PERCOLATION, FIRST-PASSAGE PERCOLATION, AND COVERING TIMES FOR RICHARDSON'S MODEL ON THE n-CUBE

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ABSTRACT:

Percolation with edge-passage probability p and first-passage percolation are studied for the n-cube $\mathcal{B}_n = \{0,1\}^n$ with nearest neighbor edges. For oriented and unoriented percolation, $p = e/n$ and $p = 1/n$ are the respective critical probabilities. For oriented first-passage percolation with i.i.d. edge-passage times having a density of 1 near the origin, the percolation time (time to reach the opposite corner of the cube) converges in probability to 1 as $n \to \infty$. This resolves a conjecture of David Aldous. When the edge-passage distribution is standard exponential, the (smaller) percolation time for unoriented edges is at least 0.88.

These results are applied to Richardson's model on the (unoriented) n-cube. Richardson's model, otherwise known as the contact process with no recoveries, models the spread of infection as a Poisson process on each edge connecting an infected node to an uninfected one. It is shown that the time to cover the entire n-cube is bounded between 1.41 and 14.05 in probability as $n \to \infty$.

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1 Introduction and notation

Percolation theory, broadly speaking, is the study of connectivity in a random medium. Grimmett (1989) gives the following example of a question addressed by percolation theory. "Suppose we immerse a large porous stone in water. What is the probability that the centre of such a stone is wetted?" The mathematical model for this is graph-theoretic. Small volume elements of the stone become vertices of the integer lattice (of dimension three in this case). Neighboring volume elements may be connected by a channel broad enough to allow the flow of water, or they may not be. Model this by letting each pair of neighboring vertices independently have a connecting edge with probability p, for some parameter p. The center of the stone is then wetted if and only if the connected component of this random subgraph containing the center extends to the surface.

Oriented percolation is a variant on this, where each edge has a particular orientation, and the water may pass only in that direction if at all. For example, the stone may be subjected to water only from above, with flow of water from one volume element to a neighboring one occurring (due to gravity) only when the latter is lower. First-passage percolation is a similar model, the difference being that the passage of water through a channel is not simply a yes or no event, but takes an amount of time depending on the breadth of the channel. Thus each edge between neighboring vertices, instead of being randomly included or excluded, is assigned a random (i.i.d.) passage time. The question is not whether, but when the center of the stone first gets wet.

Richardson's model is a stochastic process on a graph that begins with one vertex *infected* and evolves by transmission of the infection according to an i.i.d. Poisson process on each edge: if the Poisson process on the edge connecting v to w has a point of increase at time t and one of v or w was infected before time t, then both are infected after time t. Questions about Richardson's model can be reduced to questions about first-passage percolation with i.i.d. exponentially distributed passage times. In Pemantle (1988), questions about recurrence or transience of random walks in random environments on trees are essentially reduced to oriented first-passage percolation on trees. Lyons (1990) connects random walks and electrical networks with percolation on trees. These equivalences (along with the desire to generalize whenever possible) are among the reasons to study percolation on graphs such as trees and n-cubes which do not model any actual stones.

Classical percolation theory, i.e., on $\mathbf{Z}^d$, is geometric. The main arguments—counting countours, piecing together sponge crossings, viewing the process from its left edge—are all pictorial. Infinite binary trees were introduced as a way to get graphs which were in some sense limits of $\mathbf{Z}^d$ as $d \to \infty$. Trees are in general easier to study than integer lattices. For example, the first-passage percolation problem for binary trees is essentially a large deviation calculation: if every path of length n in the tree were disjoint then the passage time would be the minimum of 2^n different sums of n i.i.d. random variables; such a calculation is quantitative, requiring no picture. The argument in Pemantle (1988) consists mostly of showing that the slight overlapping of paths is inconsequential. Unfortunately, trees are locally quite different from lattices (despite being called *Bethe lattices*) and are therefore not very satisfactory in the role of limiting lattices. The n-cube is an alternative way to capture the high-dimensional limiting behavior of integer lattices without altering the local connectivity properties. The solutions presented here to percolation problems on n-cubes use large deviation and second moment estimates and are thus closer to the tree case than to the integer lattice case. In this respect, the method is very similar to the arguments used by Cox and Durrett (1983) for the analogous problem on $(\mathbf{Z}^+)^d$ in high dimensions. The results resolve affirmatively conjecture G7.1 of Aldous (1989), despite a remark there indicating that second moment methods shouldn't work. (To be fair: the second moment method bounds the probability in question away from 0 but a variance reduction trick is needed to get it equal to 1.)

Notation will be as follows. Let $\mathcal{B}_n$ be the Boolean algebra of rank n whose elements are ordered n-tuples of 0's and 1's. It has a bottom element $\hat{0} = (0, \ldots, 0)$ and a top element $\hat{1} = (1, \ldots, 1)$. It is also useful to regard elements of $\mathcal{B}_n$ as subsets of $\{1, \ldots n\}$, where the set A corresponds to the sequence with a 1 in position i if and only if $i \in A$. The collection of all A of a given cardinality k is called the k^{th} *level* of $\mathcal{B}_n$. The n-dimensional cube, or n-cube, is the graph whose vertices are the elements of $\mathcal{B}_n$ and whose edges connect each set A to $A \cup \{j\}$ for each $j \notin A$. Sometimes we must think of the edges as oriented from A to $A \cup \{j\}$. Representing the vertices of the n-cube as the standard basis in $\mathbb{R}^n$ makes each such edge parallel to the unit vector $e(j)$ connecting $\hat{0}$ to $(0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 in position j. Then a path from $\hat{0}$ to $\hat{1}$ of length n can be represented as a permutation $(\pi(1), \ldots, \pi(n))$ where the kth edge in the path is parallel to $e(\pi(k))$ and in fact connects the set $A = \{\pi(i) : i < k\}$ to the set $A \cup \{\pi(k)\}$. Similarly, a path connecting a set A to a set $B \supseteq A$ may be represented as a permutation of $B \setminus A$.

Let X_{vw} be i.i.d. random variables as $\overline{vw}$ ranges over all oriented edges of the n-cube. The *first-passage time* from $\hat{0}$ to $\hat{1}$, or *percolation time*, is defined as the minimum over paths $\gamma = (\hat{0}, v_1, v_2, \ldots, v_{n-1}, \hat{1})$ from $\hat{0}$ to $\hat{1}$ of $T_n(\gamma)$, where $T_n(\gamma)$ is the sum of $X_{v_i, v_{i+1}}$ along the edges of γ. In the application of first-passage percolation to Richardson's model, the common distribution of the X_{vw}'s is exponential with mean 1. For our basic first-passage percolation result itself, the assumption of exponential edge-passage times simplifies the derivation of certain large deviation estimates but is not necessary; we state the theorem for more general distributions of the passage time, though probably still greater generality is possible. For the integer lattice in high dimensions, Kesten (1984) gives sufficient conditions on the behavior of the common density of the edge-passage times near the origin for similar calculations to work. It is unlikely that the same conditions are sufficient for our problem; at any rate, our paper does not address this issue.

Of course, for ordinary percolation, the common distribution of the X's takes on the two values "open" and "closed" with respective probabilities p and $1-p$, for some p. The basic question then is to compute the probability that there is a path from $\hat{0}$ to $\hat{1}$ consisting only of open edges.

The rest of the paper is organized as follows. Section 2 contains a description of the second moment method, and its enhancement by a variance reduction technique, that is used to analyze oriented percolation and oriented first-passage percolation. Section 2 also presents lemmas that count oriented paths in the n-cube.

Section 3 discusses oriented percolation, showing that the threshold for $\hat{0}$ to be connected to $\hat{1}$, with positive probability, by a path of open bonds occurs when the probability that any given bond is open is e/n. Then similar methods are used to analyze oriented first-passage percolation. It is shown that the percolation time converges in probability to 1 as $n \to \infty$ for any common density of the passage times with value 1 at the origin.

Section 4 treats unoriented percolation, for which the critical probability is shown to be $1/n$.

Section 5 provides a lower bound for the unoriented first-passage time when the common distribution of the passage times is exponential by comparing the process to a branching translation process, which is similar to a branching random walk. Since the oriented first-passage time is an upper bound for the unoriented first-passage time, the unoriented first-passage time is thereby bounded asymptotically between two fairly close constants, namely, 0.88 and 1. For oriented first-passage times, lower bounds are easy (first moment calcluation); upper bounds equalling these lower bounds are more difficult (second moment estimates). For unoriented first-passage times, we do not know how to bridge the gap between the lower and upper bounds.

Section 6 applies unoriented first-passage percolation with exponential passage times to Richardson's model. The main result is that the time for the entire n-cube to become infected is bounded in probability by a constant as $n \to \infty$. Since the obvious lower bound is also a constant, this provides the right order of magnitude for the covering time, improving upon the previous best upper bound of the covering time, which was order of $\ln n$.

A future paper will address the discrete-time analogues of these problems, which are closely related to the so-called broadcasting problem discussed by Feige, et al. (1990) and others.

2 Preliminaries: the second moment method and path counting

2.1 The second moment method, with variance reduction

Aldous (1989, Lemma A15.1) gives the following lemma as the cornerstone of the so-called second moment method. The proof is a simple application of the Cauchy–Schwarz inequality.

Lemma 2.1 (Second moment method) *Let N be a nonnegative real random variable with $\mathbf{E}N^2 < \infty$. Then $\mathbf{P}(N > 0) \geq (\mathbf{E}N)^2/\mathbf{E}N^2$.* □

In our applications, the random variable $N \equiv N_n$ will be indexed by the dimension n of the cube under consideration. When the variance of N is $o((\mathbf{E}N)^2)$ as n tends to infinity, the inequality shows that $\mathbf{P}(N > 0) = 1 - o(1)$. On the other hand, if we can only show the variance of N to be $O((\mathbf{E}N)^2)$, then the conclusion is weaker, namely, that $\mathbf{P}(N > 0)$ is bounded away from 0. One of the purposes of the present work is to point out that the weaker conclusion may often be strengthened to the former by a simple variance reduction trick. Consider an auxiliary random variable Z which absorbs most of the variance of N, in the sense that the conditional second moment $\mathbf{E}(N^2 \mid Z)$ is only $(1+o(1))$ times the conditional squared first moment $(\mathbf{E}(N \mid Z))^2$ uniformly over a set of values of Z of probability $1-o(1)$. Then uniformly for those values of Z, $\mathbf{P}(N > 0 \mid Z)$ is $1-o(1)$, and hence $\mathbf{P}(N > 0)$ is also $1-o(1)$.

To apply the second moment method to oriented percolation on the n-cube, let N be the number of paths of length n from $\hat{0}$ to $\hat{1}$ consisting entirely of open edges. Then $\hat{0}$ is connected to $\hat{1}$ by an oriented path of open edges if and only if $N > 0$. For large enough values of $p = \mathbf{P}(\text{edge is open})$, the (unenhanced) second moment method will get $\mathbf{P}(N > 0)$ bounded away from 0 by showing that $(\mathbf{E}N^2)/(\mathbf{E}N)^2$ is bounded, in the following manner. A similar argument, to be detailed later, using the enhanced second moment method will yield the exact limiting value of the percolation probability $\mathbf{P}(N > 0)$.

Since $\mathbf{E}N$ is the sum over paths γ of the probability that γ consists entirely of open edges and $\mathbf{E}N^2$ is the sum over pairs of paths (γ, γ') of the probability that both paths consist entirely of open edges, the quotient $\mathbf{E}N^2/(\mathbf{E}N)^2$ would be precisely 1 if the events that γ consists entirely of open edges and that γ' consists entirely of open edges were always independent. Of course they are not independent when $\gamma = \gamma'$, but also they are not independent when γ and γ' have any edges in common, and the covariance of their indicators is greater the more edges that γ and γ' share. Thus the argument rests on showing that the number of pairs (γ, γ') sharing a lot of edges is small.

The oriented first-passage percolation problem is handled similarly, with N being the number of paths whose total passage time is at most M; if $N > 0$ with high probability, then the passage time is less than M with high probability.

2.2 Counting oriented paths

Section 2.2 contains the path-counting lemmas needed to execute the second moment method. By symmetry, it will only be necessary to consider the case where γ is the path given by $(1, \ldots, n)$ in the permutation representation. Let $f(n,k)$ be the number of paths γ' that share precisely k edges with γ. Let $F(n,k) = \sum_{n \geq l \geq k} f(n,l)$. Most of the time it will suffice to bound $F(n,k)$ and observe that $f(n,k) \leq F(n,k)$.

Lemma 2.2 *Let $K(n)$ be any function that is $o(n)$ as $n \to \infty$. Then $f(n,k) \leq F(n,k) \leq (1+o(1))(k+1)(n-k)!$ as $n \to \infty$ uniformly in k for $k \leq K(n)$. Furthermore, for $k > 0$, consider paths that agree with γ only in an initial segment and a final segment; it is these paths that matter for $F(n,k)$, in the sense that all the rest of the paths only contribute $o((k+1)(n-k)!)$, uniformly in $0 < k \leq K(n)$.*

Lemma 2.3 *Suppose $k \geq n - n^{3/4}/2$. Then $f(n,k) \leq F(n,k) \leq (n-k+1)(2n^{7/8})^{n-k}$.*

Lemma 2.4 *Suppose $k \leq n - 5e(n+3)^{2/3}$. Then, for $n \geq 25$, $f(n,k) \leq n^6(n-k)!$, and, for n so large that $\left\lceil 5e(n+3)^{2/3} \right\rceil \leq \left\lceil n^{3/4}/2 \right\rceil$,*

$$F(n,k) \leq 2n^6(n-k)! + \left\lceil 5e(n+3)^{2/3} \right\rceil (2n^{7/8})^{\left\lceil 5e(n+3)^{2/3} \right\rceil - 1}.$$

The following lemma will be needed when we apply the enhanced second moment method.

Lemma 2.5 *Consider now a Boolean lattice $\mathcal{B}_{n+2L}$ of size $n+2L$ for some positive integer L. Let x_1 and x_2 be distinct vertices in level L of $\mathcal{B}_{n+2L}$ and let y_1 and y_2 be distinct vertices in level $n+L$ of $\mathcal{B}_{n+2L}$. Assume that x_i lies below y_i for $i = 1, 2$. Let $H(n, L, k, x_1, x_2, y_1, y_2)$ be the maximum over paths γ from x_1 to y_1 of the number of* *paths from x_2 to y_2 that have at least k edges in common with γ. Let $F_1(n,L,k) = \max_{x_1,x_2,y_1,y_2} H(n,L,k,x_1,x_2,y_1,y_2)$. Then for fixed L and any function $K(n)$ that is $o(n)$ as $n \to \infty$, the following hold:*

$$F_1(n,L,k) = o((k+1)(n-k)!) \text{ uniformly in } 0 < k \le K(n); \tag{1}$$

$$F_1(n,L,k) \le (n-k+1)(2n^{7/8})^{(n-k)} \text{ for } k \ge n - n^{3/4}/2; \tag{2}$$

$$F_1(n,L,k) \le 2n^6(n-k)! + \left\lceil 5e(n+3)^{2/3} \right\rceil (2n^{7/8})^{\left\lceil 5e(n+3)^{2/3}\right\rceil - 1} \text{ for } k \le n - 5e(n+3)^{2/3}. \tag{3}$$

For the last inequality we require $\left\lceil 5e(n+3)^{2/3} \right\rceil \le \left\lceil n^{3/4}/2 \right\rceil$.

The following notation is common to the proofs of Lemmas 2.2 – 2.4. If γ' has precisely k edges in common with γ and is given by the permutation π, then let $r_1, \ldots, r_k$ be the terminal positions of the shared edges. In other words, let r_1 be minimal so that the set $\{1, \ldots, r_1 - 1\}$ is equal to the set $\{\pi(1), \ldots, \pi(r_1 - 1)\}$ and $\pi(r_1) = r_1$; let r_2 be the next such value, and so on. By convention, let r_0 always be 0 and r_{k+1} always be $n+1$. Write $\mathbf{r} = \mathbf{r}(\gamma')$ for the sequence $(r_0, \ldots, r_{k+1})$. Write $s_i = r_{i+1} - r_i$, $i = 0, \ldots, k$. For any sequence $\mathbf{r}_0 = (r_0, \ldots, r_{k+1})$ with $0 = r_0 < r_1 < \cdots < r_k < r_{k+1} = n+1$, let $C(\mathbf{r}_0)$ denote the number of paths γ' with $\mathbf{r}(\gamma') = \mathbf{r}_0$. Then it is easy to see that $C(\mathbf{r}) \le G(\mathbf{r})$, where

$$G(\mathbf{r}) = \prod_{i=0}^{k} (s_i - 1)!, \tag{4}$$

since the values $\pi(r_i+1), \ldots, \pi(r_i+s_i-1)$ must be a permutation of $\{r_i+1, \ldots, r_i+s_i-1\}$. For Lemma 2.4, the more precise bound $C(\mathbf{r}) \le G_1(\mathbf{r})$ will be necessary, where G_1 is defined by

$$G_1(\mathbf{r}) = \prod_{i=0}^{k} [(s_i - 1)! - 1 + \delta_{1,s_i}] \tag{5}$$

and δ_{1,s_i} is 1 if $s_i = 1$ and 0 otherwise. To see this inequality, recall that γ' must not have any common edges with γ strictly between r_i and r_{i+1}, and therefore, for $s_i \ne 1$, at least one permutation of $\{r_i+1,\ldots,r_{i+1}-1\}$ (namely, the identity permutation) is ruled out for the values of $\pi(r_i+1),\ldots,\pi(r_{i+1}-1)$. Note that $s_i(\mathbf{r}(\gamma'))$ can never be 2, a fact which is used liberally in the proofs of Lemmas 2.3 and 2.4. Of course, $f(n,k)$ is equal to the sum of $C(\mathbf{r})$ over all sequences $\mathbf{r}$ satisfying $0=r_0<r_1<\cdots<r_k<r_{k+1}=n+1$. Furthermore, $F(n,k)$ is at most the sum of $G(\mathbf{r})$ over all such sequences, since $G(\mathbf{r})$ counts the paths γ' with $\mathbf{r}\subseteq\mathbf{r}(\gamma')$. It remains then to get bounds on these sums. We use the following facts about factorials.

Proposition 2.6 (i) *Factorials are log-convex, i.e.,* $a!b!\le(a+j)!(b-j)!$ *for* $a\ge b\ge j\ge 0$.

(ii) $(a!-1)((a+j)!-1)\le((a-1)!-1)((a+j+1)!-1)$ *as long as* $a\ge 4$ *or* $a=3$ *and* $j\ge 1$.

(iii) $a>b>0$ *implies* $(a!-1)/(b!-1)>a!/b!>(a/e)^{a-b}$.

Proof: (i) follows from (in fact, is equivalent to) the fact that $(a+1)!/a!=a+1$ is increasing. (ii) is easy to verify. The first inequality of (iii) is trivial. To prove the second one, use Stirling's formula $a!e^{-1/(12a)}<a^ae^{-a}\sqrt{2\pi a}<a!$ to get

$$a!/(b!(a/e)^{a-b})>\frac{a^ae^{-a}\sqrt{2\pi a}}{b^be^{-b}\sqrt{2\pi b}e^{1/(12b)}}(e/a)^{a-b}=(a/b)^{b+1/2}e^{-1/(12b)},$$

and taking the logarithm of the last expression gives at least

$$(b+1/2)\ln(1+1/b)-1/(12b)>(b+1/2)(1/b-1/b^2)-1/(12b)>0$$

for $b\ge 2$. For $b=1$, the result follows directly from Stirling's formula. □

Proof of Lemma 2.2: For fixed $\mathbf{r}$, let $j=j(\mathbf{r})=\max_i(s_i-1)$. Consider separately the cases $j\le n-4k$ and $j>n-4k$. The idea is that $G(\mathbf{r})$ is small in the first case, and while G is not so small in the second case, there are not too many sequences with such large values of j. In the first case, $G(\mathbf{r})$ must be no more than $(n-4k)!(3k)!$. To see this, note that by log-convexity of factorials, the product in (4) is maximized subject to $j \le n-4k$ at some $\mathbf{r}$ for which $j(\mathbf{r})$ is equal to $n-4k$, in which case since $\sum_i (s_i - 1) = n-k$, $G(\mathbf{r})$ is at most $[\max_i(s_i-1)]![\sum_i(s_i-1) - \max_i(s_i-1)]! \le (n-4k)!(3k)!$. Meanwhile, the number of sequences $\mathbf{r}$ under consideration is at most $\binom{n}{k}$. Thus the total contribution from this case is at most

$$\begin{aligned}
& (n-4k)!(3k)!n!/(k!(n-k)!) \\
= \; & (n-k)! \frac{(n-4k)!}{(n-k)!} \frac{n!}{(n-k)!} \frac{(3k)!}{k!} \\
= \; & (n-k)! \frac{[n(n-1)\cdots(n-k+1)][3k(3k-1)\cdots(k+1)]}{(n-k)(n-k-1)\cdots(n-4k+1)} \\
\le \; & (n-k)! \left[\frac{(3k)^2 n}{(n-4k+1)^3} \right]^k
\end{aligned}$$

The term inside the square brackets converges to 0 as $n \to \infty$ uniformly in k as long as $k \le K(n) = o(n)$, so for large n and $k \le K(n)$, the contribution from the case $j(\mathbf{r}) \le n-4k$ is $o((n-k)!)$, uniformly in these k.

The second contribution is from the terms with $j > n-4k$. Since factorials are log-convex, it follows (as in the preceding case) that $G(\mathbf{r})$ can be no greater than $j(\mathbf{r})!(n-k-j(\mathbf{r}))!$. The number of sequences $\mathbf{r}$ for which $j(\mathbf{r})$ is any fixed value j_0 is at most $k+1$ times the number of sequences $\mathbf{r}$ with $s_0 - 1 = j_0$, which is in turn at most $(n-j_0-1)!/((k-1)!(n-j_0-k)!)$. So the contribution for fixed j_0 is at most

$$\frac{(k+1)j_0!(n-j_0-k)!(n-j_0-1)!}{(k-1)!(n-j_0-k)!} = \frac{(k+1)}{(k-1)!} j_0!(n-j_0-1)!.$$

Changing j_0 to j_0+1 multiplies this by $(j_0+1)/(n-j_0-1) > (n-4K(n))/(4K(n))$, which tends to infinity by assumption on $K(n)$; hence the sum is bounded by $(1+o(1))$ times the single term with the maximum value of j_0, namely, $j_0 = n-k$. The contribution from this term is at most $[(k+1)/(k-1)!](n-k)!(k-1)! = (k+1)(n-k)!$. The case $j(\mathbf{r}) = n-k$ covers precisely those sequences $\mathbf{r}$ consisting of an initial segment and a final segment; thus both statements in the lemma have been proved. $\square$

Proof of Lemma 2.3: Let $m = n-k$ be the number of positions in which the edges of γ and γ' differ and let $j(\mathbf{r})$ this time be the number of runs of consecutive positions that constitute these m edges. For example, if $n = 10$ and $k = 6$ with $(r_0, \ldots, r_{k+1}) = (0, 3, 4, 5, 8, 9, 10, 11)$ then $j(\mathbf{r}) = 2$ because the edges not shared are in two runs of consecutive positions, namely, $1, 2$ and $6, 7$. Now the inequality $F(n,k) \leq \sum_{\mathbf{r}} G(\mathbf{r})$ can be broken up as

$$F(n,k) \leq \sum_{l} g(n,k,l)h(n,k,l),$$

where $g(n,k,l)$ is an upper bound for $G(\mathbf{r})$ over sequences $0 = r_0 < \cdots < r_{k+1} = n+1$ with $j(\mathbf{r}) = l$ and $h(n,k,l)$ is the number of sequences with $j(\mathbf{r}) = l$.

As remarked before, each such run of consecutive positions contains at least two positions, since otherwise the single edge in the run would also be a shared edge. Thus $l \leq m/2$, and for fixed l, the number of $\mathbf{r}$ for which $j(\mathbf{r}) = l$ is equal to the number of ways of choosing m positions out of n in l clusters of size at least 2 each. Treating each cluster as a unit, there are $\binom{n-m+l}{l}$ ways of locating the l clusters among the $n-m$ shared edges. Since each cluster has at least two positions, there are $m-2l$ extra positions to be distributed among the l clusters. The number of ways to do this is the number of ways to drop $m-2l$ indistinguishable balls into l distinguishable boxes, which is $\binom{m-l-1}{l-1}$. Thus $h(n,k,l) = \binom{n-m+l}{l}\binom{m-l-1}{l-1}$. The first factor is increasing in l, thus maximized when $l = \lfloor m/2 \rfloor$, while the second factor is at most $\binom{m}{l}$, which is also maximized at $l = \lfloor m/2 \rfloor$. Thus, assuming for simplicity that m is even,

$$h(n,k,l) \leq \binom{n-m/2}{m/2}\binom{m}{m/2}.$$

Using again the log-convexity of factorials, $G(\mathbf{r})$ is less than or equal to the product of the factorials of the cluster sizes, which is maximized when all clusters have size 2 except for one large cluster. This gives $g(n,k,l) \leq 2^{l-1}(m-2l+2)! \leq 2^{m/2}m!$, and thus

$$\begin{aligned}
\sum_{l=0}^{m/2} g(n,k,l)h(n,k,l) &\leq (\frac{m}{2}+1)\binom{n-m/2}{m/2}\binom{m}{m/2}2^{m/2}m! \\
&\leq (\frac{m}{2}+1)\frac{n^{m/2}}{(m/2)!}\frac{m^{m/2}}{(m/2)!}2^{m/2}m! \\
&= (\frac{m}{2}+1)(2mn)^{m/2}\binom{m}{m/2} \\
&\leq (\frac{m}{2}+1)(8mn)^{m/2}.
\end{aligned}$$

The same inequality $\sum_l g(n,k,l)h(n,k,l) \leq (\frac{m}{2}+1)(8mn)^{m/2}$ can be established similarly when m is odd. Now the hypothesis that $m < n^{3/4}/2$ implies that the bound $(\frac{m}{2}+1)(8mn)^{m/2}$ is at most $(m+1)(2n^{7/8})^m$. $\Box$

Proof of Lemma 2.4: The idea this time is to define a weight $w(\mathbf{r})$ with the property that the sum of the $w(\mathbf{r})$ over all sequences $0 = r_0 < \cdots < r_{k+1} = n+1$ is less than 1. Then

$$\sum_{\mathbf{r}} C(\mathbf{r}) = \sum_{\mathbf{r}} w(\mathbf{r})[C(\mathbf{r})/w(\mathbf{r})] \leq \max_{\mathbf{r}}[C(\mathbf{r})/w(\mathbf{r})],$$

which will be shown to be as small as required. Let n_i denote $n - r_i$ and let $s_i = r_{i+1} - r_i$ as before, viewing the sequence of s_i's as a function of $\mathbf{r}$. The weight w is defined by

$$w(\mathbf{r}) = \prod_{i=0}^{k} Q(n_i, s_i),$$

where

$$Q(m,s) = \begin{cases} (m+1)/(m+3) & : \ s = 1 \\ 1/(m+3) & : \ s = 3 \\ 1/[(m+3)(m+2)] & : \ s \geq 4. \end{cases}$$

Note that $Q(m,1)+\sum_{s=3}^{m+1} Q(m,s) < 1$ for each fixed m; hence by a simple induction argument the sum of $w(\mathbf{r})$ over all sequences of a given length k is less than 1. It remains to bound for fixed n and k the quantity $\max_{\mathbf{r}}[C(\mathbf{r})/w(\mathbf{r})]$.

The procedure will be to find for each $\mathbf{r}$ a quantity $D(\mathbf{r})$ and a pair $\mathbf{r}^{(1)}, \mathbf{r}^{(2)}$ such that $C(\mathbf{r})/w(\mathbf{r}) \leq \frac{1}{2}(n+3)^2 D(\mathbf{r})$, $D(\mathbf{r}) \leq D(\mathbf{r}^{(1)}) \leq (n+3)^2 D(\mathbf{r}^{(2)})$, and $D(\mathbf{r}^{(2)}) \leq D(\mathbf{r}_0)$. This will give $C(\mathbf{r})/w(\mathbf{r}) \leq \frac{1}{2}(n+3)^4 D(\mathbf{r}_0)$, and the calculation $D(\mathbf{r}_0) = [(n-k)!-1](n-k+3)(n-k+2)$ will follow directly from the definition of D. Then

$$\frac{C(\mathbf{r})}{w(\mathbf{r})} \leq \frac{1}{2}(n+3)^6(n-k)! \leq n^6(n-k)!$$

for $n \geq 25$, as desired.

To begin, recall the bound (5) on $C(\mathbf{r})$:

$$G_1(\mathbf{r}) = \prod_{i=0}^{k}[(s_i-1)! - 1 + \delta_{1,s_i}];$$

use this together with the definition of w to get the inequality

$$C(\mathbf{r})/w(\mathbf{r}) \leq G_1(\mathbf{r}) \left[\prod_{i=0}^{k} Q(n_i,s_i)^{-1}\right].$$

Let Z_i be defined to be 1 when $s_i = 1$ and $Q(n_i,s_i)^{-1}$ otherwise. Then $\prod_i Q(n_i,s_i)^{-1} \leq \prod_i (Z_i(n_i+3)/(n_i+1)) \leq (\prod_i Z_i)[(n+3)!/(2!(n+1)!)] = \frac{1}{2}(n+3)(n+2)\prod_i Z_i$. So

$$C(\mathbf{r})/w(\mathbf{r}) \leq \frac{1}{2}(n+3)^2 D(\mathbf{r}), \tag{6}$$

where $D(\mathbf{r}) := G_1(\mathbf{r}) \prod_{i=0}^{k} Z_i$.

Now if there are $i<j$ for which $s_i \geq 5$ and $s_j \geq 4$, consider the (lexicographically) first such pair i,j, and let $\phi(\mathbf{r})$ be the sequence for which $s_i(\phi(\mathbf{r})) = 4$, $s_j(\phi(\mathbf{r})) = s_j + s_i - 4$, and $s_l(\phi(\mathbf{r})) = s_l$ for all $l \neq i,j$, where s_l always refers to $s_l(\mathbf{r})$ unless otherwise noted. Then (ii) of Proposition 2.6 shows that $G_1(\phi(\mathbf{r})) \geq G_1(\mathbf{r})$, with strict inequality if $s_j \neq 4$.

But also $\prod_i Z_i(\phi(\mathbf{r})) \geq \prod_i Z_i(\mathbf{r})$. To see this, note that each factor Z_l is increasing in n_l (even when s_l varies, as long as s_l stays at least 4 or else stays constant at 1 or constant at 3). Changing $\mathbf{r}$ to $\phi(\mathbf{r})$ changes the s_l's only by switching two of them that are both at least 4, and furthermore the values of n_l for $\phi(\mathbf{r})$ are equal to the values of n_l for $\mathbf{r}$ except when $i < l \leq j$. Since n_l is increased for $i < l \leq j$, it follows that $D(\phi(\mathbf{r})) \geq D(\mathbf{r})$.

Let $\mathbf{r}^{(1)}$ be gotten from $\mathbf{r}$ by iterating ϕ until it is no longer possible to do so. Note that $\mathbf{r}^{(1)}$ has at most one s_i greater than 4 and that, by induction, $D(\mathbf{r}^{(1)}) \geq D(\mathbf{r})$.

Now if $\max_i s_i(\mathbf{r}^{(1)}) = 4$, then let $\mathbf{r}^{(2)} = \mathbf{r}^{(1)}$; otherwise, if $s_i(\mathbf{r}^{(1)}) > 4$, let $\mathbf{r}^{(2)}$ be the sequence for which $s_i(\mathbf{r}^{(2)}) = s_k(\mathbf{r}^{(1)})$, $s_k(\mathbf{r}^{(2)}) = s_i(\mathbf{r}^{(1)})$, and $s_l(\mathbf{r}^{(2)}) = s_l(\mathbf{r}^{(1)})$ for $l \neq i, k$. Clearly, $G_1(\mathbf{r}^{(2)}) = G_1(\mathbf{r}^{(1)})$. For each $l \neq i, k$, s_l is unchanged while n_l is increased or remains the same, and so Z_l is not decreased. Since $Z_i(\mathbf{r}^{(2)}) \geq (n+3)^{-2} Z_i(\mathbf{r}^{(1)})$ and $Z_k(\mathbf{r}^{(2)}) \geq Z_k(\mathbf{r}^{(1)})$, it follows that $D(\mathbf{r}^{(2)}) \geq (n+3)^{-2} D(\mathbf{r}^{(1)})$. Summarizing the progress so far, there is a sequence $\mathbf{r}^{(2)}$ with $s_i(\mathbf{r}^{(2)}) \leq 4$ for $i < k$ such that

$$C(\mathbf{r})/w(\mathbf{r}) \leq \frac{1}{2}(n+3)^2 D(\mathbf{r}) \leq \frac{1}{2}(n+3)^2 D(\mathbf{r}^{(1)}) \leq \frac{1}{2}(n+3)^4 D(\mathbf{r}^{(2)}).$$

The last step is to compare $D(\mathbf{r}^{(2)})$ to $D(\mathbf{r}_0) = [(n-k)!-1](n-k+3)(n-k+2)$, where $\mathbf{r}_0$ is the sequence $(0, 1, 2, \ldots, k, n+1)$. Let $k_3 = k_3(\mathbf{r}^{(2)})$ be the number of $i < k$ for which $s_i(\mathbf{r}^{(2)}) = 3$ and let $k_4 = k_4(\mathbf{r}^{(2)})$ be the number of $i < k$ for which $s_i(\mathbf{r}^{(2)}) = 4$. Then $s_k(\mathbf{r}^{(2)}) = n + 1 - k - 3k_4 - 2k_3$. Now the inequalities $(a! - 1)/(b! - 1) > a!/b! > (a/e)^{a-b}$ for $a > b > 0$ from (iii) of Proposition 2.6 give

$$\begin{aligned}
& D(\mathbf{r}_0)/D(\mathbf{r}^{(2)}) \\
\geq\ & [(n-k)! - 1]/[5^{k_4}(n+3)^{2k_4}(n+3)^{k_3}((n-k-3k_4-2k_3)! - 1)] \\
\geq\ & ((n-k)/e)^{3k_4+2k_3}/[5^{k_4}(n+3)^{2k_4+k_3}] \\
\geq\ & (5(n+3)^{2/3})^{3k_4+2k_3}/[5^{k_4}(n+3)^{2k_4+k_3}]
\end{aligned}$$

$$\geq \; 5^{2k_4+2k_3}(n+3)^{k_3/3}$$

$$\geq \; 1,$$

where the inequality $(n-k)/e \geq 5(n+3)^{2/3}$ comes from the hypothesis of the lemma. This proof of $D(\mathbf{r}_0) \geq D(\mathbf{r}^{(2)})$ has assumed $n-k-3k_4-2k_3 \neq 1$, but can easily be modified to treat the contrary case.

Putting all this together gives $C(\mathbf{r})/w(\mathbf{r}) \leq n^6(n-k)!$ for all $\mathbf{r}$, proving the $f(n,k)$ part of the lemma.

Finally, using Lemma 2.3, which is valid because $n - \left\lceil 5e(n+3)^{2/3} \right\rceil + 1 \geq n - n^{3/4}/2$,

$$\begin{aligned} F(n,k) &\leq \sum_{n-5e(n+3)^{2/3} \geq l \geq k} f(n,l) + F(n, n - \left\lceil 5e(n+3)^{2/3} \right\rceil + 1) \\ &\leq 2n^6(n-k)! + \left\lceil 5e(n+3)^{2/3} \right\rceil (2n^{7/8})^{\left\lceil 5e(n+3)^{2/3} \right\rceil - 1}. \end{aligned}$$

□

In order to prove Lemma 2.5, the following comparison is needed between overlaps of pairs of paths connecting $\hat{0}$ to $\hat{1}$ and overlaps of pairs of paths connecting x_i to y_i.

Lemma 2.7 *Let x_1, x_2 be distinct vertices at level L of $\mathcal{B}_{n+2L}$ and let y_1, y_2 be distinct vertices at level $n+L$ of $\mathcal{B}_{n+2L}$, with x_1 below y_1 and x_2 below y_2. Fix an oriented path γ connecting x_1 to y_1. For $0 \leq k \leq n$ and $i = 1,2$, let $A_i(\gamma,k)$ be the set of oriented paths γ' from x_i to y_i that share exactly k edges with γ. For $i = 1,2$, let A_i [$= \cup_{k=0}^{n} A_i(\gamma,k)$] be the set of all oriented paths from x_i to y_i. Then there is a bijection ϕ from A_1 to A_2 such that* (i) *the set of edges that $\phi(\gamma')$ has in common with γ is a subset of the edges that γ' has in common with γ;* (ii) *hence if $\gamma' \in A_1(\gamma,k)$ then $\phi(\gamma') \in A_2(\gamma,j)$ for some $j \leq k$; and* (iii) *if γ' and γ share either their first or last edge, then the inclusion in (i) (and hence the inequality in (ii)) is strict.*

Proof: Viewing vertices of $\mathcal{B}_{n+2L}$ as subsets of $\{1, \ldots, n+2L\}$, the path γ is represented by a permutation of $y_1 \setminus x_1$. Assume without loss of generality that $x_1 = \{1, \ldots, L\}$ and $y_1 = \{1, \ldots, n+L\}$, and that the permutation representing γ is in fact the increasing permutation $(L+1, \ldots, L+n)$. We shall express the desired ϕ as a bijection, call it h, between permutations of $\{L+1, \ldots, L+n\}$ and permutations of $y_2 \setminus x_2$.

Let γ' connecting x_1 to y_1 be represented by a permutation $\pi = (\pi(1), \ldots, \pi(n))$ of $\{L+1, \ldots, L+n\}$. We need to define a corresponding permutation $h(\pi) = (h(\pi)(i), i = 1, \ldots, n)$ of $y_2 \setminus x_2$. The idea is this: h tries to copy π, but is required only to copy elements of $y_2 \setminus x_2$; so it replaces in corresponding order the elements of $\{L+1, \ldots, L+n\}$ that are not elements of $y_2 \setminus x_2$ by elements of $y_2 \setminus x_2$ that are not elements of $\{L+1, \ldots, L+n\}$.

The construction of h can be expressed more formally as follows. Let $I := \{1, \ldots, n\} \cap [(y_2 \setminus x_2) - L]$ denote the set of indices i, $1 \le i \le n$, such that $L + i \in y_2 \setminus x_2$; thus the points $L + i$, $i \in I$, form the intersection of $\{L+1, \ldots, L+n\}$ and $y_2 \setminus x_2$. There are $m := n - |I|$ elements in $\{L+1, \ldots, L+n\}$ that are not in $y_2 \setminus x_2$; label them in increasing order as $k_1 < k_2 < \cdots < k_m$. Label the m elements that are in $y_2 \setminus x_2$ but not in $\{L+1, \ldots, L+n\}$ as $k'_1 < k'_2 < \cdots < k'_m$.

Let $\pi = (\pi(1), \ldots, \pi(n))$ be a permutation of $\{L+1, \ldots, L+n\}$. If $\pi(i) \in y_2 \setminus x_2$, i.e., if $\pi(i) - L \in I$, then let $h(\pi)(i) = \pi(i) \in L + I$. If $\pi(i) \notin y_2 \setminus x_2$, i.e., if $\pi(i) = k_t$ for some $1 \le t \le m$, then let $h(\pi)(i) = k'_t$. It is easy to see that this yields a bijection between permutations of $\{L+1, \ldots, L+n\}$ and permutations of $y_2 \setminus x_2$. It remains to show that it has the required properties.

Let $\gamma' \in A_1$ be represented by π and $\phi(\gamma') \in A_2$ by $h(\pi)$. Then i belongs to $\mathbf{r}(\gamma')$ (defined as a subsequence of $(1, \ldots, n)$ in the obvious fashion) if and only if

$$\{\pi(1), \ldots, \pi(i-1)\} = \{L+1, \ldots, L+i-1\} \quad \text{and} \quad \pi(i) = L + i. \tag{7}$$

On the other hand, $i \in \mathbf{r}(\phi(\gamma'))$ if and only if

$$\{h(\pi)(1), \ldots, h(\pi)(i-1)\} = \{1, \ldots, L+i-1\} \setminus x_2 \quad \text{and} \quad h(\pi)(i) = L+i; \tag{8}$$

in particular, $x_2 \subset \{1 \ldots, L+i-1\}$ is necessary for (8).

We claim that $\mathbf{r}(\phi(\gamma')) \subseteq \mathbf{r}(\gamma')$, i.e., that for each $i \in \{1, \ldots, n\}$, (8) implies (7). Indeed, suppose (8) holds for a given value of i. Since $h(\pi)(i) = L+i \in \{L+1, \ldots, L+n\}$, we have $\pi(i) = h(\pi)(i) = L+i$ by our construction of $h(\pi)$. Also, $\{h(\pi)(1), \ldots, h(\pi)(i-1)\}$ is the union of some subset of $\{L+1, \ldots L+i-1\}$ and an initial segment of $\{k'_1, \ldots, k'_m\}$ (namely, $\{1, \ldots, L\} \cap \{k'_1, \ldots, k'_m\}$). Therefore $\{\pi(1), \ldots, \pi(i-1)\}$ is the union of some subset of $\{L+1, \ldots, L+i-1\}$ and an initial segment of $\{k_1, \ldots, k_m\}$, and so equals $\{L+1, \ldots, L+i-1\}$. Thus (7) is established, and the proof of (i) is complete. (ii) follows immediately from (i).

To finish the proof of the lemma, observe that since x_1 and x_2 are distinct, $\phi(\gamma')$ can never share its first edge with γ. Similarly, y_1 and y_2 are distinct, so $\phi(\gamma')$ can never share its last edge with γ. Thus if γ' shares either its first or last edge with γ, the inclusion $\mathbf{r}(\phi(\gamma')) \subseteq \mathbf{r}(\gamma')$ must be strict. $\square$

Proof of Lemma 2.5: Fix x_i, y_i $(i = 1, 2)$ and a path γ connecting x_1 and y_1. Use Lemma 2.7 to get a bijection ϕ from paths connecting x_1 and y_1 to paths connecting x_2 and y_2 with the properties stated therein. Now the interval in $\mathcal{B}_{n+2L}$ between x_1 and y_1 is isomorphic to $\mathcal{B}_n$. Hence the number of paths γ' connecting x_1 and y_1 and sharing at least k edges with γ is just $F(n,k)$, and Lemmas 2.2 – 2.4 may be used to bound this. Now since ϕ is a bijection and $\phi(\gamma')$ shares at most as many edges with γ as γ' does, this immediately gives $H(n, L, k, x_1, x_2, y_1, y_2) \leq F(n,k)$; maximizing over x_1, y_1, x_2, and y_2 and applying Lemmas 2.3 and 2.4 establishes (2) and (3). To show (1), note that if γ'' connects x_2 to y_2 and shares at least k edges with γ then $\gamma' = \phi^{-1}(\gamma'')$ either shares strictly more edges with γ or shares exactly the same edges in which case the shared edges include neither the first nor last edge. The number of γ' in the former category is at most $F(n, k+1)$, while the number of γ' in the latter category is at most $o((k+1)(n-k)!)$ uniformly in $0 < k \le K(n)$ according to the last part of Lemma 2.2. But $F(n, k+1) \le (1+o(1))(k+2)(n-k-1)! = o((k+1)(n-k)!)$ uniformly in $0 < k \le K(n)$ according to the first part of Lemma 2.2, and the desired conclusion follows. $\square$

3 Oriented percolation and oriented first-passage percolation

3.1 Oriented percolation

The first application of Lemmas 2.1 – 2.5 will be to ordinary oriented percolation. This means that the edges are independently open with probability p and closed otherwise. The problem considered here is to determine how large p should be as a function of the dimension n in order that $\hat{0}$ and $\hat{1}$ may be connected with substantial probability. It turns out that the answer is $p = e/n$ in the sense that if $p = c/n$ then for $c < e$ the probability that $\hat{0}$ is connected to $\hat{1}$ tends to 0 as $n \to \infty$, while for $c \ge e$, the probability that $\hat{0}$ is connected to $\hat{1}$ approaches a positive limit which we calculate. (It is easy to see that the limit is not 1 since the disconnection probability is at least the chance that $\hat{0}$ is isolated, namely, $(1 - c/n)^n \to e^{-c}$.) In particular, the limiting connection probability is $\doteq 0.8416$ at the critical value $c = e$. We consider oriented percolation mainly as a warm-up to the arguments used to analyze oriented first-passage percolation. We do not intend for our results to be viewed as a complete analysis of the threshold behavior of oriented percolation with edge probabilities c/n as c crosses the critical value e.

The next lemma derives a lower bound of $(e-1)^2/e^2 \doteq 0.400$ on the percolation probability in the critical region by using the unenhanced second moment method. The bound is not tight but will be a crucial ingredient to the proof of the sharper result.

Lemma 3.1 *Fix $L \in \mathbb{R}$. Let each edge of $\mathcal{B}_n$ be independently open with probability $p = e/(n+2L)$. Then*

$$\liminf_n \mathbf{P}(\hat{0} \text{ is connected to } \hat{1} \text{ by an oriented open path}) \geq \frac{(e-1)^2}{e^2}.$$

Proof: Observe that $\mathbf{E}N^2$ is the sum over pairs of paths of the probability that both paths are open. By symmetry this is the same as $n!$ times the sum when the first path is fixed, say as the path γ corresponding to the identity permutation. The probability that both γ and γ' are open depends only on the number k of edges they share and is equal to p^{2n-k}. Thus by Lemma 2.1 it suffices to find a finite upper bound for

$$\frac{\mathbf{E}N^2}{(\mathbf{E}N)^2} = n! \frac{\sum_k f(n,k) p^{2n-k}}{(p^n n!)^2} = \sum_k [f(n,k)(\frac{n+2L}{e})^k / n!]. \tag{9}$$

We bound this in three pieces, corresponding to Lemmas 2.2 – 2.4; of course we may assume n to be as large as needed. When $k < 12 \ln n$, the summand in the final sum of (9) is by Lemma 2.2 at most $(1+o(1))(k+1)e^{-k}(n+2L)^k/[n(n-1)\cdots(n-k+1)]$ uniformly in k as $n \to \infty$. This is at most $(1+o(1))(k+1)[(n+2L)/(n-12\ln n)]^k e^{-k}$ and so the sum over k is at most $(1+o(1))\left(e/\left[e-(n+2L)/(n-12\ln n)\right]\right)^2 = (1+o(1))e^2/(e-1)^2$.

The contribution from any term with $k > n - n^{3/4}/2$ is at most

$$(n-k+1)(2n^{7/8})^{n-k}[(n+2L)/e]^k/n!$$

according to Lemma 2.3. Using the inequality $n! > (n/e)^n n^{1/2}$ gives an upper bound for these terms of $(n-k+1)(2en^{-1/8}\frac{n}{n+2L})^{n-k} n^{-1/2}(1+\frac{2L}{n})^n$. The sum over k is then at most $[1/(1-2en^{-1/8}\frac{n}{n+2L})]^2 n^{-1/2}(1+\frac{2L}{n})^n = (1+o(1))e^{2L}n^{-1/2} = o(1)$ as $n \to \infty$.

Finally, consider the contribution from terms with $12 \ln n \leq k \leq n - 5e(n+3)^{2/3}$. By Lemma 2.4 this is at most

$$\sum_{12 \ln n \leq k \leq n-5e(n+3)^{2/3}} g(n,k),$$

where $g(n,k) = n^6(n-k)![(n+2L)/e]^k/n!$. Now $g(n,k+1)/g(n,k) = (n+2L)/(e(n-k))$, which is increasing in k; hence g is U-shaped in k for fixed n (and L), with its minimum at $1+\lfloor n-(n+2L)/e \rfloor$. In particular, the maximum of $g(n,k)$ for fixed n over an interval of values of k is achieved at an endpoint. Thus $\max\{g(n,k) : 12\ln n \le k \le n - 5e(n+3)^{2/3}\}$ is achieved at an endpoint. At the first endpoint, $k = \lceil 12\ln n \rceil$ and $g(n,k) = n^6 e^{-\lceil 12\ln n\rceil} \prod_{i=0}^{k-1}[(n+2L)/(n-i)] \le n^6 n^{-12}[(n+2L)/(n-12\ln n)]^{\lceil 12\ln n\rceil} = (1+o(1))n^{-6}$. At the second endpoint, $k = n - \left\lceil 5e(n+3)^{2/3} \right\rceil$ and the inequality $n! > (n/e)^n$ gives

$$
\begin{aligned}
g(n,k) &= n^6 \left\lceil 5e(n+3)^{2/3} \right\rceil! \,[(n+2L)/e]^{n-\lceil 5e(n+3)^{2/3}\rceil}/n! \\
&< n^6 \left\lceil 5e(n+3)^{2/3} \right\rceil! \,[(n+2L)/e]^{-\lceil 5e(n+3)^{2/3}\rceil}(1+2Ln^{-1})^{n-\lceil 5e(n+3)^{2/3}\rceil} \\
&= (1+o(1))e^{2L}n^6(\left\lceil 5e(n+3)^{2/3} \right\rceil/e)^{\lceil 5e(n+3)^{2/3}\rceil}\sqrt{2\pi 5en^{2/3}}[e/(n+2L)]^{\lceil 5e(n+3)^{2/3}\rceil} \\
&= (1+o(1))e^{2L}n^6\left(\frac{\left\lceil 5e(n+3)^{2/3} \right\rceil}{n+2L}\right)^{\lceil 5e(n+3)^{2/3}\rceil}\sqrt{2\pi 5en^{2/3}}
\end{aligned}
$$

by Stirling's formula. This clearly tends to 0 faster than any power of n and is thus $o(n^{-6})$. Now the terms at both ends have been shown to be at most $(1+o(1))n^{-6}$ and there are at most n terms, so the total contribution from these terms is at most $(1+o(1))n^{-5} = o(1)$.

To sum up, for n sufficiently large, $\sum_k f(n,k)[(n+2L)/e]^k/n!$ is at most the sum of the contributions from the three ranges, which was computed to be $(1+o(1))e^2/(e-1)^2 + o(1) + o(1)$. Thus by Lemma 2.1, $\mathbf{P}(\hat{0}$ is connected to $\hat{1}) \ge (1+o(1))(e-1)^2/e^2$. □

Theorem 3.2 *Let each edge of $\mathcal{B}_n$ be independently open with probability $p = c/n$. Then* $\mathbf{P}(\hat{0}$ is connected to $\hat{1}$ by an oriented open path$)$ *converges to a limit as $n \to \infty$. The limit is 0 if $c < e$ and is $(1-x(c))^2$ if $c \ge e$, where $x(c)$ is the extinction probability for a Poisson(c) Galton–Watson process, namely, the solution in $(0,1)$ to $x = e^{c(x-1)}$.*

Note that as $c \to \infty$, $x(c) = (1+o(1))e^{-c} = o(1)$, so that the limiting connection probability is $1-(1+o(1))2e^{-c} \to 1$.

Proof: There are $n!$ oriented paths from $\hat{0}$ to $\hat{1}$. Let N be the random number of these that consist entirely of open edges. For each path γ the probability that γ is open is p^n, so $\mathbf{E}N = n!p^n$. If $c < e$ then $\mathbf{E}N = n!c^n n^{-n} = (1+o(1))(c/e)^n\sqrt{2\pi n}$, which tends to 0. Since $\mathbf{P}(N > 0) \leq \mathbf{E}N$, this proves the first part.

For the second part, fix $c \geq e$. Also fix $\epsilon > 0$. Write $M = \lceil 1/\epsilon \rceil$. For $i = 1, 2, \ldots$, let A_i be the set of vertices at level i reachable from $\hat{0}$ in $\mathcal{B}_n$. For any fixed i, as $n \to \infty$, the joint distribution of $|A_0|, \ldots, |A_i|$ approaches in total variation the distribution of a Galton–Watson process with the number of offspring of each particle Poisson distributed with mean c. Because a surviving branching process proliferates, an integer $L = L(\epsilon)$ may be chosen so that $\mathbf{P}(|A_L| \geq M) \geq (1 - x(c)) - \epsilon$ for sufficiently large n, where $x(c)$ is the extinction probability for the Galton–Watson process, namely, the solution in $(0,1)$ to $x = e^{c(x-1)}$. Let B_j be A_j upside down, i.e., the set of vertices at distance j from $\hat{1}$ that can reach $\hat{1}$. Then by symmetry and independence, we have $\mathbf{P}(F) \geq (1 - x(c))^2 - 2\epsilon$, where F is the event $\{|A_L| \geq M$ and $|B_L| \geq M\}$. Now if either of the two sets A_L or B_L is empty, then $\hat{0}$ is not connected to $\hat{1}$, so the lim sup of $(1 - x(c))^2$ is established by the convergence in total variation.

For the lower bound we employ the enhanced second moment method described following Lemma 2.1, although the details here are slightly different. The variance-absorbing random variable Z is $\min(|A_L|, |B_L|)$. Uniformly in $z \geq M$, we show

$$\mathbf{P}(N > 0 | Z = z) \geq (1 - o(1)) \left[1 + \frac{e^2}{M(e-1)^2}\right]^{-1} \tag{10}$$

as $n \to \infty$. Then

$$\mathbf{P}(N > 0) \geq (1 - o(1)) \left[1 + \frac{e^2}{M(e-1)^2}\right]^{-1} \mathbf{P}(F)$$

and so

$$\liminf_n \mathbf{P}(N > 0) \geq \left[1 + \frac{e^2}{M(e-1)^2}\right]^{-1} [(1 - x(c))^2 - 2\epsilon].$$

Letting $\epsilon \downarrow 0$ gives $\liminf_n \mathbf{P}(N > 0) \geq (1 - x(c))^2$, as desired.

Henceforth tacitly conditioning on $Z = z \geq M$, we prove (10) by applying the second moment method to a truncation N' of N, as follows. First, reduce the probability that any given edge falling between levels L and $n - L$ is open from $p = c/n$ to $p = e/n$. The obvious coupling argument shows that this diminishes N stochastically. Then let $x_1, \ldots, x_M$ be an enumeration of the first M vertices of A_L and let $y_1, \ldots, y_M$ be an enumeration of the first M vertices of B_L, in some arbitrary ordering of the vertices at levels L and $n - L$, respectively. Let N_i be the number of open paths connecting x_i to y_i, and consider $N' := \sum_{i=1}^{M} N_i$. Essentially the same calculations that showed $\mathbf{E}N^2/(\mathbf{E}N)^2$ to be bounded in the proof of Lemma 3.1 will be used to show $\mathbf{E}(N')^2/(\mathbf{E}N')^2 \leq (1+o(1))\,(1 + e^2/[M(e-1)^2])$ via the two results $\mathbf{E}N_i^2/(\mathbf{E}N_i)^2 \leq (1+o(1))e^2/(e-1)^2$, uniformly in i, and $\mathbf{E}(N_iN_j)/[(\mathbf{E}N_i)(\mathbf{E}N_j)] \leq (1+o(1))$, uniformly in pairs $i \neq j$. Indeed, these last two statements imply $\mathbf{E}(N')^2 = \sum_i \mathbf{E}N_i^2 + \sum_{i \neq j} \mathbf{E}(N_iN_j) \leq (1+o(1))[e^2/(e-1)^2] \sum_i (\mathbf{E}N_i)^2 + (1+o(1)) \sum_{i \neq j} (\mathbf{E}N_i)(\mathbf{E}N_j) \leq (1+o(1))M[e^2/(e-1)^2](\mathbf{E}N_1)^2 + (1+o(1))M(M-1)(\mathbf{E}N_1)^2 \leq (1+o(1))\,(1 + e^2/[M(e-1)^2])\,(\mathbf{E}N')^2$. Now simply apply Lemma 2.1 to deduce (10) for N' and hence for N.

Working for convenience with $p = e/(n+2L)$ in $\mathcal{B}_{n+2L}$ rather than with $p = e/n$ in $\mathcal{B}_n$, and given distinct vertices x_1 and x_2 in level L and distinct vertices y_1 and y_2 in level $n + L$, let N_i denote the number of open paths from x_i to y_i, $i = 1, 2$. We must show $\mathbf{E}N_1^2/(\mathbf{E}N_1)^2 \leq (1+o(1))e^2/(e-1)^2$ and $\mathbf{E}(N_1N_2)/[(\mathbf{E}N_1)(\mathbf{E}N_2)] \leq 1 + o(1)$ as $n \to \infty$, uniformly in the choice of x_1, x_2, y_1, y_2. Now the interval from x_1 to y_1 is isomorphic to $\mathcal{B}_n$, so N_1 has the same distribution as the total number of open paths, N, in the proof of Lemma 3.1. Thus we immediately obtain $\mathbf{E}N_1^2/(\mathbf{E}N_1^2) \leq (1+o(1))e^2/(e-1)^2$ (uniformly in x_1). For the other inequality, mimic the calculation from Lemma 3.1 to get

$$\frac{\mathbf{E}(N_1N_2)}{(\mathbf{E}N_1)(\mathbf{E}N_2)} \leq n! \frac{\sum_k H(n, L, k, x_1, x_2, y_1, y_2) \left(\frac{e}{n+2L}\right)^{2n-k}}{\left[\left(\frac{e}{n+2L}\right)^n n!\right]^2} \leq \sum_k F_1(n, L, k) \left(\frac{n+2L}{e}\right)^k /n!.$$

Now break the sum into three pieces again, corresponding to values $k < 12 \ln n$, $k >$

$n - n^{3/4}/2$, and all k in between. This time use Lemma 2.5 instead of Lemmas 2.2 – 2.4, one difference being that the contribution for terms with $0 < k < 12 \ln n$ is now $o((k+1)(n-k)!)[(n+2L)/e]^k/n!$, and thus the contribution from those terms and the unit contribution from the $k = 0$ term sum to $1 + o(1)$. As before, the contribution from large k is $o(1)$, and the contribution from the first term in the bound (3) for intermediate values of k is also $o(1)$. We finish the proof of the theorem by showing that the contribution from the second term in (3) is also $o(1)$:

$$\begin{aligned}
&\sum_{12 \ln n \le k \le n - 5e(n+3)^{2/3}} \left\lceil 5e(n+3)^{2/3} \right\rceil (2n^{7/8})^{\left\lceil 5e(n+3)^{2/3} \right\rceil - 1} [(n+2L)/e]^k/n! \\
\le \;& n \left\lceil 5e(n+3)^{2/3} \right\rceil (2n^{7/8})^{\left\lceil 5e(n+3)^{2/3} \right\rceil - 1} [(n+2L)/e]^{n - \left\lceil 5e(n+3)^{2/3} \right\rceil}/n! \\
= \;& (1 + o(1)) \frac{5e^{2L+1}}{2\sqrt{2\pi}} n^{7/24} \left(\frac{2e}{n^{1/8}} \right)^{\left\lceil 5e(n+3)^{2/3} \right\rceil},
\end{aligned}$$

which vanishes at a rate faster than any power of n. □

3.2 Oriented first-passage percolation

Now consider oriented first-passage percolation (OFPP). Give each edge in $\mathcal{B}_n$ an upward orientation and assign independent, identically distributed random variables X_e with common density f to each edge e. The problem in OFPP is to determine the minimum value over oriented paths from $\hat{0}$ to $\hat{1}$ of the sum along the path of the X_e's. Under mild conditions on f, it turns out (Theorem 3.5) that this random minimum converges in probability to $1/f(0)$ as $n \to \infty$. By multiplying every edge-passage time X_e by a constant, it can be assumed without loss of generality (provided $0 < f(0) < \infty$) that $f(0) = 1$.

As Aldous (1989) points out, use of the exponential distribution $f(x) = e^{-x}$ simplifies some of the calculations involved but is not necessary. The following two lemmas, treating the exponential and more general cases, respectively, produce large deviation estimates that correspond to the probability $[e/(n+2L)]^{2n-k}$ in the proof of Theorem 3.2.

Lemma 3.3

(i) *Let S_n be the sum of $n \geq 1$ independent, identically distributed random variables Y_i, each exponential with mean 1, and let u be a real number in $[0,1]$. Then $\mathbf{P}(S_n \leq u) = (1+K_1(u,n))e^{-u}u^n/n!$ with $0 \leq K_1(u,n) \leq e/(n+1) \leq 2$.*

(ii) *Given $1 \leq k \leq n-1$, let $S'_n = \sum_{i=1}^{k} Y_i + \sum_{i=k+1}^{n} Y'_i$, where $Y_1, \ldots, Y_n, Y'_{k+1}, \ldots, Y'_n$ are independent and identically distributed. Then $\mathbf{P}(S_n \leq 1 \text{ and } S'_n \leq 1) \leq K_2 R(n,k)$, where*

$$R(n,k) = 2^{2n-2k} e^{2n-k} (2n-k)^{-(2n-k)} / \left[(n-k)^{1/2}(2n-k)^{1/2}\right]$$

and K_2 is constant. Furthermore, for $1 < k \leq n-1$, $R(n,k-1)/R(n,k) \leq K_3/n$ for some constant K_3.

Proof: The key for (i) is the standard switching relation $\mathbf{P}(S_n \leq u) = \mathbf{P}(X_u \geq n)$, where $X = (X_u)_{u \geq 0}$ is a Poisson process with unit intensity parameter. Thus $\mathbf{P}(S_n \leq u) \geq \mathbf{P}(X_u = n) = e^{-u}u^n/n!$. Moreover,

$$\begin{aligned}
\mathbf{P}(S_n \leq u) &= \sum_{m=n}^{\infty} e^{-u}\frac{u^m}{m!} = \frac{u^n}{n!} \times e^{-u} \sum_{l=0}^{\infty} \frac{u^l}{(n+l)\cdots(n+1)} \\
&\leq \frac{u^n}{n!} \times e^{-u} \sum_{l=0}^{\infty} \frac{u^l}{l!} = \frac{u^n}{n!}.
\end{aligned}$$

Using the switching relation together with this crude upper bound, we obtain

$$\begin{aligned}
\mathbf{P}(S_n \leq u) &= \mathbf{P}(X_u = n) + \mathbf{P}(X_u \geq n+1) \\
&= e^{-u}\frac{u^n}{n!} + \mathbf{P}(S_{n+1} \leq u) \\
&\leq e^{-u}\frac{u^n}{n!}(1 + e^u \frac{u}{n+1}) \\
&\leq e^{-u}\frac{u^n}{n!}(1 + \frac{e}{n+1}),
\end{aligned}$$

as desired.

For (ii), we both prove the large deviations inequality and show that it is tight. Begin by writing $\mathbf{P}(S_n, S'_n \leq 1) = \int_0^1 \mathbf{P}(S_k \in du)\, [\mathbf{P}(S_{n-k} \leq 1-u)]^2$. Using the bounds from (i) gives

$$\mathbf{P}(S_n, S'_n \leq 1) = \int_0^1 [e^{-u}u^{k-1}/(k-1)!][e^{2u-2}(1-u)^{2n-2k}][1+K_1(1-u, n-k)]^2/[(n-k)!]^2\, du,$$

and using lower and upper bounds for e^{u-2} and $K_1(1-u, n-k)$ bounds this below by

$$\frac{e^{-2}}{(k-1)![(n-k)!]^2} \int_0^1 u^{k-1}(1-u)^{2n-2k}\, du$$

and above by

$$\frac{9e^{-1}}{(k-1)![(n-k)!]^2} \int_0^1 u^{k-1}(1-u)^{2n-2k}\, du.$$

Now the integral is equal to $(k-1)!(2n-2k)!/(2n-k)!$, so $\mathbf{P}(S_n, S'_n \leq 1)$ is bounded between e^{-2} and $9e^{-1}$ times $\binom{2n-2k}{n-k}/(2n-k)!$. This can be approximated using Stirling's formula, for which it will suffice to note that the error factor of $e^{1/(12n)}$ is bounded. Thus $\mathbf{P}(S_n, S'_n \leq 1)$ is bounded between positive constant multiples of $R(n,k)$.

To see that $R(n, k-1)/R(n,k) \leq K_3/n$, note that the exact quotient is $4e[(2n-k)/(2n-k+1)]^{2n-k}(2n-k+1)^{-1}[(n-k)/(n-k+1)]^{1/2}[(2n-k)/(2n-k+1)]^{1/2}$. The factor $[(n-k)/(n-k+1)]^{1/2}$ is between $\sqrt{1/2}$ and 1, while the rest of the product equals $(1+o(1))\, 4/(2n-k) \leq (1+o(1))\, 4/n$, uniformly in k. This proves the claim and finishes that of the lemma. $\square$

The Lipschitz condition in the following lemma does not give the most general f for which the first-passage times can be calculated, but it does cover most non-pathological cases.

Lemma 3.4 *Let f be a probability density on $[0, \infty)$, and suppose that $f(0) = 1$ and that f satisfies a "global" Lipschitz condition at the origin: $|f(x) - f(0)| \leq K_4 x$ for* *some positive* $K_4 < \infty$ *and all* $x \geq 0$. *Let* T_n *be the sum of* n *independent random variables* $Z_1, \ldots, Z_n$ *with common density* f *and let* S_n *be the sum of* n *i.i.d. exponentials* $Y_1, \ldots, Y_n$ *with unit mean. Then for* $0 < u \leq 1$, $\mathbf{P}(T_n \leq u) \leq e^{(1+K_4)u}\,\mathbf{P}(S_n \leq u)$ *and* $\liminf_n \mathbf{P}(T_n \leq u)/\mathbf{P}(S_n \leq u) \geq e^{-K_4 u}$. *Similarly, if* $T_n' = \sum_{i=1}^k Z_i + \sum_{i=k+1}^n Z_i'$, *where* $Z_1, \ldots, Z_n, Z_{k+1}', \ldots, Z_n'$ *are i.i.d., then* $\mathbf{P}(T_n \leq 1 \text{ and } T_n' \leq 1) \leq K_5 R(n,k)$, *where* $K_5 := e^{2+2K_4} K_2$.

Proof: For the upper bound, note that the Radon–Nikodym derivative of Z_i with respect to Y_i at x is $f(x)/e^{-x} \leq (1+K_4 x)/e^{-x} \leq e^{(1+K_4)x}$. Thus the Radon–Nikodym derivative of the n-tuple $(Z_1, \ldots, Z_n)$ with respect to $(Y_1, \ldots, Y_n)$ at $(x_1, \ldots, x_n)$ is at most $e^{(1+K_4)\sum x_i}$, and hence the derivative of T_n with respect to S_n at x is at most $e^{(1+K_4)x}$. This establishes the upper bound for $\mathbf{P}(T_n \leq u)$. Together with (ii) of Lemma 3.3, this argument also establishes the upper bound for $\mathbf{P}(T_n \leq 1, T_n' \leq 1)$.

For the lower bound, we first establish the fact that for any fixed $u \in (0,1]$ and $\delta \in (0,u)$, $\mathbf{P}(\max_i Y_i < \delta \,|\, S_n \leq u)$ converges to 1 as $n \to \infty$. To see this, note that $\mathbf{P}(S_n \leq u, \max_i Y_i \geq \delta) \leq n\mathbf{P}(S_n \leq u, Y_1 \geq \delta) \leq n\mathbf{P}(S_{n-1} \leq u - \delta)$; part (i) of the last lemma shows that for $n \geq 2$ this is at most $3ne^{\delta}e^{-u}(u-\delta)^{n-1}/(n-1)!$. Then the lower bound from (i) of the previous lemma shows that $\mathbf{P}(\max_i Y_i \geq \delta \,|\, S_n \leq u) \leq 3e^{\delta}((u-\delta)/u)^{n-1} n^2/u$ which vanishes at an exponential rate as $n \to \infty$, proving the claim.

Now, given $K_4' > K_4$, a lower bound for the Radon–Nikodym derivative of Z_i with respect to Y_i is $1 - K_4 x \geq e^{-K_4' x}\mathbf{1}_{\{x<\delta\}}$ for an appropriate $\delta \in (0,u)$. Then $\mathbf{P}(T_n \leq u) \geq e^{-K_4' u}\mathbf{P}(S_n \leq u, \max_i Y_i < \delta)$; thus $\mathbf{P}(T_n \leq u)/\mathbf{P}(S_n \leq u)$ is at least $e^{-K_4' u}\mathbf{P}(\max_i Y_i < \delta \,|\, S_n \leq u)$ and so has a $\liminf$ of at least $e^{-K_4' u}$ by the fact in the previous paragraph. Now let $K_4' \downarrow K_4$. □

We are now ready for the main theorem for OFPP.

Theorem 3.5 *Let the edges of $\mathcal{B}_n$ be assigned i.i.d. positive random passage times with common density f, and assume that $|f(x) - 1| \leq K_4 x$ for all $x \geq 0$. Then the oriented first-passage percolation time $T = T^{(n)}$ for $\mathcal{B}_n$ converges to 1 in probability as $n \to \infty$.*

Proof: Let ϵ be small and positive. With X_{vw} being i.i.d. with common density f and γ an oriented path from $\hat{0}$ to $\hat{1}$ in $\mathcal{B}_n$, let $T_n(\gamma)$ be the sum of X_{vw} along edges $\overline{vw}$ of γ, so that the first-passage time T is just $\min_\gamma T_n(\gamma)$. Let T_n denote a random variable distributed identically to each $T_n(\gamma)$ and let S_n denote the sum of n i.i.d. exponentials of unit mean, as in the lemmas. Showing that $\mathbf{P}(T \leq 1 - \epsilon) \to 0$ is easy. Let N be the number of γ for which $T_n(\gamma) \leq 1 - \epsilon$. Then, using Lemmas 3.4 and 3.3, $\mathbf{P}(N > 0) \leq \mathbf{E}N = n!\mathbf{P}(T_n \leq 1 - \epsilon) \leq n! \exp[(1 + K_4)(1 - \epsilon)]\mathbf{P}(S_n \leq 1 - \epsilon) \leq 3 \exp[K_4(1 - \epsilon)](1 - \epsilon)^n = o(1)$ as $n \to \infty$, where S_n is the sum of n i.i.d. exponentials of mean 1.

To show that $\mathbf{P}(T \leq 1 + \epsilon)$ is bounded away from 0, one can mimic the proof of Lemma 3.1, but in order to show that this probability converges to 1, we need to find another auxiliary random variable to reduce the variance. It will be easier to work in $\mathcal{B}_{n+2}$. Let A_0 be the random set of neighbors v of $\hat{0}$ for which the edge $\overline{\hat{0}v}$ has $X_{\hat{0}v} \leq \epsilon/2$. Similarly, let A_1 be the random set of neighbors v of $\hat{1}$ for which the edge $\overline{v\hat{1}}$ has $X_{v\hat{1}} \leq \epsilon/2$. Let b' be the minimum of $|A_0|$ and $|A_1|$. Enumerate the elements of A_0 by $x_1, x_2, \ldots$ and the elements of A_1 by $y_1, y_2, \ldots$ in such a way that for $1 \leq i \leq b'$, y_i lies above x_i. This is easy to do since there is only one neighbor of $\hat{1}$ that does not lie above any given x_i. Let $b = \lceil \epsilon n/4 \rceil$. The first thing to observe is that $b' > b$ with probability converging to 1 as $n \to \infty$. This is immediate from the fact that b' is the minimum of two independent random variables that are binomial with parameters $n + 2$ and $\int_0^{\epsilon/2} f(x)\,dx \geq \int_0^{\epsilon/2} (1 - K_4 x)\,dx > \epsilon/4$. Now condition on the event that $b' > b$. It suffices to show that the probability of finding an oriented path γ connecting x_i to y_i with $T_n(\gamma) \leq 1$ for some $i \leq b$ converges (with appropriate uniformity) to 1, conditionally given $b' > b$ and the enumeration of the x_i's and y_i's. What will in fact be shown is that, uniformly over all choices of vertices $x_1, x_2, \ldots, x_b$ neighboring $\hat{0}$ and $y_1, y_2, \ldots, y_b$

neighboring $\hat{1}$ with y_i above x_i for each i, the probability of finding an oriented path of passage time at most 1 connecting some x_i to y_i tends to 1.

For this we use the second moment method. For $1 \leq i \leq b$, let N_i be the number of paths connecting x_i to y_i with passage time at most 1. Let $N = \sum_{i=1}^{b} N_i$. The interval in $\mathcal{B}_{n+2}$ from x_i to y_i is isomorphic to $\mathcal{B}_n$. It is therefore easy, using Lemmas 3.3 and 3.4, to see that

$$\mathbf{E}N = \sum_{i=1}^{b} \mathbf{E}N_i = b\,\mathbf{E}N_1 = b\,n!\,\mathbf{P}(T_n \leq 1) = b\,c_0,$$

where $c_0 = c_0(n)$ is bounded between positive constants. Now $\mathbf{E}N^2 = \sum_i \mathbf{E}N_i^2 + \sum_{i \neq j} \mathbf{E}(N_i N_j)$. If we can show that

$$\mathbf{E}N_1^2 = O(1) \quad \text{and} \quad \mathbf{E}(N_1 N_2) \leq (1 + o(1))c_0^2, \tag{11}$$

uniformly in the choice of x_1, x_2, y_1, y_2, then, it will follow that $\mathbf{E}N^2/(\mathbf{E}N)^2 \leq O(b^{-1}) + 1 + o(1)$, uniformly in the choice of the $2b$ vertices. Since b tends to infinity with n, this bound converges to 1, and so $\mathbf{P}(T \leq 1 + \epsilon) \to 1$, proving the theorem.

Each part of (11) is established in pieces, in a manner similar to the bounding of (9). For any fixed γ connecting x_1 to y_1, $\mathbf{E}N_1^2$ is given by $n!$ times the sum over γ' connecting x_1 to y_1 of $\mathbf{P}(T_n(\gamma) \leq 1$ and $T_n(\gamma') \leq 1)$. Break the sum into three ranges according to the number k of edges shared by γ and γ' as before, and additionally separate the cases $k = 0$ and $k = n$. The case $k = 0$ means $T_n(\gamma)$ is independent of $T_n(\gamma')$, so the contribution to $\mathbf{E}N_1^2$in this case is at most $(\mathbf{E}N_1)^2$; and the case $k = n$ has $\gamma = \gamma'$, so the contribution in this case is exactly $\mathbf{E}N_1$. Using Lemma 3.4 for the other three ranges and recalling that the case $k = n - 1$ is impossible, the sum can be bounded by

$$\begin{aligned}
\mathbf{E}N_1^2 &\leq (\mathbf{E}N_1)^2 + \mathbf{E}N_1 + n! \sum_{k=1}^{n-2} f(n,k) K_5 R(n,k) \\
&\leq (c_0^2 + c_0) \\
&\quad + (1 + o(1))n! \sum_{0 < k < 12 \ln n} (k+1)(n-k)! K_5 R(n,k)
\end{aligned}$$

$$
\begin{aligned}
&+n! \sum_{n-2\geq k>n-n^{3/4}/2} (n-k+1)(2n^{7/8})^{n-k} K_5 R(n,k) \\
&+n! \sum_{12\ln n\leq k\leq n-5e(n+3)^{2/3}} n^6 (n-k)! K_5 R(n,k),
\end{aligned}
$$

for large enough n. Now it will not be too hard to show that this is $c_0^2 + c_0 + O(1) = O(1)$, but before doing so, notice how similar the above bound is to a good bound on $\mathbf{E}(N_1 N_2)$. The $k=0$ term for $\mathbf{E}(N_1 N_2)$ has the same bound as above, but the $k=n$ term vanishes. The necessary changes are completed by using F_1 in place of f. This allows the $(k+1)(n-k)!$ to be replaced by $o((k+1)(n-k)!)$ according to Lemma 2.5, and hence

$$
\begin{aligned}
\mathbf{E}(N_1 N_2) \ \leq\ & c_0^2 + o\left(n! \sum_{0<k<12\ln n} (k+1)(n-k)! K_5 R(n,k) \right) \\
&+n! \sum_{n-2\geq k>n-n^{3/4}/2} (n-k+1)(2n^{7/8})^{n-k} K_5 R(n,k) \\
&+n! \sum_{12\ln n\leq k\leq n-5e(n+3)^{2/3}} 2n^6 (n-k)! K_5 R(n,k) \\
&+n! \sum_{12\ln n\leq k\leq n-5e(n+3)^{2/3}} \left\lceil 5e(n+3)^{2/3} \right\rceil (2n^{7/8})^{\lceil 5e(n+3)^{2/3}\rceil -1} K_5 R(n,k).
\end{aligned}
$$

It will be shown that the last three terms of the bound on $\mathbf{E}N_1^2$ are respectively $O(1)$, $o(1)$, and $o(1)$. This will show that $\mathbf{E}N_1^2 = O(1)$, and also that the first four terms of the bound on $\mathbf{E}(N_1 N_2)$ sum to $c_0^2 + o(1) + o(1) + o(1) = c_0^2 + o(1)$. For large enough n, the fifth term of the bound on $\mathbf{E}(N_1 N_2)$ is bounded by

$$
\begin{aligned}
K_5 n! \ &\times\ n \left\lceil 5e(n+3)^{2/3} \right\rceil (2n^{7/8})^{\lceil 5e(n+3)^{2/3}\rceil -1} R(n, n - \left\lceil 5e(n+3)^{2/3} \right\rceil) \\
&= \ (1+o(1)) K_5 \left(\frac{5e\pi}{2}\right)^{1/2} n^{11/24} \\
&\quad \times \left(1 + \frac{\left\lceil 5e(n+3)^{2/3} \right\rceil}{n}\right)^{-\left(n + \lceil 5e(n+3)^{2/3}\rceil\right)} \left(\frac{8e}{n^{1/8}}\right)^{\lceil 5e(n+3)^{2/3}\rceil},
\end{aligned}
$$

which vanishes at a rate faster than any power of n. Thus $\mathbf{E}(N_1 N_2) = c_0^2 + o(1)$, completing the proof of the theorem via (11).

The three estimates for the bound on $\mathbf{E}N_1^2$ are now routine calculations. Plugging in the value of $R(n,k)$ and using Stirling's formula gives for the second term in the bound

$$
\begin{aligned}
& (1+o(1))n! \sum_{0<k<12\ln n} (k+1)(n-k)! K_5 R(n,k) \\
= \;& (1+o(1))K_5 n^n e^{-n}\sqrt{2\pi n} \sum_{1\le k<12\ln n} (k+1)(n-k)^{n-k} e^{-(n-k)} \sqrt{2\pi(n-k)}\, 2^{2n-2k} e^{2n-k} \\
& \qquad \times (2n-k)^{-(2n-k)}(n-k)^{-1/2}(2n-k)^{-1/2} \\
= \;& (1+o(1))K_5 2^{1/2}\pi \sum_{1\le k<12\ln n} (k+1)\left[(n-k)^{n-k} n^n 2^{2n-2k}(2n-k)^{-(2n-k)}\right].
\end{aligned}
$$

Note that if the sum here were to contain a $k=0$ term, that term would equal 1. Furthermore, changing k to $k+1$ multiplies the part of the summand inside square brackets by

$$
(n-k-1)^{-1}[(n-k-1)/(n-k)]^{n-k} 2^{-2}[(2n-k-1)/(2n-k)]^{-(2n-k)}(2n-k-1).
$$

Now $[(n-k-1)/(n-k)]^{n-k} \to e^{-1}$, while $[(2n-k-1)/(2n-k)]^{-(2n-k)} \to e$ and $(2n-k-1)/(n-k-1) \to 2$, all uniformly over $k < 12\ln n$. Thus the successive ratios are $(1+o(1))\frac{1}{2}$ uniformly over k in the range of summation. Therefore the sum is at most $(1+o(1))\sum_{k=1}^{\infty}(k+1)2^{-k} = (1+o(1))3 = O(1)$, establishing the first bound.

For the third term in the bound on $\mathbf{E}N_1^2$, let $m = n-k$. Then plugging in for $R(n,k)$ and using Stirling's formula yields

$$
\begin{aligned}
& n! \sum_{n-2\ge k>n-n^{3/4}/2} (n-k+1)(2n^{7/8})^{n-k} K_5 R(n,k) \\
= \;& (1+o(1))K_5 \sum_{n-n^{3/4}/2<k\le n-2} (n-k+1)(2n^{7/8})^{n-k} n^n e^{-n}\sqrt{2\pi n}\, 2^{2n-2k} e^{2n-k} \\
& \qquad \times (2n-k)^{-(2n-k)}(n-k)^{-1/2}(2n-k)^{-1/2} \\
\le \;& (1+o(1))K_5(2\pi)^{1/2} \sum_{2\le m<n^{3/4}/2} (m^{1/2}+m^{-1/2})((2e)^{8/7}n)^{7m/8} n^n 2^{2m}(n+m)^{-(n+m)}
\end{aligned}
$$

$$\leq \ (1+o(1))K_5(2\pi)^{1/2} \sum_{2\leq m<n^{3/4}/2} 2m(8e)^m n^{-m/8},$$

For the first inequality here we used $\sqrt{n}(2n-k)^{-1/2} = 1+o(1)$ uniformly over $k > n-n^{3/4}/2$, and for the second we used $m^{1/2}+m^{-1/2} \leq 2m$ and $(n+m)^{-(n+m)} \leq n^{-(n+m)}$. Changing m to $m+1$ multiplies the term by $8e(1+1/m)n^{-1/8}$, which vanishes in the limit uniformly in m; thus the sum is dominated by the $m=2$ term, whose value is a constant times $n^{-1/4}$, and is thus $O(n^{-1/4}) = o(1)$.

Finally, to bound the fourth term in the bound on $\mathbf{E}N_1^2$, plug in to get

$$\begin{aligned}
& n! \sum_{12\ln n\leq k\leq n-5e(n+3)^{2/3}} n^6(n-k)!K_5R(n,k) \\
= \ & (1+o(1))2\pi K_5 \sum_{12\ln n\leq k\leq n-5e(n+3)^{2/3}} n^6(n-k)^{n-k}e^{-(n-k)}\sqrt{n-k}\, n^n e^{-n}\sqrt{n}2^{2n-2k}e^{2n-k} \\
& \qquad \times(2n-k)^{-(2n-k)}(n-k)^{-1/2}(2n-k)^{-1/2} \\
\leq \ & (1+o(1))2\pi K_5 \sum_{12\ln n\leq k\leq n-5e(n+3)^{2/3}} n^6(n-k)^{n-k}n^n 2^{2n-2k}(2n-k)^{-(2n-k)},
\end{aligned}$$

since $\sqrt{n}(2n-k)^{-1/2} \leq 1$.

The sum here is at most n times its largest term. Let $r = k/n$ and rewrite the typical summand as $n^6[(4-4r)^{1-r}/(2-r)^{2-r}]^n = n^6h(r)^n$, say. Now we find the maximum of $h(r)$ on $[0,1]$. Taking logs gives

$$\ln h(r) = (1-r)\ln(4-4r) - (2-r)\ln(2-r),$$

so that

$$(d/dr)\ln h(r) = \ln(2-r) - \ln(4-4r).$$

This increases from $-\ln 2$ to ∞ as r increases from 0 to 1, so the maximum of $\ln h(r)$ over the interval $(12\ln n)/n \leq r \leq 1-5e(n+3)^{2/3}/n$ is achieved at one of the endpoints, at least for large n. Again for large enough n, we can, for any $\delta > 0$, get the derivative of $\ln h$ on $[0,(12\ln n)/n]$ to be bounded above by $-\ln 2+\delta < -0.693+\delta$, so choosing $\delta = 0.003$ makes the derivative of $\ln h$ bounded above by -0.69 on this interval. Similarly, for large enough n the derivative on $[1-5e(n+3)^{2/3}/n, 1]$ is bounded below by 1. Noting that $h(0)=h(1)=1$, it follows that the value of $n^6 h(r)^n$ at $r=(12\ln n)/n$ is at most $[e^{(-0.69)(12\ln n)/n}]^n n^6 = n^{-2.28}$, and the value at $1-5e(n+3)^{2/3}/n$ is at most $[e^{-5e(n+3)^{2/3}/n}]^n n^6 = e^{-5e(n+3)^{2/3}} n^6 < n^{-2.28}$ for large n. Thus the sum under consideration is at most $n^{-1.28}$.

Putting all of this together gives $\mathbf{E}N_1^2 \le c_0^2 + c_0 + O(1) + O(n^{-1/4}) + O(n^{-1.28}) = O(1)$, as desired. $\square$

4 Unoriented percolation

In Section 5 we shall consider the first-passage time to $\hat{1}$ for unoriented first-passage percolation on $\mathcal{B}_n$. For completeness, in this section we treat ordinary unoriented percolation and argue that the critical probability is $1/n$, as put forth in the following theorem:

Theorem 4.1 *Let each edge of $\mathcal{B}_n$ be independently open with probability $p = c/n$, $0 < c < \infty$. Then* $\mathbf{P}(\hat{0}$ is connected to $\hat{1}$ by an (unoriented) open path$) \to (1-x(c))^2$, *where $x(c)$ is, as in Theorem 3.2, the extinction probability for a Poisson(c) Galton–Watson process.*

Proof: Write $\theta_n \equiv \theta_n(c)$ for the percolation probability in question. We first note that $\limsup_n \theta_n \le (1-x(c))^2$ by a branching process approximation similar to that in the second paragraph of the proof of Theorem 3.2; we omit the details.

For the lower bound we may restrict attention to the case $c > 1$; it is precisely for these values of c that $y(c) := 1 - x(c) > 0$. Let $0 < \epsilon < y(c)/4$. We rely heavily on a result of Ajtai, Komlós, and Szemerédi (1982): $\mathbf{P}(F) \geq 1 - o(1)$, where F is the event that (a) there is exactly one component in the random graph formed by the open edges that has at least $(y(c) - \epsilon)2^n$ vertices, and (b) all the other components are of size at most $\epsilon 2^n$.

For $v \in B_n$, let A_v denote the event $\{y$ is connected by an open path to at least $(y(c) - 2\epsilon)2^n$ vertices$\}$. By a simple application of the FKG inequality (Fortuin, Ginibre, and Kasteleyn (1971)), the indicators of the events A_v are pairwise positively correlated. Furthermore, conditionally given F, we have by symmetry

$$\mathbf{P}(A_v) = \mathbf{P}(v \in \text{ the unique giant component (GC)}) = 2^{-n}\mathbf{E}(\text{size of GC}) \geq y(c) - \epsilon;$$

thus, unconditionally, $\mathbf{P}(A_v) \geq y(c) - 2\epsilon$ for sufficiently large n. By FKG,

$$\mathbf{P}(A_{\hat{0}} \cap A_{\hat{1}}) \geq (y(c) - 2\epsilon)^2,$$

and so

$$\begin{aligned}\mathbf{P}(\hat{0} \text{ and } \hat{1} \text{ are in the same component} \,|\, F) &\geq \mathbf{P}(\hat{0} \text{ and } \hat{1} \text{ are in the GC} \,|\, F) \\ = \mathbf{P}(A_{\hat{0}} \cap A_{\hat{1}} \,|\, F) &\geq \frac{\mathbf{P}(A_{\hat{0}} \cap A_{\hat{1}}) - (1 - \mathbf{P}(F))}{\mathbf{P}(F)} \geq (y(c) - 3\epsilon)^2,\end{aligned}$$

and hence $\theta_n \geq (y(c) - 4\epsilon)^2$, for sufficiently large n. Let $\epsilon \downarrow 0$ to complete the proof. $\Box$

We close this section by noting that for $c < 1$ there is a more elementary proof that $\theta_n(c) \to 0$. For $x \in B_n$, let $g(x)$ denote the probability that x is connected to $\hat{0}$ by an open path. Clearly, for $x \neq \hat{0}$, $g(x)$ equals the probability that there is a neighbor y of x such that y is connected to $\hat{0}$ by an open path not containing x and the edge $\{y, x\}$ is open. Hence

$$g(\hat{0}) = 1,\ g(x) \leq p \textstyle\sum_{y \sim x} g(y) \text{ for } x \neq \hat{0},\ g(x) \leq 1 \text{ for all } x, \tag{12}$$

where the sum is over vertices y adjacent to x.

Repeatedly applying (12), we find $g(x) \leq pn = c$ for $x \neq \hat{0}$, $g(x) \leq c^2$ for $d(\hat{0}, x) \geq 2$, $g(x) \leq c^3$ for $d(\hat{0}, x) \geq 3$, ..., and finally $\theta_n = g(\hat{1}) \leq c^n \to 0$, as desired.

5 Richardson's growth model and unoriented percolation

Consider the following model for the spread of disease. Individuals are located at vertices of an n-cube, with edges modelling pairs of individuals in frequent contact. One individual, $\hat{0}$, is infected at time 0 and the rest are healthy. Independently for each edge between an infected individual and an uninfected one, there is a constant small probability per small unit of time that the contact between those two individuals will cause the uninfected one to become infected. It is easy to construct from this description a stochastic model for the growing set of infected individuals. The model is a continuous time Markov chain on the space of subsets of $\mathcal{B}_n$ which jumps from A to $A \cup \{v\}$ at rate $k(A, v)$, where $k(A, v)$ is the number of infected neighbors of v, i.e., the number of neighbors of v in A. This Markov chain is called Richardson's growth model.

Interesting questions about this model are (1) When should we expect $\hat{1}$ to become infected? and (2) What is the cover time, i.e., when should we expect all the vertices to become infected? In this section we discuss the first question, giving limiting upper and lower of 1 and 0.88, respectively. These are obtained by proving and then exploiting the fact that the infection time for $\hat{1}$ in Richardson's model has the same distribution as the first-passage time to $\hat{1}$ in *unoriented* first-passage percolation on $\mathcal{B}_n$. The cover time question is addressed in Section 6.

The following lemma reduces the problem of when $\hat{1}$ first becomes infected to unoriented first-passage percolation with exponentially distributed edge-passage times. Since the oriented percolation time is always at least as great as the unoriented percolation time (the minimum over paths directed away from $\hat{0}$ must be at least the minimum over all paths), the upper bound of Theorem 5.2 for the infection time of $\hat{1}$ is immediate.

Lemma 5.1 *Let the edges $\{v, w\}$ of the undirected graph $\mathcal{B}_n$ be assigned independent* *exponential random variables* $X_{v,w} = X_{w,v}$ *of mean* 1. *Define the* infection time *of a vertex* v*, denoted* $T_n(v)$*, to be* $\inf \sum_i X_{v_i v_{i+1}}$*, where the* inf *is over all paths from* $v_0 = \hat{0}$ *to* v*. Let* $A(t) = \{v \in \mathcal{B}_n : T_n(v) \leq t\}$*. Then the random map* A *from* $[0, \infty)$ *to subsets of* $\mathcal{B}_n$ *has the same law as Richardson's model.*

Proof: Fix n and $t \geq 0$ and $h \geq 0$, and let $\mathcal{F}_t = \sigma\langle A(s) : s \leq t\rangle$ denote the σ-field generated by $A(s)$, $s \leq t$. We show that the conditional distributions $\mathcal{L}(A(t+h) \,|\, \mathcal{F}(t))$ are of the appropriate form.

For each connected set B containing $\hat{0}$, let $E_B := \{A(t) = B\}$; note that the events E_B partition the sample space. We show that A is a time homogeneous Markov chain by showing that

$$\sum_B \mathbf{1}(\omega \in E_B)\, \mathbf{P}(A(1+h) = C \,|\, A(1) = B) \text{ serves as } \mathbf{P}(A(t+h) = C \,|\, \mathcal{F}(t))(\omega). \quad (13)$$

Define the random subset $A_B(s)$ of $\mathcal{B}_n$ to be $A(s)$ if $A(t) = B$ and $\emptyset$ otherwise. Then one checks easily that, for any event D, $\sum_B \mathbf{1}(\omega \in E_B)\, \mathbf{P}(D \,|\, \mathcal{F}_B(t))(\omega)$ serves as $\mathbf{P}(D \,|\, \mathcal{F}(t))(\omega)$, where $\mathcal{F}_B(t) := \sigma\langle A_B(s) : s \leq t\rangle$. Clearly $\mathcal{F}_B(t)$ is contained in $\mathcal{G}_B(t) := \sigma\langle X_{v,w} : (v,w) \in (B,B); \mathbf{1}(T_B(v) + X_{v,w} > t) : (v,w) \in (B, B^c)\rangle$, where the notation $(v,w) \in (B_1, B_2)$ means that $\{v, w\}$ is an edge in $\mathcal{B}_n$ with $v \in B_1$ and $w \in B_2$, and where $T_{B_1}(v)$ is defined just as is $T_n(v)$ with the additional requirement that all vertices along a path belong to B_1.

Now if $A(t) = B$, then for $y \notin B$, $T_n(y) = t + \inf\{T_B(v) + X_{v,w} - t + T_{B^c}(w,y) : (v,w) \in (B, B^c)\}$. It now follows easily from the mutual (unconditional) independence of all the edge-passage times and the memoryless property of the exponential distribution for each $X_{v,w}$ with $(v,w) \in (B, B^c)$ that there exists a conditional probability distribution

$$\mathcal{L}(A(t+h) \,|\, X_{v,w} = x_{v,w} \text{ for } (v,w) \in (B,B) \text{ and } T_B(v) + X_{v,w} > t \text{ for } (v,w) \in (B, B^c))$$

depending neither on t nor on the particular values $x_{v,w}$ provided $T_B(v) \leq t$ for all $v \in B$ for these values of $X_{v,w}$, $(v,w) \in (B,B)$. It then follows that there is a version of the conditional probability $\mathbf{P}(A(t+h) = C \mid \mathcal{G}_B(t))$ such that

$$\mathbf{P}(A(t+h) = C \mid \mathcal{G}_B(t))(\omega) = \mathbf{P}(A(1+h) = C \mid A(1) = B), \ \omega \in E_B.$$

Since $E_B \in \mathcal{F}_B(t) \subseteq \mathcal{G}_B(t)$, the same is true with $\mathcal{G}_B(t)$ replaced by $\mathcal{F}_B(t)$. Hence (13) is established, and so A is a Markov chain.

Just as for Richardson's model, it is clear that, with probability one, $A(t)$ increases from $\{\hat{0}\}$ at $t = 0$ to $\mathcal{B}_n$ for all large t, adding at each jump time τ a single vertex w_0 with the property $T_n(v) + X_{v,w_0} = \tau$ for a vertex v satisfying $(v, w_0) \in (A(\tau-), A(\tau-)^c)$. We complete the proof by showing that, conditionally given $A(t) = B$, the process jumps from B to $B \cup \{w_0\}$ with the appropriate rate.

Indeed, fix w_0 and suppose that $(v, w_0) \in (B, B^c)$ for precisely k values of v, say, $v_1, \ldots, v_k$. In light of our discussions so far, it is enough to note that $P(T_n(v_j) + X_{v_j,w_0} \leq t + h \text{ for some } j \mid X_{v,w} = x_{v,w} \text{ for } (v,w) \in (B,B) \text{ and } T_B(v) + X_{v,w} > t \text{ for } (v,w) \in (B,B^c)) = 1 - e^{-kh} = kh + o(h)$ as $h \downarrow 0$. Thus the process jumps from B to $B \cup \{w_0\}$ at rate k. □

Theorem 5.2 *For any $\epsilon > 0$, the probability of finding $\hat{1}$ infected by time $1+\epsilon$ in Richardson's model on $\mathcal{B}_n$, beginning with only $\hat{0}$ infected at time 0, tends to 1 as $n \to \infty$.*

Proof: Theorem 3.5 and Lemma 5.1. □

We doubt whether this result is sharp, since there is no reason why the unoriented percolation time should be as great as the oriented percolation time. The next theorem, based on a calculation by R. Durrett (personal communication), gets a lower bound for the unoriented first-passage time by comparing to a branching translation process

(BTP). This is a process, started with a single particle at $\hat{0}$, for which each existing particle generates offspring at rate n, where the offspring are each displaced from the parent by an independent uniform random step $e(j)$. Letting $Z(x,t)$ be the number of particles at x at time t, the process $(Z(x,t) : x \in \mathcal{B}_n,\, t \geq 0)$ is formally defined by the transition rates $(Z(x)) \to (Z(x) + \delta_{xy})$ at rate $\sum_{w:\, d(y,w)=1} Z(w)$, where δ_{xy} is 1 if $x = y$ and 0 otherwise. It is easy to couple BTP to Richardson's model so that the set of infected vertices in Richardson's model is always a subset of the set of populated vertices in BTP. Thus the first time τ_n that $\hat{1}$ is populated in BTP is stochastically less than the first infection time T_n of $\hat{1}$ in Richardson's model.

Theorem 5.3 (Durrett) *As $n \to \infty$, the time τ_n of first population of $\hat{1}$ in BTP converges in probability to* $\ln(1+\sqrt{2}) \doteq 0.88$. *Consequently,* $\mathbf{P}(T_n \leq \ln(1+\sqrt{2}) - \epsilon) \to 0$.

Proof: The lower bound will be gotten by a routine first moment calculation. The upper bound in probability for BTP (which is not necessary for the result on Richardson's model) requires a second moment calculation and a little more work. Fix n and write $m_1(x,t)$ for $\mathbf{E}Z(x,t) =$ the expected number of particles at x at time t in BTP starting from a single particle at $\hat{0}$. We remark for later that this is also the expected number of offspring at $y + x$ at time $s + t$ of a particle at y at time s that are born to the particle after time s, where the addition in $y + x$ is taken, as usual, to be coordinatewise mod 2 addition. Since $\mathbf{P}(Z(\hat{1},t) > 0) \leq m_1(\hat{1},t)$, the lower bound in probability will follow from showing that $m_1(\hat{1},t) \to 0$ as $n \to \infty$ for any $t < \ln(1+\sqrt{2})$. Viewing vertices of $\mathcal{B}_n$ as sets, we write $|x|$ for the cardinality of x; the differential equation for $m_1(x,t)$ is easily seen to be

$$\frac{d\, m(x,t)}{dt} = \sum_{y:\, d(x,y)=1} m(y,t) \tag{14}$$

with initial conditions $m(x,0) = \delta_{\hat{0},x}$. Let

$$p(x,t) = \left(\frac{1-e^{-2t}}{2}\right)^{|x|} \left(\frac{1+e^{-2t}}{2}\right)^{n-|x|}$$

be the probability that a simple random walk with rate n started at $\hat{0}$ is at x at time t. Then, as may be verified by a variety of probabilistic and analytic arguments, the unique solution to (14) is given by

$$m_1(x,t) = e^{nt} p(x,t).$$

Putting $x = \hat{1}$ gives

$$m_1(\hat{1},t) = \left(\frac{e^t - e^{-t}}{2} \right)^n .$$

Since $(e^t - e^{-t})/2$ is increasing in t and equal to 1 at $t = \ln(1+\sqrt{2})$, it follows that for $t < \ln(1+\sqrt{2})$, $m_1(\hat{1},t)$ tends to 0 as $n \to \infty$. Hence $\mathbf{P}(\tau_n \le t) \to 0$, as desired.

The upper bound in probability on τ_n is gotten by a now familiar sort of argument. Fix $\epsilon > 0$. First the second moment method is used to show that $\liminf_n \mathbf{P}(\tau_n \le \ln(1+\sqrt{2}) + \epsilon) \ge 1/12$. Then the initial branching of the process is used to show that with just 2ϵ more time units, there are actually many independent chances of no less than $1/12$ each to get $\hat{1}$ populated, and hence the probability that this occurs is near 1.

Begin with

$$\mathbf{P}(\tau_n \le t) = \mathbf{P}(Z(\hat{1},t) > 0) \ge \frac{(\mathbf{E}\, Z(\hat{1},t))^2}{\mathbf{E}(Z(\hat{1},t)^2)} = \frac{m_1(\hat{1},t)^2}{m_2(\hat{1},t)},$$

where $m_2(x,t) := \mathbf{E}(Z(x,t)^2)$. To calculate the value of $m_2(x,t)$ in terms of $m_1(x,t)$, write $Z(x,t)^2$ as $Z(x,t)$ plus twice the number of unordered pairs of distinct particles at x at time t. Each such pair of particles has a well-defined time s at which their ancestral lines first split apart. Say that at time s a particle p_1 at vertex y gave birth to a particle p_2 at vertex $y + e(i)$, and that both particles are descendants of p_1 but only one is a descendant of p_2. For fixed y and i and interval $[s, s+ds)$, the expected number of such pairs is $m_1(y,s)\, ds\, m_1(x-y, t-s)\, m_1(x-y-e(i), t-s)$, so summing over y and i and integrating over s gives

$$\frac{m_2(\hat{1},t)}{m_1(\hat{1},t)^2} = \frac{1}{m_1(\hat{1},t)} + \sum_i 2 \int_0^t ds \sum_y \frac{m_1(y,s) m_1(\hat{1}-y, t-s) m_1(\hat{1}-y-e(i), t-s)}{m_1(\hat{1},t)^2}. \tag{15}$$

Now fix $t = \ln(1+\sqrt{2}) + \epsilon$. The first term tends to 0 as $n \to \infty$, so it suffices to show that the lim sup of the sum on i is at most 12.

Substituting $m_1(x,t) = e^{nt} p(x,t)$ into the sum on i in (15) yields

$$2\int_0^t ds \sum_{y,i} e^{-ns} \frac{p(y,s)p(\hat{1}-y,t-s)p(\hat{1}-y-e(i),t-s)}{p(\hat{1},t)^2}. \tag{16}$$

Next, plug in the value for $p(x,t)$. At the same time, group together all y on the same level of $\mathcal{B}_n$, i.e., all y with $|y| = k$ for each k. Then $|y+e(i)|$ will equal either $k+1$ or $k-1$; since $p(\hat{1}-x,t)$ increases with $|x|$, we get an upper bound by replacing $|y+e(i)|$ by $k+1$. This gives an upper bound for the integrand of

$$ne^{-ns}\frac{1+e^{-2(t-s)}}{1-e^{-2(t-s)}}$$
$$\times \sum_{k=0}^{n} 2^{-n}\binom{n}{k}[(1-e^{-2s})(1+e^{-2(t-s)})^2]^k[(1+e^{-2s})(1-e^{-2(t-s)})^2]^{n-k}(1-e^{-2t})^{-2n}.$$

The sum over k is just the binomial expansion of

$$\left(\frac{(1-e^{-2s})(1+e^{-2(t-s)})^2 + (1+e^{-2s})(1-e^{-2(t-s)})^2}{2(1-e^{-2t})^2}\right)^n,$$

and simplifying this yields

$$\left(\frac{1}{2(1-e^{-2t})^2}(2+2e^{-4(t-s)}-4e^{-2t})\right)^n = \left(1+\frac{e^{-4(t-s)}-e^{-4t}}{(1-e^{-2t})^2}\right)^n,$$

which gives a bound for the integrand in (16) of

$$n\,\frac{1+e^{-2(t-s)}}{1-e^{-2(t-s)}}e^{-ns}\left(1+\frac{e^{-4(t-s)}-e^{-4t}}{(1-e^{-2t})^2}\right)^n. \tag{17}$$

We need a better bound on the integrand when s is near t: the factor $\dfrac{1+e^{-2(t-s)}}{1-e^{-2(t-s)}}$ blows up like $(t-s)^{-1}$, which is not integrable. Note that in the case $k=n$ it is not possible to have $|y+e(i)| = k+1$. Thus for the $k=n$ term, the factor $\frac{1+e^{-2(t-s)}}{1-e^{-2(t-s)}}$ can be replaced by its reciprocal. This reduces the integrand significantly when the $k=n$ term is the dominant term in the sum. The ratio of the $k=n$ term to the entire above sum on k is

$$\begin{aligned}
& [(1-e^{-2s})(1+e^{-2(t-s)})^2]^n / [(1-e^{-2s})(1+e^{-2(t-s)})^2 + (1+e^{-2s})(1-e^{-2(t-s)})^2]^n \\
\geq\ & [(1-e^{-2s})/((1-e^{-2s}) + (1+e^{-2s})(2(t-s))^2)]^n \\
\geq\ & [1-K(t-s)^2]^n \\
\geq\ & 1-K(t-s)
\end{aligned}$$

for some constant K when $t-s \leq 1/n$. ($K=15$ will do when $n \geq 2$.) Also, $[\frac{1+e^{-2(t-s)}}{1-e^{-2(t-s)}}]^{-2} = (t-s)^2 + O((t-s)^3)$. Putting this all together, a better bound for the integrand in (16), uniformly for s satisfying $t-s \leq 1/n$, is

$$(1+o(1))K(t-s)n\frac{1+e^{-2(t-s)}}{1-e^{-2(t-s)}}e^{-ns}\left(1+\frac{e^{-4(t-s)}-e^{-4t}}{(1-e^{-2t})^2}\right)^n. \tag{18}$$

In order to make use of (17) and (18), examine the function

$$G(s,u) = \ln\left[e^{-s}\left(1+\frac{e^{-4(u-s)}-e^{-4u}}{(1-e^{-2u})^2}\right)\right].$$

This is convex in s for the values of u we are interested in, which may be seen by differentiating twice with respect to s: writing $c \equiv c_u = e^{-4u}/(1-e^{-2u})^2$ gives

$$\begin{aligned}
\frac{\partial^2 G(s,u)}{\partial s^2} &= \frac{\partial^2}{\partial s^2}\left[-s+\ln(1+c(e^{4s}-1))\right] \\
&= \frac{\partial}{\partial s}\left(-1+\frac{4ce^{4s}}{1+c(e^{4s}-1)}\right) \\
&= \frac{16c(1-c)e^{4s}}{[1+c(e^{4s}-1)]^2},
\end{aligned}$$

which is positive for all s whenever $c < 1$. This is true if and only if $u > \ln\sqrt{2}$ and hence for $u = \ln(1+\sqrt{2}) + \epsilon$. In particular, the maximum of $G(s,u)$ over $s \in [0,u]$ is achieved at an endpoint. But $G(0,u) = 0$ and

$$G(u,u) = \ln\left[e^{-u}(1 + \frac{1-e^{-4u}}{(1-e^{-2u})^2})\right] = \ln\left[2e^u/(e^{2u}-1)\right].$$

This is decreasing in u and has value 0 when $u = \ln(1+\sqrt{2})$. Thus when $u = t = \ln(1+\sqrt{2}) + \epsilon$, $G(u,u)$ has a negative value which we shall call $-V(\epsilon)$. Now we bound the upper bound for $(m_2(\hat{1},t) - m_1(\hat{1},t))/m_1(\hat{1},t)^2$ given by display (16) in three pieces:

$$\begin{aligned} & [m_2(\hat{1},t) - m_1(\hat{1},t)]/m_1(\hat{1},t)^2 \\ \leq\ & 2\int_0^{1/2} \frac{1+e^{-2(t-s)}}{1-e^{-2(t-s)}} n e^{nG(s,t)}\, ds && (19) \\ +\ & 2\int_{1/2}^{t-1/n} \frac{1+e^{-2(t-s)}}{1-e^{-2(t-s)}} n e^{nG(s,t)}\, ds && (20) \\ +\ & (1+o(1))2\int_{t-1/n}^{t} \frac{1+e^{-2(t-s)}}{1-e^{-2(t-s)}} nK(t-s) e^{nG(s,t)}\, ds. && (21) \end{aligned}$$

For the first piece we calculate the value of $G(1/2,t)$, getting a constant less than $-1/4$. Thus by convexity, $G(s,t) \leq -s/2$ for $0 \leq s \leq 1/2$. Now for $0 \leq s \leq 1/2$ and any ϵ, the factor $\frac{1+e^{-2(t-s)}}{1-e^{-2(t-s)}}$ is at most 3, so the contribution from (19) is at most $2\int_0^{1/2} 3ne^{-ns/2}\, ds = 12(1-e^{-n/4}) < 12$.

For the second piece, bound $G(s,t)$ on $1/2 \leq s \leq t$ by its values at the endpoints; for small ϵ the greater value is the value at the right endpoint, namely, $-V(\epsilon)$. The value of $\frac{1+e^{-2(t-s)}}{1-e^{-2(t-s)}}$ on $1/2 \leq s \leq t - 1/n$ is bounded by its maximum, which is achieved at $s = t - 1/n$ and has a value of at most $n+1$. Thus the contribution from (20) is at most $2\int_{1/2}^{t-1/n} (n+1)ne^{-nV(\epsilon)}\, ds < 2t(n+1)ne^{-nV(\epsilon)}$, and this tends to 0 as $n \to \infty$.

To bound the third piece, expand the integrand in powers of $(t-s)$ to compute the integral as $(1+o(1))\int_{t-1/n}^{t}(t-s)^{-1}nK(t-s)e^{nG(s,t)}\,ds \leq (1+o(1))Ke^{-nV(\epsilon)}$, which tends to 0 as $n\to\infty$. Thus the entire integral is bounded by $12(1+o(1))$, and $\liminf_n \mathbf{P}(\tau_n \leq \ln(1+\sqrt{2})+\epsilon) \geq 1/12$.

Finally, to show that $\mathbf{P}(\tau_n \leq \ln(1+\sqrt{2})+3\epsilon) \to 1$, let A be the set of particles at distance 1 from $\hat{0}$ at time ϵ. Then $|A| > n^{1/2}$ with probability approaching 1 as $n\to\infty$. The offspring of elements of A now act independently from time ϵ to time $\ln(1+\sqrt{2})+2\epsilon$, each particle at some y with $d(\hat{0},y)=1$ having probability at least $(1+o(1))\frac{1}{12}$ of having a descendant at the antipodal point to y at time $\ln(1+\sqrt{2})+2\epsilon$, according to the calculation just completed. Letting B be the set of particles at sites that neighbor $\hat{1}$ at time $\ln(1+\sqrt{2})+2\epsilon$, $\mathbf{P}(|B| > n^{1/4})$ is at least $\mathbf{P}(|A| > n^{1/2})\mathbf{P}(X > n^{1/4})$, where X is a binomial with parameters $\lceil n^{1/2}\rceil$ and $(1+o(1))\frac{1}{12}$. This probability also tends to 1 as $n\to\infty$. Finally, $\mathbf{P}(\tau_n \leq \ln(1+\sqrt{2})+3\epsilon \mid |B| > n^{1/4}) \geq 1-e^{-\epsilon n^{1/4}}$, which tends to 1 as $n\to\infty$, proving the theorem. □

6 Covering times in Richardson's model

This section answers affirmatively the question of whether the time until the entire n-cube $\mathcal{B}_n$ is infected is bounded in probability as $n\to\infty$. The constant upper bound given here is $4\ln(4+2\sqrt{3})+6 \doteq 14.04$. This is by no means sharp, but on the other hand we also produce a lower bound on the cover time of $\frac{1}{2}\ln(2+\sqrt{5})+\ln 2 \doteq 1.41$. Since the infection time of $\hat{1}$ is bounded between $\ln(1+\sqrt{2}) \doteq 0.88$ and 1 in probability, this means that the lim inf in probability of the cover time is strictly greater than the lim sup in probability of the time to reach the farthest vertex. Perhaps the cover time has a limit in probability, but we do not venture a guess as to what the limit should be.

6.1 The upper bound

The following statement of duality in Richardson's model will be helpful. The proof can be found in any introduction to the contact process, such as Durrett (1988). The intuition is to think of $A_2(t-s)$ as the set of vertices that would be able to infect $\hat{1}$ by time t if they were infected at time s.

Lemma 6.1 *Let $(A_1(t))$ be Richardson's model on $\mathcal{B}_n$ as defined above and let $(A_2(t))$ be an independent copy with the difference that $A_2(0)$ is set to be $\{\hat{1}\}$ instead of $\{\hat{0}\}$. Then $\mathbf{P}(\hat{1} \in A_1(t)) = \mathbf{P}(A_1(s) \cap A_2(t-s) \neq \emptyset)$ for any $0 \leq s \leq t$.* $\square$

Theorem 6.2 *Let $(A(t))$ be Richardson's model on $\mathcal{B}_n$ starting with only $\hat{0}$ infected at time 0. Let $c := 2\ln(4+2\sqrt{3}) + 3 \doteq 7.02$. Then for any $\epsilon > 0$ there exists $N = N(\epsilon)$ and $h(\epsilon) > 0$ such that for $n \geq N$, $\mathbf{P}(\hat{1} \in A(c+\epsilon)) \geq 1 - (4+h(\epsilon))^{-n}$.*

We defer the proof of Theorem 6.2 in order to present the resulting cover time upper bound.

Corollary 6.3 *For any $\epsilon > 0$, $\mathbf{P}(A(2c+\epsilon) = \mathcal{B}_n) \to 1$ as $n \to \infty$.*

Proof of Corollary 6.3: Let $N \equiv N(\epsilon/2)$ be chosen as in the statement of the previous theorem and pick any $n \geq 2N$. Let x be any element of $\mathcal{B}_n$. First suppose that the distance from $\hat{0}$ to x is $d \geq n/2$. Then the sublattice with top element x and bottom element $\hat{0}$ is a Boolean algebra of rank at least N, and the induced process on the sublattice (i.e., the process with no infections allowed except on the sublattice) is still Richardson's model. By the previous theorem, x is infected by time $c + \epsilon/2$ with probability at least $1 - (4+h(\epsilon/2))^{-d} \geq 1 - (4+h(\epsilon/2))^{-n/2}$.

On the other hand, suppose x is at distance less than $n/2$ to $\hat{0}$. Then after time $c+\epsilon/2$ the top element $\hat{1}$ is infected with probability at least $1-(4+h(\epsilon/2))^{-n}$ and, conditioned on that, the probability that x is infected another $c+\epsilon/2$ time units later is at least $1-(4+h(\epsilon/2))^{-n/2}$ by the previous argument (since the distance from x to $\hat{1}$ is at least $n/2$). Thus each x fails to be infected at time $2c+\epsilon$ with probability at most $(4+h(\epsilon/2))^{-n}+(4+h(\epsilon/2))^{-n/2}$, and summing over all x gives at most $(1+h(\epsilon/2)/4)^{-n/2}+2^{-n}$, which tends to 0 as $n\to\infty$. □

Notice that the reason we get $2c$ instead of c as an upper bound in probability for the cover time is that the proof of Corollary 6.3 gives better upper bounds on the probability that a vertex is uninfected the further it is from $\hat{1}$. It is unlikely that the bounds reflect the true state of affairs. In particular, we suspect that the random time until a vertex x is infected is stochastically increasing in $|x|$. This would immediately imply an upper bound in probability of $c \doteq 7.02$ for the covering time. (In fact, the bound could then be lowered to $\tilde{c} := 2\ln(2+\sqrt{2})+3 \doteq 5.46$ by establishing the modification $\mathbf{P}(\hat{1}\in A(\tilde{c}+\epsilon)) \geq 1-(2+\tilde{h}(\epsilon))^{-n}$ of Theorem 6.2.) More generally, we have the following conjecture.

conjecture 1 *Let G be a graph with distinguished vertex x. Let H be the graph $G\times\{0,1\}$, with edges between (x,i) and (y,i) for neighbors x,y of G and $i=0,1$ and edges between $(x,0)$ and $(x,1)$ for all $x\in G$. For $z\in H$, let $T(z)$ be the time that z is first infected in Richardson's model on H beginning with a single infection at $(x,0)$. Then, for any $y\in G$, $T(y,0)$ is stochastically smaller than $T(y,1)$.*

We now turn to the proof of Theorem 6.2.

Proof of Theorem 6.2: Throughout the proof we use the notation x' for the complement of a vertex x (viewing x as a subset of $\{1,\ldots,n\}$) and S' for $\{x'\in\mathcal{B}_n : x\in S\}$ when $S\subseteq\mathcal{B}_n$. Let $t_0=0$, $t_1=\ln(4+2\sqrt{3})+\epsilon/7$, $t_2=\ln(4+2\sqrt{3})+2\epsilon/7$, $t_3=\ln(4+2\sqrt{3})+1+3\epsilon/7$, and $t_4=\ln(4+2\sqrt{3})+2+4\epsilon/7$. By Lemma 6.1 it suffices to find h for which $\mathbf{P}(A_1(t_4) \cap A_2(t_3) \neq \emptyset) \geq 1 - (4 + h(\epsilon))^{-n}$ for large n. The method will be to watch the evolutions of A_1 and A_2 and look for vertices x for which simultaneously $x \in A_1(t)$ and $x' \in A_2(t)$. In particular, we will show that for $i = 1, 2, 3$, certain subsets D_i of $\{x : x \in A_1(t_i) \text{ and } x' \in A_2(t_i)\}$ (actually, of a slight modification of this for D_3) are sufficiently large, and then we will argue that each x in D_3 has an independent chance of becoming an element of $A_1(t_4) \cap A_2(t_3)$.

For $1 \leq i, j \leq n$ let $\eta_{ij} = \{i, j\} \in \mathcal{B}_n$. Let $n_1 = \lfloor n - 2\log_2 n \rfloor$ and let $S_2 \subseteq \mathcal{B}_n$ be the set $\{\eta_{ij} : i < j \leq n_1\}$. Let $S_1 = \{\{i\} : i \leq n_1\}$ be the set of elements at level 1 of $\mathcal{B}_n$ beneath S_2. For each $x \in S_2$ let $T_x \subseteq \mathcal{B}_n$ be the set $\{\, y : x \cap \{1, \ldots, n_1\} = y \cap \{1, \ldots, n_1\} \,\}$. Note that T_x and T_y are disjoint for distinct $x, y \in S_2$; similarly for T'_x and T'_y.

Several of the arguments below will involve the monotonicity of Richardson's model: forbidding some edges to pass the infection at various times decreases $A(t)$ and hence can only increase the infection time to any vertex. It can therefore only increase all infection times to suppose for A_1 that from time t_0 to time t_1 infections may occur only in rank 1 of $\mathcal{B}_n$, from time t_1 to time t_2 infections may occur only in rank 2 (and in fact later the set of allowed infections will be further restricted), and from time t_2 to time t_3 infections may occur only between x and y when the symmetric difference $x \triangle y$ is the singleton $\{i\}$ for some $i > n_1$ (in other words, infections from t_2 to t_3 may occur only when x and y are neighbors both in T_z for some z). Also suppose dually for A_2 that infections between times t_0 and t_1 occur only in level $n-1$, that infections between times t_1 and t_2 occur only in level $n-2$, and that infections between times t_2 and t_3 occur only between elements of the same T'_z. Finally, suppose that between times t_3 and t_4, infections in A_1 spread only between vertices x and y for which $x \cap \{n_1+1, \ldots, n\} = y \cap \{n_1+1, \ldots, n\}$.

Let $U_1 = S_1 \cap A_1(t_1)$, let $V_1 = S'_1 \cap A_2(t_1)$, and let $D_1 = U_1 \cap V'_1$. The first claim is that there exist $\delta_1(\epsilon)$ and $h_1(\epsilon)$, both positive, for which $\mathbf{P}(|D_1| \leq \delta_1(\epsilon)n) \leq (4 + h_1(\epsilon))^{-n}$ for sufficiently large n. This is just a large deviation calculation. $|D_1|$ is the sum of n_1 i.i.d. Bernoulli random variables, each equalling 1 with probability $p = (1 - e^{-t_1})^2$ and 0 with probability $q = 2e^{-t_1} - e^{-2t_1}$. Let $0 < a < p$ and write $b = 1 - a$. By choosing the optimal value $\ln(\frac{bp}{aq})$ for $\theta > 0$, we find from the moment generating function inequality

$$\mathbf{P}(|D_1| \le an_1) = \mathbf{P}(e^{-\theta|D_1|} \ge e^{-\theta a n_1}) \le e^{\theta a n_1}\mathbf{E}e^{-\theta|D_1|}$$

that $\mathbf{P}(|D_1| \le an_1) \le [(p/a)^a(q/b)^b]^{n_1}$. As $a \downarrow 0$, $(p/a)^a(q/b)^b \downarrow q = 2e^{-t_1} - e^{-2t_1} < 2(4+2\sqrt{3})^{-1} - (4+2\sqrt{3})^{-2} = (2-\sqrt{3}) - (7/4-\sqrt{3}) = 1/4$. Thus there exists $\delta_0(\epsilon) > 0$ and $h_0(\epsilon) > 0$ such that $\mathbf{P}(|D_1| \le \delta_0(\epsilon)n_1) \le (4+h_0(\epsilon))^{-n_1}$. Since $n_1/n \to 1$, this implies the existence of $h_1(\epsilon)$ such that for any fixed $\delta_1 < \delta_0(\epsilon)$, $\mathbf{P}(|D_1| \le \delta_1 n) \le (4 + h_1(\epsilon))^{-n}$ for sufficiently large n.

Now let $U_2 = S_2 \cap A_1(t_2)$, let $V_2 = S_2' \cap A_2(t_2)$, and let $D_2 = U_2 \cap V_2'$. The next claim is that there exist $\delta_2(\epsilon)$ and $h_2(\epsilon)$, both positive, for which $\mathbf{P}(|D_2| \le \delta_2 n^2) \le (4 + h_2(\epsilon))^{-n}$ for sufficiently large n. Between times t_1 and t_2 the only infections allowed in A_1 involve vertices in S_1 infecting vertices in S_2. For convenience, restrict further the allowed infections by requiring that $\{i\}$ may infect $\{i,j\}$ only if $i < j$ and $j - i < n_1/2$ or $i > j$ and $i - j > n_1/2$. Note that each $\{i,j\}$ in S_2 can be infected by i or j but not both. The exception is $|j-i| = n_1/2$; in that case $\{i,j\}$ cannot be infected at all. Then for each $x \in D_1$, there is a set $S(x)$ of $\lceil n_1/2 \rceil - 1$ vertices in S_2 that x can infect between times t_1 and t_2, and these sets are disjoint as x varies over S_1. For each $x \in D_1$, the number of vertices infected by x by time t_2 whose complements have been infected by x' by time t_2 in the process A_2 (with the dual restrictions) is a binomial random variable with parameters $\lceil n_1/2 \rceil - 1$ and $(1-e^{-\epsilon/7})^2$. Now the probability that this binomial is less than half its mean is exponentially small in n, say $\le e^{-\alpha n}$, so conditioning on $|D_1| > \delta_1 n$, the probability that no more than $\lceil \delta_1 n \rceil /2$ of these i.i.d. binomials are greater than half their means is at most $2^{\lceil \delta_1 n \rceil} e^{-\alpha \delta_1 n^2/2}$. This is smaller than $(4+h_1(\epsilon))^{-n}$ for sufficiently large n, and thus we have shown that $\mathbf{P}(|D_2| \le (1-e^{-\epsilon/7})^2 \lceil \delta_1 n \rceil (\lceil n_1/2 \rceil - 1)/4) < 2(4+h_1(\epsilon))^{-n}$ for sufficiently large n. Choosing $\delta_2 < (1-e^{-\epsilon/7})^2 \delta_1/8$ and $h_2 < h_1$ proves the second claim.

Condition until the last sentence of this paragraph on D_2. Between times t_2 and t_3 the spread of infection in A_1 is confined to each T_z, so the spread of infection is independent on each T_z. The same goes for the propagation of A_2 on each T'_z. On each T_z, the process is just a Richardson's model on a cube of dimension $n-n_1$; hence uniformly for $x \in D_2$ and $y \in T_x$ with $\frac{1}{4}(n-n_1) \leq d(x,y) \leq \frac{3}{4}(n-n_1)$, the probability that y is infected by time $t_3 = t_2+(1+\epsilon/7)$ tends to 1 for large n by Theorem 5.2. Similarly, the probability that $\hat{y}$ is dual infected by time t_3 tends uniformly to 1, where $\hat{y}$ is the element of T'_x that agrees with y in the last $n-n_1$ places. (Note $d(x', \hat{y}) \geq \frac{1}{4}(n-n_1)$.) Thus the probability that y is infected by t_3 and $\hat{y}$ is dual infected by t_3 also tends uniformly to 1. Thus uniformly for $x \in D_2$, the expected cardinality of $V_x := \{y \in T_x : y \text{ is } A_1\text{-infected by } t_3 \text{ and } \hat{y} \text{ is } A_2\text{-infected by } t_3\}$ is at least $(1-o(1))|T_x| \geq (1-o(1))n^2$. Then, since we always have $|V_x| \leq |T_x|$, $\mathbf{P}(|V_x| \leq n^2/2) \leq 1/2$ for large n, so $\mathbf{P}(|V_x| \leq n^2/2 \text{ for all } x \in D_2) \leq (1/2)^{|D_2|}$. Combining this with the previous claim about the distribution of $|D_2|$ shows that the (now unconditional) probability of the event F that there is some $x \in D_2$ for which $|V_x| \geq n^2/2$ is at least $1-(4+h_2(\epsilon))^{-n} - (1/2)^{\delta_2 n^2} \geq 1-(4+h_3(\epsilon))^{-n}$ for $h_3 < h_2$ and sufficiently large n.

Finally, condition on F. Let x be an element of D_2 with $|V_x| \geq n^2/2$ and let $D_3 = V_x$. It suffices to show that with high probability there is some $y \in D_3$ such that $\hat{y}$ is infected at time t_4, since we already know that $\hat{y}$ is dual infected at time t_3. Recall that the only way that infection spreads between times t_3 and t_4 is between neighboring vertices that intersect $\{n_1+1, \ldots, n\}$ equally. This effectively breaks $\mathcal{B}_n$ into 2^{n-n_1} fibers on which propagation of the infection is independent and behaves like a Richardson's model of dimension n_1. Each element of V_x is in a different fiber, and uniformly for $y \in V_x$ the probability that infection passes from y to the opposite corner $\hat{y}$ of the fiber in the time $t_4 - t_3 = 1+\epsilon/7$ tends to 1 for large n by Theorem 5.2 and in particular is eventually greater that 1/2. Thus the conditional probability given F that some $\hat{y}$ is infected at time t_4 is at least $1-(1/2)^{n^2/2}$ for large enough n.

Combining all the conditional probabilities and using Lemma 6.1 yields a probability of at least $1-(4+h_3(\epsilon))^{-n}-(1/2)^{n^2/2}$ that $\hat{1}$ is infected by time $t_3+t_4 = 2\ln(4+2\sqrt{3})+3+\epsilon$. Pick $h(\epsilon) < h_3(\epsilon)$. Then when n is large enough, our bound is at least $1-(4+h(\epsilon))^{-n}$ and the proof is finsihed. □

6.2 The lower bound

Theorem 6.4 *For any $\epsilon > 0$, $\mathbf{P}(A(\frac{1}{2}\ln(2+\sqrt{5})+\ln 2-\epsilon) = \mathcal{B}_n) \to 0$ as $n \to \infty$.*

Proof: First stochastically dominate Richardson's model at time $t = \frac{1}{2}\ln(2+\sqrt{5})-\epsilon/2$ by the corresponding value of a branching translation process, as in Section 5. Let x be any vertex with $d(\hat{0},x) \geq n/2$. Then in the notation of the proof of Theorem 5.3, $\mathbf{P}(x \in A(t)) \leq e^{nt}p(x,t) \leq e^{nt}((1-e^{-2t})/2)^{n/2}((1+e^{-2t})/2)^{n/2} = e^{nt}((1-e^{-4t})/4)^{n/2} = [(e^{2t}-e^{-2t})/4]^n$. Now since $e^{2t} < \sqrt{5}+2$, it follows that $e^{2t}-e^{-2t} < \sqrt{5}+2-(\sqrt{5}-2) = 4$, so $\mathbf{P}(x \in A(t)) \to 0$ uniformly in such x as $n \to \infty$. Since at least half the vertices of $\mathcal{B}_n$ satisfy $d(\hat{0},x) \geq n/2$, it follows that $\mathbf{P}(|A(t)^c| \geq 2^{n-2}) \to 1$.

Now condition on $|A(t)^c| \geq 2^{n-2}$. The process $\{A(s) : s \geq t\}$ is stochastically dominated by the process $\{Z(u) : u \geq t\}$ for which $Z(t) = A(t)$ and transitions from S to $S \cup \{y\}$ occur at rate n for all S and $y \notin S$. Now for each $x \in A(t)^c$, $\mathbf{P}(x \in Z(t+s)) = 1-e^{-ns}$, and furthermore these events are independent as x varies. Thus $\mathbf{P}(Z(t+s) = \mathcal{B}_n \mid |A(t)^c| \geq 2^{n-2}) \leq (1-e^{-ns})^{2^{n-2}}$. Plugging in $s = \ln 2-\epsilon/2 < \ln(2-\epsilon/2)$ (for $\epsilon < 2$) and using $1-\epsilon < e^{-\epsilon}$ gives

$$\mathbf{P}(Z(t+s) = \mathcal{B}_n \mid |A(t)^c| \geq 2^{n-2}) \leq [e^{-(2-\epsilon/2)^{-n}}]^{2^{n-2}} = e^{-(\frac{2}{2-\epsilon/2})^n/4},$$

which tends to 0 as $n \to \infty$. The theorem now follows readily. □

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