j) - \eta'(j)|\,.$$

With this metric, for any j_0, the set $\{(\eta, \delta);\ \forall j > j_0 \quad \eta(j) = 0\}$ is compact. By Lemma 2.3, it follows that the laws of (ξ_n, Δ_n) are tight. Thus, the Cesàro averages $n^{-1}\sum_{j=1}^{n} \mathsf{Law}(\xi_n, \Delta_n)$ are tight and have a subsequential weak* limit π_0. Since the transition probabilities K_λ^* are continuous, π_0 must be stationary for K_λ^*. Passing to an ergodic component yields a stationary ergodic Markov chain (K_λ^*, π) (Rosenblatt 1971). Inducing the Markov system (K_λ^*, π) on the subset $\{\mathsf{flag}^* < 0\}$ of the state space (see Petersen (1983)) yields a measure π' for which the increments $Z_{k+1} - Z_k$ form a stationary ergodic sequence.

A little thought shows that $Z_{k+1} - Z_k$ is either -1, 0, or 1. Let R_k denote the cardinality of the range $\{Z_1, \ldots, Z_k\}$. From (3.2), it follows that

$$\lim_{k\to\infty} Z_k/k = \lim_{k\to\infty} R_k/k = \mathbf{P}_{\pi'}[Z_k \ne 0 \text{ for all } k > 0] \quad \text{a.s.}, \tag{3.3}$$

where the second equality uses Theorem 6.3.1 of Durrett (1991). Hence by the ergodic theorem and (3.2),

$$\mathbf{E}_{\pi'}[Z_2 - Z_1] \ge \frac{2h(\lambda)}{2+\lambda}\,. \tag{3.4}$$

By the tower proof of Kac's lemma (Petersen 1983), we have

$$\mathbf{E}_\pi[\Delta_1] = \pi\{\mathsf{flag}^* < 0\}\mathbf{E}_{\pi'}[Z_2 - Z_1]\,.$$

By Kac's lemma itself, the reciprocal of $\pi\{\mathsf{flag}^* < 0\}$ is the expected return time to $\{\mathsf{flag}^* < 0\}$. If the return time is not 1, then $\mathsf{flag}^*(\xi_1) = 0$ and the expected additional time needed to return to $\{\mathsf{flag}^* < 0\}$ is, by Lemma 2.4, at most $(1+2\lambda)/(\lambda-1)$. Hence the total expected return time is bounded by $1 + (1+2\lambda)/(\lambda-1) = 3\lambda/(\lambda-1)$. Combined with (3.4), this yields

$$s_\lambda := \mathbf{E}_\pi[\Delta_1] \ge \frac{2(\lambda-1)}{3\lambda(2+\lambda)}h(\lambda)\,. \tag{3.5}$$

Note that this is the lower bound in the statement of Theorem 1.1.

Let p_0 be the probability that RW_λ in G_1^+ started from $(1, \mathbf{0})$ does not return to the infinite set of states where the marker is at the origin. By the obvious coupling to G_1^+, the probability that the partial sums $\sum_{j=1}^n \Delta_j$ remain positive is p_0 starting from any state in $\{\mathsf{flag}^* < 0\}$. To see that $p_0 > 0$, let

$$T := \inf\{n \ge 1\,;\ \sum_{j=1}^n \Delta_j \le 0\}\,,$$

which corresponds on Γ to the first time that the marker is not to the right of its starting point. According to the Maximal Ergodic Lemma (Durrett (1991), Theorem 6.2.2),

$$\mathbf{E}_\pi\left[\Delta_1 \mathbf{1}_{\{T<\infty\}}\right] \le 0\,,$$

whence $p_0 = \mathbf{P}[T = \infty] > 0$ by (3.5).

We use this to transfer the bound (3.5) to G_1 by coupling RW_λ on G_1 to the Markov chain K_λ^* on Γ^* with initial distribution π conditioned on $\{\mathsf{flag}^* < 0\}$. Let $\langle X_0, X_1, \ldots\rangle$ be the Markov chain RW_λ on G_1 starting from any state with $M(X_0) > \mathsf{flag}_R(X_0) \ge 0$, and define

$$\widetilde{T} := \inf\{n \ge 1\,;\ M(X_n) \le M(X_0)\}\,.$$

Let (ξ_0, Δ_0) have the conditional distribution $(\pi \mid \mathsf{flag}^*(\xi_0) < 0)$. For $1 \le n < \widetilde{T}$, define $\Delta_n := M(X_n) - M(X_{n-1})$. This produces by (3.1) a sequence $\langle \xi_n\,;\ 0 \le n < \widetilde{T}\rangle$; if $\widetilde{T}$ is finite, then continue the chain (ξ_n, Δ_n) independently by using the kernel K_λ^*. Note that with this coupling, $T = \widetilde{T}$, and in particular, $\mathbf{P}[\widetilde{T} = \infty] = p_0$. Thus, on the event $\{\widetilde{T} = \infty\}$, the speed of the marker equals

$$\lim_{n\to\infty} M(X_n)/n = \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n \Delta_j = \mathbf{E}_\pi[\Delta_1] = s_\lambda \quad \text{a.s.} \tag{3.6}$$

by the ergodic theorem.

Let $N(k)$ be the first $n > k$ such that

$$M(X_n) > \mathsf{flag}_R(X_n) \geq 0 \quad \text{or} \quad M(X_n) < \mathsf{flag}_L(X_n) \leq 0\,.$$

Then $N(k) < \infty$ a.s. and by the coupling argument of the preceding paragraph,

$$\mathbf{P}\big[\lim_{n\to\infty} |M(X_n)|/n = s_\lambda \bigm| X_0, X_1, \ldots, X_{N(k)}\big] \geq p_0\,.$$

Therefore the speed of the marker equals s_λ almost surely by the Lévy 0-1 law.

We calculate the speed s'_λ of RW_λ by using (1.2) and the above coupling. Let

$$U_n := \big(\xi_n(0) - \xi_{n-1}(0)\big)\mathbf{1}_{\{\Delta_n = 0\}}\,.$$

Let $X_n = (M(X_n), \eta_n)$ be the Markov chain RW_λ on G_1 starting from any state with $M(X_0) > \mathsf{flag}_R(X_0) \geq 0$. On the event $\{\widetilde{T} = \infty\}$, the coupling gives

$$\frac{1}{n}\sum_{k\in\mathbb{Z}} \eta_n(k) = \frac{1}{n}\sum_{k\in\mathbb{Z}} \eta_0(k) + \frac{1}{n}\sum_{i=1}^{n} U_i \to \mathbf{E}_\pi[U_1] \quad \text{a.s.} \tag{3.7}$$

as $n \to \infty$ by the ergodic theorem. The bound in Lemma 2.3 and the Borel-Cantelli lemma imply that

$$\frac{1}{n}\big(\mathsf{flag}_R(X_n) - M(X_n)\big)^+ \to 0 \quad \text{a.s.} \tag{3.8}$$

Combining equations (1.2), (3.6), (3.7), and (3.8), we arrive at

$$\lim_{n\to\infty} |X_n|/n = s_\lambda + \mathbf{E}_\pi[U_1] =: s'_\lambda \tag{3.9}$$

a.s. on $\{\widetilde{T} = \infty\}$. Using symmetry and the Lévy 0-1 law as before shows that this equation holds a.s. and completes the proof. ∎

Remark. An alternative expression for $\mathbf{E}_\pi[U_1]$ is $\mathbf{E}_\pi[\xi_0(0)\mathbf{1}_{\{T=\infty\}}]$, which shows that this expectation is strictly positive. We omit the argument.

§4. Continuity of Speed and Outward-biased Random Walks.

PROPOSITION 4.1. *Both almost sure limits*

$$s_\lambda = lim_{n\to\infty}|M(X_n)|/n \quad and \quad s'_\lambda = lim_{n\to\infty}|X_n|/n$$

are continuous for $\lambda \in [1, \varphi)$. *As* $\lambda \uparrow \varphi$, *both speeds tend to 0.*

Note that by the criterion of Nash-Williams (1959), RW_φ is recurrent.

Proof. The continuity at $\lambda = 1$ follows from the trivial bounds

$$s'_\lambda \le 2s_\lambda = 2\mathbf{E}_\pi[M(X_1) - M(X_0)] \le 2\frac{\lambda - 1}{1 + 2\lambda}\,.$$

For the continuity at points in $(1, \varphi)$, we need to make explicit the dependence of π on λ, so denote the stationary measures by π_λ. It follows from the positivity of s_λ that these measures are unique, but we do not need that fact. Instead, we may rely simply on the fact that s_λ and s'_λ are the same for all stationary measures since they have the values determined on G_1. By Lemma 2.3, any collection $\{\pi_\lambda\,;\ 1 < \lambda_{\min} \le \lambda < \varphi\}$ is tight. Since K^*_λ is continuous in λ, it follows that as $\lambda \to \lambda_0 \in (1, \varphi)$, the measures π_λ have a weak*-limit point π_{λ_0} on Γ^* which is $K^*_{\lambda_0}$-stationary. Therefore, $s_\lambda \to s_{\lambda_0}$ and $s'_\lambda \to s'_{\lambda_0}$ by the definitions (3.5) and (3.9).

The proof that the speeds tend to 0 as $\lambda \uparrow \varphi$ requires another approach. As above, it suffices to show that the speed of the marker $s_\lambda \to 0$ as $\lambda \uparrow \varphi$. Now this speed can be estimated in Γ^* by inducing on the set $\{\mathsf{flag}^* < 0\}$. Recall that Z_k denotes the change in position of the marker after k visits to $\{\mathsf{flag}^* < 0\}$. From (3.3), we have

$$s_\lambda \le \lim_{k\to\infty} Z_k/k = \mathbf{P}_{\pi'}[Z_k \ne 0 \text{ for all } k > 0]\,.$$

Checking the definitions shows that this last probability equals the probability for RW_λ to escape from Θ in G_1^+. By Doyle and Snell (1984), this escape probability is the ratio of the effective conductance from Θ to infinity divided by the sum of the conductances incident to Θ. We bound the effective conductance by shorting all vertices at the same distance from Θ (see Doyle and Snell (1984), Chapter 6). The shorted graph is equivalent to a graph on $\mathbb{N}$ with the edge conductance between n and $n+1$ equal to λ^{-n} times the number of edges in G_1^+ at distance n from Θ. Since this last number is at most $c\varphi^n$ for some constant c, the effective resistance is at least

$$\sum_{n\ge 0} \frac{(\lambda/\varphi)^n}{c} = \frac{\varphi}{c(\varphi - \lambda)}\,.$$

This finally gives the bound

$$s_\lambda \le \frac{c(\varphi - \lambda)}{2\varphi}. \quad \blacksquare$$

We now turn to outward-biased random walks on G_1 and show that they escape at a logarithmic rate from the identity.

Proof of Proposition 1.2. It suffices to show this on G_1^+. Let L_k be the position of the marker after it has moved k times. This stochastically dominates an asymmetric simple random walk on $\mathbb{N}$, where the latter has probability $\lambda/(1+\lambda)$ of moving to the right. The expected time for the asymmetric walk to reach l is $O(\lambda^{-l})$ (see Chung (1960), p. 65). Therefore, the same is true for $\langle L_k \rangle$, whence the Borel-Cantelli lemma shows that

$$\liminf_k \frac{\max\{L_j\,;\ j \le k\}}{\log k} > 0 \quad \text{a.s.}$$

Since the flag grows linearly in $\max\{L_j\,;\ j \le k\}$, the lower bound of the proposition follows.

For the upper bound, note that from an initial state $(0, \eta)$ with $\eta(j) = 1$ and $j \ge 1$, the number of visits of the marker to 0 before the first visit to j is geometric with mean at least $c\lambda^{-j}$ for some positive constant c. Therefore, from any state (j, η), the time until the marker is at $j+1$ is at least $c\lambda^{-j}$ with probability bounded below by some $\alpha > 0$, since the probability is bounded below that the next state has $\eta(j) = 1$ and subsequently the marker visits 0 before returning to j.

Consequently, the probability that the marker reaches k from $k - \sqrt{k}$ in less than $c\lambda^{-(k-\sqrt{k})}$ steps is at most $(1-\alpha)^{\sqrt{k}}$. The Borel-Cantelli lemma then yields that

$$\limsup_{n\to\infty} \frac{M(X_n)}{\log n} < \infty \quad \text{a.s.}$$

This completes the proof (on G_1^+). $\blacksquare$

REMARK. The above argument shows that $\limsup_{n\to\infty} M(X_n)/\log n = 1/|\log \lambda|$ a.s.

QUESTIONS: The groups G_k, defined like G_1 but with $\mathbb{Z}^k$ in place of $\mathbb{Z}$, also play an important role in Kaimanovich and Vershik (1983): for $k \ge 3$, these yielded the first examples of a symmetric finitary measure on an amenable group which admits non-constant bounded harmonic functions. The speed of simple random walk on G_k is 0 iff $k \le 2$. Is it true that the speed of RW_λ is positive on G_k for $1 < \lambda < \operatorname{gr} G_k$? What can one say about the asymptotic shape of the configuration on G_k? Is the speed of RW_λ positive for $1 < \lambda < \operatorname{gr} G$ when G is an arbitrary finitely generated group?

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