Diagonals of Rational Fractions

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First introduced to study properties of Hadamard's product [6], the diagonal of a power series could appear to be a somewhat artificial notion. However Furstenberg proved [16] that, over a finite field, the diagonals of rational fractions are exactly the algebraic functions. Our aim is to explain why, over $\mathbb{Q}$, they make a very interesting class of functions, sharing most of the properties of algebraic ones.

1 Introduction

Recently, both physicists and combinatorialists have encountered a lot of powers series with integer coefficients that are D-finite, i.e. solutions of linear differential equations with polynomial coefficients. They all appear to be *diagonals of rational fractions* (DRFs) [5]. For instance, [1] gives a (complete in some sense) list of 403 differential equations of "Calabi-Yau type", and the (unique) analytic solution near 0 of each one can be shown (more or less easily) to be a DRF.[1]

The main aim of this paper is to explain why this could be a general fact. Although they are seemingly elementary objects, DRFs will actually lead us into algebraic geometry, somewhere between Weil's conjectures and motives theory.

We will limit to working over the field $\mathbb{Q}$. Actually, a large number of the results remain true over any field of characteristic zero but we are mainly interested in the arithmetic aspect of the topic and more precisely in a "for almost all p" theory ("*almost all*" meaning "all but a finite number"). This means that we are concerned with properties that can be expressed through reductions modulo p^h, for almost all primes p and all $h \geq 1$. For instance, over fields of finite characteristic, DRFs are exactly the algebraic functions. This implies that DRFs have algebraic reductions modulo p for almost all p.[2] In addition, there are actually non-algebraic DRFs!

In some respects, it would be more convenient to replace $\mathbb{Q}$ with a number field but this would introduce technical difficulties that would obscure things without introducing new ideas.

2 Diagonals of rational fractions

2.1 Definition

For $F(x_1, \ldots, x_n) = \sum_{(s_1,\ldots,s_n)\in\mathbb{N}^n} a_{s_1\cdots s_n}\, x_1^{s_1}\cdots x_n^{s_n}$ in $\mathbb{Q}[[x_1, \ldots, x_n]]$, let's set:

$$\mathrm{Diag}(F) \stackrel{\mathrm{def}}{=} \sum_{s=0}^{\infty} a_{s\cdots s}\, x^s \quad \in \mathbb{Q}[[x]]\,.$$

A power series f in $\mathbb{Q}[[x]]$ is said to be a *diagonal of a rational fraction (DRF) of n variables* if there is a rational fraction $F = \frac{P}{Q} \in \mathbb{Q}(x_1, \ldots, x_n)$ with Q *invertible* in $\mathbb{Q}[[x_1, \ldots, x_n]]$ (i.e. with $Q(0, \ldots, 0) \neq 0$), such that $f = \mathrm{Diag}(F)$.

We will denote by $\mathcal{D}_n$ the set of DRFs of n-variables and by $\mathcal{D} = \bigcup_{n\geq 1} \mathcal{D}_n$ the set of all DRFs. It is almost obvious that $\mathcal{D}_n \subset \mathcal{D}_m$ for $n < m$.

Remark 1. For a given DRF f, the power series F, such that $f = \mathrm{Diag}(F)$, is far from unique. For instance, for any $F \in x_1\mathbb{Q}[[x_1, x_2]]$, setting $G(x_1, x_2) = F(x_1x_2^2, x_2)$, we get $\mathrm{Diag}(G) = 0$.

Considering algebraic functions instead of rational ones does not increase the set of diagonals, as shown by the following proposition.

Proposition 2 ([15])**.** Let $F \in \mathbb{Q}[[x_1, \ldots, x_n]]$ be algebraic over $\mathbb{Q}(x_1, \ldots, x_n)$. Then, its diagonal belongs to $\mathcal{D}_{2n}$.

When $n = 1$, this is a consequence of an explicit formula due to Furstenberg [16]: let $f \in x\mathbb{Q}[[x]]$ such that $P(x, f(x)) = 0$ with $P'_y(0,0) \neq 0$. Then,

$$f^m(x) = \mathrm{Diag}\left(\frac{y^m\, P(xy, y)}{P'_y(xy, y)}\right), \quad (m \geq 0). \tag{1}$$

General algebraic functions in a single variable can be reduced to that case by separating roots. However, even if theoretically easy, this is algorithmically a rather complex process.

When $n = 1$, the converse of Proposition 2 is also true: any $f \in \mathcal{D}_2$ is algebraic over $\mathbb{Q}(x)$. This is easily seen because residues of a rational fraction are algebraic over the field of its coefficients (see Equation (3) below).

However, in general, elements of $\mathcal{D}_n$ are not algebraic for $n > 2$: Furstenberg [16] pointed out the function

$$\sum_{s=0}^{\infty} \binom{2s}{s}^2 x^s = (1-4x)^{-1/2} \star (1-4x)^{-1/2},$$

which is both in $\mathcal{D}_3$ and not algebraic over $\mathbb{Q}(x)$.

2.2 Properties

DRFs can be characterised by their coefficients.

Proposition 3 ([24] Theorem 15.1)**.** $\sum u_n\, x^n \in \mathbb{Q}[[x]]$ is a DRF if and only if the sequence u_n is a binomial sum, i.e. it can be obtained from *binomial coefficients* and *geometric sequences* by means of *affine changes of index* and *finite sums* (see [24] page 111 for a precise definition).

A representative example is given by Apery's numbers $a(s) = \sum_{k=0}^{s} \binom{s}{k}^2 \binom{s+k}{k}^2$, for which one has

$$\sum_{s=0}^{\infty} a_s x^s = \mathrm{Diag}\, \frac{1}{(1-x_1-x_2)(1-x_3-x_4) - x_1x_2x_3x_4}.$$

1 Some of them (#242 , #244, ...) involve harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$ in their coefficients but this is only an artefact, as shown by Maillard's formula

$$\sum_{k=1}^{n} k \binom{2n-2k}{n-k}\binom{2k}{k}(H_{n-k} - H_k) = \binom{2n}{n}(2n+1) - 4^n(n+1)\,.$$

2 Actually, DRFs have algebraic reductions modulo p^h for almost all p and all $h \geq 1$, "bad" primes being the (finitely many) ones involved in the denominator of some coefficients

However, the most striking fact is the stability of the set $\mathcal{D}$ under many operations.

Proposition 4. $\mathcal{D}$ is stable under:
1. Sum, and (Cauchy) product ($\mathcal{D}$ is a sub-algebra of the $\mathbb{Q}$-algebra $\mathbb{Q}[[x)]]$).
2. Derivative $f \mapsto \frac{d}{dx} f$.
3. Hadamard $\sum a_s x^s \star \sum b_s x^s = \sum a_s b_s x^s$ and Hurwitz $\sum a_s x^s \mathsf{H} \sum b_s x^s = \sum_{s,t} \binom{s+t}{s} a_s b_t x^{s+t}$ products.
4. "Decimations" $\psi_{r,d} : \sum a_s x^s \mapsto \sum a_{r+ds} x^s$.
5. Algebraic changes of variable: $f \mapsto f \circ g$ for $g \in x\mathbb{Q}[[x]]$ algebraic over $\mathbb{Q}(x)$.

Points 1 to 4 are rather easily checked. Point 5 is more subtle and is based on Equation (1). Complete proofs can be found in [10].

The following statements can also be made.

Proposition 5. The invertible elements of $\mathcal{D}$:
1. For the Cauchy product are the algebraic functions.
2. For the Hadamard product are rational functions.

The proof of 1 is based on two rather deep results: D-finiteness of both f and $1/f$ implying algebraicity of f [17] and Grothendieck's conjecture being true for first order linear differential equations over curves [13].

A proof of 2 for $\mathcal{D}_2$ is given in [4] and is easy to generalise. We now state the properties we will mainly be concerned with. Recall that "for almost all" (notation faa) means for all but a finite number. A property that is true for almost all p will be said to be *global*.

Proposition 6. Any DRF $f \in \mathbb{Q}[[x]]$ is:
1. *Globally bounded:* f has a non-zero radius of convergence (in $\mathbb{C}$) and $\exists\, c, d \in \mathbb{N}^*$, $d\, f(c\, x) \in \mathbb{Z}[[x]]$.
2. *D-finite:* $\exists\, L \in \mathbb{Z}[x][\frac{d}{dx}]$, $L \neq 0$, such that $L(f) = 0$.

The proof of 1 is straightforward but the proof of 2 is much deeper. It was first proved as a corollary of Theorem 23. Then, more elementary and direct proofs were given [25]. The last step was to find algorithms to compute the differential equation L. An overview on these topics can be found in [24] where the most efficient known algorithms are given.

Remark 7. The (non-equivalent) absolute values on the field $\mathbb{Q}$ are known to be the classical one and the p-adic ones (p prime). For each, $\mathbb{Q}$ has a completion, namely $\mathbb{R}$ and $\mathbb{Q}_p$. The completion $\mathbb{Z}_p$ of $\mathbb{Z}$ coincides with the unit disc of $\mathbb{Q}_p$. Property 6.1 means firstly that the function f has a non-zero radius of convergence in all these completions and secondly that, for almost all prime p (i.e. for p not dividing d or c), it belongs to $\mathbb{Z}_p[[x]]$ and hence is a bounded (by 1) function on $\mathbb{Z}_p$.

Let A be any (finite) ring. A power series $\sum a_s x^s \in A[[x]]$ is said to be *p-automatic* if there is a deterministic finite p-automaton[3] that returns a_n when inputting the digit sequence of n in base p. For people not interested in automata, let's just say that a power series $f \in \mathbb{Z}/p^h\mathbb{Z}[[x]]$ is p-automatic if and only if it is, in some sense, algebraic over $\mathbb{Z}/p^h\mathbb{Z}(x)$ (see Theorems 40 and 41 for precise statements).

Proposition 8. Any DRF f is *globally automatic*: for almost all p and for all h, $f \mod p^h$ is p-automatic.

For f in $\mathcal{D}_2$, this is proved in [7] and is easily generalised. It is connected with Property *4* of Proposition **4**.

2.3 Conjectures

The properties of Proposition 6 could be characteristic of DRFs; we will give a definition and set a working hypothesis.

Definition 9. Any power series in $\mathbb{Q}[[x]]$ satisfying Properties 6-1 and 6-2 will be said to be a *pseudo-diagonal*.

Conjecture 10. Every pseudo-diagonal is a DRF.

In Section 4, this conjecture will be proved to be true under a large number of circumstances, including examples introduced by physicists and combinatorialists. However, at the end of the section, we will see that it remains widely open for some (hypergeometric) functions.

Remark 11. It is natural to imagine seemingly more general conjectures, for instance, replacing the field $\mathbb{Q}$ by any number field or replacing the field $\mathbb{Q}(x)$ by any finite extension, i.e. working on a curve instead of the projective line. In [10], both extensions are shown to be consequences of Conjecture 10.

Proposition 8 suggests a weak form of the conjecture.

Conjecture 12 (weak)**.** Every pseudo-diagonal is globally automatic.

Assuming the seemingly out of reach geometric Conjecture 19, we will give an almost-proof of Conjecture 12. To have an actual proof, a better understanding of the p-adic cohomology for families of varieties over a finite field is still needed.

Conjecture 10 is reminiscent of a more classical one.

Conjecture 13 (Grothendieck)**.** If, for almost all p, $L \in \mathbb{Z}[x][\frac{d}{dx}]$ has a complete set of solutions in $\mathbb{F}_p[x]$ then all its solutions are algebraic over $\mathbb{Q}(x)$.

A first difference between Conjectures 10 and 13 is that Conjecture 13 needs to know *all* solutions of the differential equation L and hence, when proved for a particular set S of differential equations (for instance Picard–Fuchs ones as Katz did) then it is not proved for all the sub-equations (i.e. right factors) of the $L \in S$.

On the other side, Conjecture 10 requires a solution in $\mathbb{Z}_p[[x]]$ and not only in $\mathbb{F}_p[x]$, but it can be seen that both requirements are almost equivalent.

There is another deep but less visible difference between the two conjectures. Actually, in Conjecture 10, the underlying differential equation has singularities involving logarithms (with non-finite local monodromy); they cannot be cancelled by any algebraic pullback as is done for Conjecture 13.

3 A p-automaton $\mathcal{M} = \{M, m_0, e_i, s\}$ is built from a finite set M, an input element $m_0 \in M$, maps $\{e_i : M \to M\}_{0 \le i < p}$ satisfying $e_0(m_0) = m_0$ and an output map $s : M \to A$.
When inputting $n = n_h p^h + \cdots + n_1 p + n_0$ $(0 \le n_i < p)$, it outputs $s \circ e_{n_0} \circ e_{n_1} \circ \cdots \circ e_{n_h} (m_0) \in A$.

3 Integral representations

3.1 G-functions

Definition 14. A G-function is a $f \in \mathbb{Q}[[x]]$ such that:
1. f is D-finite;
2. f has a non-zero radius of convergence in $\mathbb{C}$; and
3. its radii of convergence $\mathtt{Ray}_p(f)$ in $\mathbb{Q}_p$ satisfy

$$\prod_{p \text{ prime}} \mathtt{Ray}_p^{\leq 1}(f) > 0,$$

where $\mathtt{Ray}_p^{\leq 1}(f) = \min(1, \mathtt{Ray}_p(f))$.

Remark 15. Any pseudo-diagonal f is a G-function because, by globally boundedness, it satisfies the condition

$$(\forall p)\ \mathtt{Ray}_p(f) > 0\,, \quad (\text{faa}\, p)\ \mathtt{Ray}_p^{\leq 1}(f) = 1. \tag{2}$$

Definition 16 (Galočkin condition)**.** For $L \in \mathbb{Z}[x][\frac{d}{dx}]$, one can define, for each prime p, a *generic radius of convergence* $\mathtt{Ray}_p(L)$. Roughly speaking, it is the minimum of the radii of convergence of solutions of L near any point of the unit disc (of any extension of $\mathbb{Q}_p$). Galočkin's condition requires that $\prod_p \mathtt{Ray}_p(L) > 0$.

Now, we have a marvellous theorem, which was first proved by D. and G. Chudnowsky [12] (actually, the proof contained a mistake that was later corrected by Y. André). It is yet more remarkable when it is completed, as here, with a previous result of N. Katz [21]. As far as we know, it is the only theorem allowing one to go from properties of a single solution to properties of the (minimal) differential equation.

Theorem 17 (Chudnovsky)**.** Let f be a G-function. The *minimal* $L \in \mathbb{Z}[x][\frac{d}{dx}]$ such that $L(f) = 0$ fulfils Galočkin's condition. In particular, L only has regular singular points with rational exponents.

3.2 Geometric conjecture

The following conjecture is classical even if it is not precise.

Conjecture 18. G-functions come from geometry.

A weak form of this conjecture says that any G-function satisfies Condition (2) of remark 15. Such results seem entirely out of reach at the moment.

We will use the following avatar of Conjecture 18.[4]

Conjecture 19. Any pseudo-diagonal power series has an integral representation

$$f(x) = \int_\gamma F(x,\, x_1, \ldots, x_n)\, dx_1 \wedge \cdots \wedge dx_n,$$

where
- F is an algebraic function of $x,\, x_1,\, \ldots,\, x_n$; and
- γ is an n-cycle on the variety V of F ("not depending" on x).

In other words $f(x) = \int_\gamma \omega(x)$, where
- ω is an n-differential form on a smooth quasi-projective variety $V \to S$ of relative dimension n, defined over an open set $S \in \mathbb{P}_1$ that contains a punctured disc centred on 0; and
- γ is an n-cycle on V (actually on V_x and not depending on x up to homotopy).

3.3 Integral representations for DRFs

Insisting on giving a special role to one of the variables, we give it the index 0, hence considering DFRs in $n+1$ variables.

For $F = \frac{P}{Q} \in \mathbb{Q}[[x_0, \ldots, x_n]]$, a repeated application of the residue theorem easily shows that:

$$\mathrm{Diag}\Big(F\Big)(x) = \frac{1}{(2i\pi)^n} \int_\gamma \omega(x), \tag{3}$$

- $\gamma = \prod_{i=1}^n \gamma_i$, $\gamma_i = \{|x_i| = \varepsilon\}^{\circlearrowleft}$ (*evanescent cycle*); and
- $\omega = F\Big(\frac{x}{x_1\cdots x_n},\, x_1, \ldots, x_n\Big)\, \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$.

In other words, ω is an n-differential form on $V \to S$ with

$$\begin{aligned} V &: \{(x_0, \ldots, x_n)\,;\ x_0 \cdots x_n\, Q(x_0, \ldots, x_n) \neq 0\} \subset \mathbb{A}_S^{n+1}, \\ V \to S &: (x_0, \ldots, x_n) \mapsto x = x_0 \cdots x_n \qquad \text{(onto map)}. \end{aligned}$$

Remark 20. As Q is invertible in $\mathbb{Q}[[x_0, \ldots, x_n]]$, one has $Q(0, \ldots, 0) \neq 0$ and, in particular, $\exists \eta$ such that $Q(x_0, \ldots, x_n) \neq 0$ for $|x_i| < \eta$. Now, for $\varepsilon < \eta$ and $0 < |x| \leq \varepsilon^{n+1}$, $\gamma \subset V$ because, on γ,

$$0 < |x_0| = \left|\frac{x}{x_1 \cdots x_n}\right| = |x|\varepsilon^{-n} \leq \varepsilon < \eta\,.$$

3.4 De Rham cohomology

Sections 3.4 and 3.5 briefly summarise a rather abstract and multiform theory. The reader ready to accept finiteness properties of spaces $H^n_{\mathrm{DR}}(V)$ and $\mathcal{H}^n_{\mathrm{DR}}(V/S)$ can skip these sections. He will find the down-to-earth objects we are concerned with in Remarks 24 and 26. Actually, these remarks generally contain all we need.

Let V be an n-dimensional smooth algebraic complex variety, let V_{an} be the n-dimensional underlying analytic complex variety and let $V_\mathbb{R}$ be the $2n$-dimensional underlying real differential variety.

Each of them is endowed with the sheaves (of $\mathbb{C}$-vector spaces) of m-differential forms ($\Omega^m = \bigwedge^m \Omega^1$):

$$\Omega^m(V) \overset{\text{def}}{=} \{\textit{algebraic } m\text{-differential forms on } V\},$$

$$\Omega^m(V_{an}) \overset{\text{def}}{=} \{\textit{holomorphic } m\text{-differential forms on } V_{an}\},$$

$$\Omega^m(V_\mathbb{R}) \overset{\text{def}}{=} \{C^\infty\ m\text{-differential forms on } V_\mathbb{R}\}.$$

Definition 21. The *de Rham cohomology of* $V_\mathbb{R}$ is the cohomology $H^i_{\mathrm{DR}}(V_\mathbb{R})$ of the global section complex:

$$0 \longrightarrow \Gamma\Omega^0(V_\mathbb{R}) \overset{d}{\longrightarrow} \Gamma\Omega^1(V_\mathbb{R}) \longrightarrow \cdots \longrightarrow \Gamma\Omega^{2n}(V_\mathbb{R}) \longrightarrow 0.$$

By Poincaré's lemma, this complex is a resolution of the constant sheaf $\mathbb{C}$. Hence, $H^i_{\mathrm{DR}}(V_\mathbb{R}) = H^i(V(\mathbb{C}), \mathbb{C})$ and dimensional finiteness of $H^i_{\mathrm{DR}}(V_\mathbb{R})$ follows.

To avoid considering global sections, it is possible to replace the simple complex of vector spaces by a complex of

4 *Why limit ourself to the "central dimension" n?*
Actually, introducing more general cohomology spaces would not be an actual generalisation in view of the following statements.
Theorem (Lefschetz). *Let V be an n-dimensional complex projective algebraic variety.*
- *For W a hyperplane section such that V − W is smooth,*
$$H^i_{\mathrm{DR}}(V) \to H^i_{\mathrm{DR}}(W) \text{ is } \begin{cases} \textit{an isomorphism for} & i < n-1, \\ \textit{injective for} & i = n-1. \end{cases}$$
- *For* $r \geq 1$, *one has* $H^{n-r}_{\mathrm{DR}}(V) \overset{\sim}{\to} H^{n+r}_{\mathrm{DR}}(V)$.

sheaves but at the cost of looking at its *hypercohomology*. Sheaves of C^∞-differential forms being fine, we have

$$H^i_{\mathrm{DR}}(V_{\mathbb{R}}) = \mathbb{H}^i(\Omega^0(V_{\mathbb{R}}) \to \cdots \to \Omega^{2n}(V_{\mathbb{R}}))\ .$$

This leads us to give the following definitions.

Definition 22. The *analytic and algebraic de Rham cohomologies* are given by

$$H^i_{\mathrm{DR}}(V_{an}) = \mathbb{H}^i(\Omega^0(V_{an}) \to \cdots \to \Omega^n(V_{an}))\ ,$$
$$H^i_{\mathrm{DR}}(V) = \mathbb{H}^i(\Omega^0(V) \to \cdots \to \Omega^n(V))\ .$$

There is an analytic Poincaré lemma such that:

$$H^i_{\mathrm{DR}}(V_{an}) = H^i(V(\mathbb{C}), \mathbb{C}) = H^i_{\mathrm{DR}}(V_{\mathbb{R}})$$

and $H^i_{\mathrm{DR}}(V_{an})$ is finite dimensional. But there is no algebraic Poincaré lemma. To obtain dimensional finiteness of $H^i_{\mathrm{DR}}(V)$, we have to use a much deeper statement.

Theorem 23 (Grothendieck comparison theorem)**.** The natural map $H^i_{\mathrm{DR}}(V) \to H^i_{\mathrm{DR}}(V_{an})$ is an isomorphism.

Remark 24. In particular, when V is an *affine variety*,

$$H^n(V(\mathbb{C}), \mathbb{C}) = H^n_{\mathrm{DR}}(V) = \Omega^n(V)/d(\Omega^{n-1}(V))\ .$$

3.5 Relative de Rham cohomology

Actually, we are not looking at one variety but at a family of varieties. However, definitions can be made in the same way by just replacing cohomology with *higher direct images* ([18] III-8). More precisely, let $f : V \to S$ be a smooth morphism of smooth algebraic varieties over $\mathbb{Q}$ and let $\Omega^m(V/S)$ be the algebraic S-differential forms on V ([18] II-8) enjoying the following characteristic property:

$$f^*\big(\Omega^1(S)\big) \longrightarrow \Omega^1(V) \longrightarrow \Omega^1(V/S) \longrightarrow 0\ .$$

Then, the relative de Rham cohomology is defined by

$$\mathcal{H}^i_{\mathrm{DR}}(V/S) \overset{\mathrm{def}}{=} \mathbf{R}^i f_*(\Omega^\bullet(V/S))\ .$$

Proposition 25 ([22])**.** *$\mathcal{H}^i_{\mathrm{DR}}(V/S)$ is a sheaf on S endowed with an integrable connection called the Gauss-Manin connection.*

Remark 26. Roughly speaking, it is a differentiation under the integral sign. To compute $\frac{d}{dx}\int\omega$ with $\omega = f(x, x_1, \ldots, x_\ell)$ $dx_1 \wedge \cdots \wedge dx_n$, where variables $x_{n+1}, \ldots, x_\ell$ are linked by $\ell - n$ relations $g_j(x, x_1, \ldots, x_\ell) = 0$, one assumes $\frac{d}{dx}(x_i) = \frac{d}{dx}(dx_i) = 0$ for $1 \le i \le n$ and computes the other derivatives $\frac{d}{dx}(x_i)$ $(i > n)$ by means of the relations $g_j = 0$. The point is that the value so computed for $\frac{d}{dx}(\omega)$ does not depend on the choice of the "independent" variables $x_1, \ldots, x_n$ up to an *exact differential form* (i.e. it lies in $d(\Omega^{n-1}(V))$ and so disappears when integrating on a cycle γ).

Now, $\mathcal{H}^n_{\mathrm{DR}}(V/S) \otimes \mathbb{Q}(x)$ is a $\mathbb{Q}(x)$ finite dimensional vector space endowed with an action of the derivative $\frac{d}{dx}$. The *solutions* of this *module with connection* are given by $\int_\gamma$ for the various cycles γ in V.

For a given differential form $\omega \in \Omega^n(V)$, there is $L_\omega \in \mathbb{Z}[x][\frac{d}{dx}]$ such that $L_\omega(\omega) \in d(\Omega^{n-1}(V))$. The *periods* $\int_\gamma \omega(x)$ are solutions of L_ω (over S).

The *minimal equation* of $\int_\gamma \omega(x)$ is the minimal monic $L_{\omega,\gamma} \in \mathbb{Q}[x][\frac{d}{dx}]$ such that $L_{\omega,\gamma}(\int_\gamma \omega(x)) = 0$.

4 Toward a proof of Conjecture 10

4.1 To be or not to be evanescent

Let f be pseudo-diagonal. By Conjecture 19, it has an integral representation $f(x) = \int_\gamma \omega(x)$ for $\omega \in \Omega^n(V/S)$. The function f being defined on a punctured disc centred on 0, S contains this punctured disc. By *Hironaka's theorem on the resolution of singularities* [19], we can extend S to 0 in such a way that V_0 is a normal crossings divisor. By the *semi-stable reduction theorem* [23], up to ramifying x and replacing $\mathbb{Q}$ by some finite extension, one can even suppose V_0 to be reduced.

Practically, this means we can choose, for each point P in V_0, *local coordinates* $x_0, x_1 \ldots, x_n$ such that, near P, the "equation" of V is $x_0\, x_1 \cdots x_r = x \quad (0 \le r \le n)$, i.e. $x_i = 0$ $(0 \le i \le r)$ are the (local) equations of divisors contained in V_0.

Now, the integral representation of f can be written

$$f(x) = \int_\gamma F(x_0, \ldots, x_n)\, \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_r}{x_r} \wedge dx_{r+1} \wedge \cdots \wedge dx_n,$$

where F is an algebraic function.

Remark 27. Coefficients of F are algebraic numbers. A Galois argument shows that they can be taken in $\mathbb{Q}$.

Remark 28. The differential ω could have multiple poles along some of the divisors. An easy but rather tedious computation shows that these multiple poles can be eliminated by adding an exact differential form to ω.

Finally, when γ is (homotopic to) the evanescent cycle (see 3.3) around P, one finds

$$f(x) = \mathrm{Diag}\left(x_{r+1} \cdots x_n\, F(x\, x_{r+1} \cdots x_n, x_1, \ldots, x_n)\right)\ .$$

It remains to decide wether this is the case. For that purpose, we have to introduce filtrations.

4.2 Filtrations

The space $\mathcal{H}^n_{\mathrm{DR}}(V/S)$ is filtered by the order of poles along V_0. ω has weight at least $n + k$ if, near any $P \in V_0$, it can be written:

$$\omega = F(x_0, \ldots, x_n)\, \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_k}{x_k} \wedge dx_{k+1} \wedge \cdots \wedge dx_n$$

for some algebraic F with $F(0, \ldots, 0) \ne 0$.

The space $\mathcal{S}$ of solutions (near 0) of the Picard–Fuchs equation L_ω is endowed with the so-called *monodromy filtration* that we now define. Let T be the monodromy operator (turning once anticlockwise around 0) and let $N \overset{\mathrm{def}}{=} T - 1$. Then, $N(\log x) = 2i\pi$ and $N(f) = 0$ for any analytic function f. Then, N, acting on $\mathcal{S}$, is a nilpotent operator. The monodromy filtration is entirely determined by asking the operator N to decrease the weight by 2. More explicitly, when $f \in \mathcal{S}$ is associated to exactly k solutions involving logarithms, i.e.

$$f\ ,\quad \{f\ \log(x) + \cdots\}\ , \ldots\ ,\quad \{f\ \log^k(x) + \cdots\},$$

then these solutions have the following weights

$$-k\ ,\quad -k+2\quad ,\ldots,\quad +k.$$

The key point is the following statement:

Theorem 29 ([27])**.** *The two filtrations on $\mathcal{H}^n_{\mathrm{DR}}(V/S)$ and $\mathcal{S}$ just defined are set in duality through $(\gamma, \omega) \to \int_\gamma \omega(x)$.*

$$M_k\,(\mathcal{H}^n_{\mathrm{DR}}(V/S)) = \mathtt{Ann}\ M_{n-k+1}\ (\mathcal{S}).$$

Corollary 30 ([11])**.** Solutions of the Picard–Fuchs equation L_ω with minimum weight (i.e. $-n$) are DRFs.

In particular, minimal weighted solutions of any Picard–Fuchs equation are globally bounded. In some respect, this looks like a p-adic analogue to the Deligne weight-monodromy conjecture [14].

4.3 The case of hypergeometric functions

Definition 31. The Pochhammer symbol is defined by

$$(a)_0 = 1\,, \quad (a)_s = a(a+1)(a+2)\dots(a+s-1), \quad (s \geq 1)$$

and the (generalised) hypergeometric functions are

$${}_{n+1}F_n(\mathbf{a};\mathbf{b};x) = \sum_{s=0}^{\infty} \frac{(a_1)_s \dots (a_{n+1})_s}{(b_1)_s \dots (b_n)_s} \frac{x^s}{s!},$$

where $\mathbf{a} = (a_1, \dots, a_{n+1})$ and $\mathbf{b} = (b_1, \dots, b_n)$ are in $\mathbb{Q}^n$.

They are D-finite, satisfying $L_{\mathbf{a};\mathbf{b}}(f) = 0$ with

$$L_{\mathbf{a};\mathbf{b}} = x\prod_{i=1}^{n+1}\left(x\frac{d}{dx} + a_i\right) - x\frac{d}{dx}\prod_{i=1}^{n}\left(x\frac{d}{dx} + b_i - 1\right).$$

An integral representation can be given recursively:[5]

$${}_1F_0(a_1;;x) = (1-x)^{a_1}\,,\ {}_{n+1}F_n\,(\mathbf{a}, a_{n+1}; \mathbf{b}, b_n; x) =$$

$$\text{cste} \int_0^1 t^{a_{n+1}-1}(1-t)^{b_n - a_{n+1}-1}\ {}_nF_{n-1}\,(\mathbf{a};\mathbf{b};t\,x)\,dt$$

(the integration path seems not to be a cycle but it is classical to close it by coming back from 1 to 0 on the other side of the cut).

Then, it is a pseudo-diagonal if and only if it is globally bounded.

Remark 32. For hypergeometric functions ${}_{n+1}F_n$, the relative dimension is n and the differential equation is of order $n+1$. A solution f of the equation is of minimum weight, namely $-n$, if and only if there is a solution $f\ \log^n x + \cdots$. In that case, f is the unique analytic solution near 0 and the monodromy has maximum nilpotent order, namely $N^n \neq 0$. This is exactly the MUM (Maximum Unipotent Monodromy) condition required to be a Calabi-Yau differential equation. It seems likely that the convenient condition should be a "solution with minimum weight" instead of MUM. Unfortunately, it is much harder to check the minimum weight condition than the MUM condition because one almost always knows the order of the differential equation but the relative dimension of the integral representation is rarely known.

5 *Another integral representation for* ${}_3F_2$: let $N \geq 2$, let V_x be defined by equations

$$x_1^N + y_1^N + 1 = 0\,,\ x_2^N + y_2^N + 1 = 0\,,\ x_3^N + y_3^N + 1 = 0\,,$$
$$x_1 x_2 x_3 = x$$

(hence of relative dimension 2) and let

$$\omega(x) = x_1^p y_1^q x_2^r y_2^s y_3^t \frac{dx_1}{x_1}\frac{dx_2}{x_2}\,.$$

Then, for a suitable cycle γ, one has

$$\int_\gamma \omega(x) = \text{cste}\ {}_3F_2\left(\frac{-t}{N}, \frac{-p-q}{N}, \frac{-r-s}{N}; \frac{N-r}{N}, \frac{N-p}{N}; x^N\right).$$

In this setting, the fiber V_0 is the union of three families of N divisors with almost normal crossing. For instance, the divisors of the first family have the following equations:

$$x_1 = 0\,,\ y_1^N + 1 = 0\,,\ x_2^N + y_2^N + 1 = 0\,,\ x_3^N + y_3^N + 1 = 0\,.$$

Now, it is easy to check that the monodromy weight ${}_{n+1}F_n(\mathbf{a};\mathbf{b};x)$ is minus the number of integers amongst the b_i. Hence, Corollary 30 says that ${}_{n+1}F_n(\mathbf{a};\mathbf{b};x)$ is a DRF when $\mathbf{b} = (1,\dots,1)$. But this can be checked straightforwardly:

$${}_{n+1}F_n(\mathbf{a};1,\dots,1;x) = (1-x)^{a_1} \star \cdots \star (1-x)^{a_{n+1}}\,.$$

The following theorem proves Conjecture 10 for functions of weight 0.

Theorem 33 ([3])**.** Let a_i and b_i be in $\frac{1}{N}\mathbb{Z}$ and $b_i \notin \mathbb{Z}$. Then, the following conditions are equivalent:

1. ${}_{n+1}F_n(\mathbf{a};\mathbf{b};x)$ is globally bounded and of weight 0.
2. $(\forall k,\ (k,N) = 1)$, $\exp(2i\pi\,k\,a_i)$ and $\exp(2i\pi\,k\,b_i)$ are intertwined on the trigonometric circle.
3. The monodromy group of $L_{\mathbf{a};\mathbf{b}}$ is finite.
4. The function ${}_{n+1}F_n(\mathbf{a};\mathbf{b};x)$ is algebraic over $\mathbb{Q}(x)$.

In the general case (weight $k \in]-n, 0[$), it is still easy to decide whether ${}_{n+1}F_n(\mathbf{a};\mathbf{b};x)$ is globally bounded ([9]). We know many examples. The older one is ${}_3F_2(\frac{1}{9},\frac{4}{9},\frac{5}{9};\frac{1}{3},1;x)$ but there are a lot of others: $(\frac{1}{7},\frac{2}{7},\frac{4}{7};\frac{1}{2})$, $(\frac{1}{11},\frac{2}{11},\frac{6}{11};\frac{1}{2})$, etc. Moreover, one knows several globally bounded ${}_4F_3$ with weight -2 or -1.[6]

Now, the open question is to decide whether or not these pseudo-diagonals are actually DRFs.

5 A quasi-proof of Conjecture 12

5.1 Proof overview

In this section, p will be a fixed prime.

The p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}_p$ can be extended to $\mathbb{Q}_p(x)$ by setting

$$\left|\frac{\sum a_i x^i}{\sum b_i x^i}\right|_p \overset{\text{def}}{=} \frac{\max|a_i|}{\max|b_i|}\,.$$

This absolute value is usually called the "*Gauss norm*". Let E_p be the completion of $\mathbb{Q}_p(x)$ for $|\cdot|_p$.

Definition 34. An *E_p-differential module $\mathcal{H}$* is a finite dimensional E_p-vector space endowed with a $\mathbb{Q}_p$-linear action of $\frac{d}{dx}$ satisfying, for $a \in E_p$ and $m \in \mathcal{H}$,

$$\frac{d}{dx}(am) = \frac{d}{dx}(a)m + a\frac{d}{dx}(m)\,.$$

A morphism $\theta : M \to N$ of E_p-differential modules is a morphism of E_p-modules such that $\theta(\frac{d}{dx}m) = \frac{d}{dx}\theta(m)$.

For instance, the E_p-differential module $E_p[\frac{d}{dx}]/E_p[\frac{d}{dx}].L$ is associated to each $L \in E_p[\frac{d}{dx}]$. The cyclic base theorem says that this is an equivalence of categories.

Definition 35. Let $\mathcal{H}$ be an E_p-differential module. Its *Frobenius* $\mathcal{H}^\varphi$ is its inverse image by the change of variable $x \to x^p$. When the E_p-differential modules $\mathcal{H}$ and $\mathcal{H}^\varphi$ are isomorphic, then $\mathcal{H}$ will be said to *have a Frobenius*. If G is the

6 Given an example, one can get others by "permutation": $a_i \to k\,a_i$, $b_i \to k\,b_i$ $(k,N) = 1$. The number of distinct examples so obtained is difficult to foresee:
$(\frac{2}{9},\frac{4}{9},\frac{5}{9};\frac{1}{3},1) \to (\frac{2}{9},\frac{5}{9},\frac{7}{9};\frac{2}{3},1) \to (\frac{1}{9},\frac{7}{9},\frac{8}{9};\frac{1}{3},1) \to (\frac{4}{9},\frac{5}{9},\frac{8}{9};\frac{2}{3},1)$
but $(\frac{1}{9},\frac{4}{9},\frac{7}{9};\frac{1}{3},1) \to (\frac{2}{9},\frac{5}{9},\frac{8}{9};\frac{2}{3},1)$. Let's notice that when listing such examples, one can limit oneself to the *primitive* case, excluding Hadamard products of simpler ones.

matrix of $\frac{d}{dx}$ in some base of $\mathcal{H}$, this happens if and only if there exists an invertible matrix $H \in \mathrm{Gl}(E_p)$ such that $px^{p-1}G(x^p)\,H = \frac{d}{dx}(H) + H\,G$.

Roughly speaking, when $L \in E_p[\frac{d}{dx}]$, its "Frobenius" is the $L^\varphi \in E_p[\frac{d}{dx}]$ such that $L^\varphi(f(x^p)) = 0$ as soon as $L(f(x)) = 0$.

Remark 36. Most of the E_p-differential modules $\mathcal{H}$ we will encounter "come from" $\mathbb{Q}(x)$-differential modules $\mathcal{H}_0$, namely $\mathcal{H} = E_p \otimes_{\mathbb{Q}(x)} \mathcal{H}_0$. Then, the Frobenius is still defined over $\mathbb{Q}(x)$: $\mathcal{H}^\varphi = E_p \otimes_{\mathbb{Q}(x)} \mathcal{H}_0^\varphi$ but the isomorphism between $\mathcal{H}$ and $\mathcal{H}^\varphi$ comes very rarely from a $\mathbb{Q}(x)$-morphism.

Now, let $V, S, \omega \in \mathcal{H}^n_{\mathrm{DR}}(V/S)$, γ and $f(x) = \int_\gamma \omega(x)$ be defined as in Conjecture 19. We will use the following notation:
- $V \to V_p \stackrel{\text{def}}{=}$ modulo p reduction of V (it does exist faa p),
- $\mathcal{H}^n_p(V_p/S_p) \stackrel{\text{def}}{=}$ "*p-adic relative cohomology*" of V_p (to be defined).

Then, to deduce Conjecture 12 from Conjecture 19, it would be enough to prove the following points:
1. $\mathcal{H}^n_p(V_p/S_p)$ is an E_p-module with connection and there is an injection $\mathcal{H}^n_p(V_p/S_p) \to \mathcal{H}^n_{\mathrm{DR}}(V/S) \otimes E_p$. By functoriality, the Frobenius $V_p \to V_p$ endows the E_p-differential module $\mathcal{H}^n_p(V_p/S_p)$ with a Frobenius.
2. Let $f \in \mathbb{Z}_p[[x]]$ be a solution of some differential equation $E_p[\frac{d}{dx}]$ with a Frobenius (i.e. the corresponding differential module has a Frobenius). Then, for all $h > 1$, it is p-automatic modulo p^h.

We will explain how to prove Point 1 in Section 5.2 and we will prove Point 2 in Section 5.3.

5.2 Cohomology of varieties in characteristic p

Point 1 just gives a precise sense to the imprecise sentence "any Picard Fuchs equation is endowed with a Frobenius for almost all p". The prototype result of this type is the following.

Theorem 37. Dwork-Katz theory [20]. Let $V \subset \mathbb{P}^{n+1}_{\mathbb{Q}(x)}$ be a *non-singular and in general position* hypersurface with (homogenous) equation

$$F(x, X_0,..., X_{n+1}) = 0\,, \quad F \in \mathbb{Q}[x][X_0,...,X_{n+1}]$$

and $R(x) := \texttt{resultant}\{X_i \frac{\partial F}{\partial X_i}\}_{0\le i\le n+1} \neq 0$ (i.e. the $X_i \frac{\partial F}{\partial X_i}$ have no common zero). Then, $\mathcal{H}^n_p(V_p/S_p)$ (and $\mathcal{H}^n_p((\mathbb{P}^{n+1} - V)_p/S_p)$) can be defined, with "good" p being those verifying $|R|_p = 1$, as
- $\mathcal{H}^n_p(V_p^\emptyset/S_p) \sim \mathcal{L}/(D_x\mathcal{L} + \sum_i D_{X_i}\mathcal{L})$,
- $\mathcal{H}^n_p((\mathbb{P}^{n+1} - V)_p^\emptyset/S_p) \sim \mathcal{L}/(\sum_i D_{X_i}\mathcal{L})$, where $V^\emptyset = V \bigcap \{X_0 \cdots X_{n+1} \neq 0\}$ and D is the derivative $\frac{d}{dx}$ "twisted" by $\mathrm{e}^{-\pi x F}$.

Remark 38. As we know V/S only through Conjecture 18, it is hazardous to presume its properties. Nevertheless, by removing some points if necessary, we can assume V and S to be smooth affine varieties and, using Bertini's theorem, we can also assume the morphism $V \to S$ to be smooth. So, the problem is not really the non-singular property but the hypersurface assumption. Dwork proposed generalisations of Theorem 37 concerning some particular complete intersections but, to this day, we do not have a satisfactorily general statement.

To totally agree with our approach in Section 5.1, a *functorial* p-adic cohomology theory for varieties in characteristic p would be needed. For smooth affine varieties, it does exist, namely the Monsky-Washnitzer theory [26]. The case of general varieties is also solved in [2] using the clever notion of "special modules". Unfortunately, there is not yet any relative theory. Among the numerous difficulties to be overcome, there is the non-validity of Bertini's theorem in characteristic p.

5.3 Frobenius and p-automaticity

Firstly, there is an elementary statement.

Proposition 39. For any solution f of a differential equation with Frobenius, there exist d and $a_i \in E_p$ such that f is a solution of a difference equation in the Frobenius:

$$\sum_{i=0}^{d} a_i(x)\, f(x^{p^i}) = 0\,, \qquad (a_0\, a_d \neq 0). \qquad (C_d)$$

Then, $f(x^p)$ being not far p-adically from f^p, one gets:

Theorem 40 ([8] 6.5)**.** Any $f \in \mathbb{Z}_p[[x]]$ satisfies a relation (C_d) if and only if it is a limit (for the Gauss norm) of functions in $\mathbb{Z}_p[[x]]$ *algebraic* over E_p.

Finally, there is the p-adic generalisation of a classical statement over finite fields.

Theorem 41 ([8] 8.1)**.** When $f \in \mathbb{Z}_p[[x]]$ is *algebraic* over E_p, then, for all $h \geq 1$, it is *p-automatic* modulo p^h.

Summarising the three statements, we get that any $f \in \mathbb{Z}_p[[x]]$ solution of a differential equation with Frobenius (as pseudo-diagonals should be for almost all p) is p-automatic modulo p^h for all h.

Remark 42. The proof does not make any distinction between globally bounded functions and locally bounded functions (i.e. whose denominators of coefficients can contain infinitely many primes, each one with bounded powers). For instance, the function ${}_3F_2\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}; \frac{5}{3}, 1; x\right)$ is so proved to be globally automatic.

6 Self-adjoint differential equations

Definition 43. Let $L^* = \sum(-1)^i \frac{d^i}{dx^i} f_i(x)$ be the adjoint of $L = \sum f_i(x) \frac{d^i}{dx^i}$. Then, L is *self-adjoint* when the modules associated to L and L^* are *isomorphic*.

For irreducible L, self-adjointness means existence of M and Q such that $\deg M < \deg L$ and $L\,M = Q\,L^*$.[7]

To be self-adjoint is a rather hard constraint but when computing irreducible factors of differential equations satisfied by DRFs, they (almost) all appear to be self-adjoint! As the (irreducible) minimal differential equation of ${}_3F_2(\frac{1}{9}, \frac{4}{9}, \frac{5}{9}; \frac{1}{3}, 1; x)$ is not self-adjoint, it is natural to ask if that is an indication it is not a DRF.

7 As $\hom(L, L^*) = L \otimes L = \mathrm{ext}^2(L) \oplus \mathrm{sym}^2(L)$, L is self-adjoint if and only if $\mathrm{ext}^2(L)$ or $\mathrm{sym}^2(L)$ has a rational non-zero solution. It can be seen in its Galois differential group.

The first answer is that the DRF ${}_3F_2(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; 1, 1; x)$ also has a non-self-adjoint irreducible differential equation.

To go further, we call on the Deligne-Steenbrink-Zucker theorem, saying that Gauss-Manin connections are "variations of *polarised* mixed Hodge structures". In particular, the associated graded modules are Gauss-Manin connections of smooth and projective varieties. They are self-adjoint by Poincaré duality. Turning back to the Picard–Fuchs differential equation we started with, this means that it is a product of not necessarily irreducible self-adjoint factors. Now, for rather small examples, it is no more surprising that these self-adjoint factors are very often irreducible.

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