# The Dimension of the Brownian Frontier is Greater Than 1.

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Bishopwithout loss of generality$File : Ulasy.fd 1995/03/18 v2.2a LaTeX symbol font definitions$[1],
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**Abstract**

Consider a planar Brownian motion run for finite time. The *frontier* or "outer boundary" of the path is the boundary of the unbounded component of the complement. Burdzy (1989) showed that the frontier has infinite length. We improve this by showing that the Hausdorff dimension of the frontier is strictly greater than 1. (It has been conjectured that the Brownian frontier has dimension 4/3, but this is still open.) The proof uses Jones's Traveling Salesman Theorem and a self-similar tiling of the plane by fractal tiles known as Gosper Islands.

Keywords: Brownian motion, frontier, outer boundary, Hausdorff dimension, self-similar tiling, traveling salesman

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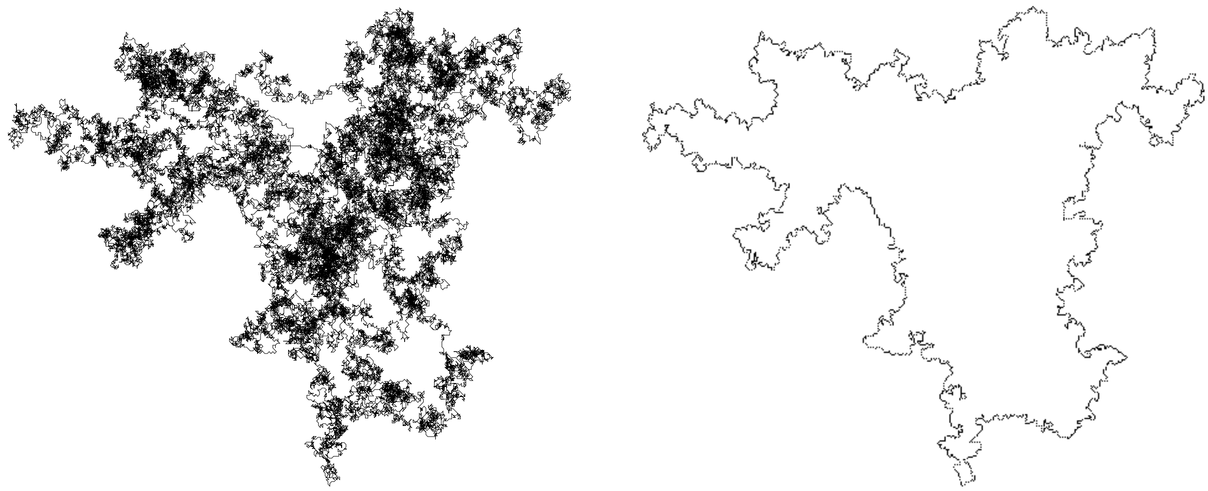

Figure 1: A Brownian path and its frontier

# 1 Introduction

Let $K$ be any compact, connected set in the plane. The complement of $K$ has one unbounded component and its topological boundary is called the *frontier* of $K$, denoted $\text{frontier}(K)$. The example we are most interested in is when $K$ is the range of a planar Brownian motion run for a finite time (see Figure 1). In this case, Mandelbrot (1982) conjectured that the Hausdorff dimension $\dim(\text{frontier}(K))$ is $4/3$. (See Falconer (1985) for the definition of Hausdorff dimension in any metric space.) Rigorously, the best proven upper bound on the dimension is $3/2 - 1/(4\pi^2) \approx 1.475$ by Burdzy and Lawler (1990). Burdzy (1989) proved that $\text{frontier}(K)$ has infinite length; our main result improves this to a strict dimension inequality:

**Theorem 1.1** *Let $B[0,t]$ denote the range of a planar Brownian motion, run until time $t > 0$. There is an $\epsilon > 0$ such that with probability 1, the Hausdorff dimension $\dim(\text{frontier}(B[0,1]))$ is at least $1 + \epsilon$. Moreover, with probability 1,*

$$\inf_{t>0} \inf_{V} \dim\Big(\text{frontier}(B[0,t]) \cap V\Big) \geq 1 + \epsilon\,,$$

*where the inner infimum is over all open sets $V$ that intersect $\text{frontier}(B[0,t])$.*

*Remarks:* The uniformity in $t$ implies that $\dim(\mathrm{frontier}(B[0,\tau])) \geq 1+\epsilon$ almost surely for any positive random variable $\tau$ (which may depend on the Brownian motion). We also note that our proof shows that the frontier can be replaced in the statement of the theorem by the boundary of any connected component of the complement $B[0,t]^c$. (One can also infer this from the statement of the theorem by using conformal invariance of Brownian motion). As explained at the end of Section 6, The result also extends to the frontier of the planar Brownian bridge (which is a closed Jordan curve by Burdzy and Lawler (1990)).

Bishop and Jones (1994) proved that if a compact, connected set is "uniformly wiggly at all scales", then it has dimension strictly greater than 1. Here we adapt this to a stochastic setting in which the set is likely to be wiggly at each scale, given the behavior at previous scales. The difficulty is in handling statistical dependence.

**Definitions:** Let $G$ be a compact set in the plane with complement $G^c$, and let $\eta > 0$. Denote by $\mathbf{core}(G,\eta)$ the set $\{z \in G : \mathrm{dist}(z, G^c) > \eta \cdot \mathrm{diam}(G)\}$. Say that the compact set $K$ **$\eta$-surrounds** $G$ if $\mathrm{core}(G,\eta)$ is disjoint from the unbounded component of $(K \cap G)^c$.

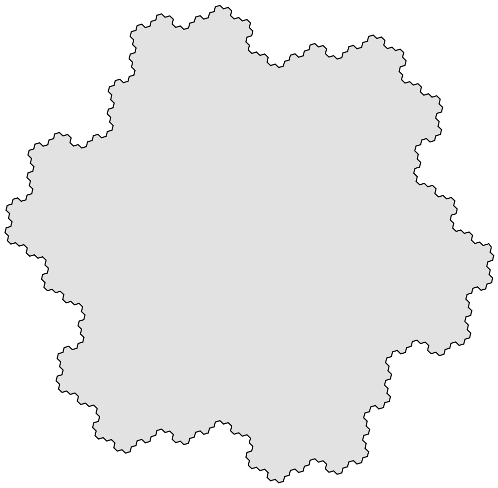

Figure 2: The Gosper island

The collection of compact planar sets equipped with the Hausdorff metric is a metric space (see Falconer (1985)). A **random compact set** is a measurable mapping $\mathbf{F}$ from a probability

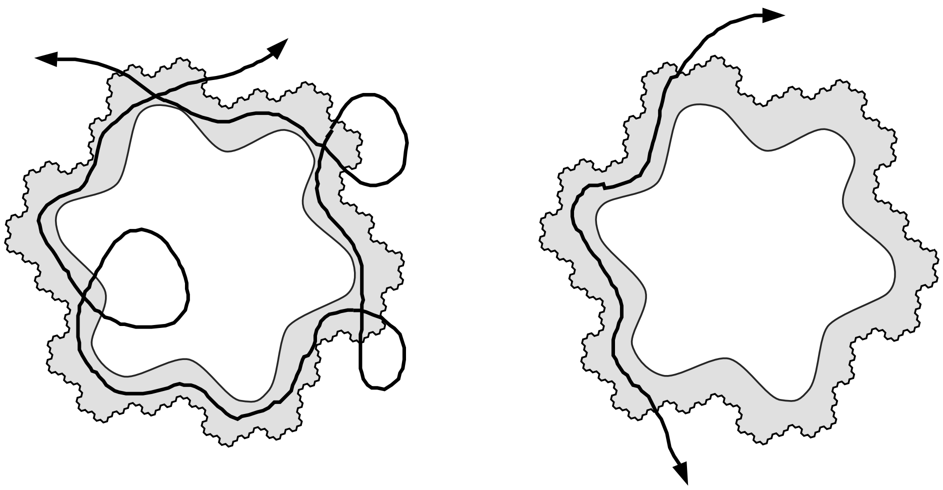

Figure 3: A Brownian motion which surrounds the core and one which misses it.

space to this metric space with its Borel $\sigma$-field. The preimage of the Borel $\sigma$-field under $\mathbf{F}$ is denoted $\sigma(\mathbf{F})$.

**Theorem 1.2** *Let $G_0$ be the* **Gosper Island***, defined in the next section and illustrated in Figure 2. There exists an absolute constant $\eta_0 > 0$ with the following property. Suppose that $c_0 > 0$, and $K$ is a random compact connected subset of the plane such that for all images $G = z + rG_0$ of $G_0$ with $r$ complex and $0 < |r| < 1$ and $z$ in the plane:*

$$\mathbf{P}\Big[K \ \ \eta_0\text{-surrounds } G \ \ \underline{\text{or}} \ \ K \cap \text{core}(G, \eta_0) = \emptyset \ \Big| \ \sigma(K \setminus G^\circ)\Big] > c_0, \tag{1}$$

*where the conditioning is on the $\sigma$-field generated by the random set $K \setminus G^\circ$ and $G^\circ$ is the interior of $G$. Then there is an $\epsilon > 0$, depending only on $c_0$, such that*

$$\dim(\text{frontier}(K)) \geq 1 + \epsilon$$

*with probability 1. More generally,* $\dim(\text{frontier}(K) \cap V) \geq 1 + \epsilon$ *for any open $V$ intersecting* frontier$(K)$.

Remarks: **1.** In fact, the proof in Section 3 shows that with probability 1, for any connected component $\Omega$ of $K^c$ and any open $V$ intersecting $\partial\Omega$, there is a John domain $\Omega_J \subset \Omega$ with closure

$\overline{\Omega_{\mathrm{J}}}$ contained in $V$, such that $\dim(\partial\Omega \cap \partial\Omega_{\mathrm{J}}) \geq 1+\epsilon$.
**2.** The constant $\eta_0$ will be chosen in the next section to ensure that no "macroscopic" line segment can be wholly contained within a $2\eta_0$-neighborhood of the Gosper Island's boundary $\partial G_0$.

The appearance of the Gosper Island might seem strange at this point, but is explained as follows. The hypothesis on $K$ that guarantees "wiggliness" should be local to handle dependence (thus it must hold inside each $G$ conditioned on $K \cap G^c$). If $K$ $\eta_0$-surrounds $G$, or $K \cap \mathrm{core}(G, \eta_0) = \emptyset$, then $\mathrm{frontier}(K)$ cannot intersect $\mathrm{core}(G, \eta_0)$. Having thus controlled $\mathrm{frontier}(K)$ inside $G$, away from the boundary of $G$, we must worry about how $\mathrm{frontier}(K)$ behaves near boundaries of cells $G$, as these run over a partition of the plane. If a small neighborhood of the union of the boundaries of cells $G$ of a fixed size contains no straight line segments of length comparable to $\mathrm{diam}\,(G)$, then no significant flatness can be introduced near cell boundaries. To apply the argument with the same constants on every scale, we need a self-similar tiling where tile boundaries have no straight portions; the Gosper Island yields such a tiling.

Proving Theorem 1.2 is the main effort of the paper and is organized as follows. Section 2 summarizes notation and useful facts about the Gosper Island. We also discuss the notion of a Whitney decomposition with respect to these tiles. Section 3 constructs a random tree of Whitney tiles for $K$ and reduces Theorem 1.2 to a lower bound on the expected growth rate of the tree, via some general propositions on random trees. In Section 4 we state a variant of Jones's Traveling Salesman Theorem adapted to the current setting. In Section 5 this theorem is used to derive the required lower bound on the expected growth rate of the "Whitney tree" mentioned above, which then finishes the proof of Theorem 1.2. In Section 6 we verify that the range of planar Brownian motion, killed at an independent exponential time, satisfies the hypothesis of Theorem 1.2; this easily yields Theorem 1.1.

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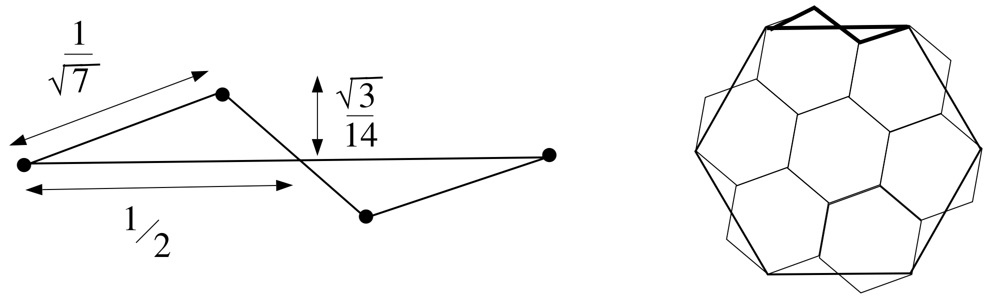


Figure 4: Substitution defining Gosper island

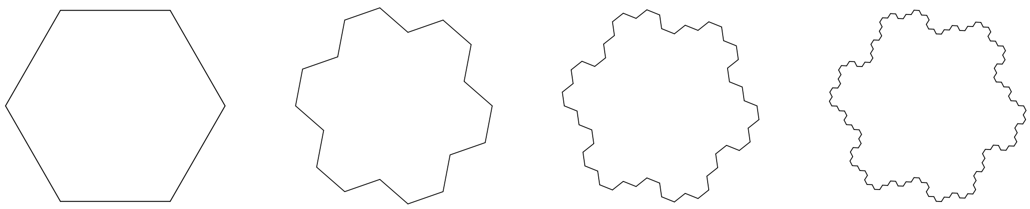

Figure 5: First four generation of the construction

## 2 Gosper Islands and Whitney tiles

The standard hexagonal tiling of the plane is not self-similar, but can be modified to obtain a self-similar tiling. Replacing each hexagon by the union of seven smaller hexagons (of area 1/7 that of the original – see Figure 4) yields a new tiling of the plane by 18-sided polygons; denote by $d_1$ the Hausdorff distance between each of these polygons and the hexagon it approximates. Applying the above operation to each of the seven smaller hexagons yields a 54-sided polygon with Hausdorff distance $7^{-1/2} \cdot d_1$ from the 18-sided polygon, which also has translates that tile the plane. Repeating this operation (properly scaled) ad infinitum, we get a sequence of polygonal tilings of the plane, that converge in the Hausdorff metric to a tiling of the plane by translates of a compact connected set $G_0$ called the "Gosper Island" (see Gardner (1976) and Mandelbrot (1982)).

**Notation:** We normalize $G_0$ to be centered at the origin and have diameter 1. Denote by $\mathcal{D}_0$ the set of translates of $G_0$ that form a tiling of the plane (depicted in Figure 6). This tiling is

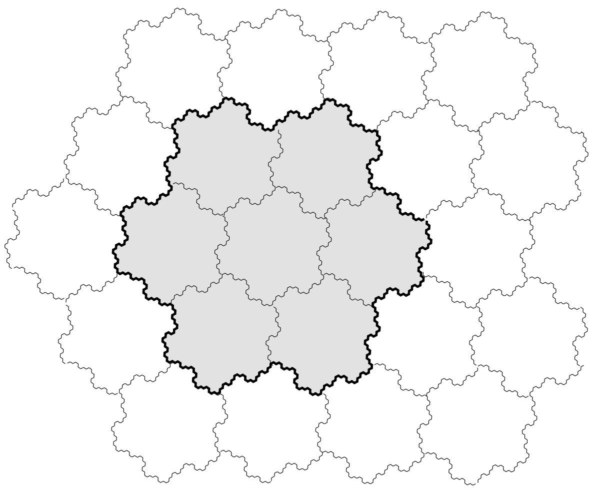

Figure 6: A self-similar tiling of the plane

**self-similar**, i.e., there is a complex number $\lambda$ with $|\lambda| > 1$ such that for each tile $G \in \mathcal{D}_0$, the homothetic image $\lambda \cdot G$ is the union of tiles in $\mathcal{D}_0$. (For the tiling by Gosper Islands, $|\lambda| = 7^{1/2}$.) For each integer $n$, we denote by $\mathcal{D}_n$ the scaled tiling $\{\lambda^{-n} \cdot G \ : G \in D_0\}$, and let $\mathcal{D} = \cup_{n=0}^{\infty} \mathcal{D}_n$. If $G \in \mathcal{D}_n$ we say that $G$ is a tile of index $n$ and write $||G|| = n$. Every tile $G \in \mathcal{D}_n$ is contained in a unique tile of $\mathcal{D}_{n-1}$, denoted $\text{parent}(G)$. Each tile $G$ is centrally symmetric about a "center point" $z$; for any non-zero, complex $\theta$, denote by $\theta \odot G = z + \theta \cdot (G - z)$ the expansion of $G$ by a factor $\theta$ around $z$.

We record several simple properties of the tiling by Gosper Islands, which will be useful later.

1. There is some minimal distance $d_0$ between any two nonadjacent tiles of $\mathcal{D}_0$.

2. There is an $\eta_0 > 0$ such that any line segment of length $d_0$ must intersect $\text{core}(G, 2\eta_0)$ for some $G \in \mathcal{D}_0$. (The existence of $\eta_0$ follows by a compactness argument from the fact that $\partial G_0$ contains no straight line segments.)

3. The Gosper Island $G_0$ contains an open disk centered at the origin which in turn contains $\lambda^{-1} G_0$.

4. The blow-up $\lambda^3 \odot G$ contains $\lambda \odot \text{parent}(G)$ for any $G \in \mathcal{D}$ (see Figure 7).

5. If $||G|| = ||G'|| - 1$ for neighboring tiles $G$ and $G'$, then $\lambda \odot G$ contains $\lambda \odot G'$. (See Figure 7.)

6. The blow-up $\lambda \odot G$ is contained in $\bigcup(\lambda \odot G'')$ where the union is over all neighbors $G''$ of $G$ of index $||G||$.

7. The boundary of $G_0$ is a Jordan curve. To see this note that when we replace each segment or length $r$ by the three segments of the next generation, they remain within distance $r\sqrt{3}/14$ of the segment. Thus the limiting arc is within

$$r\frac{\sqrt{3}}{14}\sum_{n=0}^{\infty}(\frac{1}{\sqrt{7}})^n = \frac{r\sqrt{21}}{14(\sqrt{7}-1)} \approx r(0.198892),$$

of the segment. If $I_1, I_2, I_3$ are consecutive segments of length $r$ then $\text{dist}(I_1, I_3) = r$, so this shows the limiting arcs corresponding to them are at least distance $r/2$ apart. Thus the boundary of the Gosper Island is a Jordan curve, indeed, is the image of the unit circle under a map $f$ satisfying

$$\frac{1}{C} \leq \frac{|f(x) - f(y)|}{|x - y|^\alpha} \leq C,$$

where $\alpha = \frac{1}{2}\log 7/\log 3$.

8. For any $\eta > 0$, there is a topological annulus with a rectifiable boundary, which separates $\text{core}(G_0, \eta)$ from the boundary $\partial G_0$ of the Gosper island.
(By the previous property, the interior $G_0^\circ$ of $G_0$ is simply connected, so this annulus can be obtained, for instance, by applying the Riemann mapping theorem.)

**Definitions:** Let $K$ be a compact connected subset of the plane. We say that $G \in \mathcal{D}$ is a **Whitney tile** for $K$ if $\lambda \odot G$ is disjoint from $K$, but $\lambda \odot \text{parent}(G)$ intersects $K$. (See Figure 7.) Let $W_K$ denote the set of Whitney tiles for $K$. This collection is called a Whitney decomposition of $K^c$, since it decomposes $K^c$ into a countable union of tiles (disjoint except for their boundaries) each with diameter comparable to its distance from $K$. See Figure 8.

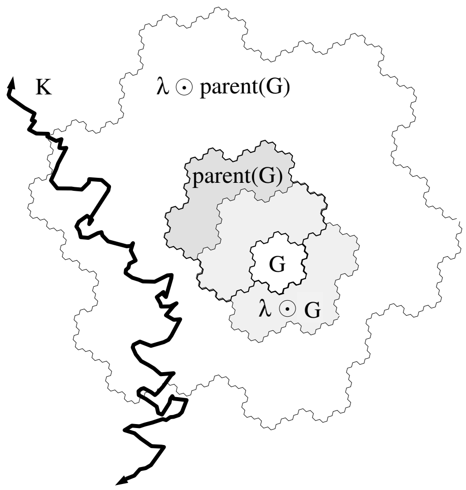


Figure 7: $G$ is a Whitney tile if $K$ misses $\lambda \odot G$, but hits $\lambda \odot \text{parent}(G)$.

A chain of adjacent tiles $\{G_1, G_2, \ldots, G_j\}$ in $W_K$ such that $G_i \subset \lambda^5 \odot G_1$ and $||G_i|| \geq ||G_{i-1}||$ for all $i \in \{2, \ldots, j\}$ is called a **Whitney chain** (see Figure 8). Given $G \in W_K$, define $W_K^G \subset W_K$ to be the set of tiles $G'$ such that there is a Whitney chain $\{G_1, G_2, \ldots, G_j\}$ with $G_1 = G$ and $G_j = G'$.

Note the following property of the Whitney decomposition, which holds for any connected component $\Omega$ of $K^c$:

$$\text{For any open } V \text{ intersecting } \partial\Omega\text{, there is a tile } G_* \in W_K \text{ with } \lambda^5 \odot G_* \subset V. \tag{2}$$

**Lemma 2.1** *If $G_1, G_2 \in W_K$ are adjacent then $||G_1|| - ||G_2||$ is 0 or $\pm 1$.*

Proof: Suppose $||G_1|| - ||G_2|| \geq 2$. Let $G$ be the tile of index $||G_2|| + 1$ that contains $G_1$, and observe that $G$ is adjacent to $G_2$. Then by Property 5 of the Gosper tiling, $\lambda \odot \text{parent}(G_1) \subset \lambda \odot G \subset \lambda \odot G_2$ and maximality of $G_1$ is violated. $\square$

**Lemma 2.2** *Suppose $\mathcal{C} \subset W_K \cap \mathcal{D}_n$ is a collection of tiles whose union topologically surrounds a smaller Whitney tile $G \in W_K \cap \mathcal{D}_{n+k}$ where $k > 0$. Then $\mathcal{C}$ surrounds a point of $K$.*

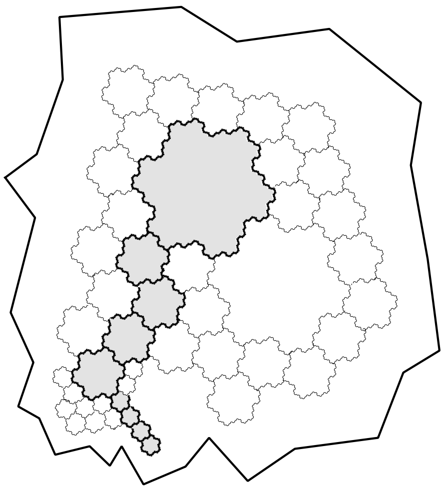

Figure 8: Whitney decomposition and a chain of tiles

PROOF: The $k$-fold parent of $G$ is a tile $D \in \mathcal{D}_n$ which is surrounded by $\mathcal{C}$. Applying Property 6 inductively shows that the union of $\lambda \odot G'$ for $G' \in \mathcal{C}$ surrounds whatever part of $\lambda \odot D$ it does not contain. Thus maximality of $G$ implies that $\lambda \odot D$ intersects $K$, and any point of intersection is surrounded by $\mathcal{C}$. $\square$

**Lemma 2.3** *Suppose that $G \in W_K$ and that there is a Whitney chain from some larger tile outside $\lambda^5 \odot G$ to $G$. For any $n > ||G||$ define $\mathrm{Wall}(G,n)$ to be the set $W_K^G \cap \mathcal{D}_n$. (See Figure 9.) Let*

$$E_n = \bigcup \{D \, : \, D \in \mathrm{Wall}(G,n)\} \cup \partial(\lambda^5 \odot G) \, .$$

*Then $E_n$ is a connected set which topologically separates $G$ from $K$. Furthermore, if $\Gamma$ is a Jordan curve separating $G$ from the complement of $\lambda^5 \odot G$, then every connected component of $\bigcup\{D \, : \, D \in \mathrm{Wall}(G,n)\}$ intersecting the domain bounded by $\Gamma$ also intersects $\Gamma$.*

PROOF: By Lemma 2.1 any path which connects $G$ to $K$ must hit Whitney tiles of every index larger than $||G||$. Thus any such path either hits $\mathrm{Wall}(G,n)$ or must leave $\lambda^5 \odot G$, proving that $E_n$ separates $G$ from $K$. Connectedness follows from the last assertion of the lemma for $\Gamma = \partial(\lambda^5 \odot G)$, so it remains only to prove the last assertion.

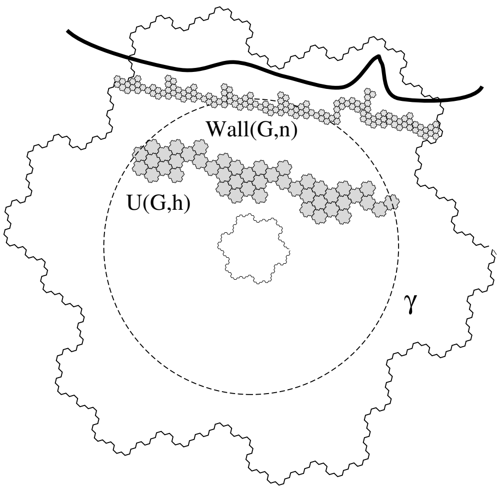


Figure 9: The sets Wall(G,n) and U(G,h).

Suppose to the contrary that there is a component $U$ of $\bigcup\{D : D \in \mathrm{Wall}(G,n)\}$ which intersects the domain bounded by $\Gamma$ but is disjoint from $\Gamma$ itself. The union of all Whitney tiles which are in the unbounded component of $U^c$ and are adjacent to $U$ is a connected set. By Lemma 2.1 all of these tiles have index $n-1$ or $n+1$. By connectedness and Lemma 2.1, they must all have a single index. Suppose they all have index $n+1$. Since tiles in $U$ can be connected to $G$ by Whitney chains which don't cross any tile of index $n+1$, this means $G$ is in a bounded component of the complement of $U$, which contradicts our assumption that $G$ could be connected by a Whitney chain to a larger tile outside $\lambda^5 \odot G$. Thus the adjacent tiles must all have index $n-1$. But then by Lemma 2.2 these adjacent tiles must also surround a point of $K$, which implies that $K$ is not connected, another contradiction. □

**Lemma 2.4** *For any Whitney tile $G$ there is a Whitney chain of tiles starting at $G$, staying inside $\lambda \odot \mathrm{parent}(G)$ and containing tiles of all generations larger than $||G||$. In particular, Wall(G,n) hits $\lambda \odot \mathrm{parent}(G)$ for all $n \geq ||G||$.*

PROOF: We claim it is enough to show that given a Whitney tile $G$ there is another Whitney tile $G'$ such that

1. $||G'|| = ||G|| + 1$.

2. $G' \subset \lambda \odot \mathrm{parent}(G)$.

3. $G'$ is connected to $G$ by a chain of tiles in $\lambda \odot \mathrm{parent}(G)$, all of generation $||G||$.

4. $\lambda \odot \mathrm{parent}(G') \subset \lambda \odot \mathrm{parent}(G)$.

Given these conditions we easily construct the desired chain by induction.

So suppose $G$ is a Whitney tile for $K$. By definition, $K$ intersects $\lambda \odot \mathrm{parent}(G)$ and hence it must intersect one of the children of $\lambda \odot \mathrm{parent}(G)$. Let $\widehat{G}$ denote one such child and let $\widetilde{G}$ denote the center tile of $\widehat{G}$ (i.e., the unique child which does not hit $\partial\widehat{G}$). Note that $||\widetilde{G}|| = ||G||$, and that $\widetilde{G}$ cannot be a Whitney tile (since $K$ hits $\widehat{G} = \lambda \odot \widetilde{G}$). By checking a small number of cases, we see that there is a chain $G = G_0, \ldots G_s = \widetilde{G}$, of adjacent tiles in generation $||G||$ which are disjoint from $\partial(\lambda \odot \mathrm{parent}(G))$.

Let $G_j$ be the first tile in the chain which is not a Whitney tile (there is one since $G$ is Whitney but $\widetilde{G}$ is not). Let $G'$ be a Whitney subtile of $G_j$ which is adjacent to $G_{j-1}$. Since $G_{j-1}$ is a Whitney tile of generation $||G||$ Lemma 2.1 implies $||G'|| = ||G|| + 1$. This is condition (1). Conditions (2) and (3) are obvious by construction. Finally, condition (4) holds because $G_j$ is not adjacent to $\partial(\lambda \odot \mathrm{parent}(G))$, so

$$\lambda \odot \mathrm{parent}(G') = \lambda \odot G_j \subset \lambda \odot \mathrm{parent}(G).$$

This proves the lemma. $\square$

The next two lemmas are needed in order to show that if "major portions" of a wall of Whitney tiles can be covered by a thin strip, then $K$ must intersect the core of an appropriate tile $G''$ without $\eta$-surrounding it; the latter event is controlled by the hypothesis of Theorem 1.2.

**Lemma 2.5** *Fix $G' \in \mathcal{D}$ and $\beta \in (0, \eta_0)$. Let $\widehat{U}$ be any connected set intersecting both $\partial(\lambda^5 \odot G')$ and $\lambda^3 \odot G'$. Suppose that $\widehat{U} \cap (\lambda^5 \odot G')$ is contained in an infinite open strip of width $2\beta\mathrm{diam}(G')$. Then there is a tile $G''$ contained in $\lambda^5 \odot G'$ and of the same index as $G'$, such that $\widehat{U}$ intersects* $\mathrm{core}(G'', 2\eta_0 - 2\beta)$.

Proof: Pick a point $x \in \widehat{U} \cap (\lambda^3 \odot G')$ and choose $y \in \widehat{U} \cap \partial(\lambda^4 \odot G')$ connected to $x$ inside $\widehat{U} \cap (\lambda^4 \odot G')$. By Property 1 of the tiling, the segment $\overline{xy}$ has length at least $d_0 \cdot \mathrm{diam}(G')$, so by Property 2 of the tiling, there is a tile $G''$ of the same index as $G'$, such that $\overline{xy}$ contains some point $z \in \mathrm{core}(G'', 2\eta_0)$. By Property 3 and convexity of disks, $z \in \lambda^5 \odot G'$, and therefore $G'' \subset \lambda^5 \odot G'$. Observe that $\mathrm{dist}(z, \widehat{U}) < 2\beta \mathrm{diam}(G')$, for if not, removing the open disk centered at $z$ of radius $2\beta \mathrm{diam}(G')$ from the infinite strip would contradict the connectedness of $\widehat{U}$. This observation implies the assertion of the lemma. $\square$

Let $G$ be any Whitney tile with $\lambda^5 \odot G$ not containing all of $K$. We will also assume (as in Lemma 2.3) that there is a Whitney chain from some larger tile outside $\lambda^5 \odot G$ to $G$. Let $\gamma$ be a circle centered at center$(G)$ which separates $\lambda^4 \odot G$ from $\partial(\lambda^5 \odot G)$. (Such a circle exists by Property 3 of the tiling.) For any positive integer $h$, let $U(G, h)$ be the union of all tiles $D \in W_K^G$ of index $||G|| + h$ such that $D$ intersects the disk bounded by $\gamma$ (see Figure 9).

**Lemma 2.6** *Choose $a_2$ so that $\lambda^{3-a_2} \leq \eta_0/2$. With $G$ as above, let $G'$ be a tile in $W_K^G$ with $||G|| < ||G'|| < ||G|| + h - a_2$ such that $\lambda^5 \odot G'$ is contained in the disk bounded by $\gamma$. Suppose that $U(G, h) \cap (\lambda^5 \odot G')$ is covered by an open strip of width $2\beta \mathrm{diam} G'$ with $\beta < \eta_0/4$. Then there is a tile $G'' \subset \lambda^5 \odot G'$ of the same index as $G'$, such that $K$ intersects $\mathrm{core}(G'', \eta_0)$ without $\eta_0$-surrounding $G''$.*

Proof: By Lemma 2.4 $G'$ can be connected by a Whitney chain in $\lambda^3 \odot G'$ to at least one tile $\widetilde{G}$ of generation $|G| + h$. Since $G'$ is (by assumption) connected by a Whitney chain to $G$, so is $\widetilde{G}$, and hence this tile is in $U(G, h)$. Thus $U(G, h)$ hits $\lambda^3 \odot G'$. Now Lemma 2.3 implies that there is a connected component of $U(G, h)$ which hits $\lambda^3 \odot G'$ and also hits $\partial(\lambda^5 \odot G')$. Let $\widehat{U}$ be a connected component of $U(G, h) \cap \lambda^5 \odot G'$ which hits $\lambda^3 \odot G'$. Now apply Lemma 2.5 with $G'$ and $\widehat{U}$ and let $G''$ be the tile given by the conclusion of the lemma.

Since $2\eta_0 - 2\beta > 3\eta_0/2$, we can pick a point $u$ in $U(G, h) \cap \mathrm{core}(G'', 3\eta_0/2)$. This clearly prevents $K$ from $\eta_0$-surrounding $G''$. For any Whitney tile $D$ of index $||G|| + h$, the blow-up $\lambda^3 \odot D$ intersects $K$ (by Property 4 of the tiling). Since $U(G, h) \cap (\lambda^5 \odot G')$ is a union of tiles of index $||G|| + h$, it follows that

$$\mathrm{dist}(u, K) < |\lambda|^{3-||G||-h} \leq |\lambda|^{3-a_2-||G'||} \leq \frac{\eta_0}{2}\mathrm{diam}(G'')\,,$$

by the choice of $a_2$. Therefore $K$ intersects $\mathrm{core}(G'', \eta_0)$. $\square$

# 3 A tree of Whitney tiles

Fix a compact, connected $K \subset \mathbb{C}$, a tile $G_*$, and a positive integer, $h$. We construct a tree $\widehat{T} = T(K, G_*, h)$ of Whitney tiles. If $G_* \notin W_K$ then $\widehat{T}$ is empty. If $G_* \in W_K$ then we take The root of $\widehat{T}$ to be $G_*$, and the remaining generations of $\widehat{T}$ are defined recursively as follows.

Assume $\widehat{T}$ has been defined up to generation $n$ and for each $G$ in $\widehat{T}_n$, the $n^{th}$ generation of $\widehat{T}$, define $\widetilde{T}_{n+1}(G)$ to be the set of tiles $D$ with the following properties:

1. $||D|| = ||G_*|| + (n+1)h$;
2. $D \in W_K^G$;
3. $\lambda^5 \odot D \subset \lambda^5 \odot G$.

Let $T_{n+1}(G)$ be a subcollection of $\widetilde{T}_{n+1}(G)$ which has maximal cardinality among all subcollections $\mathcal{C}$ for which the expanded tiles $\{\lambda^6 \odot D : D \in \mathcal{C}\}$ are disjoint. By maximality, $\bigcup\{\lambda^7 \odot D : D \in T_{n+1}(G)\}$ contains all tiles in $\widetilde{T}_{n+1}(G)$ and therefore

$$|\widetilde{T}_{n+1}(G)| \leq |\lambda|^{14}|T_{n+1}(G)|\,. \tag{3}$$

The children of $G$ in $\widehat{T}$ are defined to be the collection $T_{n+1}(G)$. Some trivial inductive observations are that $\widehat{T}_n \subset \mathcal{D}_{nh+||G_*||}$, that each $G \in \mathcal{D}_n$ is connected to $G_*$ by a Whitney chain, and that the sets $\lambda^5 \odot G$ are disjoint as $G$ runs over any $\widehat{T}_n$.

**Some tree terminology:** Let $V$ be a countable set.

(i) A mapping $\mathbf{T}$ from a measure space $\mathcal{S}$ to the set of trees on the vertex set $V$ is **measurable** with respect to a $\sigma$-field $\mathcal{F}$ on $\mathcal{S}$, if for any pair $\{v, v'\} \subset V$, the event $[\{v, v'\}$is an edge of $\mathbf{T}$ ] is in $\mathcal{F}$.

(ii) For any tree $T$ with vertex set contained in $V$, and any element $v \in V$, define $\mathrm{trunc}_v(T)$ to be null if $v$ is not a vertex of $T$, and otherwise let $\mathrm{trunc}_v(T)$ be $T$ with the part below $v$ removed; more precisely, the vertices of $\mathrm{trunc}_v(T)$ are the vertices of $T$ not separated from the root by $v$, and the edges are the edges of $T$ spanning pairs of vertices in this smaller vertex set.

For any $G \in \mathcal{D}$, let $\mathcal{F}_G$ denote the $\sigma$-field generated by the events $\{D \cap K \neq \emptyset\}$ for all tiles $D$ for which either $||D|| \leq ||G||$ or the interior of $D$ is disjoint from $\lambda^5 \odot G$.

**Lemma 3.1** *Let $G_*$ and $G$ be fixed tiles with $||G|| = nh + ||G_*||$. the mapping $\mathrm{trunc}_G \circ \widehat{T}$ from the underlying probability space to the space of trees on $\mathcal{D}$, is measurable with respect to $\mathcal{F}_G$.*

Recall $\mathrm{trunc}_G \circ \widehat{T}$ is the

PROOF: Consider an event of the form $\left\{\{D, D'\} \text{ is in the edge set of } \mathrm{trunc}_G(\widehat{T})\right\}$, where $D$ and $D'$ are tiles of index $mh + ||G_*||$ and $(m+1)h + ||G_*||$ respectively,
with $\lambda^5 \odot D' \subset \lambda^5 \odot D$. If $m \geq n$ and $\lambda^5 \odot D$ is not disjoint from $\lambda^5 \odot G$, then the edge $\{D, D'\}$ cannot be in $\mathrm{trunc}_G(\widehat{T})$. If $m < n$ or $\lambda^5 \odot D$ is disjoint from $\lambda^5 \odot G$ then the event that $\{D, D'\}$ is an edge of $\widehat{T}$ is the union of events witnessed by particular Whitney chains of tiles, all tiles being either disjoint from $\lambda^5 \odot G$ or of index at most $||G||$, so the event is measurable with respect to $\mathcal{F}_G$. $\square$

The next lemma requires the traveling salesman theorem described in the next section, so its proof is delayed until Section 5.

**Lemma 3.2** *Assume the random set $K$ satisfies the hypotheses of Theorem 1.2. Fix any tile $G_*$ and $h > 0$ and let $\widehat{T}$ be the random tree $T(K, G_*, h)$. There are constants $c_1, c_2 > 0$ such* *that for any tile $G \in \mathcal{D}_{nh+||G_*||}$,*

$$\mathbf{E}[\#\widehat{T}_{n+1}(G) \mid \mathcal{F}_G] \geq c_1 h|\lambda|^h - c_2|\lambda|^h \tag{4}$$

*on the event that $G_* \in W_K$, the tile $G$ is in $\widehat{T}_n$, and $\lambda^5 \odot G_*$ does not contain $K$. The constants $c_1$ and $c_2$ depend only on $c_0$.*

To prove Theorem 1.2, we also need two general lemmas concerning trees. Define the **boundary** $\partial T$ of the infinite rooted tree $T$ to be the set of infinite self-avoiding paths from the root. The next lemma is implicit in Hawkes (1981) and can be found in a stronger form in Lyons (1990). For convenience, we include the short proof.

**Lemma 3.3** *Let $T$ be an infinite rooted tree. Given constants $C > 0$ and $\theta > 1$, define a metric on $\partial T$ by*

$$\mathrm{dist}(\xi, \xi') = C\theta^{-n} \text{ if } \xi \text{ and } \xi' \text{ share exactly } n \text{ edges.} \tag{5}$$

*Suppose that independent percolation with parameter $p \in (0,1)$ is performed on $T$, i.e., each edge of $T$ is erased with probability $1-p$ and retained with probability $p$, independently of all other edges. If*

$$\dim(\partial T) < \alpha = \frac{\log(1/p)}{\log \theta}$$

*then with probability 1, all the connected components of retained edges in $T$ are finite.*

Proof: It suffices to show that the connected component of the root is finite almost surely. For any vertex $v$ of $T$, denote by $|v|$ the number of edges between $v$ and the root. By the dimension hypothesis and the definition of the metric on $\partial T$, there must exist cut-sets $\Pi$ in $T$ for which the $\alpha$-dimensional cut-set sum

$$\sum_{v \in \Pi} \theta^{-|v|\alpha} = \sum_{v \in \Pi} p^{|v|}$$

is arbitrarily small. But for any cutset $\Pi$, the right-hand side is the expected number of vertices in $\Pi$ which are connected to the root after percolation; this expectation bounds the probability that the connected component of the root is infinite. $\square$

The next lemma formalizes the notion of a random tree which "stochastically dominates" the family tree of a branching process. We require the analogue of a filtration in our setting.

**Definition:** Let $V$ be a countable set and let $T$ be a random tree with vertex set contained in $V$, i.e., $T$ is a measurable mapping from some probability space $\langle \mathcal{S}, \mathcal{A}, \mathbf{P} \rangle$ to the set of trees on the vertex set $V$. Say that $\sigma$-fields $\{\mathcal{F}_v : v \in V\}$ on $\mathcal{S}$ form a **tree-filtration** if for any $v, w \in V$ and any $A \in \mathcal{F}_v$, the event $A \cap \{w \text{ is a descendant of } v \text{ in } T\}$ is $\mathcal{F}_w$-measurable.

**Lemma 3.4** *Let $V$ be a countable set and let $T$ be a random tree with vertex set contained in $V$. We assume that $T$ is rooted at a fixed $v_* \in V$. Assume that $b > 1$ and a tree-filtration $\{\mathcal{F}_v : v \in V\}$ exists such that* $\mathrm{trunc}_v(T)$ *(defined before Lemma 3.1) is $\mathcal{F}_v$-measurable for each $v \in V$, and the conditional expectation*

$$\mathbf{E}(\textit{number of children of } v \textit{ in } T \,|\, \mathcal{F}_v) \geq b.$$

*If every vertex of $T$ has at most $M$ children and at least $m$ children, $m \geq 0$, then*

1. *The probability that $T$ is infinite is at least $1 - q > 0$, where $q$ is the unique fixed point in $[0, 1)$ of the polynomial*
$$\psi(s) = s^m + \frac{b - m}{M - m}(s^M - s^m)\,.$$
*(Observe that $q = 0$ when $m > 0$.)*

2. $\mathbf{P}(T \textit{ is infinite } | \mathcal{F}_{v_*}) \geq 1 - q$.

3. *If $\partial T$ is endowed with the metric (5), then* $\dim(\partial T) \geq \log b / \log \theta$ *with probability at least $1 - q$ ($\theta$ is as in (5)).*

PROOF: **1.** Let $|T_n|$ be the size of the $n^{th}$ generation $T_n$ of $T$ and let $\psi_n(s)$ denote the $n$-fold iterate of $\psi$. We claim that for $s \in [0, 1]$,

$$\mathbf{E} s^{|T_n|} \leq \psi_n(s). \tag{6}$$

When $n = 1$, convexity of $x \mapsto s^x$ implies that

$$s^{|T_1|} \leq s^m + \frac{|T_1| - m}{M - m}(s^M - s^m)$$

and the claim follows by taking expectations:

$$\mathbf{E}s^{|T_1|} \leq s^m + \frac{\mathbf{E}|T_1| - m}{M - m}(s^M - s^m) \leq \psi_1(s)$$

since $s^M - s^m \leq 0$. For $n > 1$ proceed by induction. Let $|T_{n+1}(v)|$ be the number of children of $v$ if $v \in T_n$ and zero otherwise, and use the argument from the $n = 1$ case to see that $\mathbf{E}(s^{|T_{n+1}(v)|} \,|\, \mathcal{F}_v) \leq \psi(s)$ on the event $v \in T_n$ (which is an event in $\mathcal{F}_v$). Giving $V$ an arbitrary linear order (denoted "$<$"), we have in particular

$$\mathbf{E}(s^{|T_{n+1}(v)|} \,|\, T_n \text{and } T_{n+1}(w) \text{ for } w < v \text{ in } T_n) \leq \psi(s)$$

for $v \in T_n$. Since

$$s^{|T_{n+1}|} = \prod_{v \in T_n} s^{|T_{n+1}(v)|}\,, \quad \text{this yields} \quad \mathbf{E}(s^{|T_{n+1}|} \,|\, T_n) \leq \psi(s)^{|T_n|}\,.$$

Taking expectations and applying the induction hypothesis with $\psi(s)$ in place of $s$ gives

$$\mathbf{E}s^{|T_{n+1}|} \leq \mathbf{E}\Big(\psi(s)^{|T_n|}\Big) \leq \psi_n(\psi(s)) = \psi_{n+1}(s)\,,$$

proving the claim (6).

From (6) we see that $\mathbf{P}(|T_n| = 0) \leq \mathbf{E}q^{|T_n|} \leq \psi(q) = q$, establishing the first conclusion of the lemma.

**2.** By copying the derivation of (6), inserting an extra conditioning on $\mathcal{F}_{v_*}$, one easily verifies that $\mathbf{E}(s^{|T_n|} \,|\, \mathcal{F}_{v_*}) \leq \psi_n(s)$, and the rest of the argument is the same as in the first part.

**3.** Let $T'(v)$ be the connected component of the subtree of $T$ below $v$ after removing each vertex of $T$ below $v$ independently with probability $1 - p$. For $p > 1/b$, let $q_p \in (0, 1)$ solve $q_p = 1 + (bp/M)(q_p^M - 1)$. We apply the second part of the lemma to $T'(v)$ conditioned on $\mathcal{F}_v$ to see that

$$\mathbf{P}(T'(v) \text{ is infinite } \,|\, \mathcal{F}_v) \geq 1 - q_p$$

for $v \in T$. By Lemma 3.3, the event $\{\dim(\partial T) < |\log p|/\log\theta\}$ is contained up to null sets in the event $\{T'(v) \text{ is finite for all } v \in T_n\}$. Thus

$$\begin{aligned}
\mathbf{P}\Big(\dim(\partial T) < |\log p|/\log\theta \,\Big|\, T_n\Big) &\leq \mathbf{P}\Big(\cap_{v\in T_n} T'(v) \text{ finite } \Big| T_n\Big) \\
&= \prod_{v\in T_n} \mathbf{P}\Big(T'(v) \text{ finite } \Big| T_n\,,\, T'(w) \text{ finite for all } w < v \text{ in } T_n\Big) \\
&\leq q_p^{|T_n|}
\end{aligned}$$

since each event conditioned on is in the corresponding $\mathcal{F}_v$. Taking expectations yields

$$\mathbf{P}\Big(\dim(\partial T) < \frac{|\log p|}{\log\theta}\Big) \leq \psi_n(q_p).$$

Since $q_p < 1$ for each $p > 1/b$, this goes to $q$ as $n \to \infty$, proving the last conclusion of the lemma.

$\square$

Proof of Theorem 1.2: Let $G_*$ be a fixed tile and let $h$ be a positive integer (to be specified later in the proof). Denote by $A$ the event that $G_* \in W_K$ and $\lambda^5 \odot G_*$ does not contain $K$. On the event $A$, let $\widehat{T} = T(K, G_*, h)$ and let $\Omega_*$ be the connected component of $K^c$ that contains $G_*$. Define a metric on $\partial\widehat{T}$ by

$$\operatorname{dist}(\xi,\xi') = |\lambda|^{-(n+1)h-||G_*||} \text{ if } \xi \text{ and } \xi' \text{ share exactly } n \text{ edges.}$$

Each $\xi = (G_1, G_2, \ldots) \in \partial\widehat{T}$ defines a unique limiting point $\phi(\xi) \in \partial\Omega_*$ which is the decreasing limit of the set $\lambda^5 \odot G_n$. If $\xi = (G_1, G_2, \ldots)$ and $\xi' = (G_1', G_2', \ldots)$ share exactly $n$ edges, then by definition of $\widehat{T}_{n+1}$, the expanded tiles $\lambda^6 \odot G_{n+1}$ and $\lambda^6 \odot G_{n+1}'$ are disjoint. Since $\phi(\xi) \in \lambda^5 \odot G_{n+1}$ and $\phi(\xi') \in \lambda^5 \odot G_{n+1}'$, it follows from Property 1 of the tiling that

$$|\phi(\xi) - \phi(\xi')| \geq d_0 |\lambda|^{-(n+1)h-||G_*||}.$$

Thus

$$|\phi(\xi) - \phi(\xi')| \geq d_0 \cdot \operatorname{dist}(\xi,\xi')$$

and since the range of $\phi$ is included in $\partial\Omega \cap \lambda^5 \odot G_*$, it follows that

$$\dim(\partial\Omega \cap \lambda^5 \odot G_*) \geq \dim(\partial\widehat{T}). \tag{7}$$

From Lemma 3.4 and the conclusion of Lemma 3.2, we see that

$$\dim(\partial T(K, G_*, h)) \geq \frac{\log(c_1 h|\lambda|^h - c_2|\lambda|^h)}{h \log|\lambda|} \tag{8}$$

with probability 1, on the event that $G_* \in W_K$ and $\lambda^5 \odot G_*$ does not contain $K$. Choose $h$ to maximize the RHS of (8). Since the maximum is greater than 1, there is an $\epsilon > 0$ for which

$$\dim(\partial \widehat{T}(K, G_*, h)) \geq 1 + \epsilon$$

with probability 1 on this event. Finally, let $\Omega$ be any connected component of $K^c$. By property (2) of the Whitney decomposition, for any open $V$ intersecting $\partial\Omega$, there is a tile $G_* \in W_K$ with $\lambda^5 \odot G_* \subset V$, and the theorem follows from (7). $\square$

**Remark:** A planar domain $\Omega$ is called a **John domain** if there is a base point $z_0 \in \Omega$ and a constant $C > 0$ so that any point $x \in \Omega$ can be joined to $z_0$ by a curve $\gamma_x \subset \Omega$ so that $\operatorname{dist}(z, \partial\Omega) \geq C|x - z|$ for any $z \in \gamma_x$. John domains were introduced by Fritz John in 1961, and some basic facts about them can be found in Näkki and Väisälä (1994).

With the notation of the above proof, if $G_* \in W_K$ is contained in a component $\Omega$ of $K^c$, choose for every tile $G \neq G_*$ in the tree $T(K, G_*, h)$, a Whitney chain leading to $G$ from its unique ancestor in the previous generation of the tree. For each tile $G'$ in this chain, there is an open disk containing it which is contained in $\lambda \odot G'$ (by Property 3 of the tiling). The union of all these open disks as $G'$ runs over the chosen Whitney chain for $G$ and $G$ runs over $T(K, G_*, h)$, is a John domain $\Omega_{\mathrm{J}}$ satisfying $\dim(\partial\Omega \cap \partial\Omega_{\mathrm{J}}) \geq 1 + \epsilon$.

# 4 The traveling salesman theorem

Given a set $E$ in the plane and another bounded plane set $S$, we define

$$\beta_E(S) = (\operatorname{diam}(S))^{-1} \inf_{L \in \mathcal{L}} \sup_{z \in E \cap S} \operatorname{dist}(z, L),$$

where $\mathcal{L}$ is the set of all lines $L$ intersecting $S$.

**Theorem 4.1 (Jones 1990)** *If $E \subset \mathbb{C}$ then the length of the shortest connected curve $\Gamma$ containing $E$ is bounded between (universal) constant multiples of*

$$\mathrm{diam}(E) + \sum_{Q} \beta_E(3 \odot Q)^2 \mathrm{diam}(Q),$$

*where the sum is over all dyadic squares in the plane and $3 \odot Q$ is the union of a 3 by 3 grid of congruent squares with $Q$ as the central square.*

A simpler proof of this Theorem, and an extension to higher dimensions, are given in Okikiolu (1992). The theorem easily implies that the length $|\Gamma|$ of any curve $\Gamma$ which passes within $r$ of every point of $E$ satisfies

$$\mathrm{diam}(E) + \sum_{\mathrm{diam}(Q) \geq r} \beta_E(3 \odot Q)^2 \mathrm{diam}(Q) \leq c_3 |\Gamma|, \tag{9}$$

where the sum is over all dyadic squares in the plane with diameter at least $r$. For every set $S$, there is a dyadic square $Q$ of side length at most $2\mathrm{diam}(S)$ for which $S \subset 3 \odot Q$. Picking $S = \lambda^5 \odot G$ for some tile $G$ and $Q$ accordingly, we get

$$\beta_E(\lambda^5 \odot G)^2 \mathrm{diam}(G) \leq 9|\lambda|^{-10} \beta_E(3 \odot Q)^2 \mathrm{diam}(Q)$$

and since each expanded square $3 \odot Q$ contains a bounded number of expanded tiles $\lambda^5 \odot G$ for tiles $G$ with $\sqrt{2}|\lambda|^5 \mathrm{diam}(G) \geq \mathrm{diam}(Q)$, it follows that the length of any curve passing within $r$ of every point of $E$ satisfies

$$|\Gamma| \geq c_4 \left( \mathrm{diam}(E) + \sum_{\mathrm{diam}(G) \geq r} \beta_E(\lambda^5 \odot G)^2 \mathrm{diam}(G) \right). \tag{10}$$

We require the following corollary, which uses an idea from Bishop and Jones (1994).

**Corollary 4.2** *Let $\gamma$ be a Jordan curve with length denoted $|\gamma|$ and let $\mathcal{C}$ be a collection of Whitney tiles of index $n$. Let $U$ denote $\bigcup_{D \in \mathcal{C}} D$ and suppose that $\gamma \cup U$ is connected. Then there is a constant $c_5$ such that the cardinality of $\mathcal{C}$ is at least*

$$c_5 |\lambda|^n \left( -|\gamma| + \sum_{G' \in \Xi(\mathcal{C})} \beta_U(\lambda^5 \odot G')^2 \mathrm{diam}(G') \right),$$

*where $\Xi(\mathcal{C})$ is the collection of tiles $G'$ of index at most $n$ for which $\lambda^5 \odot G'$ intersects $U$.*

Proof: Let $\mathcal{C}_\circ$ be the collection of circles of radius $r := |\lambda|^{-n}$ centered at points center$(D)$ for $D \in \mathcal{C}$. Since neighboring tiles in $\mathcal{C}$ give rise to intersecting circles in $\mathcal{C}_\circ$, we see that $\Gamma := \gamma \cup \bigcup_{\Theta \in \mathcal{C}_\circ} \Theta$ is connected and passes within $r$ of every point of $\gamma \cup U$. Furthermore, any connected finite union of closed curves is a closed curve, and hence $\Gamma$ is a curve of length at most $|\gamma| + 2\pi \#(\mathcal{C})|\lambda|^{-n}$. Combining this with (10) shows that

$$\#(\mathcal{C}) \geq \frac{|\lambda|^n}{2\pi} \Big( c_4 \sum_{\mathrm{diam}(G') \geq r} \beta_U(\lambda^5 \odot G')^2 \mathrm{diam}(G') - |\gamma| \Big).$$

Since all tiles in $\Xi(\mathcal{C})$ have diameter at least $r$, this proves the lemma. $\square$

# 5 Expected offspring in the Whitney tree

Proof of Lemma 3.2: Fix $G_*$ and $h$ as in the statement of the lemma and let $G$ be any tile in $\widehat{T}_n$. Let $\gamma$ be the circle separating $|\lambda|^4 \odot G$ from $\partial(|\lambda|^5 \odot G)$, which was defined before the statement of Lemma 2.6. Let $\mathcal{C}$ be the collection of tiles $D \in W_K^G$ of index $||G|| + h$ intersecting the disk bounded by $\gamma$. The union of all tiles in $\mathcal{C}$ is the set $U = U(G, h)$ defined before Lemma 2.6. We want to show that the expected cardinality of $\widehat{T}_{n+1}(G)$ is large. Since the cardinality of $\widehat{T}_{n+1}(G)$ is at least $|\lambda|^{-14}$ times the cardinality of $\widetilde{T}_{n+1}(G)$ by (3), and $\widetilde{T}_{n+1}(G)$ is a superset of $\mathcal{C}$, it suffices to show that

$$\mathbf{E}(\#\mathcal{C} \,|\, \mathcal{F}_G) \geq c_1' h |\lambda|^h - c_2' |\lambda|^h \,.$$

To do this, we will apply Corollary 4.2 to $\mathcal{C}$, so that the set $U$ defined in that corollary is the same as $U(G, h)$ defined above.

We will be able to bound from below the summands in Corollary 4.2 for most, but not all, "intermediate-sized" tiles $G'$. Pick an integer $a_3 > 1$ so that $|\lambda|^{3-a_3} < d_0$, where $d_0$ is the minimal distance between nonadjacent tiles in $\mathcal{D}_0$. Let $\widetilde{\gamma}$ be a circle concentric with $\gamma$, with a smaller radius: $\mathrm{rad}(\widetilde{\gamma}) = \mathrm{rad}(\gamma) - |\lambda|^{5-a_3}$ (see Figure 10).

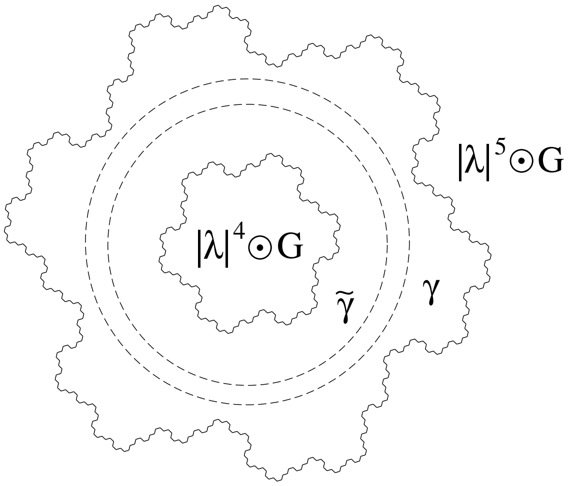


Figure 10: The circles $\gamma$ and $\widetilde{\gamma}$

For $a_3 < j < h$, let $W_K^G(j)$ be the set of tiles $G' \in W_K^G$ such that $||G'|| = ||G|| + j$ and $\lambda^5 \odot G'$ intersects the disk bounded by $\widetilde{\gamma}$ . For any such tile $G'$ the blow-up $\lambda^5 \odot G'$ is contained in the disk bounded by $\gamma$.

Fix any tile $G' \in W_K^G(j)$ with $a_3 < j \le ||G|| + h - a_2$, where $a_2$ was specified in Lemma 2.6. That lemma implies

$$\mathbf{P}\left(\beta_U(\lambda^5 \odot G') \ge \frac{\eta_0}{4} \,\Big|\, \mathcal{F}_G \text{ and } W_K^G(j)\right) \tag{11}$$

$$\ge \quad \mathbf{P}\Big[\bigcap_{G''} \Big(K \;\; \eta_0\text{-surrounds } G'' \;\; \underline{\text{or}} \;\; K \cap \text{core}(G'', \eta_0) = \emptyset\Big) \,\Big|\, \mathcal{F}_G \text{ and } W_K^G(j)\Big],$$

where the intersection is over all tiles $G'' \subset \lambda^5 \odot G'$ such that $||G''|| = ||G'||$. The set of such tiles $G''$ for a fixed $G'$ has cardinality $|\lambda|^{10}$. Enumerating these and multiplying conditional probabilities using the hypotheses of Lemma 3.2 (since the $\sigma$-fields $\mathcal{F}_G$ and $\sigma(W_K^G(j))$ are contained in $\sigma(K \setminus G''^\circ)$ ) gives a lower bound of $c_0^{|\lambda|^{10}}$ for (11), and implies that

$$\mathbf{E}\Big(\beta_U(\lambda^5 \odot G')^2 \,\Big|\, \mathcal{F}_G \text{and } W_G^K(j)\Big) \ge \left(\frac{\eta_0}{4}\right)^2 c_0^{|\lambda|^{10}} = c_6 > 0\,.$$

(This is the definition of $c_6$). Since $\gamma$ is outside $\lambda^4 \odot G$, the distance from $\gamma$ to $\lambda^3 \odot G$ is at least

$|\lambda|^3 d_0 \cdot \mathrm{diam}(G)$, by Property 1 of the tiling. Therefore the distance from $\tilde{\gamma}$ to $\lambda^3 \odot G$ is at least $(|\lambda|^3 d_0 - |\lambda|^{5-a_3}) \cdot \mathrm{diam}(G)$, which is greater than $d_0 \cdot \mathrm{diam}(G)$ by the choice of $a_3$. (Recall that for Gosper islands, $|\lambda| = \sqrt{7}$.)

By applying Lemma 2.3 to each of the connected components of of $W_K^G(j)$, we see that the union of $\tilde{\gamma}$ with all the tiles in $W_K^G(j)$ is a connected set (note that a component of $W_K^G(j)$ not hitting $\tilde{\gamma}$ is the same as a component of $\mathrm{Wall}(G, |\,G\,| + j)$ missing $\tilde{\gamma}$, but hitting its interior). By Lemma 2.4, $W_K^G(j)$ must intersect $\lambda \odot \mathrm{parent}(G)$ and hence $\lambda^3 \odot G$. Since $W_K^G(j)$ intersects both $\lambda^3 \odot G$ and $\tilde{\gamma}$, it follows that its cardinality is at least $d_0|\lambda|^j$. Thus for each $j \in (a_3, h - a_2]$ we have

$$\mathbf{E}\Big( \sum_{G' \in W_K^G(j)} \beta_U(\lambda^5 \odot G')^2 \,\Big|\, \mathcal{F}_G \text{ and } W_G^K(j) \Big) \geq c_6 d_0 |\lambda|^j .$$

By Corollary 4.2,

$$\mathbf{E}(\#\mathcal{C} \,|\, \mathcal{F}_G) \geq c_5 |\lambda|^{(n+1)h + ||G_*||} \left( -|\gamma| + \sum_{j=a_3+1}^{h-a_2} |\lambda|^{-nh - ||G_*|| - j} c_6 d_0 |\lambda|^j \right).$$

Summing gives

$$\mathbf{E}(\#\mathcal{C} \,|\, \mathcal{F}_G) \geq c_5 \left( c_6 d_0 (h - a_2 - a_3)|\lambda|^h - |\gamma| \cdot |\lambda|^{(n+1)h + ||G_*||} \right)$$

which proves the lemma since $|\gamma| \geq 2\pi |\lambda|^{4 - nh - ||G_*||}$. $\square$

# 6 The Brownian frontier: Proof of Theorem 1.1

Let $\mathbf{P}_x$ denote the law of a planar Brownian motion $\{B(t)\}_{t \geq 0}$ started at $x$. We use $\mathbf{P}_0$ unless indicated explicitly otherwise. Let $\tau_{\mathrm{exp}}$ be a positive random variable, independent of the Brownian motion, which is exponential of mean 1 (i.e., its density is $e^{-t}$). We will verify (1) for $K = B[0, \tau_{\mathrm{exp}}]$. By Brownian scaling, this will imply the first assertion of Theorem 1.1.

**Notation:** for any compact planar set $S$, denote by $\tau_S = \min\{t \geq 0 : B(t) \in S\}$ the first hitting time of $S$, which is almost surely finite if $S$ has positive logarithmic capacity. Given

$\eta \in (0, 1/10)$, let $J_0$ be a rectifiable closed Jordan curve, which is the exterior boundary of a topological annulus separating $\mathrm{core}(G_0, \eta)$ from $\partial G_0$. (Here the constant $1/10$ can be replaced by any constant smaller than the inradius of $G_0$, and the existence of $J_0$ is guaranteed by Property 8 of the tiling.) For the rest of this section, consider a homothetic image $G = z_{\mathrm{cen}} + rG_0$ of the Gosper Island $G_0$, with $r \in (0, 1)$ and $z_{\mathrm{cen}}$ in the plane. Also, denote by $J = z_{\mathrm{cen}} + rJ_0$ the image of $J_0$ in $G$. We must obtain estimates which are uniform in the location and scale of $G$, as well as in the structure of the Brownian range outside $G$.

**Lemma 6.1** *For every $x \in \mathrm{core}(G, \eta)$ ,*

$$\mathbf{P}_x\Big(B[0, \tau_J] \;\; \eta\text{-surrounds } G\Big) \geq c_7(\eta) > 0\,. \tag{12}$$

*Furthermore, there exists $c_8(\eta) > 0$ such that*

$$\mathbf{P}_x\Big(B[0, \tau_J] \;\; \eta\text{-surrounds } G \;\text{ and } \tau_J < \tau_{\exp} < \tau_{\partial G}\Big) \geq c_8(\eta)\mathbf{P}_x(\tau_{\exp} < \tau_{\partial G})\,. \tag{13}$$

Proof: The first estimate is immediate for $G_0$, and the general case follows by scaling. For the second, observe that by Brownian scaling, $\inf_{y \in J} \mathbf{P}_y(\tau_{\partial G} > \mathrm{diam}(G)^2)$ is a positive constant depending only on $J_0$, hence only on $\eta$. Also, clearly $\mathbf{P}(\tau_J < 1) > 1/2$ and therefore

$$\mathbf{P}(\tau_J < \tau_{\exp} < \tau_J + \mathrm{diam}(G)^2) > \frac{e^{-2}}{2}\mathrm{diam}(G)^2\,, \;\; \text{since } \mathrm{diam}(G) < 1.$$

Applying (12), lack of memory of exponential variables, and the strong Markov property of Brownian motion at the stopping time $\tau_J$, then shows that the left-hand side of (13) is at least a constant multiple of $\mathrm{diam}(G)^2$. On the other hand, for any $x \in G$ we have $\mathbf{P}_x(\tau_{\exp} < \tau_{\partial G}) \leq \mathbf{E}_x(\tau_{\partial G}) \leq \mathrm{diam}(G)^2$. This completes the proof. $\square$

**Lemma 6.2** *There exists $c_9(\eta) > 0$ such that for any tile $G$, for any $x \in \mathrm{core}(G, \eta)$ and any $A \subset \partial G$,*

$$\mathbf{P}_x\Big(B[0, \tau_{\partial G}] \;\; \eta\text{-surrounds } G \text{ and } B(\tau_{\partial G}) \in A\Big) \geq c_9(\eta)\mathbf{P}_x(B(\tau_{\partial G}) \in A).$$

PROOF: Recall that $J$ is a Jordan curve of finite length separating $\text{core}(G,\eta)$ from $\partial G$. The Harnack principle (see, e.g., Bass (1995, Theorem 1.20)) implies that there is a constant $c_{10} = c_{10}(J_0)$ such that for any $y, z \in J$ and for any $A \subset \partial G$,

$$\mathbf{P}_y(B(\tau_{\partial G}) \in A) \geq c_{10}\mathbf{P}_z(B(\tau_{\partial G}) \in A). \tag{14}$$

Therefore for any $x \in \text{core}(G,\eta)$ ,

$$\mathbf{P}_x(B[0,\tau_J] \;\; \eta\text{-surrounds } G \text{ and } B(\tau_{\partial G}) \in A)$$

$$= \mathbf{E}_x\Big(\mathbf{1}_{\{B[0,\tau_J] \;\; \eta\text{-surrounds } G\}} \cdot \mathbf{P}_{B(\tau_J)}[B(\tau_{\partial G}) \in A]\Big) \tag{15}$$

Applying the Harnack inequality (14) with $y = B(\tau_J)$ and then invoking the estimate (12) from the previous lemma, we find that the expression (15) is at least $c_7(\eta)c_{10}\mathbf{P}_z(B(\tau_{\partial G}) \in A)$, for any $z \in J$. Finally, taking $z = B(\tau_J)$ and averaging with respect to $\mathbf{P}_x$ using the strong Markov property gives

$$\mathbf{P}_x(B[0,\tau_J] \;\; \eta\text{-surrounds } G\text{and } B(\tau_{\partial G}) \in A) \geq c_7(\eta)c_{10}\mathbf{P}_x(B(\tau_{\partial G}) \in A)\,,$$

for any Borel set $A \subset \partial G$. This proves the lemma. $\square$

Given $\eta > 0$, we abbreviate $\tau_{\mathrm{C}} = \tau_{\text{core}(G,\eta)}$ and partition the Brownian trajectory into three pieces:

1. Until the first time $\tau_{\mathrm{C}}$ that the path visits $\text{core}(G,\eta)$.
   Formally, define $B^{(1)}(t) = B(t \wedge \tau_{\mathrm{C}})$ for $t \geq 0$, where $t \wedge s$ is shorthand for $\min\{t,s\}$.

2. From time $\tau_{\mathrm{C}}$ until the next visit to $\partial G$, denoted $\tau_{\mathrm{C},\partial G} = \min\{t \geq \tau_C \,:\, B(t) \in \partial G\}$.
   Define $B^{(2)}(t) = B((t + \tau_{\mathrm{C}}) \wedge \tau_{\mathrm{C},\partial G})$ for $t \geq 0$.

3. After time $\tau_{\mathrm{C},\partial G}$.
   Denote $B^{(3)}(t) = B(t + \tau_{\mathrm{C},\partial G})$ for $t \geq 0$.

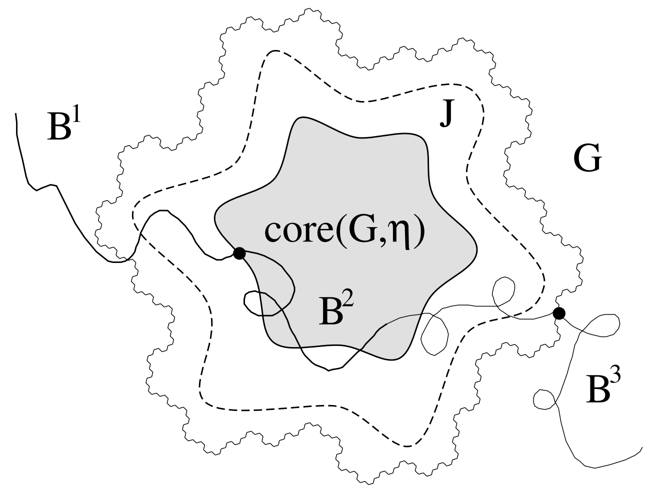


Figure 11: The partition of the Brownian trajectory

The idea now is that $B^{(1)}$ and $B^{(3)}$ determine the Brownian range outside $G^{\circ}$, and $B^{(2)}$ has a substantial chance of $\eta$-surrounding $G$, even when we condition on its endpoints. However, we still have to take the exponential killing into account. Define the random variable

$$I = \begin{cases} 1 & \text{if} \quad \tau_{\exp} < \tau_{\mathrm{C}} \\ 2 & \text{if} \quad \tau_{\mathrm{C}} \le \tau_{\exp} < \tau_{\mathrm{C},\partial G} \\ 3 & \text{if} \quad \tau_{\mathrm{C},\partial G} \le \tau_{\exp} \end{cases}$$

that indicates in which part of the motion the exponential killing occurred. Finally, define

$$\widetilde{\tau}_{\exp} = \begin{cases} \tau_{\exp} & \text{if} \quad I = 1 \\ \tau_{\mathrm{C}} & \text{if} \quad I = 2 \\ \tau_{\exp} - \tau_{\mathrm{C},\partial G} & \text{if} \quad I = 3 \end{cases}$$

**Proposition 6.3** *For any $\eta \in (0, 1/10)$ there is a constant $c_0 = c_0(\eta) > 0$ such that for all homothetic images $G = z_{\mathrm{cen}} + rG_0$ of $G_0$ with $r \in (0,1)$ and $z_{\mathrm{cen}}$ in the plane:*

$$\mathbf{P}\Big(B[0,\tau_{\exp}] \ \eta\text{-surrounds } G \ \underline{\text{or}} \ \ B[0,\tau_{\exp}] \cap \mathrm{core}(G,\eta) \ = \emptyset \ \Big| \ \mathcal{A}_G\Big) > c_0 \tag{16}$$

*where the conditioning is on the $\sigma$-field $\mathcal{A}_G$ generated by $I$, $B^{(1)}$, $B^{(3)}\mathbf{1}_{\{I=3\}}$ and $\widetilde{\tau}_{\exp}$.*

PROOF: On the event $\{I = 1\}$, the set $B[0, \tau_{\exp}]$ is disjoint from $\mathrm{core}(G, \eta)$ .
To handle the case $\{I = 2\}$, we use the strong Markov property at time $\tau_{\mathrm{C}}$ and apply the estimate (13) to $B^{(2)}$. Denoting $\tau_{\mathrm{C},J} = \min\{t \geq \tau_C : B(t) \in \partial G\}$, this gives

$$\mathbf{P}_0\Big(B[\tau_{\mathrm{C}}, \tau_{\mathrm{C},J}] \;\; \eta\text{-surrounds } G \;\text{ and } \tau_{\mathrm{C},J} < \tau_{\exp} < \tau_{\mathrm{C},\partial G} \,\Big|\, I \geq 2;\, B^{(1)}\Big)$$

$$\geq c_8(\eta)\mathbf{P}_0\Big(\tau_{\mathrm{C}} < \tau_{\exp} < \tau_{\mathrm{C},\partial G} \,\Big|\, I \geq 2;\, B^{(1)}\Big)\,.$$

This proves (16) on the event $I = 2$.

Only the case $I = 3$ remains. By using the strong Markov property at time $\tau_{\mathrm{C}}$ and applying Lemma 6.2 to $B^{(2)}$, we see that for any $A \subset \partial G$,

$$\mathbf{P}_0\Big(B[\tau_C, \tau_{\mathrm{C},\partial G}] \;\; \eta\text{-surrounds } G \text{ and } B(\tau_{\mathrm{C},\partial G}) \in A \,\Big|\, B^{(1)}\Big) \geq c_9(\eta)\mathbf{P}_0\Big(B(\tau_{\mathrm{C},\partial G}) \in A \,\Big|\, B^{(1)}\Big)\,.$$

In other words,

$$\mathbf{P}_x\Big(B[\tau_C, \tau_{\mathrm{C},\partial G}] \;\; \eta\text{-surrounds } G \,\Big|\, B^{(1)},\, B(\tau_{\mathrm{C},\partial G})\Big) \geq c_9(\eta)\,.$$

An application of the strong Markov property at time $\tau_{\mathrm{C},\partial G}$ shows that this lower bound is still valid if we insert an additional conditioning on $I \geq 2$ and on $B^{(3)}$.
Finally, since $\mathbf{P}_0(I = 3 \,|\, I \geq 2) \geq e^{-1}/2$ and $\tilde{\tau}_{\exp}$ is conditionally independent of $B[0, \tau_{\mathrm{C},\partial G}]$ given $I = 3$, this completes the proof of the proposition. □

To obtain the uniformity in Theorem 1.1, we will need the following general observation.

**Lemma 6.4** *Let $\Gamma : [0, \infty) \to \mathbb{C}$ be any continuous path and let $t > 0$. For any open disk $U$ intersecting* $\mathrm{frontier}(\Gamma[0, t])$ *such that $\Gamma(t) \notin \overline{U}$, there is a $\delta > 0$ such that for any $s \in [t, t + \delta]$, we have*

$$U \cap \mathrm{frontier}(\Gamma[0, t]) \supset U \cap \mathrm{frontier}(\Gamma[0, s]) \neq \emptyset\,. \tag{17}$$

PROOF: By hypothesis $U$ intersects the unbounded component, $\Omega$ of $\Gamma[0, t]^c$, so there is a point $u \in U$ and an unbounded curve starting from $u$ and contained in $\Omega$. Using the convexity of $U$, we can append to this curve a line-segment connecting $u$ to a nearest point $x$ on $\Gamma[0,t]$, and thus obtain an unbounded curve $\psi$ starting at $x$ and contained in $\Omega \cup \{x\}$. Choose $\delta > 0$ small enough so that $\Gamma[t, t+\delta]$ is disjoint from the curve $\psi$. This gives the right-hand side of (17) for $s \in [t, t+\delta]$. If we also require that $\Gamma[t, t+\delta]$ is disjoint from $U$, then the left-hand side of (17) follows from the general fact that

$$\text{frontier}(\Gamma[0,s]) \subset \text{frontier}(\Gamma[0,t]) \cup \Gamma[t,s]\,.$$

$\square$

As noted by the referee the containment in (17) is actually an equality for small enough $\delta$. However, we won't need this fact so we omit the proof.

Proof of Theorem 1.1: The random set $B[0, \tau_{\text{exp}}] \setminus G^\circ$ is completely determined by the variables generating the $\sigma$-field $\mathcal{A}_G$ defined in Proposition 6.3, so the proposition implies that $K = B[0, \tau_{\text{exp}}]$ satisfies the hypothesis (1) of Theorem 1.2.

Since $B[0, \tau_{\text{exp}}]$ has the same distribution as $\sqrt{\tau_{\text{exp}}} \cdot B[0,1]$, this establishes the first assertion of Theorem 1.1.

For $t > 0$, Let $A_t$ be the event that $\dim(\text{frontier}(B[0,s]) \cap V) \geq 1 + \epsilon$ simultaneously for all open disks $V$ that intersect frontier$(B[0,t])$ and have rational centers and radii. Theorem 1.2 and Proposition 6.3 give $\mathbf{P}_0(A_1) > 0$, and we must show that $\mathbf{P}_0(\cap_{t>0} A_t) = 1$. Denote by $A_{\mathbb{Q}} = \cap_{s \in \mathbb{Q}_+} A_s$ the intersection over all positive rational times. Now Brownian scaling and countable additivity imply that $\mathbf{P}_0(A_{\mathbb{Q}}) = 1$, so it suffices to prove that $A_{\mathbb{Q}} \subset A_t$ for all $t > 0$. Fix $t > 0$ and an open disk $V$ that intersects frontier$(B[0,t])$. Since frontier$(B[0,t])$ is connected, it must intersect some (random) open disk $U = U(V,t)$ with rational center and radius such that $U \subset V$ and $B(t) \notin \overline{U}$. By the previous lemma, there is a rational $s$ such that

$$\text{frontier}(B[0,t]) \cap U \supset \text{frontier}(B[0,s]) \cap U \neq \emptyset\,.$$

This implies that $A_{\mathbb{Q}} \subset A_t$, and completes the proof of the theorem. $\square$

Finally, we consider the **planar Brownian bridge** $B_{\text{br}}$, which may be defined either by conditioning the Brownian path to return to the origin, or by $B_{\text{br}}(t) = B(t) - tB(1)$ for $t \in [0,1]$.

For every $t < 1$, the restrictions $B_{\rm br}|_{[0,t]}$ and $B|_{[0,t]}$ have mutually absolutely continuous laws (these laws are measures on the space of continuous maps from $[0,t]$ to the plane.) Therefore by Theorem 1.1, for every *fixed* $t \in (0,1)$,

$$\dim\Big(\text{frontier}(B_{\rm br}[0,t])\Big) \geq 1+\epsilon \quad \text{a.s.} \tag{18}$$

Consider a sequence of annuli $\{A_n\}$ of modulus $2^{-n}$ around the origin. The probability that $B_{\rm br}$ surrounds the origin in $A_n$ is bounded away from 0, so the Blumenthal 0–1 law implies that with probability 1, there is some rational $t < 1$ such that $\text{frontier}(B_{\rm br}[0,1]) = \text{frontier}(B_{\rm br}[0,t])$ (see Burdzy and Lawler (1990)). Thus by (18), with probability 1,

$$\dim\Big(\text{frontier}(B_{\rm br}[0,1])\Big) \geq \inf_{t\in\mathbb{Q}\cap(0,1)} \dim\Big(\text{frontier}(B_{\rm br}[0,t])\Big) \geq 1+\epsilon\,.$$

# 7 Concluding remarks

It can be shown (Krzysztof Burdzy, personal communication) that $\dim(\text{frontier}(B[0,1]))$ is almost surely constant; this fact is not required for the arguments in this paper. The conjecture that the Brownian frontier has dimension 4/3 is related to well-known conjectures concerning self-avoiding random walks, which in turn are a model for long polymer chains. In that context, the exponent 4/3 first appeared in the non-rigorous considerations of Flory (1949); see also de Gennes (1991).

Theorem 1.2 is stated for general random sets, rather than just Brownian motion, in view of potential applications to the ranges and level-sets of other stochastic processes. Besides the range of Brownian motion, another natural random set that satisfies the hypothesis of Theorem 1.2 is the support of **super-Brownian motion**, i.e. the intersection of all closed sets that are assigned full measure by this measure-valued diffusion throughout its lifetime. (For the definitions see, e.g., Dawson, Iscoe, and Perkins (1989).) Equivalently, this random set may be characterized as the set of points ever visited by the path-valued process constructed by Le-Gall (1993). (This process is often referred to as "The Brownian snake".). We are grateful to Steve Evans for enlightening discussions of super-Brownian motion.

For the special case when $K$ is the range of planar Brownian motion, it seems likely that methods directly adapted to this case will yield better estimates for $\dim(\text{frontier}(K))$ than those obtainable by our methods. Indeed, Gregory Lawler has informed us that immediately after he learned of our Theorem 1.1 (but without seeing its proof), he proved (using completely different methods) that the dimension of the Brownian frontier can be expressed in terms of the "double disconnection exponent" of Brownian motion. This allowed Lawler to deduce that $\dim(\text{frontier}(B[0,1])) > 1.01$ a.s., by invoking recent estimates of Werner (1994) on disconnection exponents. We refer the reader to Lawler's forthcoming paper for this and several other striking results on the Brownian frontier.

Finally, we note an application to simple random walk on the square lattice $\mathbb{Z}^2$. Given a finite subset $S$ of $\mathbb{Z}^2$, say that a lattice point $x \in S$ is on the **outer boundary** of $S$ if $x$ is adjacent to some point in the unbounded component of $\mathbb{Z}^2 \setminus S$. We remark that using the strong approximation results of Auer (1990) and our construction of the Whitney tree in Section 3, it is easy to derive the following.

**Corollary 7.1** *Let $\{S(k)\}$ denote simple random walk on $\mathbb{Z}^2$, and let $\epsilon > 0$ be as in Theorem 1.1. Then for every $\epsilon_1 < \epsilon$ we have*

$$\lim_{n\to\infty} \mathbf{P}\{\textit{There are more than } n^{(1+\epsilon_1)/2} \textit{ points on the outer boundary of } S[0,n]\} = 1.$$

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