

The probability that Brownian motion almost contains a line

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ABSTRACT:

Let R_ϵ be the Wiener sausage of radius ϵ about a planar Brownian motion run to time 1. Let P_ϵ be the probability that R_ϵ contains the line segment $[0, 1]$ on the x -axis. Upper and lower estimates are given for P_ϵ . The upper estimate implies that the range of a planar Brownian motion contains no line segment.

1 Introduction

Let $\{B_t : t \geq 0\} = \{(X_t, Y_t)\}$ be a 2-dimensional Brownian motion, started from the origin unless otherwise stated. Let $R = B[0, 1]$ be the range of this trajectory up to time 1. Brownian motion is one of the most fundamental stochastic processes, applicable to a wide variety of physical models, and consequently its properties have been widely studied. Some people prefer to study simple random walk, but questions in the discrete case are nearly always convertible to the continuous case: rescale the simple random walk $\{X_n : n \in \mathbf{Z}\}$ in space and time to give a trajectory $\{N^{-1/2}X_n : n \in N^{-1}\mathbf{Z}\}$ and replace the question of whether this trajectory covers or avoids a lattice set $A \subseteq (N^{-1}\mathbf{Z})^2$ by the question of whether the Wiener sausage $\{B_t + x : 0 \leq t \leq 1, |x| \leq N^{-1/2}\}$ covers or avoids the set $A + [-N^{-1/2}, N^{-1/2}]^2$.

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Focusing on the random set R is a way of concentrating on the geometric properties of Brownian motion and ignoring those having to do with time. The sorts of things we know about R are as follows. Its Hausdorff dimension is 2, and we know the exact gauge function for which the Hausdorff measure is almost surely positive and finite [12]. Studies from 1940 to the present of intersections of independent Brownian paths (or equivalently multiple points of a single path) show that two paths are always disjoint in dimensions 4 and higher, two paths intersect in three dimensions but three paths do not, while in 2 dimensions any finite number of independent copies of R have a common intersection with positive probability [10, 5, 9].

Questions about which sets R intersects are reasonably well understood via potential theory. For example, Kakutani showed in 1944 that in any dimension

$$\mathbf{P}[R \cap S \neq \emptyset] > 0 \Leftrightarrow \text{cap}_G(S) > 0,$$

where cap_G is the capacity with respect to the Green's kernel [7]. There are versions of this result for intersections with several independent copies of R [6, 14] and quantitative estimates exist (up to computable constant factors) for $\mathbf{P}[R \cap S \neq \emptyset]$ [1]. Not so well understood are the dual questions to these, namely covering probabilities. The most interesting questions seem to arise in dimension 2. In the discrete setting, one may ask for the radius L of the largest disk about the origin covered by a simple random walk up to time N . The answer is roughly $\exp(c\sqrt{\log N})$, in the sense that $\mathbf{P}[\ln L \geq t\sqrt{\log N}]$ is bounded away from 0 and 1 for fixed t as $N \rightarrow \infty$; see [8, 13, 11]. Another type of covering question is whether R contains any set from a certain class. For example, it is known that R contains a self-avoiding path of Hausdorff dimension 2 [2] and a self-avoiding path of dimension less than $3/2$ [3], but it is not known whether the infimum of dimensions of self-avoiding paths contained in R is 1. A modest start on an answer would be to show something that seems obvious, namely that R almost surely contains no line segment.

Despite the seeming self-evidence of this assertion, I know of no better proof than via the following Wiener sausage estimates. Let

$$R_\epsilon = \{x \in \mathbf{R}^2 : |y - x| \leq \epsilon \text{ for some } y \in R\}$$

be the Wiener sausage of radius ϵ . Let

$$\mathbf{P}_\epsilon = \mathbf{P}[R_\epsilon \supseteq [0, 1] \times \{0\}]$$

be the probability that the Wiener sausage contains the unit interval on the x -axis. An easy upper bound for \mathbf{P}_ϵ is given by $\mathbf{P}[(1/2, 0) \in R_\epsilon]$ which is order $1/|\log \epsilon|$. Omer Adelman (personal communication) has obtained an upper bound that is $o(\epsilon^2)$, though this was never written down; see Remark 2 following Theorem 1.2. An obvious lower bound of order $\exp(-c/\epsilon)$ is gotten from large deviation theory by forcing the Brownian motion to hit each of ϵ^{-1} small disks in order. The main result of this paper is to improve the upper and lower bounds on \mathbf{P}_ϵ so as to be substantially closer to each other.

Theorem 1.1 *There exist positive constants c_1, c_2, c_3 and c_4 such that for all sufficiently small ϵ the following four inequalities hold:*

$$\mathbf{P}_\epsilon \leq c_1 \exp(-|\log \epsilon|^2/c_2 \log^2 |\log \epsilon|) \tag{1.1}$$

and

$$\exp(-c_4 |\log \epsilon|^4) \leq \mathbf{P}_\epsilon; \tag{1.2}$$

if Ξ is the Lebesgue measure of the intersection of R_ϵ with any fixed line segment of length at most 1 contained in $[0, 1]^2$, then for any $\theta \in (0, 1)$,

$$\mathbf{P}[\Xi \geq \theta] \leq \exp(-|\log \epsilon|^2 \theta^2/c_3 \log^2 |\log \epsilon|) \tag{1.3}$$

and

$$\exp(-c_4 |\log \frac{1-\theta}{3}|^2 |\log \epsilon|^2) \leq \mathbf{P}[\Xi \geq \theta]. \tag{1.4}$$

This implies

Theorem 1.2 *A 2-dimensional Brownian motion, run for infinite time, almost surely contains no line segment. In fact it intersects no line in a set of positive Lebesgue measure.*

Remarks:

1. A discrete version of these results holds, namely that simple random walk run for time N^2 covers the set $\{(1, 0), \dots, (N, 0)\}$ with probability between $\exp(-C_4(\log N)^4)$ and $C_1 \exp(-\log^2 N/C_2 \log^2(\log N))$; similarly the probability of covering at least θN of these N points is between

$$\exp\left(-C_4 \left|\log \frac{1-\theta}{3}\right|^2 \log^2 N\right)$$

and

$$\exp(-\log^2 N \theta^2 / C_3 \log^2 \log N).$$

The proof is entirely analogous (though the details are a little trickier) and is available from the author.

2. The proof of Theorem 1.2 requires only the upper bound $\mathbf{P}_\epsilon = o(\epsilon^2)$, which is weaker than what is provided by Theorem 1.1. Thus Adelman's unpublished bound, if correct, would also imply Theorem 1.2. However, since both Adelman's proof and the proof of Theorem 1.1 are non-elementary (Adelman discretizes as in Remark 1 and then sums over all possible orders in which to visit the N points), the question of Y. Peres which motivated the present article is still open: can you find an elementary argument to show $\mathbf{P}_\epsilon = o(\epsilon^2)$?

This section concludes with a discussion of why Theorem 1.2 follows from the upper estimate on \mathbf{P}_ϵ . The next section begins the proof of the upper bounds (1.1) and (1.3), first replacing time 1 by a hitting time and then proving some technical but routine lemmas on the quadratic variation of a mixture of stopped martingales. Section 3 finishes the proof of the upper bounds and Section 4 proves the lower bounds (1.2) and (1.4).

PROOF OF THEOREM 2 FROM THEOREM 1: It suffices by countable additivity to show that $R \cap [0, 1]^2$ almost surely intersects no line in positive Lebesgue measure. Let $m(\cdot)$ denote 1-dimensional Lebesgue measure; then it suffices to see for each $\theta > 0$ that $\mathbf{P}[H_\theta] = 0$, where H_θ is the event that for some line l , $m(R \cap [0, 1]^2 \cap l) \geq \theta$. From now on, fix $\theta > 0$.

Let \mathcal{L}_ϵ be the set of line segments whose endpoints are on the boundary of $[0, 1]^2$ and

have coordinates that are integer multiples of ϵ . The set \mathcal{L}'_ϵ of halves of line segments in \mathcal{L}_ϵ has twice the cardinality and contains segments of maximum length $\sqrt{2}/2$. For any line l intersecting the interior of $[0, 1]^2$ and any ϵ there is a line segment $l_\epsilon \in \mathcal{L}_\epsilon$ whose endpoints are within ϵ of the respective two points where l intersects $\partial[0, 1]^2$. If $m(R \cap [0, 1]^2 \cap l) \geq \theta$ then $m(R_\epsilon \cap l_\epsilon)$ will be at least $\theta - 2\epsilon$. Thus on the event H_θ , every $\epsilon > 0$ satisfies

$$m(R_\epsilon \cap l) \geq \theta - 2\epsilon \text{ for some } l \in \mathcal{L}_\epsilon,$$

and hence

$$m(R_\epsilon \cap l) \geq \theta/2 - \epsilon \text{ for some } l \in \mathcal{L}'_\epsilon.$$

By (1.3), when $\epsilon < \theta/3$, the probability of this event is at most

$$2|\mathcal{L}_\epsilon| \exp(-|\log \epsilon|^2 (\theta/6)^2 / c_3 \log^2 |\log \epsilon|).$$

Since $|\mathcal{L}_\epsilon|$ is of order ϵ^{-2} , this goes to zero as $\epsilon \rightarrow 0$ which implies $\mathbf{P}[H_\theta] = 0$, proving the theorem. \square

2 Change of stopping time and martingale lemmas

The proof of the upper bound (1.3), which implies the other upper bound (1.1), comes in three pieces. It is convenient to stop the process at the first time the y -coordinate of the Brownian motion leaves the interval $(-1, 1)$. The first step is to justify such a reduction. A reduction is made at the same time to the case where l is the line segment $[0, 1] \times \{0\}$. The proof then proceeds by representing the measure $Z := m(R_\epsilon \cap l)$ as the final term in a martingale $\{Z_t : 0 \leq t \leq 1\}$, where Z_t is simply $\mathbf{E}(Z | \mathcal{F}_t)$ and \mathcal{F}_t is the natural filtration of the Brownian motion. The second step is to identify the quadratic variation of this martingale. This involves a few technical but essentially routine lemmas. The final step, carried out in Section 3, is to obtain upper bounds for the quadratic variation which lead directly to tail bounds on $Z - \mathbf{E}Z$.

Let l_0 be the line segment $[0, 1] \times \{0\}$; here is why it suffices to prove (1.3) for the line l_0 . Let l be any line of length at most 1 contained in $[0, 1]^2$. Let σ be the first time l is hit.

Let l_1 and l_2 be the two closed line segments whose union is l and whose intersection is B_σ . If the measure of $R_\epsilon \cap l$ is at least θ then the measure of $R_\epsilon \cap l_j$ is at least $\theta/2$ for some j , and the probability of this is at most twice the probability of $R_\epsilon \cap l_0$ having measure at least $\theta/2$. Thus if c_3 works in (1.3) for l_0 , then $5c_3$ works for arbitrary l .

The more serious reduction is to define a stopping time

$$\tau = \inf\{t : |Y_t| \geq 1\}$$

and to replace the event $\{\Xi \geq \theta\}$ with the event $\{\Xi \geq \theta\} \cap \{\tau \geq 1\}$. The value of this replacement will be seen in the next section, where some estimates depend on stopping at a hitting time. To carry out this reduction, let G_1 be the event $\{\tau \geq 1\}$, let $K = \mathbf{P}[G_1]^{-1}$, and let $G_2^{(\theta)}$ be the event $\{\Xi \geq \theta\}$. The requisite lemma is:

Lemma 2.1

$$\mathbf{P}[G_2^{(\theta)}] \leq K\mathbf{P}[G_1 \cap G_2^{(\theta)}].$$

PROOF: I must show that

$$\mathbf{P}[G_2^{(\theta)} | G_1] \geq \mathbf{P}[G_2^{(\theta)}]. \quad (2.5)$$

The law of Brownian motion up to time 1, conditioned on the event G_1 is a time inhomogenous Markov process (an h -process) and is representable as a Brownian motion with drift. By translational symmetry in the x -direction, the drift, $(h_1(x, y, t), h_2(x, y, t))$ must be toward the x -axis, i.e., $h_1 = 0$ and $yh_2 \leq 0$. Given an unconditioned Brownian motion, $\{(X_t, Y_t)\}$ with $Y_0 \in [0, 1]$, we may in fact construct this h -process $\{W_t = (X_t, V_t)\}$ by defining $V_0 = Y_0$ and define $\{V_t\}$ by

$$dV_t = \text{sgn}(V_t)dY_t + h(Y_t, t)dt.$$

Then (X, V) is distributed as a Brownian motion conditioned to stay inside the strip until time 1 and is coupled so that $|V_t| \leq |Y_t|$ for all $t \leq \tau \wedge 1$. The coupling is constructed such that for all $t \leq 1$ and $\epsilon > 0$,

$$\{x : |B_s - (x, 0)| < \epsilon \text{ for some } s \leq t\} \subseteq \{x : |W_s - (x, 0)| < \epsilon \text{ for some } s \leq t\}.$$

Equation (2.5) follows directly. \square

Let, with the natural identification of l_0 and $[0, 1]$,

$$A_t = B[0, t \wedge \tau]_\epsilon \cap l_0 = \{x \in [0, 1] : |B_s - (x, 0)| \leq \epsilon \text{ for some } s \leq t \wedge \tau\}.$$

For achieving stochastic upper bounds on $m(A_1)$, one may take advantage of the trivial inequality $m(A_1) \leq m(A_\infty)$. Thus let $M_t = \mathbf{E}(m(A_\infty) | \mathcal{F}_t)$. On the event $G_1 \cap G_2^{(\theta)}$, clearly $M_1 \geq \theta$. By the previous lemma, the inequality (1.3) will follow from the inequality

$$\mathbf{P}[M_1 \geq \theta] \leq \exp(-|\log \epsilon|^2 \theta^2 / c'_3 \log^2 |\log \epsilon|) \quad (2.6)$$

with $c_3 > c'_3$ and ϵ small enough to make up for the factor of K .

Now comes the second step, namely identifying the quadratic variation of the martingale M . This step is just the rigorizing (Lemma 2.2) of the following intuitive description of M .

If $I(x, t, \epsilon)$ is the indicator function of the event that the Brownian motion, killed at time τ , has visited the ϵ -ball around x by time t , then by the Markov property, if τ has not yet been reached,

$$\begin{aligned} M_t &= m(A_t) + \int_{[0,1] \setminus A_t} \mathbf{P}_{B_t}[x \in A_\infty] dx \\ &= m(A_t) + \int_{[0,1]} [1 - I(x, t, \epsilon)] \mathbf{E}_{B_t} I(x, \infty, \epsilon) dx. \end{aligned}$$

Thus the quadratic variation $d \langle M \rangle_t$ is given by

$$dt \left| \nabla \int_{x \in [0,1] \setminus A_t} g_\epsilon(B_t, x) dx \right|^2,$$

where

$$g_\epsilon(y, x) = \mathbf{E}_y[I(x, \infty, \epsilon)]$$

and ∇ denotes the gradient with respect to the y .

The formal statement of this is:

Lemma 2.2 *Let D be the closure of a connected open set $D^\circ \subset \mathbf{R}^d$ and let $(\mathcal{S}, \mathcal{B}, \mu)$ be a probability space. Suppose that ∂D is nonpolar for Brownian motion started inside D and that $f : D \times \mathcal{S} \rightarrow [0, 1]$ has the following properties:*

- (i) $f(x, \alpha)$ is jointly measurable, continuous in x for each α , and $f(\cdot, \alpha)$ is identically zero on ∂D ;
- (ii) There is a constant C such that $f(\cdot, \alpha)$ is Lipschitz on D with constant C for all α ;
- (iii) For each α , let $D_\alpha = \{x \in D : f(x, \alpha) \in (0, 1)\}$; then for each α $f(\cdot, \alpha)$ is C^2 on D_α and its Laplacian vanishes there.

Let $\{B_t\}$ be a Brownian motion started from some $B_0 \in D^\circ$. For each α , let $\tau_\alpha = \inf\{t : f(B_t, \alpha) \in \{0, 1\}\}$ be the first time B_t exits D_α (in particular, τ_α is at most $\tau_{\partial D}$, the exit time of D°). Then

$$M_t := \int f(B_{t \wedge \tau_\alpha}, \alpha) d\mu(\alpha)$$

is a continuous $\{\mathcal{F}_t\}$ -martingale with quadratic variation

$$\langle M \rangle_t = \int_0^t \left| \int_{\mathcal{S}} \mathbf{1}_{s < \tau_\alpha} \nabla f(B_s, \alpha) d\mu(\alpha) \right|^2 ds.$$

The proof of the lemma requires two elementary propositions, whose proofs are routine and relegated to the appendix.

Proposition 2.3 *With the notation and hypotheses of Lemma 2.2, for each $\alpha \in \mathcal{S}$ let $M_t^{(\alpha)} = f(B_{t \wedge \tau_\alpha}, \alpha)$. Then $M^{(\alpha)}$ is a continuous $\{\mathcal{F}_t\}$ -martingale for each α and the brackets satisfy*

$$[M^{(\alpha)}, M^{(\beta)}]_t = \int_0^t \mathbf{1}_{s < \tau_\alpha} \mathbf{1}_{s < \tau_\beta} \nabla f(B_s, \alpha) \cdot \nabla f(B_s, \beta) ds.$$

□

Proposition 2.4 *Let $\{M_t^{(\alpha)}\}$ be continuous $\{\mathcal{F}_t\}$ -martingales as α ranges over the non-atomic measure space $(\mathcal{S}, \mathcal{B}, \mu)$, jointly measurable in t and α and taking values in $[0, 1]$. Suppose that the infinitesimal covariances are all bounded, that is, for some C' , all $\alpha, \beta \in \mathcal{S}$ and all $t > s$,*

$$\mathbf{E} \left[(M_t^{(\alpha)} - M_s^{(\alpha)})(M_t^{(\beta)} - M_s^{(\beta)}) \mid \mathcal{F}_s \right] \leq C'(t - s).$$

Then $M := \int M^{(\alpha)} d\mu(\alpha)$ is a continuous martingale with

$$\langle M \rangle_t = \int_0^t \left[\int \int_{\mathcal{S}^2} d[M^{(\alpha)}, M^{(\beta)}]_s d\mu(\alpha) d\mu(\beta) \right].$$

□

PROOF OF LEMMA 2.2: Applying Propositions 2.3 and 2.4 to the collection $\{M_t^{(\alpha)}\}$ shows that M is a continuous martingale with covariance

$$\langle M \rangle_t = \int_0^t \left[\int \int_{\mathcal{S}^2} \mathbf{1}_{s < \tau_\alpha} \mathbf{1}_{s < \tau_\beta} \nabla f(B_s, \alpha) \cdot \nabla f(B_s, \beta) d\mu(\alpha) d\mu(\beta) \right] ds. \quad (2.7)$$

If $s < \tau_\alpha, \tau_\beta$ then $B_s \in D_\alpha \cap D_\beta$, so $\nabla f(B_s, \alpha)$ and $\nabla f(B_s, \beta)$ exist and have modulus at most C by the Lipschitz condition. Thus

$$\int \int_{\mathcal{S}^2} \mathbf{1}_{s < \tau_\alpha} \mathbf{1}_{s < \tau_\beta} |\nabla f(B_s, \alpha)| |\nabla f(B_s, \beta)| d\mu(\alpha) d\mu(\beta) < \infty$$

which implies absolute convergence of the inner integral in (2.7) and thus its factorization as

$$\int \int_{\mathcal{S}^2} \mathbf{1}_{s < \tau_\alpha} \mathbf{1}_{s < \tau_\beta} \nabla f(B_s, \alpha) \cdot \nabla f(B_s, \beta) d\mu(\alpha) d\mu(\beta) = \left| \int_{\mathcal{S}} \mathbf{1}_{s < \tau_\alpha} \nabla f(B_s, \alpha) d\mu(\alpha) \right|^2,$$

which proves the lemma. □

3 Proof of upper bounds

Let $D \subseteq \mathbf{R}^2$ be the strip $\{(x_1, x_2) : |x_2| \leq 1\}$. Let $\{B_t\}$ be a 2-dimensional Brownian motion, with $B_t = (X_t, Y_t)$ as before, and let τ be the hitting time of ∂D . Recall the notation

$$A_t = \{x \in l_0 : |B_s - (x, 0)| \leq \epsilon \text{ for some } s \leq t \wedge \tau\}.$$

Fix $\epsilon > 0$ and for $\alpha \in \mathbf{R}$ define

$$f(x, \alpha) = \mathbf{P}_x[(\alpha, 0) \in A_\infty].$$

Observe that for $t < 1$,

$$\begin{aligned} M_t &= m(A_t) + \mathbf{1}_{t < \tau} \int_{[0,1] \setminus A_t} \mathbf{P}_{B_t}[(\alpha, 0) \in A_\infty] d\alpha \\ &= \int_{l_0} f(B_{t \wedge \tau_\alpha}, \alpha) d\alpha, \end{aligned}$$

since the integrand in the last expression is 1 if $\alpha \in A_t$, zero if $\alpha \notin A_t$ and $\tau \leq t$, and $\mathbf{P}_{B_t}[(\alpha, 0) \in A_\infty]$ otherwise.

Lemma 2.2 is applicable to M and f , with $(\mathcal{S}, \mathcal{B}, \mu) = (l_0, \text{Borel}, m)$, provided conditions (i) - (iii) are satisfied. Checking these is not too hard. Joint measurability, in fact joint continuity, are clear, as is the vanishing of each $f(\cdot, \alpha)$ on ∂D . Condition (iii) is immediate from the Strong Markov Property. By translation invariance it suffices to check (ii) for $\alpha = 0$. This is Lemma 3.4 below. The conclusion of Lemma 2.2 is that M_t is a continuous martingale with

$$\langle M \rangle_1 = \int_0^1 \left| \int_0^1 \mathbf{1}_{s < \tau_\alpha} \nabla f(B_s, \alpha) d\alpha \right|^2 ds, \quad (3.8)$$

where τ_α is the first time Brownian motion hits either ∂D or the ball of radius ϵ about the point $(\alpha, 0)$.

The inequality (2.6) will follow from equation (3.8) along with some lemmas, stated immediately below.

Lemma 3.1 *For any continuous martingale $\{M_t\}$ and any real u and positive t and L ,*

$$\mathbf{P}[M_t \geq u] \leq \exp(-(u - M_0)^2/2L) + \mathbf{P}[\langle M \rangle_t > L].$$

Lemma 3.2

$$d \langle M \rangle_t \leq 4 \left| \int_0^\infty \nabla f((x, (B_t)_2), 0) dx \right|^2.$$

Lemma 3.3 *There are constants c_5, c_6 and c_7 independent of ϵ for which*

$$(i) \quad f((0, y), 0) \leq c_5 \frac{|\log |y||}{|\log \epsilon|},$$

$$(ii) \quad f((|\alpha|, 0), 0) \leq c_6 \frac{e^{-\alpha}(1 + (\log(1/\alpha))^+)}{|\log \epsilon|},$$

and

$$(iii) \quad \int_{-\infty}^{\infty} f((0, y), \alpha) d\alpha \leq c_7 \frac{1 - y}{|\log \epsilon|}.$$

Lemma 3.4 *Fix $\epsilon > 0$. For $x \in \mathbf{R}$ and $y \in (-1, 1)$, let $\phi(x, y) = \mathbf{P}_{(x, y)}[\sigma < \tau]$, where σ is the hitting time on the ϵ -ball around the origin and τ is the first time $|Y_t| = 1$. Then ϕ is Lipschitz.*

Lemma 3.5 *There is a constant c_0 for which*

$$d \langle M \rangle_t \leq c_0 \left(\frac{1 + |\log |Y_t||}{|\log \epsilon|} \right)^2.$$

Lemma 3.6 *Let $L(x, t)$ be local time for a Brownian motion and let $U = \sup_x L(x, 1)$. Then for some constant c_8 , $\mathbf{P}[U \geq u] \leq c_8 \exp(-u^2/2c_8)$.*

PROOF OF (2.6): First, by Lemma 3.3,

$$M_0 = \int_0^1 f((0, 0), \alpha) d\alpha \leq \frac{c_7}{|\log \epsilon|}.$$

Next, following the notation of the Lemma 3.6, let H be the event $\{U \leq |\log \epsilon|\}$. According to Lemma 3.6 there is a constant c_8 for which

$$\mathbf{P}[H^c] \leq c_8 \exp(-(\log \epsilon)^2/2c_8).$$

Use Lemma 3.5 to get

$$\langle M \rangle_1 \leq c_0 \left[|\log \epsilon|^{-2} + \int_0^1 \frac{\log^2 |Y_t|}{|\log \epsilon|^2} dt \right] = c_0 |\log \epsilon|^{-2} \left[1 + \int_{-1}^1 \mathcal{L}(y, 1) \log^2 |y| dy \right]$$

where \mathcal{L} is the local time function for the y -coordinate of the Brownian motion. On the event H , this is maximized when $\mathcal{L}(y, 1) = |\log \epsilon|$ on the interval $[-|\log \epsilon|/2, |\log \epsilon|/2]$ and has value at most

$$c_0 |\log \epsilon|^{-2} [2(\log |2 \log \epsilon|)^2].$$

Setting

$$L = c_0 |\log \epsilon|^{-2} [2(\log |2 \log \epsilon|)^2]$$

thus gives $\mathbf{P}[\langle M \rangle_1 > L] \leq c_8 \exp(-(\log \epsilon)^2/2c_8)$. Plugging this value of L into Lemma 3.1 yields

$$\mathbf{P}[M_1 \geq \theta] \leq \exp\left(-\frac{(\theta - K|\log \epsilon|^{-1})^2}{2L}\right) + \mathbf{P}[\langle M \rangle_1 > L]$$

where $K|\log \epsilon|^{-1}$ is an elementary estimate for the expected measure of l_0 covered by the ϵ -sausage to time 1. This is good enough to prove (2.6) when $|\log \epsilon|$ is small compared to θ . \square

It remains to prove the six lemmas.

PROOF OF Lemma 3.1: Routine; see the appendix. \square

PROOF OF Lemma 3.2: First, it is evident that $f((x, y), \alpha)$ depends only on $|x - \alpha|$ and $|y|$. I claim that f is decreasing in $|x - \alpha|$ and $|y|$. This may be verified by coupling. Let $\alpha \leq x \leq x'$ and $0 \leq y \leq y'$ and couple two Brownian motions beginning at (x, y) and (x', y') by letting the x -increments be opposite until the two x -coordinates meet, then identical, and the same for the y -coordinates. Formally, if $W_t = (X_t, Y_t)$ is a Brownian motion, let $B_t = (x, y) + \int_0^t dW_s$ and $B'_t = (x', y') + \int_0^t [(1 - 2 \cdot \mathbf{1}_{s < \tau_x})dX_s + (1 - 2 \cdot \mathbf{1}_{s < \tau_y})dY_s]$, where $\tau_x = \inf\{t : (B_t)_1 \geq (x' + x)/2\}$ and $\tau_y = \inf\{t : (B_t)_2 \geq (y' + y)/2\}$. This coupling has the property that $|B_t - (\alpha, 0)| \leq |B'_t - (\alpha, 0)|$ for all t , thus showing that $f((x', y'), \alpha) \leq f((x, y), \alpha)$.

Use this claim to see that the signs of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are constant as α varies over $(-\infty, x)$, and are also constant when α varies over (x, ∞) . Hence

$$\left| \int \mathbf{1}_{t < \tau_\alpha} \nabla f((x, y), \alpha) d\mu(\alpha) \right|$$

$$\begin{aligned}
&\leq \left| \int_{\alpha \leq x} \mathbf{1}_{t < \tau_\alpha} \nabla f((x, y), \alpha) d\mu(\alpha) \right| + \left| \int_{\alpha \geq x} \mathbf{1}_{t < \tau_\alpha} \nabla f((x, y), \alpha) d\mu(\alpha) \right| \\
&\leq \left| \int_{\alpha \leq x} \nabla f((x, y), \alpha) d\mu(\alpha) \right| + \left| \int_{\alpha \geq x} \nabla f((x, y), \alpha) d\mu(\alpha) \right| \\
&\leq 2 \left| \int_0^\infty \nabla f((x, y), 0) dx \right|
\end{aligned}$$

by translation invariance. Plugging this into the conclusion of Lemma 2.2 gives the desired result. \square

PROOFS OF LEMMA 3.3 AND 3.4: Routine; see the appendix. \square

PROOF OF LEMMA 3.5: Lemma 3.2, which gives estimates in terms of the gradient of f , may be turned into an estimate in terms of f itself as follows. First break down into the two coordinates:

$$d < M >_t \leq 4 \left| \int_0^\infty \frac{\partial f}{\partial x}((x, y), 0) dx \right|^2 + 4 \left| \int_0^\infty \frac{\partial f}{\partial y}((x, y), 0) dx \right|^2,$$

where $y = Y_t$. From the fundamental theorem of calculus, the square root of the first component is

$$2 \left| \int_0^\infty \frac{\partial f}{\partial x}((x, y), 0) dx \right| = 2(f((0, y), 0) - f((\infty, y), 0)) = 2f((0, y), 0). \quad (3.9)$$

To estimate the second component, assume without loss of generality that $y > 0$ and write

$$2 \left| \int_0^\infty \frac{\partial f}{\partial y}((x, y), 0) dx \right| = -\frac{\partial}{\partial y} \int_{-\infty}^\infty f((0, y), \alpha) d\alpha. \quad (3.10)$$

For $\epsilon \in (0, 1 - y)$, run a Brownian motion from $(0, y + \epsilon)$ until it hits one of the horizontal lines $\mathbf{R} \times \{1\}$ or $\mathbf{R} \times \{y\}$. The probability it hits $\mathbf{R} \times \{y\}$ first is $1 - \epsilon/(1 - y)$. Sending ϵ to zero and using the Strong Markov Property gives

$$-\frac{\partial}{\partial y} \int_{-\infty}^\infty f((0, y), \alpha) d\alpha = (1 - y)^{-1} \int_{-\infty}^\infty f((0, y), \alpha) d\alpha. \quad (3.11)$$

Combining (3.11) and (3.9) and using Lemma 3.3 gives $d < M >_t \leq 4c_5^2(\log^2 |y|/\log^2 \epsilon) + c_7^2/\log^2 \epsilon$, proving the lemma with $c_0 = 4c_5^2 + c_7^2$. \square

PROOF OF LEMMA 3.6: This is well known. For example, it is stated and proved as Lemma 1 of [4]. \square

4 Proof of lower bounds

In this section, (1.2) and (1.4) are proved simultaneously. For (1.4) I assume that $\theta \leq 1 - 3\epsilon$, since otherwise this case is covered by (1.2). Let N be a integer parameter to be named later. Define a sequence of balls $\{\mathcal{C}_j : 1 \leq j \leq N^2\}$, each of radius $1/N$ and centered on the x -axis with ordinates

$$1/N, 2/N, \dots, (N-1)/N, 1, (N-1)/N, \dots, 1/N, 0, 1/N, \dots, 1, \dots, 1/N, 0, \dots$$

running back and forth along the unit interval a total of N times. Let H_N be the event that

$$B_{j/N^2} \in \mathcal{C}_j \text{ for all } j \leq N^2.$$

Let $c_9 = \log \mathbf{P}_{(-1/N, 0)}[B_{1/N^2} \in \mathcal{C}_1]$. By the Markov property,

$$\mathbf{P}[B_{j/N^2} \in \mathcal{C}_j | B_{i/N^2} \in \mathcal{C}_i : 1 \leq i < j] \geq \min_{z \in \mathcal{C}_{j-1}} \mathbf{P}_z[B_{1/N^2} \in \mathcal{C}_j],$$

which is easily seen to be minimized when $j = 1$ and $z = (-1/N, 0)$, leading to

$$\mathbf{P}[H_N] \geq \exp(-c_9 N^2). \quad (4.12)$$

Next, bound from below the conditional probabilities of the events $\{R_\epsilon \supseteq [0, 1]\}$ and of $\{\Xi \geq \theta\}$ given H_N (recall Ξ is the measure of $R_\epsilon \cap ([0, 1] \times \{0\})$). Let $b = \lceil 1/\epsilon \rceil$ be the least integer exceeding $1/\epsilon$. For $1 \leq j \leq b$, let $z_j = (j\epsilon, 0)$. If $z_j \in R_{\epsilon/2}$ then $[z_j - \epsilon/2, 1 \wedge (z_j + \epsilon/2)] \times \{0\} \subseteq R_\epsilon$; thus $\Xi \geq \epsilon(\#\{j : z_j \in R_{\epsilon/2}\} - 1)$ and if every z_j is in $R_{\epsilon/2}$ then $\{R_\epsilon \supseteq [0, 1]\}$. The following routine lemma is proved in the appendix.

Lemma 4.1 *There is a universal constant K such that if q_1, q_2 and q_3 are points in \mathbf{R}^2 at pairwise distances of no more than $3/N$ and $\{W_t : 0 \leq t \leq 1/N^2\}$ is a Brownian bridge with $W_0 = q_1$ and $W_{1/N^2} = q_2$, then*

$$\mathbf{P}[|W_s - q_3| < \delta \text{ for some } s \leq 1/N^2] \geq \frac{1}{K|\log \delta|}$$

for all $N \geq 1$ and $\delta < 1/2$. □

Continuing the proof of the lower bounds, fix j and condition on the σ -field $\sigma(B_{i/N^2} : 1 \leq i \leq N^2)$. On the event H_N there are N values of i for which the pairwise distances between $B_{i/N^2}, B_{(i+1)/N^2}$ and z_j are all at most $3/N$. It follows from Lemma 4.1 that on H_N ,

$$\mathbf{P}[z_j \notin R_{\epsilon/2}] \leq \left(1 - \frac{1}{K|\log(\epsilon/2)|}\right)^N.$$

Choosing $N = \lceil \gamma K |\log(\epsilon/2)| \rceil$ now gives

$$\mathbf{P}[z_j \notin R_{\epsilon/2} | H_N] \leq e^{-\gamma}.$$

When $e^{-\gamma} \leq (1-\theta)/3$ this gives $\mathbf{E}[\#\{j < b : z_j \notin R_{\epsilon/2}\} | H_N] \leq (1-\theta)(b-1)/3$ and hence

$$\mathbf{P}[\#\{j < b : z_j \notin R_{\epsilon/2}\} \geq (1-\theta)b - 1 | H_N] \leq \frac{(1-\theta)(b-1)/3}{(1-\theta)b - 1}$$

which is at most $1/2$ when $\theta \leq 1 - 3\epsilon$. Thus choosing $\gamma = |\log(1-\theta)/3|$ gives

$$\mathbf{P}[\Xi \geq \theta] \geq \mathbf{P}[\#\{j < b : z_j \notin R_{\epsilon/2}\} < (1-\theta)b - 1] \geq \frac{1}{2} \mathbf{P}[H_N] \geq \frac{1}{2} \exp(-c_9 \gamma^2 K^2 |\log \epsilon/2|^2).$$

This is at least $\exp(-c_4 |\log \epsilon|^2 |\log(1-\theta)/3|^2)$ when $c_4 = 2c_9 K^2$ and ϵ is small enough, proving (1.4).

Choosing instead $N = \lceil \gamma K |\log(\epsilon/2)| |\log \epsilon| \rceil$ makes

$$\mathbf{P}[z_j \notin R_{\epsilon/2} | H_N] \leq \epsilon^\gamma$$

and for $\gamma = 2$ this makes $\mathbf{P}[R_\epsilon \supseteq [0, 1] \times \{0\} | H_N]$ at least $1 - \lceil \epsilon^{-1} \rceil \epsilon^2 \geq \exp(-c |\log \epsilon|^4)$ for any c and small ϵ . For such a γ , (4.12) gives

$$\begin{aligned} \mathbf{P}[H_N] &\geq \exp(-c_9 (2K |\log \epsilon|)^4) \\ &\geq \exp(-(1/2)c_4 |\log \epsilon|^4) \end{aligned}$$

for $c_4 = 32K^4 c_9$ and ϵ small enough; thus $\mathbf{P}[R_\epsilon \supseteq [0, 1] \times \{0\}] \geq \exp(-c_4 |\log \epsilon|^4)$, proving (1.2). \square

5 Appendix: proofs of easy propositions

PROOF OF PROPOSITION 2.3: Continuity is clear, and the vanishing of the Laplacian of f shows that each $M^{(\alpha)}$ is a local martingale, hence a martingale since it is bounded. In fact the formula for the brackets is clear too, but to be pedantic we substitute for f a sequence of functions genuinely in C^2 and apply Itô's formula. For $\epsilon > 0$ let $D_{\alpha,\epsilon} = \{x \in D : f(x, \alpha) \in [\epsilon, 1 - \epsilon]\}$. Use a partition of unity to get functions $f^{\alpha,\epsilon} : \mathbf{R}^2 \rightarrow [0, 1]$ with continuous, uniformly bounded second derivatives and such that $f^{\alpha,\epsilon} = f$ on $D_{\alpha,\epsilon}$. Let $\tau_{\alpha,\epsilon} = \inf\{t : B_t \notin D_{\alpha,\epsilon}\}$. The martingale

$$M_t^{\alpha,\epsilon} := f(B_{t \wedge \tau_{\alpha,\epsilon}}, \alpha)$$

is a continuous martingale and it is clear from Itô's formula that the covariances satisfy

$$\mathbf{E} \left[(M_t^{\alpha,\epsilon} - M_s^{\alpha,\epsilon})(M_t^{\beta,\epsilon} - M_s^{\beta,\epsilon}) \mid \mathcal{F}_s \right] = \int_s^t \mathbf{1}_{r < \tau_{\alpha,\epsilon}} \mathbf{1}_{r < \tau_{\beta,\epsilon}} \nabla f(B_r, \alpha) \cdot \nabla f(B_r, \beta) dr. \quad (5.13)$$

Now fix α and β and let ϵ go to zero. Since $\tau_{\alpha,\epsilon} \uparrow \tau_\alpha$ almost surely it follows that $M_r^{\alpha,\epsilon} \rightarrow M_r^{(\alpha)}$ almost surely and $\mathbf{1}_{r < \tau_{\alpha,\epsilon}} \uparrow \mathbf{1}_{r < \tau_\alpha}$ almost surely. The same is true with β in place of α . The expression inside the expectation (resp. integral) on the left (resp. right) side of (5.13) is bounded, hence sending ϵ to zero simply removes it typographically from both sides, and establishes the lemma. \square

PROOF OF PROPOSITION 2.4: The integral of bounded continuous functions is continuous. To check that M is a martingale, first observe that

$$\int \mathbf{E}(|M_t^{(\alpha)} - M_s^{(\alpha)}| \mid \mathcal{F}_s) d\mu(\alpha) \leq \mu(\mathcal{S}) \sqrt{C'(t-s)}$$

and hence one may interchange an integral against μ with an expectation given \mathcal{F}_s to get

$$\mathbf{E} \left[\int (M_t^{(\alpha)} - M_s^{(\alpha)}) d\mu(\alpha) \mid \mathcal{F}_s \right] = \int \mathbf{E}(M_t^{(\alpha)} - M_s^{(\alpha)} \mid \mathcal{F}_s) d\mu(\alpha).$$

Thus,

$$\mathbf{E}(M_t - M_s \mid \mathcal{F}_s) = 0.$$

To compute the quadratic variation, first observe that

$$\mathbf{E} \int \int \mathbf{E} \left[|(M_t^{(\alpha)} - M_s^{(\alpha)})(M_t^{(\beta)} - M_s^{(\beta)})| | \mathcal{F}_s \right] d\mu(\alpha) d\mu(\beta) \leq 1.$$

This justifies the third of the following equalities, the second being justified by absolute integrability:

$$\begin{aligned} [(M_t - M_s)^2 | \mathcal{F}_s] &= \mathbf{E} \left[\left(\int (M_t^{(\alpha)} - M_s^{(\alpha)}) d\mu(\alpha) \right)^2 | \mathcal{F}_s \right] \\ &= \mathbf{E} \left[\int \int_{\mathcal{S}^2} (M_t^{(\alpha)} - M_s^{(\alpha)})(M_t^{(\beta)} - M_s^{(\beta)}) d\mu(\alpha) d\mu(\beta) | \mathcal{F}_s \right] \\ &= \int \int_{\mathcal{S}^2} \mathbf{E} \left((M_t^{(\alpha)} - M_s^{(\alpha)})(M_t^{(\beta)} - M_s^{(\beta)}) | \mathcal{F}_s \right) d\mu(\alpha) d\mu(\beta) \\ &= \int \int_{\mathcal{S}^2} \left([M^{(\alpha)}, M^{(\beta)}]_t - [M^{(\alpha)}, M^{(\beta)}]_s \right) d\mu(\alpha) d\mu(\beta) \\ &= \int_s^t \left[\int \int_{\mathcal{S}^2} d[M^{(\alpha)}, M^{(\beta)}]_r d\mu(\alpha) d\mu(\beta) \right] \end{aligned}$$

again by absolute integrability. □

PROOF OF LEMMA 3.1: Let $\tau = \inf\{s : \langle M \rangle_s \geq L\}$. Then

$$\exp(\lambda M_{s \wedge \tau} - \frac{1}{2} \lambda^2 \langle M \rangle_{s \wedge \tau})$$

is a martingale and hence

$$\begin{aligned} \mathbf{E} \exp(\lambda(M_{t \wedge \tau} - M_0)) &\leq \mathbf{E} \exp \left(\lambda(M_{t \wedge \tau} - M_0) - \frac{\lambda^2}{2} (\langle M \rangle_{t \wedge \tau} - L) \right) \\ &\leq \exp\left(\frac{\lambda^2}{2} L\right). \end{aligned}$$

By Markov's inequality,

$$\begin{aligned} \mathbf{P}[M_{t \wedge \tau} - M_0 \geq u - M_0] &\leq \exp(-\lambda(u - M_0)) \mathbf{E} \exp(\lambda(M_{t \wedge \tau} - M_0)) \\ &\leq \exp\left(\frac{\lambda^2}{2} L - \lambda(u - M_0)\right). \end{aligned}$$

Plugging in $\lambda = (u - M_0)/L$ gives

$$\mathbf{P}[M_{t \wedge \tau} \geq u] \leq \exp(-(u - M_0)^2/2L)$$

which proves the lemma since $\{M_t \geq u\} \subseteq \{M_{t \wedge \tau} \geq u\} \cup \{< M >_t > L\}$. \square

PROOF OF LEMMA 3.3: Assume without loss of generality that $\epsilon < 1/4$. Let \mathcal{C}_ϵ be the circle of radius ϵ centered at the origin, and τ_ϵ the hitting time of this circle. These are useful stopping times, since for $a < |z| < b$,

$$\mathbf{P}_z[\tau_a < \tau_b] = \frac{\log(b/|z|)}{\log(b/a)}.$$

The minimal value $\mathbf{P}_z[\tau < \tau_{\mathcal{C}_{1/2}}]$ as z ranges over points of modulus $3/4$ is some constant p_0 . Given a Brownian motion, construct a continuous time non-Markovian birth and death process $\{X_t\}$ taking values in $\{0, 1, 2, 3\}$ by letting X_t be 0, 1, 2 or 3 according to whether Brownian motion has most recently hit $\mathcal{C}_\epsilon, \mathcal{C}_{1/2}, \mathcal{C}_{3/4}$ or ∂D first respectively. Let $\{X_n\}$ be the discrete time birth and death process consisting of successive values of $\{X_t\}$. Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Then

$$\mathbf{P}(X_{n+1} = 0 \mid \mathcal{F}_n, X_n = 1) = \frac{\log(3/2)}{\log(3/4\epsilon)} \text{ and}$$

$$\mathbf{P}(X_{n+1} = 3 \mid \mathcal{F}_n, X_n = 2) \geq p_0.$$

Couple this to a Markov birth and death chain whose transition probability from 2 to 3 is precisely p_0 , to see that for this chain, the probability of hitting 0 before 3 given $X_1 = 1$ is at most $K/|\log \epsilon|$ for some constant K . This implies that for $y > 1/2$,

$$f((0, y), 0) \leq \mathbf{P}_{(0,y)}[\tau_{1/2} < \tau] \frac{K}{|\log \epsilon|}.$$

Since $\mathbf{P}_{(0,y)}[\tau_{1/2} < \tau] < 2(1 - y) < |\log y|$, this verifies (i) in the case $y > 1/2$. For $y \leq 1/2$, use

$$f((0, y), 0) \leq \mathbf{P}_{(0,y)}[\tau_{1/2} < \tau_\epsilon] \frac{K}{|\log \epsilon|} + \mathbf{P}_{(0,y)}[\tau_\epsilon < \tau_{1/2}]$$

to verify (i) in the remaining cases.

This also verifies (ii) in the case $\alpha < 1/2$. For $1/2 \leq \alpha \leq 2$, the numerator on the RHS of (ii) is bounded away from 0 and the LHS is decreasing in α , so c_6 may be chosen a little larger to make (ii) hold. Finally, for $\alpha > 2$, note that

$$\mathbf{P}_{(k+1,y)}[\sigma_k < \tau] < 1/3 < e^{-1},$$

for any $y \in [0, 1]$, where σ_k is the hitting time on the vertical line $x = k$. Applying this successively gives the factor $e^{-\alpha}$ and verifies (ii).

To see (iii) for $y \leq \epsilon$, use monotonicity to get

$$f((0, \epsilon), \alpha) \leq f((0, 0), \alpha) = f((\alpha, 0), 0)$$

and integrate (ii). Finally, for $\epsilon < y < 1$, let T be the hitting time on the horizontal line $y = \epsilon$ and observe that

$$\int_{-\infty}^{\infty} f((0, y), \alpha) d\alpha = \mathbf{P}[T < \tau] \int_{-\infty}^{\infty} f((0, \epsilon), \alpha) d\alpha.$$

□

PROOF OF LEMMA 3.4: First let (x, y) be a point inside the strip $|y| \leq 1 - \epsilon$ but outside the ball $x^2 + y^2 < 4\epsilon^2$. Let V be the circle of radius $\epsilon/2$ centered at (x, y) . If (x', y') is another point inside V and $\mu_{x',y'}$ is the hitting distribution on V starting from (x', y') , then the total variation distance from $\mu_{x',y'}$ to uniform is $O(\epsilon^{-1}|(x, y) - (x', y')|)$ as $(x', y') \rightarrow (x, y)$. The strong Markov property shows that $|\phi(x', y') - \phi(x, y)|$ is bounded by this total variation distance; the truth of this statement in the limit $(x', y') \rightarrow (x, y)$ implies that ϕ is Lipschitz with a possibly greater constant in a neighborhood of (x, y) . Inverting in the circle $x^2 + y^2 = \epsilon^2$ and using a similar argument shows that ϕ is also Lipschitz in a neighborhood of ∞ on the complex sphere C^* , so compactness of the region $\{|y| \leq 1 - \epsilon\} \setminus \{x^2 + y^2 < 4\epsilon^2\}$ in C^* shows that ϕ is Lipschitz in that region.

The two other cases are also easy. If $\epsilon \leq |(x, y)| \leq 2\epsilon$ then let τ be the first time that $|(X_y, Y_t)| = \epsilon$ or 2ϵ . Let $\mu_{x,y}$ be the subprobability hitting measure on the outer circle $V \stackrel{\text{def}}{=} \{(u, v) : u^2 + v^2 = 4\epsilon^2\}$, i.e., $\mu(B) = \mathbf{P}[(X_\tau, Y_\tau) \in B \cap V]$. If (x', y') is another point of

modulus between ϵ and 2ϵ , then $|\phi(x', y') - \phi(x, y)|$ is at most the total variation distance between $\mu_{x,y}$ and $\mu_{x',y'}$. Again, since ϵ is fixed, an easy computation shows this to be $O(|(x, y) - (x', y')|)$. The case where $y > 1 - \epsilon$ is handled the same way, except there the outer circle is replaced by the line $y = 1 - 2\epsilon$. \square

PROOF OF LEMMA 4.1: Assume without loss of generality that q_3 is the origin and dilate time by N^2 and space by N , so that we have a Brownian bridge $\{X_t : 0 \leq t \leq 1\}$ started at p_1 and stopped at p_2 , with $|p_1|, |p_2| \leq 3$. We use the following change of measure formula for the law of this Brownian bridge. Let $\tau \leq 1$ be any stopping time, let \mathbf{P}_x denote the law of Brownian motion from x and \mathbf{Q} denote the law of the bridge from q_1 at time 0 to q_2 at time 1. Then for any event $G \in \mathcal{F}_\tau$,

$$\mathbf{Q}(G) = \frac{\int \mathbf{1}_G \exp(|q_2 - B_\tau|^2/2(1 - \tau)) d\mathbf{P}_{q_1}}{\exp(|q_2 - q_1|^2/2)}.$$

The probability of the bridge entering the dilated $N\delta$ -ball around the origin is of course at least the probability it enters the δ -ball around the origin before time $3/4$. If $\{B_t\}$ is an unconditioned Brownian motion starting from p_1 and τ is the first time $\{B_t\}$ enters the δ -ball around the origin, then $\mathbf{P}[\tau \leq 3/4]$ is at least $K_1/|\log \delta|$ for a universal constant C_1 . Conditioned on τ and B_τ , the density of B_1 at q_2 is at least $(2\pi)^{-1}e^{-18}$. Plugging into the change of measure formula proves the lemma. \square

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