Planar first-passage percolation times are not tight

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We consider first-passage percolation on the two-dimensional integer lattice $\mathbf{Z}^2$ with passage times that are IID exponentials of mean one; see Kesten (1986) for an overview. It has been conjectured, based on numerical evidence, that the variance of the time $T(0,n)$ to reach the vertex $(0,n)$ is of order $n^{2/3}$. Kesten (1992) showed that the variance of $T(0,n)$ is at $O(n)$. He also noted that the variance is bounded away from zero. This note improves the lower bound on the variance of $T(0,n)$ to $C \log n$. Simultaneously and independently, Newman and Piza have achieved the same result for $\{0,1\}$-valued passage times. Their methods (Newman and Piza 1993) extend to more general passage times, while ours work only for exponential times. On the other hand, our theorem shows that the variance comes from fluctuations of nonvanishing probability in the sense that, as $n \to \infty$, the law of $T(0,n)$ is not tight about its median. Very recently, Newman and Piza showed that the $\log n$ may be improved to a power of n for directions in which the shape is not flat (it is not known whether the shape can be flat in any direction; see Durrett and Liggett (1981) for a relevant example). As pointed out to us by Harry Kesten, in the exponential case this may also be obtained via the method given here.

Theorem 1 *Let* $\mathbf{v}$ *be any unit vector in* $\mathbb{R}^2$ *and let* $\mathbf{v}^{(\mathbf{n})}$ *be the vector in* $\mathbf{Z}^2$ *whose coordinates are the integer parts of the coordinates of* $n\mathbf{v}$*. Let* $T(0, \mathbf{v}^{(\mathbf{n})})$ *denote the passage time from the origin to the vertex* $\mathbf{v}^{(\mathbf{n})}$*, under IID mean 1 exponential passage times on the edges of* $\mathbf{Z}^2$*. Then* $Var(T(0, \mathbf{v}^{(\mathbf{n})})) \geq C \log n$ *and in fact any intervals* $[a_n, b_n]$ *with* $b_n - a_n = o(\log n)^{1/2}$ *satisfy* $\mathbf{P}(T(0, \mathbf{v}^{(\mathbf{n})}) \in [a_n, b_n]) \to 0$.

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Remark: The theorem extends to Richardson's model with other passage time distributions (restart all clocks after an edge is crossed).

PROOF: We compute the conditional distribution of $T(0, \mathbf{v^{(n)}})$ given a σ-field $\mathcal{F}$ and show that with probability $1 - o(1)$, this conditional distribution is close to a normal with variance at least $C \log n$; clearly this implies the conclusion of the theorem.

Let $\mathcal{F}$ be the σ-field determined by the order in which vertices are reached. Formally, if $T(v)$ is the passage time from the origin to the vertex v, then $\mathcal{F}$ is the σ-field generated by the events $T(v) < T(w)$ for $v, w \in \mathbf{Z}^2$. Let $V_0, V_1, V_2, \ldots$ be the vertices of $\mathbf{Z}^2$ listed in the order they are reached, so $(V_0, V_1, \ldots)$ is an $\mathcal{F}$-measurable random sequence. Let $\mathcal{C}_n = \{V_0, V_1, \ldots, V_n\}$ be the cluster of the first n elements to be reached from the origin, and let Y_n be the number of edges connecting elements $\mathcal{C}_{n-1}$ to elements of $\mathbf{Z}^2 \setminus \mathcal{C}_{n-1}$. The key observation is that the conditional joint distribution of the variables $T(V_n) - T(V_{n-1})$ given $\mathcal{F}$ is identical to a sequence of independent exponentials with means $1/Y_n$. This is in fact an immediate consequence of the lack of memory of the exponential distribution and of the fact that the minimum of n independent exponentials of mean 1 is an exponential of mean $1/n$. This observation leads to

Lemma 2

$$\liminf_{n\to\infty} (\log n)^{-1} \sum_{j=1}^{n} \frac{1}{Y_j^2} \geq c_0 \quad a.s.$$

for some positive constant c_0.

Assuming this lemma for the moment, define $M(n)$ to satisfy $V_{M(n)} = \mathbf{v^{(n)}}$. Let

$$\mu_n = \mathbf{E}(T(V_n) \mid \mathcal{F}) = \sum_{j=1}^{n} \frac{1}{Y_j}$$

and

$$\sigma_n^2 = \mathrm{Var}(T(V_n) \mid \mathcal{F}) = \sum_{j=1}^{n} \frac{1}{Y_j^2}.$$

The Lindeberg-Feller theorem implies that the conditional distribution of the variable $(T(\mathbf{v^{(n)}}) - \mu_{M(n)})/\sigma_{M(n)}$ converges weakly to a standard normal whenever $\sigma_{M(n)} \to \infty$. Subadditivity implies that that $T(\mathbf{v^{(n)}})/n \to c_1 = c_1(\mathbf{v})$ almost surely, and the shape theorem (Cox and Durrett 1981) implies that $c_1 > 0$ and that the number of vertices N_t reached by time t is almost surely $(c_2 + o(1))t^2$. (Equivalently, $T(V_n) = (c_2^{-1/2} + o(1))n^{1/2}$.) From this it follows that

$$\frac{M(n)}{n^2} \to c_2 c_1^2 \ \text{ a.s.,} \tag{1}$$

and hence from Lemma 2 that

$$\liminf (\log n)^{-1} \sigma^2_{M(n)} \geq 2c_0 \ \text{ a.s.}$$

Thus a conditional distribution of $T(\mathbf{v^{(n)}})$ is close to a normal with variance at least $2c_0 + o(1)$, which establishes the theorem.

To prove Lemma 2, we first observe from the isoperimetric inequality that the distribution of $T(V_n)$ is stochastically bounded by the sum of independent exponentials of means $cj^{-1/2}$, $j = 1, \ldots, n$. Thus the variables $n^{-1/2}T(V_n)$ are dominated by a variable in L^1 and hence

$$\frac{\mu_n}{n^{1/2}} = \mathbf{E}(n^{-1/2}T(V_n) \,|\, \mathcal{F}) \to c_2^{-1/2}$$

almost surely. Lemma 2 then follows from a fact about sequences of real numbers:

Lemma 3 *Let $x_1, x_2, \ldots$ be positive real numbers with $S_n = \sum_{j=1}^n x_j$ and suppose that* $\liminf n^{-1/2} S_n = c$. *Then*

$$\liminf (\log n)^{-1} \sum_{j=1}^n x_j^2 \geq \frac{c^2}{4}.$$

Proof: It suffices to show that the condition

$$S_n \geq an^{1/2} - b \ \text{ for all } n$$

implies $\liminf (\log n)^{-1} \sum_{j=1}^n x_j^2 \geq a^2/4$, since one may then take $a = c - \epsilon$ for arbitrarily small ϵ. Also, replacing x_1 with $x_1 + b$, we may assume without loss of generality that $b = 0$. Define

$$q_n = n^{1/2} - (n-1)^{1/2} \geq \frac{1}{2} n^{-1/2}.$$

Assuming $S_n \geq an^{1/2}$ for all n, we show that $\sum_{j=1}^n x_j^2 \geq a^2 \log n/4$. Rearranging the terms $\{x_j : 1 \leq j \leq n\}$ in decreasing order does not change $\sum_{j=1}^n x_j^2$ and only increases each S_j, so we may assume without loss of generality that these terms appear in decreasing order. Summing by parts three times we obtain

$$
\begin{aligned}
\sum_{j=1}^n x_j^2 &= S_n x_n + \sum_{k=1}^{n-1} S_k (x_k - x_{k+1}) \\
&\geq a \left[n^{1/2} x_n + \sum_{k=1}^{n-1} k^{1/2} (x_k - x_{k+1}) \right] \\
&= a \sum_{j=1}^n q_j x_j \\
&= a \left[q_n S_n + \sum_{k=1}^{n-1} (q_k - q_{k+1}) S_k \right] \\
&\geq a \left[q_n n^{1/2} + \sum_{k=1}^{n-1} (q_k - q_{k+1}) k^{1/2} \right].
\end{aligned}
$$

Summing once more by parts and using the definition of q_k we see that this is equal to $a \sum_{k=1}^n q_k^2$, which is at least $a^2 \log n/4$. This proves the lemma and hence the theorem. □

When the asymptotic shape has a finite radius of curvature in the direction $\mathbf{v}$, Newman and Piza have shown, using results of Kesten (1992) and Alexander (1992), that the minimizing path from 0 to $\mathbf{v}^{(\mathbf{n})}$ deviates from a straight line segment by at most cn^α for some $\alpha < 1$, with probability $1 - o(1)$ as $n \to \infty$. Thus the time $T'(0, \mathbf{v}^{(\mathbf{n})})$ to reach $\mathbf{v}^{(\mathbf{n})}$, in a new percolation where only bonds in a strip of width n^α are permitted, differs from $T(0, \mathbf{v}^{(\mathbf{n})})$ by $o(1)$ in total variation. The shape theorem for $\mathbf{Z}^2$ implies that the number of sites reached in the new percolation by this time is $O(n^{1+\alpha})$. Defining $M'(n), \mu'_n, Y'_n$ and σ'_n analogously to $M(n), \mu_n, Y_n$ and σ_n but for the new percolation, we have $\mu'_n = \sum_{k=1}^n (1/Y'_n)$, while now $M'(n) = O(n^{1+\alpha})$. By Cauchy-Schwartz – no summing by parts is needed – it follows that $(\sigma'_{M'(n)})^2 \geq cn^{1-\alpha}$, and applying Lindeberg-Feller as before proves the extension mentioned before Theorem 1.

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