COEFFICIENT ASYMPTOTICS OF MULTIVARIABLE ALGEBRAIC POWER SERIES AND RATIONAL POWER SERIES WITH PSEUDO MULTIPLE POINTS

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ABSTRACT

COEFFICIENT ASYMPTOTICS OF MULTIVARIABLE ALGEBRAIC POWER SERIES AND RATIONAL POWER SERIES WITH PSEUDO MULTIPLE POINTS

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Analytic combinatorics in several variables (ACSV) generalizes the coefficient extraction of generating functions in one variable to several variables. Current developments in ACSV mostly concern rational or meromorphic generating functions by first representing coefficients via the multivariate Cauchy integral formula and then using Morse-theoretic homology arguments to deform the integral chain so that the integral becomes a sum of saddle point integral. Coefficient asymptotics are previously known in the case when critical points of the Morse function are smooth points [PW02], multiple points [PW04, BMP24b], and quadratic cone points [BP11]. We generalize the result for multiple points to pseudo multiple points and show that these two kinds of points are similar under mild conditions. The complexity hierarchy of ACSV goes up from rational functions to algebraic functions. By embedding the coefficient for an algebraic generating function as an elementary diagonal of a rational generating function with one more variable, [GMRW22] shows that the problem can be reduced to the well-known case of rational generating functions. We take a different approach, by lifting the torus in the Cauchy integral formula to the surface of the defining polynomial of the algebraic function, taking advantage of the covering space property of the surface. This leads to a similar computation to [GMRW22], avoids the Morse-theoretic homology arguments, and brings the transparency one level up.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

$\mathsf{amoeba}(Q)$	the amoeba of a polynomial Q
br	branching locus, the set of $\mathbf{z}$ where roots of $P(\mathbf{z},\cdot)$ coalesce
$\widetilde{\mathsf{br}}$	vertical branching locus, the set of $(\mathbf{z}, f)$ on $\mathcal{V}_P$ such that $\partial P/\partial f = 0$
$\mathbb{C}_*$	$\mathbb{C} - \{0\}$
$\mathcal{C}_B$	The imaginary fiber in component B
$\mathcal{C}_{\boldsymbol{\sigma}}$	The imaginary fiber in the component at which the critical point $\boldsymbol{\sigma}$ is the minimizer of $h_{\hat{\mathbf{r}}}$.
d	dimension of the multiarray
$\mathrm{H}_d(X)$	d-th singular homology group of X with complex coefficients
$\mathrm{H}^d(X)$	d-th singular cohomology group of X with complex coefficients
$\mathrm{H}^d_{\mathrm{dR}}(X)$	d-th de Rham cohomology group of X with complex coefficients
hom	leading homogeneous term
$h_{\hat{\mathbf{r}}}$	height function $h_{\hat{\mathbf{r}}}(\mathbf{z}) = -\hat{\mathbf{r}} \cdot \mathrm{Relog}(\mathbf{z})$
$R[\mathbf{z}]$	$R[z_1, \cdots, z_d]$, ring of polynomials of $\mathbf{z}$ with coefficients in a commutative ring R
$R[[\mathbf{z}]]$	$R[[z_1, \cdots, z_d]]$, ring of formal power series of $\mathbf{z}$ with coefficients in a commutative ring R
$k\{\mathbf{z}\}$	$k\{z_1, \cdots, z_d\}$, ring of convergent power series of $\mathbf{z}$ with coefficients in $k = \mathbb{R}$ or $\mathbb{C}$
$\mathcal{M}$	$\mathbb{C}^d_* - \mathcal{V}$, the domain of holomorphy for $\omega := \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, d\mathbf{z}$
$\mathbf{m}$	$(m_1, \cdots, m_s)$ denoting the multiplicities of s factors
$[n]$	the set $\{1, 2, \cdots, n\}$
$\mathcal{O}_{\mathbf{p}}$	ring of analytic germs at $\mathbf{p}$
$\mathcal{O}$	ring of analytic germs at $\mathbf{0}$
${}_d\mathcal{O}_{\mathbf{p}}$	ring of analytic germs at $\mathbf{p} \in \mathbb{C}^d$
${}_d\mathcal{O}$	ring of analytic germs at $\mathbf{0} \in \mathbb{C}^d$
ω	a holomorphic form on $\mathcal{M}$
$\tilde{Q}$	the square-free part of Q, i.e. the generator of the radical ideal of $\langle Q \rangle$.
$\mathbf{r}$	multi-index $(r_1, \cdots, r_d)$
$\lvert\mathbf{r}\rvert$	$\lvert r_1\rvert + \cdots + \lvert r_d\rvert$
$\hat{\mathbf{r}}$	the normalized direction vector $\mathbf{r}/\lvert\mathbf{r}\rvert = (\hat{r}_1, \cdots, \hat{r}_d)$
$\mathrm{Relog}(\mathbf{z})$	$(\log\lvert z_1\rvert, \cdots, \log\lvert z_d\rvert)$
$\mathcal{S}$	a stratum in a Whitney stratification
$\mathcal{S}(\mathbf{p})$	the stratum containing $\mathbf{p}$ in a Whitney stratification
T	a torus in the domain of convergence $\mathcal{D}$ of the series for a generating function
$\mathcal{T}_{\mathbf{p}}$	a torus around the d hyperplanes L_i at $\mathbf{p}$, $\{\mathbf{z} \in \mathbb{C}^d_* : \lvert L_1(\mathbf{z})\rvert = \cdots = \lvert L_s(\mathbf{z})\rvert = \epsilon\}$ in Chapter 6.3

$T(\mathbf{w})$	$\{\mathbf{z} \in \mathbb{C}^d : \lvert z_i \rvert = \lvert w_i \rvert\}$, the torus centered at zero that passes through $\mathbf{w}$
$\tau_{\boldsymbol{\sigma}}$	linking torus at the critical point $\boldsymbol{\sigma}$
$\mathcal{V}$	analytic hypersurface, normally referring to the singular variety of the function F
$\widetilde{\mathcal{V}}$	The surface defined by the minimal polynomial P of an algebraic function F in Chapter 4
$\mathcal{V}_*$	$\mathcal{V} \cap \mathbb{C}_*^d$
$\mathcal{V}_Q$	$\{\mathbf{z} \in \mathbb{C}^d : Q(\mathbf{z}) = 0\}$
$\mathbf{z}$	$(z_1, \cdots, z_d)$
$d\mathbf{z}$	the holomorphic volume form $dz_1 \wedge \cdots \wedge dz_d$
$\lvert\mathbf{z}\rvert$	$(\lvert z_1 \rvert, \cdots, \lvert z_d \rvert)$
$\mathbf{z}^{\mathbf{r}}$	$z_1^{r_1} \cdots z_d^{r_d}$
$[\mathbf{z}^{\mathbf{r}}]F(\mathbf{z})$	coefficient of $\mathbf{z}^{\mathbf{r}}$ in the power (or Laurent) series $F(\mathbf{z})$
$\mathbf{0}$	$(0, \cdots, 0) \in \mathbb{C}^d$
$\mathbf{1}$	$(1, \cdots, 1) \in \mathbb{C}^d$
∇	gradient
$\nabla_{\log}$	logarithmic gradient, $(\nabla_{\log} f)(\mathbf{z}) = (z_1 f_{z_1}, \cdots, z_d f_{z_d})$

CHAPTER 1

INTRODUCTION

Counting seems like an easy task for most of people. Once we have the concept of numbers in the very early stage of childhood, we start to count things with our ten fingers. When the number of things is not very large, we are often confident on our manual counting. We may make mistakes the first time, but it won't take long to count the second and the third time to see if numbers match. Counting can also be a hard mission when the number gets large or when we need to count things that we can't see with our eyes. For example, count the number of ways a rook on the bottom-left corner can move to the top-right corner on a chess board, if only right and up moves are allowed. We can't see by our eyes these many possible choices, but need to enumerate all possible moves and then count them up. Manual enumeration takes time and is prone to missing some cases. Things can even go worse when the number gets large. For example, let's count the number of binary trees with n nodes. If counting when $n = 10$ is interesting for some talented brains, doing it for n equal to a million is definitely an intimidating and dull job. Indeed, in this paper, we don't care about what exact number is when n is large. After all, the number of binary trees with n nodes will go to infinity as n goes to inifinty. Therefore, we are more interested in how the number grows, or to use the mathematical phrase, the asymptotic behavior of the number.

Analytic Combinatorics makes counting easier and much more elegant. We first encode these counting numbers into a formal series, an algebraic object. Then we treat the algebraic object as an analytic object called the generating function by analyzing its singularities. These singularities give us enough information about the numbers encoded by the generating function. For example, Let c_n be the number of binary trees with n nodes. We can encode c_n into a formal power series $F(z) = \sum_{n=0}^{\infty} c_n z^n$. There is nothing fancy here until we discover a recursive relation. A binary tree with n nodes consists of the root node and its two child trees (possibly empty) of size i and $n - i - 1$. In other words, $c_n = \sum_{i=0}^{n-1} c_i c_{n-i-1}$. We let $c_0 = 1$ because there is only one empty tree. By the recursive relation, we can compute $c_1 = 1, c_2 = 2, c_3 = 5$ and so on. With a computer, we can use this recursive relation to calculate c_n for arbitrarilly large n if the processor of the computer allows. However, we still don't know the growth order of c_n as $n \to \infty$. Is it exponentially growing or just polynomially growing? Can we determine the growth order to high precision? Notice that $F^2(z) = \sum_{n=0}^{\infty} d_n z^n$ where $d_n = \sum_{i=0}^{n} c_i c_{n-i}$. By the previous recursive relation, $d_n = c_{n+1}$ and therefore $zF^2(z) + c_0 = F(z)$ by comparision of coefficients. In particular, $F(z)$ is a solution of f in the equation $zf^2 - z + 1 = 0$. Since $c_0 = F(0) = 1$, we know that $F(z)$ is

$$F(z) = \frac{1 + \sqrt{1-4z}}{2z}$$

and so the formal power series $\sum_{n=0}^{\infty} c_n z^n$ converges to the above function with a radius of convergence $1/4$. The algebraic object becomes an analytic object. From the radius of convergence we can immediately deduce that as $n \to \infty$, $\limsup \frac{1}{n} \log c_n \to \log 4$. The generalized binomial theorem tells us that $c_n = \frac{1}{n+1}\binom{2n}{n}$ and Sterling's formula gives an estimate of $O(\frac{4^n}{\sqrt{\pi n^3}})$. To get the exact order of growth on c_n, one can use Cauchy integral forumula on $F(z)$ to see that

$$c_n = \int_T F(z) z^{-n-1} \, dz$$

where $T = \{z : |z| = \epsilon\}$ for some $\epsilon < 1/4$. In particular, $F(z)$ is an algebraic generating function and [FS09, Chapter VII.7] discusses how to extract coefficient asymptotics. Chapter 4.4.1 provides a new method to estimate c_n. It gives $c_n = \frac{4^n}{\sqrt{\pi}}\left(n^{-3/2} + O(n^{-5/2})\right)$. The growth order can be calculated up to any precision and thus it gives a more precise answer than the Sterling's formula. For this problem, the method in Chapter 4.4.1 may be overkill, but for the other examples in Chapter 4.4, the method is the best known.

Any power series representing an analytic function in a small neighborhood of the origin has coefficeint growth that is at most exponential. That is, the coefficient c_n of $F(z) = \sum_{n=0}^{\infty} c_n z^n$ is asymptotic to $A^n s(n)$ where A is called the exponential growth and $s(n)$ is called the subexponential growth where $\frac{1}{n} \log s(n) \to 0$. In the above particular example, $A = 4$ and $s(n) = \sqrt{\pi} n^{-3/2} + O(n^{-5/2})$. The exponential growth is the roughest estimate one can get and the location of the singularity with the smallest modulus determines the exponential growth. If the first singularity is p, then $|p|$ is the expoential growth. This is called the first principle of coefficient asymptotics in [FS09, Chapter IV.1]. The second principal of coefficient asymptotics is that the nature of the singularity determines the subexponential factor $s(n)$. For example, if the singularity is a pole, then $s(n)$ is a polynomial [FS09, Theorem VI.10]. If the singularity is an algebraic branching point (e.g. $z = 1/4$ in the previous example), then the subexponential growth is a Puiseux series [FS09, Theorem VII.8].

In Analytical Combinatorics in Several Variables (ACSV), we try to generalize the above two principles to multivariate generating functions. [FS09, Part A] and [PWM24, Part I] introduce symbolic methods to get the generating function for interesting combinatorical objects. We are more interested in extracting coefficient asymptotics of these generating functions once given. The singularities of these functions tell us enough information on the growing order of the coefficient. The masterpiece [FS09] is on analytic combinatorics in one variable, that is, when the generating function is univariate. In ACSV, we work on a multivariate generating function $F(\mathbf{z}) = \sum_{\mathbf{r}\in\mathbb{Z}^d} a_{\mathbf{r}}\mathbf{z}^{\mathbf{r}}$ that converges in some domain to an analytic function. Here $\mathbf{r}$ is a multi-index $(r_1, \cdots, r_d)$ and $\mathbf{z}^{\mathbf{r}}$ stands for $z_1^{r_1}\cdots z_d^{r_d}$. The multi-index $\mathbf{r}$ can either be all non-negative thus giving a convergent power series, or of mixed signs thus giving a convergent Laurent series. ACSV is a natural generalization of single variable analytic combinatorics with which combinatorists are often more familiar. Indeed, ACSV gives us much more freedom to enumerate things not based just on one index n, but on an arbitrarily high dimensional multi-index $\mathbf{r}$. For example, we can enumerate the number of ways to pick r_1 people out of $r_1 + r_2$ people. These numbers can be embedded as coefficients into the power series $F(z_1, z_2) = \sum_{(r_1,r_2)\in\mathbb{N}^2} a_{r_1,r_2} z_1^{r_1} z_2^{r_2} = \frac{1}{1-z_1-z_2}$.

The coefficient $a_{\mathbf{r}}$ can be extracted using the multivariate Cauchy integral formula

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_T \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}} \tag{1.1}$$

where T is a torus in the open domain of convergence in $\mathbb{C}^d$ of the series for F and $d\mathbf{z}/\mathbf{z}$ is the logarithmic volume form $dz_1 \wedge \cdots \wedge dz_d / \prod_{j=1}^d z_j$. We are interested in asymptotic for $a_{\mathbf{r}}$ as $\mathbf{r} \to \infty$ with the **direction** $\hat{\mathbf{r}} := \mathbf{r}/|\mathbf{r}|$ varying in a compact neighborhood of a given unit vector in $\mathbb{RP}^{d-1}$.

If $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$ is a rational function, meaning P and Q are coprime polynomials, then the singular variety of $F(\mathbf{z})$ is exactly the complex algebraic hypersurface $\mathcal{V} := \{\mathbf{z} : Q(\mathbf{z}) = 0\}$. By finding critical points on $\mathcal{V}_*$, we can often represent the integral (1.1) as a sum of saddle point integrals at these critical points. [PW02] first analyzed the case when $\mathcal{V}$ is smooth. When $\mathcal{V}$ is an algebraic variety with singularities, things are more involved. If $\mathcal{V}$ is locally smooth at a critical point, then we can still use the method in [PW02]. If it's not a smooth point, then it can be a so-called multiple point (see Definition 5.2 and 5.8), a cone point, or a type of point get to be analyzed. In particular, [PW04] analyzes the case of multiple points. [BP11] uses results from hyperbolic PDEs to analyze the case of quadratic cone points. [BMP24b], though published much later than [PW04], analyze the case when $\mathcal{V}$ is a hyperplane arrangement, a subcase of multiple points. Some other cases are analyzed in [RW07, BMP24a]. The difficulty of extracting the growth order of $a_{\mathbf{r}}$ increases when we move from rational functions to algebraic functions, meaning a d-variate function $F(\mathbf{z})$ satisfying $P(\mathbf{z}, F(\mathbf{z})) = 0$ where P is a $(d+1)$-variate polynomial. [RW07, RW12]

This paper has two main results on two parts of ACSV theory. The first result is a new method of determining coefficient asymptotics of algebraic generating functions. [RW07, RW12, GMRW22] previously proposed a method of embedding a d-variate algebraic generating function as a diagonal in a $(d+1)$-variate rational function and then apply well-developed ACSV theory on rational functions. Instead of embedding the algebraic function, we lift the torus T in equation (1.1) to the variety $\widetilde{\mathcal{V}} := \{(\mathbf{z}, f) : P(\mathbf{z}, f) = 0\}$ in $\mathbb{C}^{d+1}$ and integrate $\mathbf{z}^{-\mathbf{r}} f \, d\mathbf{z}/\mathbf{z}$ over the lifted torus. We will introduce the result in Chapter 1.3 and discuss it in detail in Chapter 4. The second result is on rational generating functions when the critical point is a minimal pseudo multiple point. Even though there is already theory in ACSV for minimal multiple points, determining whether or not a point $\mathbf{p}$ is a multiple point is notoriously hard because we need to factor an analytic function Q, locally determining $\mathcal{V}$ at $\mathbf{p}$, in the ring of analytic germs at $\mathbf{p}$. Factorization in this ring is not computationally feasible. We thus invent the term, pseudo multiple point, for points $\mathbf{p}$ where $\mathsf{hom}(Q, \mathbf{p})$ factors into linear factors. This condition is a necessary condition for $\mathbf{p}$ to be a multiple point. Determining whether or not $\mathbf{p}$ is a pseudo multiple point is a much easier task because $\mathsf{hom}(Q, \mathbf{p})$ is a homogeneous polynomial and we can easily factorize it in the polynomial ring. With some mild conditions, we show in Chapter 6.2 that all pseudo multiple points in $\mathbb{C}^2$ are multiple points. For arbitrary dimensions, we show in 6.3 that with a much stronger conditions, a pseudo multiple point can be treated as a multiple point for the sake of coefficient asymptotics.

We begin with a brief introduction to analytic combinatorics in several variables (ACSV) before we state the main result of this paper. Readers familiar with ACSV can feel free to skip the overview. The following overview by no means covers all aspects of ACSV but we try our best to give a concise yet accurate depiction of the big picture. For more details on ACSV, one can refer to [PWM24] with more general Morse-theoretic arguments or the introductory text [Mel21] which focuses more on smooth point asymptotics and its applications on lattice path enumeration.

1.1. From One Variable to Several Variables

Let $\mathcal{A}$ be a collection of combinatorial objects about which we are interested. Let $\phi : \mathcal{A} \to \mathbb{N}^d$ be a map that sends each element $x \in \mathcal{A}$ into a d-vector with non-negative numbers on each coordinate. The map ϕ is called a **weight map** and it partitions the collection $\mathcal{A}$ into $\mathcal{A}_{\mathbf{r}} := \{x \in \mathcal{A} : \phi(x) = \mathbf{r}\}$ indexed by $\mathbf{r} \in \mathbb{N}^d$. If each $\mathcal{A}_{\mathbf{r}}$ is finite, then we call $\mathcal{A}$ together with the weight map ϕ a **combinatorial class** in d variables. We then define the formal power series of the combinatorial class $\mathcal{A}$ by

$$F(\mathbf{z}) = \sum_{\mathbf{r} \in \mathbb{N}^d} |\mathcal{A}_{\mathbf{r}}| \mathbf{z}^{\mathbf{r}}$$

where $|\mathcal{A}_{\mathbf{r}}|$ is the size of the finite set $\mathcal{A}_{\mathbf{r}}$. More generally, given a d-array $\{a_{\mathbf{r}}, \mathbf{r} \in \mathbb{N}^d\}$, we can define the formal power series $F(\mathbf{z}) = \sum_{\mathbf{r} \in \mathbb{N}^d} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$. This formal power series is called the generating functions for the array $\{a_{\mathbf{r}}\}$. We study the case when this formal power series converges in a neighborhood of $\mathbf{0} \in \mathbb{C}^d$ so that it is both an algebraic object and an analytic object.

The current stage of ACSV theory is mainly on rational (or meromorphic) generating functions, which is the subject of study in [PWM24]. Chapter 5 and 6 also consider rational functions only. Therefore, our introduction chapter to ACSV are based mainly on rational functions. We directly talk about methods on algebraic functions in Chapter 3 and 4 without an introduction. In particular, the embedding method in Chapter 3 requires ACSV theory on rational functions. The lifting method we develop in Chapter 4, however, bypasses most of topological arguments used in ACSV theory on rational functions, and only involves saddle point integral.

Notation

There is a convention in ACSV that we use bold font letters to represent the vector and use the normal letters to represent scalars. For example, $\mathbf{z}$ stands for the multivariable $(z_1, \cdots, z_d)$ and $\mathbf{r}$ stands for the multi-index $(r_1, \cdots, r_d)$. Here the dimension d is often implicitly understood in each particular case and we reserve the letter d for generic dimensions of our combinatorial class $\mathcal{A}$. Another simplified notation is $\mathbf{z}^{\mathbf{r}}$ which stands for $z_1^{r_1} \cdots z_d^{r_d}$ and so $\mathbf{z}$ and $\mathbf{r}$ must match dimensions in this expression. We also define a norm on the multi-index $\mathbf{r}$ by $|\mathbf{r}| = |r_1| + \cdots |r_d|$. The coefficient of $\mathbf{z}^{\mathbf{r}}$ in the power series $F(\mathbf{z})$ will be denoted by $[\mathbf{z}^{\mathbf{r}}]F(\mathbf{z})$. To simplify notations, when $d = 2$ or 3, we often use x, y, z and r, s, t to represent z_1, z_2, z_3 and r_1, r_2, r_3.

For example, we can consider $\mathcal{A}$ to be the collection of all colorings on n objects by d colors. A natural choice for the weight map ϕ is to let $\phi(x)$ be $\mathbf{r} = (r_1, \cdots, r_d)$ where r_i is the number of apperances of color i in the particular coloring x on $|\mathbf{r}|$ objects. Then $\mathcal{A}_{\mathbf{r}}$ is finite and in particular, $|\mathcal{A}_{\mathbf{r}}| = \binom{|\mathbf{r}|}{r_1, \cdots, r_d}$, the multinomial coefficient.

Directions

We are interested in the asymptotic behavior of $a_{\mathbf{r}}$ in the power series $\sum_{\mathbf{r} \in \mathbb{N}^d} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ as $\mathbf{r} \to \infty$, that is $r_i \to \infty$ for every i. Here $a_{\mathbf{r}}$ stands for $a_{r_1, \cdots, r_d}$, the coefficient of $z_1^{r_1} \cdots z_d^{r_d}$. When the dimension $d = 1$, $F(x) = \sum_{n=0}^{\infty} a_n x^n$ and there is no ambiguity for n going to infinity. However, when $d > 1$, there are multiple directions for $\mathbf{r} \to \infty$. For example, when $d = 2$, the multi-index $\mathbf{r}_n = (n, 2n)$ and $\mathbf{r}'_n = (n, 3n)$ both go to infinity as $n \to \infty$.

Definition 1.1 (Direction). *A direction is a ray in $\mathbb{R}^d$ defined by all positive multiples of a fixed non-zero vector in $\mathbb{R}^d$. In other words, a direction is an element in the projective real space $\mathbb{RP}^{d-1}$.*

In the practice of ACSV, we often say the direction of a multi-index $\mathbf{r}$ is the normalized vector $\mathbf{r}/|\mathbf{r}|$ where $|\mathbf{r}|$ is chosen to be $|r_1| + \cdots + |r_d|$, and we denote the direction of $\mathbf{r}$ by $\hat{\mathbf{r}}$. We can also scale $|\mathbf{r}|$ by any real multiple so that $\hat{\mathbf{r}}$ also gets scaled.

The notion of direction provides us a way to say how $\mathbf{r}$ goes to the infinity. The direction of $\mathbf{r}$ is not fixed often. For example, let $\mathbf{r}_n = (n+1, 2n+1)$. The direction $\hat{\mathbf{r}}_n := \left(\frac{n+1}{3n+2}, \frac{2n+1}{3n+2}\right)$ differs by the number n. On the other hand, $\hat{\mathbf{r}}_n$ varies in a bounded set and finally converges to $(1, 2)$. Results in ACSV often allows $\mathbf{r} \to \infty$ with $\hat{\mathbf{r}} := \mathbf{r}/|\mathbf{r}|$ either fixed or to vary in a compact set. For simplicity, in this introduction chapter, we fix the direction $\hat{\mathbf{r}}$ to be a vector in $\mathbb{Z}^d$ and then let $\mathbf{r} = n\hat{\mathbf{r}}$ so that $\mathbf{r} \in \mathbb{Z}^d$.

Definition 1.2 (Diagonal). *We say that* $[\mathbf{z}^{n\hat{\mathbf{r}}}]F(\mathbf{z})$ *is the* **$\hat{\mathbf{r}}$-diagonal** *of* $F(\mathbf{z})$. *In particular, if* $\hat{\mathbf{r}} = \mathbf{1} := (1, \cdots, 1)$, *then we call it the* **main diagonal**.

We assume without loss of generality that components of $\hat{\mathbf{r}}$ are non-zero. If some coordinates of $\hat{\mathbf{r}}$ are zero, for example, $\hat{\mathbf{r}} = (\hat{r}_1, \cdots, \hat{r}_{d-1}, 0)$, then the $\hat{\mathbf{r}}$-diagonal of $F(\mathbf{z})$ is equal to the $(\hat{r}_1, \cdots, \hat{r}_{d-1})$-diagonal of the $(d-1)$-variate function $F(z_1, \cdots, z_{d-1}, 0)$.

Asymptotic Expansion

We often use asymptotic expansion to describe the asymptotic behavior of the coefficient $a_{\mathbf{r}}$.

Definition 1.3 (multivariate asymptotic expansion). *The asymptotic expansion*

$$a_{\mathbf{r}} \approx \sum_{i=0}^{\infty} c_i g_i(\mathbf{r}) \tag{1.2}$$

holds on a compact set of direction $D \subset \mathbb{RP}^{d-1}$ *if each* $c_i \in \mathbb{C}$, *each* $g_i = o(g_{i+1})$, *and* $a_{\mathbf{r}} - \sum_{i=0}^{M-1} c_i g_i(\mathbf{r}) = O(g_M)$ *for each* M *as* $\mathbf{r} \to \infty$ *with* $\hat{\mathbf{r}} \in D$. *It is an uniform asymptotic expansion on* D *if the implied constants can be chosen indepdent of the sequence* $\mathbf{r}$ *as long as* $\hat{\mathbf{r}} \in D$.

We adopt the following notations henceforth.

- $f = O(g)$ *if* $\lim_{\mathbf{r}\to\infty} |f(\mathbf{r})/g(\mathbf{r})| < \infty$ *as* $\hat{\mathbf{r}} \in D$.
- $f = o(g)$ *if* $\lim_{\mathbf{r}\to\infty} |f(\mathbf{r})/g(\mathbf{r})| = 0$ *as* $\hat{\mathbf{r}} \in D$.
- f *is exponentially decaying if* $f = O(e^{-c|\mathbf{r}|})$ *for some* $c > 0$. f *is exponentially smaller than* g *if* f/g *is exponentially decaying.*

- *f is super-exponentially decaying if $f = O(e^{-c|\mathbf{r}|})$ for any $c > 0$. f is super-exponentially smaller than g if f/g is super-exponentially decaying.*

Remark. When $f = O(g)$, there exists a constant C and a sufficiently large constant $K > 0$ such that when $|\mathbf{r}| > K$, $|f(\mathbf{r})| \leq C|g(\mathbf{r})|$. Here, C is called an implied constant and we can decrease C by increase K. For two sequences $\{\mathbf{r}_1\}$ and $\{\mathbf{r}_2\}$ both converging to ∞ with $\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2 \in D$, implied constants in (1.2) are different. That is, for each M, there exist four constants C_1, C_2, K_1, K_2 depending on M such that for $j = 1$ or 2,

$$|a_{\mathbf{r}} - \sum_{i=0}^{M-1} c_i g_i(\mathbf{r}_j)| \leq C_j |g_M(\mathbf{r}_j)|$$

when $|\mathbf{r}_j| > K_j$. If the asymptotic expansion (1.2) is uniform, then implied constants can be chosen independent of the specific sequence $\mathbf{r}$. In other words, for any M, there exist constants C, K depending on M such that

$$|a_{\mathbf{r}} - \sum_{i=0}^{M-1} c_i g_i(\mathbf{r})| \leq C |g_M(\mathbf{r})|$$

when $|\mathbf{r}| > K$. □

When $F = P/Q$ is a rational generating function with convergent power series at the origin and $\mathcal{V}_Q$ is smooth, the asymptotic expansion of coefficient $a_{\mathbf{r}}$ takes the form

$$a_{\mathbf{r}} \approx \sum_{\mathbf{w} \in \mathsf{crit}} \mathbf{w}^{-\mathbf{r}} \sum_{\ell=0}^{\infty} C_{\mathbf{w},\ell} |\mathbf{r}|^{-d/2-\ell} \tag{1.3}$$

as $|\mathbf{r}| \to \infty$. Here crit is a set of critical points defined later on $\mathcal{V}_Q$. The quantity $\mathbf{w}^{-\mathbf{r}}$ in the first summation controls the exponential growth and terms in the second summation control the polynomial growth order. This kind of exp-poly growth is ubiquitous in ACSV.

Singularities

Currently, ACSV theory has mainly been developed for multivariate rational generating functions $F(\mathbf{z})$ which have convergent power series at the origin. In particular, they satisfy the following three conditions.

(i) The power series $F(\mathbf{z})$ is convergent in some bounded domain containing the origin.

(ii) $F(\mathbf{z})$ have singularities.

(iii) $F(\mathbf{z})$ can be analytically continued around its singularities.

These three conditions are important for us to apply methods discussed below. If a generating function $F(\mathbf{z})$ is nowhere analytic and so condition (i) fails, then we can't apply the analytic methods. If $F(\mathbf{z})$ is entire and thus condition (ii) fails, then we can't apply singularity analysis. If $F(\mathbf{z})$ cannot be analytically continued beyond its singularities, then Darboux's method given in [Odl95, Chapter 11.2] can obtain partial information on the asymptotics. Actually, many results in ACSV do not require the first condtion. It suffices for $F(\mathbf{z})$ to be a convergent Laurent series $\sum_{\mathbf{r}\in\mathbb{Z}^d} a_{\mathbf{r}}\mathbf{z}^{\mathbf{r}}$ on some domain $\mathcal{D} \in \mathbb{C}^d$. Later in Chapter 2.4, we will see that $\text{Relog}(\mathcal{D})$ is an open convex subset of $\mathbb{R}^d$.

The goal of ACSV is to systemize the coefficient extraction method for multivariate generating functions. Though the univariate analytical combinatorics is well developed in the last centuries, little work has been done on multivariate cases. Some precursors in the multivariate cases include [BR83], [GR92], [BM93], [BR96], [BR99], [Hwa95], [Hwa98a], and [Hwa98b]. More recent works start from [PW02] on meromorphic generating functions with smooth poles and covers mainly rational or meromorphic generating functions. The ACSV project begins to adopt a general framework involving both analytical methods and topological arguments. If $F(\mathbf{z})$ is a meromorphic function, then $F(\mathbf{z})$ is analytical continuable in $\mathbb{C}^d$ except on a set $\mathcal{V}$ of complex dimension $d-1$. We call $\mathcal{V}$ the **singular variety**. In particular, if $F(\mathbf{z})$ is a rational function of two coprime polynomials $P(\mathbf{z})$ and $Q(\mathbf{z})$, then $\mathcal{V} = \mathcal{V}_Q$.

Definition 1.4 (singular variety)**.** *A point $\mathbf{w} \in \mathbb{C}^d$ is a singularity of f if f can be analytically continued to an open set with $\mathbf{w}$ on its boundary but cannot be analytically continued to an open set containing $\mathbf{w}$. The set of all singularities of f is called **singular variety** and is denoted by $\mathcal{V}$.*

Proposition 1.5. *[PWM24, Lemma 6.31] If $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$ is a rational function with coprime polynomials P and Q, then the singular variety of F is $\mathcal{V}_Q := \{\mathbf{z} \in \mathbb{C}^d : Q(\mathbf{z}) = 0\}$.*

Domain of Convergence

Definition 1.6 (Domain of convergence)**.** *The domain of convergence of a Laurent series* $\sum_{\mathbf{r}\in\mathbb{Z}^d} a_{\mathbf{r}}\mathbf{z}^{\mathbf{r}}$ *is its open domain of absolute convergence, that is, the interior of the set of points* $\mathbf{z} \in \mathbb{C}_*^d$ *such that* $\sum_{\mathbf{r}\in\mathbb{Z}^d} |a_{\mathbf{r}}\mathbf{z}^{\mathbf{r}}|$ *converges.*

In other words, we say a Laurent series converges in $\mathcal{D}$ if it uniformly converges in any compact subsets of $\mathcal{D}$. We require $\mathbf{z} \in \mathbb{C}_*^d$ where $\mathbb{C}_* := \mathbb{C} - \{0\}$ because some $\mathbf{r} \in \mathbb{Z}^d$ may have negative coordinates and we will have a problem of dividing by zero otherwise.

ACSV on rational functions

We give a brief introduction to ACSV theory on rational functions, which are the main subject of study in ACSV in the past twenty years. Methods of coefficient extraction for algebraic functions, for example the method in Chapter 4, are still under developing at an early stage. Most of the literature we talk about previously and results in Chapter 6 are on rational functions.

From now on, let's assume that we have a d-variate rational generating function $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$ of two coprime d-variate polynomials $P(\mathbf{z})$ and $Q(\mathbf{z})$. Moreover, for the sake of simplicity, let's assume that $F(\mathbf{z})$ has a convergent power series $\sum_{\mathbf{r}\in\mathbb{N}^d} a_{\mathbf{r}}\mathbf{z}^{\mathbf{r}}$ with the domain of convergence $\mathcal{D}$ containing the origin. In general, ACSV theory works when $F(\mathbf{z})$ has a convergent Laurent series $\sum_{\mathbf{r}\in\mathbb{Z}^d} a_{\mathbf{r}}\mathbf{z}^{\mathbf{r}}$.

Cauchy Integral Formula and Exponential Growth

The first step to extract coefficient $a_{\mathbf{r}}$ is to use the multidimensional Cauchy integral formula to represent $a_{\mathbf{r}}$ as an integral over a d-dimensional torus T around the origin, that is,

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_T \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}}.$$

Here we choose $T := \{\mathbf{z} \in \mathbb{C}^d : |z_i| = c_i\}$ with sufficiently small c_i so that T is in the domain of convergence $\mathcal{D}$. The d-form $d\mathbf{z}/\mathbf{z}$ is the logarithmic volume form $dz_1 \wedge \cdots \wedge dz_d / \prod_{j=1}^d z_j$.

The maximum modulus principle implies that

$$|a_{\mathbf{r}}| \leq c_1^{-r_1} \cdots c_d^{-r_d} \max_{\mathbf{z} \in T} |F(\mathbf{z})|$$

When $d = 1$, the above inequality becomes $|a_n| \leq c^{-n} \max_{|z|=c} |F(z)|$ and thus the radius of $\mathcal{D}$ often determines an upper bound on the exponential growth order of $a_{\mathbf{r}}$. For example, the Catalan generating function in the very beginning of Chapter 1 has a convergence domain of radius 1/4. Therefore, the exponential growth order of its coefficient is at most 4. In this particular case, the exponential growth order is exactly 4.

When $d > 1$, we introduce the height function $h_{\hat{\mathbf{r}}}$ defined by the direction $\hat{\mathbf{r}}$ of $\mathbf{r}$.

Definition 1.7 (height functions)**.** *For a fixed direction $\hat{\mathbf{r}} \in \mathbb{R}^d$, define the height function $h_{\hat{\mathbf{r}}} : \mathbb{C}_*^d \to \mathbb{R}$ by*

$$h_{\hat{\mathbf{r}}}(\mathbf{z}) = -\hat{\mathbf{r}} \cdot \operatorname{Relog}(\mathbf{z}) = -\sum_{i=1}^{d} \hat{r}_i \log |z_i|$$

The above inequality becomes $|a_{\mathbf{r}}| \leq \exp(n h_{\hat{\mathbf{r}}}(\mathbf{w})) \max_{\mathbf{z} \in T} |F(\mathbf{z})|$ where $\mathbf{w}$ is any point on T. Later in Chapter 2.4 we will see that $B := \operatorname{Relog}(\mathcal{D})$ is a convex subset of $\mathbb{R}^d$. An upper bound for the exponential growth order of $a_{\mathbf{r}}$ is given by a minimizer of $h_{\hat{\mathbf{r}}}$ on the boundary of B.

ACSV theory continues exploring growth order of $a_{\mathbf{r}}$ beyond this naive upper bound on exponential growth order. In Example 1.10, this naive upper bound is not sharp at all. To do this, we need to do some topological deformation as the next step.

Topological Deformation When $\mathcal{V}$ Is Smooth

The simplest case is when $\mathcal{V} = \mathcal{V}_Q$ is smooth. In other words, $\nabla \widetilde{Q}$ and $\widetilde{Q}$ are not simultaneously zero. Here $\widetilde{Q}$ is the square-free part of Q. In this particular case, we follow the outline listed below.

(i) Expand T to a larger torus T' where the integral (1.1) over T' is super exponenitally decaying. That is

$$\frac{1}{|\mathbf{r}|} \log \left| \int_{T'} \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}} \right| \to -\infty \text{ as } |\mathbf{r}| \to \infty.$$

(ii) The expansion from T to T' is a $(d+1)$-chain which can be chosen to intersect the singular variety $\mathcal{V}_*$ transversely and and the intersection is a $(d-1)$-chain γ on $\mathcal{V}_*$.

(iii) There is a $(d-1)$-residue form ω on $\mathcal{V}_*$ of the d-form $\mathbf{z}^{-\mathbf{r}}F(\mathbf{z})\frac{d\mathbf{z}}{\mathbf{z}}$.

(iv) By residue theorem we have

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_T \mathbf{z}^{-\mathbf{r}}F(\mathbf{z})\frac{d\mathbf{z}}{\mathbf{z}} = \left(\frac{1}{2\pi i}\right)^d \int_{T'} \mathbf{z}^{-\mathbf{r}}F(\mathbf{z})\frac{d\mathbf{z}}{\mathbf{z}} + \left(\frac{1}{2\pi i}\right)^{d-1} \int_\gamma \omega$$

where the first integral on the right hand side decays super exponentially.

(v) Use Morse theory to deform γ on $\mathcal{V}_*$ so that the integral $\int_\gamma \omega$ becomes a sum of saddle point integrals $\int_{\gamma_i} \omega$ where γ_i maximizes its height at a critical point $\mathbf{z}$.

The existence of such a T' in step (i) is justified in Proposition 2.36. The $(d-1)$-chain γ in step (ii) is the so-called intersection class $\mathbf{INT}[T, T'; \mathcal{V}_*]$ (see Definition 2.9). More precisely, the intersection class is the homology class $[\gamma]$ in $\mathrm{H}_{d-1}(\mathcal{V}_*)$. The residue form ω in step (iii) is defined in Proposition 2.12. The residue theorem in step (iv) is given in 2.16.

The last step is the most complicated step where we need to apply Morse theory (see Chapter 2.3) on $\mathcal{V}$. The integral $\int_\gamma \omega$ only depends on homology classes of γ in $\mathrm{H}_{d-1}(\mathcal{V}_*)$ by Proposition 2.2. Morse theory tells us that the topology of the level sets (defined by the height function $h_{\hat{\mathbf{r}}}$) of $\mathcal{V}_*$ only changes at **critical points** of $h_{\hat{\mathbf{r}}}$. When these critical points are non-degenerate, we will have one generator $[\gamma_i]$ of the singular homology group $\mathrm{H}_{d-1}(\mathcal{V})$ at each critical point. The job is then to replace $[\gamma]$ by an integer sum of these generators $[\gamma_i]$. In particular, these γ_i can be chosen so that $h_{\hat{\mathbf{r}}}$ attains maximum at the critical point along γ_i.

Topological Deformation When $\mathcal{V}$ Is Not Smooth

When $\mathcal{V}$ is not smooth, we often can decompose $\mathcal{V}$ into a union of smooth manifolds in different dimensions. We require a specific type of stratification of $\mathcal{V}$, the **Whitney stratification** (see Chapter 5.2.1). Indeed, when $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$ is a rational function, $\mathcal{V} = \mathcal{V}_Q$ is an algebraic variety and a Whitney stratification always exists and can be computed by computer algebra systems.

In such a case, critical points are characterized on each stratum and they can be determined by algebraic method (see Chapter 5.2.2). If a critical point $\mathbf{z}$ is not smooth, then the geometry of $\mathcal{V}$ near $\mathbf{z}$ can be wild. If $\mathcal{V}$ looks like a union of several complex manifolds near $\mathbf{z}$, the point $\mathbf{z}$ is called a **multiple point** and they are briefly discussed in Chapter 5. Full details on multiple point can be found on [PWM24, Chapter 10]. [BMP24b] gives a complete solution of the coefficient extraction problem for generating functions with poles on hyperplane arrangements, a particular case where multiple points can arise, without explicit use of Morse theory. When $\mathbf{z}$ is a multiple point, the second step is almost similar but instead of taking residues, we will take iterated residues. The coefficient $a_{\mathbf{r}}$ will be represented as a $(d-k)$-dimensional integral of k-variable residues where k is the codimension of the stratum where $\mathbf{z}$ lies.

Beyond multiple points, the critical point $\mathbf{z}$ can also be a **cone point** where $\mathbf{z}$ is an isolated quadratic singularity of $\mathcal{V}$. [BP11] studies this particular case using knowledges from harmonic analysis in [ABG70]. Little is known for $\mathbf{z}$ not being these types of points. For example, we coin the term **pseudo multiple point** in Chapter 6 where $\mathcal{V}$ satisfies a necessary condition (Proposition 6.2) of $\mathbf{z}$ being a multiple point. Under some assumptions, we show that we can treat pseudo multiple points as multiple points.

In contrast to the case when $\mathcal{V}$ is smooth, we first apply Morse theory and then take residues. By Proposition 2.2, the integral

$$\left(\frac{1}{2\pi i}\right)^d \int_T \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}}$$

will not change if we replace T by another d-chain in the same holomogy class of the singular homology group $\mathrm{H}_d(\mathcal{M})$ where $\mathcal{M} := \mathbb{C}_*^d - \mathcal{V}$.

Stratified Morse theory describes homology generators of $\mathrm{H}_d(\mathcal{M})$ in Chapter 5.2.4 by Theorem 5.28. They are given by critical points of $h_{\hat{\mathbf{r}}}$ on each stratum of the Whitney stratification aforementioned. Replace $[T]$ by an integer sum of homology generators in $\mathrm{H}_d(\mathcal{M})$. When these critical points are nice enough (i.e. nondegenerate and transvese), we can often put the integral over the homology generators given by these critical points in the form of a saddle point integral and use methods in

[PWM24, Chapter 5] to evaluate it.

More detailed introduction on the case when $\mathcal{V}$ is not smooth can be found in Chapter 1.2.2. A complete treatment of such a case is in Chapter 5. Actually, this topological deformation is more general and we can even use it when $\mathcal{V}$ is smooth (see Example 1.8).

One-dimensional Example Using ACSV

Let's use the following example where $d = 1$ and F is a rational function with convergent power series to see how ACSV works. Definitely we don't need ACSV for one-dimensional cases but it is easier to see what we mean in the previous verbose introduction.

Example 1.8. *Let $F(z) = 1/Q$ where $Q = (z-1)(z-2)$. F admits a convergent power series $\sum_{n\geq 0} a_n z^n$ with domain of convergence $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $T = \{z \in \mathbb{C} : |z| = \epsilon\}$ where $\epsilon < 1$ and we can use Cauchy integral formula to represent the coefficient a_n by*

$$a_n = \frac{1}{2\pi i}\int_T z^{-n} F(z)\frac{dz}{z}$$

Since $F(z)$ is a rational function, its coefficient asymptotics are at most exponential. In particular, $a_n \sim A^n s(n)$ where A is the exponential growth and $s(n)$ is the subexponential growth. The singular variety $\mathcal{V}$ of $F(z)$ is the set containing two isolated points $z = 1$ and $z = 2$.

We follow the case when $\mathcal{V}$ is smooth first. We can let T' to be a circle of radius $M > 2$. The integral of $z^{-n}F(z)\frac{dz}{z}$ over T' does not change if we let $M \to \infty$. On the other hand, by the maximum modulus principle, the absolute value of the integral is bounded above by $M^{-n}\max_{z\in T'} F(z)$ and thus

$$\frac{1}{n}\log\left|\int_{T'} \mathbf{z}^{-n}F(z)\frac{dz}{z}\right| \leq -\log M.$$

Letting $M \to \infty$, we see that the integral over T' decays super exponentially. The expansion of T to T' cross the singular variety $\mathcal{V}_*$ and leave a 0-chain $\gamma := \{1, 2\}$ on $\mathcal{V}_*$. The 0-form residue ω of $z^{-n}F(z)\ \frac{dz}{z}$ is a function on $\mathcal{V}$ that maps 1 to $\operatorname{Res}(z^{-n-1}F(z); z = 1)$ and 2 to $\operatorname{Res}(z^{-n-1}F(z); z =$

2). *By residue theorem*

$$a_n = \frac{1}{2\pi i}\int_{T'} F(z)\frac{dz}{z} - \operatorname{Res}\left(z^{-n-1}F(z); z=1\right) - \operatorname{Res}\left(z^{-n-1}F(z); z=2\right).$$

We can also follow the more general method mostly applied when $\mathcal{V}$ is not smooth. It also works

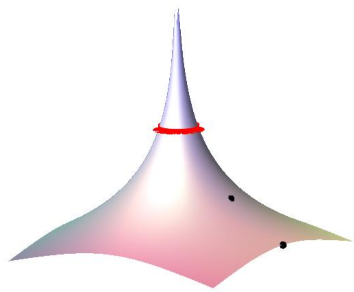

Figure 1.1: The part of the complex plane (real and imaginary parts between -2 and 2) pictured by the height function $h(\mathbf{z}) = -\log|\mathbf{z}|$. The red circle is the torus $T = \{|z| = 1/2\}$ at the height $\log 2$. The two black dots are the singularities $z = 1$ and $z = 2$ at height 0 and $-\log 2$ respectively. The origin of the complex plane is at infinite height.

when $\mathcal{V}$ is smooth. The space $\mathcal{M} := \mathbb{C}_* - \{1, 2\}$ is called the domain of holomorphy of the 1-form $z^{-n}F(z)\frac{dz}{z}$ in the integrand. The homology group $\mathrm{H}_1(\mathcal{M})$ has three generators, a small circle around 1, a small circle around 2 and a large circle T' of radius $M > 2$. We choose the first two of them explicitly to be $C_1 = \{z \in \mathbb{C} : |z-1| = 0.1\}$ and $C_2 = \{z \in \mathbb{C} : |z-2| = 0.1\}$. Then $[T] = [T'] - [C_1] - [C_2]$ in $\mathrm{H}_1(\mathcal{M})$ where C_1 and C_2 are oriented positively.

Therefore,

$$\frac{1}{2\pi i}\int_T z^{-n}F(z)\frac{dz}{z} = \frac{1}{2\pi i}\int_{T'} z^{-n}F(z)\frac{dz}{z} - \frac{1}{2\pi i}\int_{C_1} z^{-n}F(z)\frac{dz}{z} - \frac{1}{2\pi i}\int_{C_2} z^{-n}F(z)\frac{dz}{z}$$

Residue theorem says that these integrals over C_1 and C_2 are just residues at $z = 1$ and $z = 2$.

$$\frac{1}{2\pi i}\int_T z^{-n}F(z)\frac{dz}{z} = \frac{1}{2\pi i}\int_{T'} z^{-n}F(z)\frac{dz}{z} - \operatorname{Res}\left(z^{-n-1}F(z); z=1\right) - \operatorname{Res}\left(z^{-n-1}F(z); z=2\right)$$

The maximum modulus principle tells that the modulus of the first integral on the right hand side is less than or equal to $r^{-n}\max_{z\in T'}|F(z)|$. The radius r of T' can be arbitrarily large because there are no more singularities beyond modulus 2. Since $F(z)$ is a rational function, $\max_{z\in T'}|F(z)|$ can grow at most polynomially as $r\to\infty$. For n larger than some fixed number, $\int_{T'} z^{-n}F(z)\frac{dz}{z} = 0$ because we can let r be arbitrarily large. In this particular case, because $\max_{z\in T'}|F(z)|$ decays in the order $O(r^{-2})$, we see that for any n, the first integral on the right hand side is zero. Since both singularities are simple poles, these two residues are equal to -1 and 2^{-n-1}. Therefore, we have

$$a_n = 1 - 2^{-n-1}$$

Indeed, here are the first ten coefficients for the convergent power series $F(z)$,

$$1/2, 3/4, 7/8, 15/16, 31/32, 63/64, 127/128, 255/256, 511/512, 1023/1024.$$

$\square$

This simple example in univariate case enlightens us the path to the multivariate case. The first observation is that we want to enlarge the torus T to T' so that the integral over T' contribute exponentially smaller order to the asymptotics. We have already seen that we cannot do this for free. The price to pay is these residues when the deformation of the torus T crosses the singularities of F. The second observation is that there are some singularity points contributing to the asymptotics. In Example 1.8, all singularity points contribute to the asymptotics. We call $z = 1$ the minimal point because it is at the boundary of the domain of convergence $\mathcal{D}$. This point determines the exponenital growth of a_n by the first principle of analytical combinatorics. When $d = 1$, all singularities are isolated and expanding T to a larger torus must cross all singularities. This is however not the case when $d > 2$.

Contributing points

The intricate complex geometry in higher dimensions makes visualization harder and there are several problems we encounter when we try to transfer our observations in the univariate case to the multivariate case. Again, let's assume that $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$ is a rational function with convergent power series at the origin where P, Q are coprime polynomials in $\mathbb{R}[\mathbf{z}]$. The singularities of F are the $(d-1)$-complex-dimensional variety $\mathcal{V} = \mathcal{V}_Q := \{\mathbf{z} \in \mathbb{C}^d : Q(\mathbf{z}) = 0\}$. The torus T is $\{\mathbf{z} \in \mathbb{C}^d : |z_i| = \epsilon_i\} \subseteq \mathcal{D}$. Let $\hat{\mathbf{r}} \in \mathbb{N}^d$ and we consider the asymptotic behavior of $a_{\mathbf{r}}$ where $\mathbf{r} = n\hat{\mathbf{r}}$ and $n \to \infty$. Now there is one immediate problem:

Which particular points on $\mathcal{V}$ are contributing to the asymptotics?

After all, the variety $\mathcal{V}$ is of positive dimension and contains infinitely many singular points.

Since we do the integral at the level of homology and these homology generators are contributed by critical points of $h_{\hat{\mathbf{r}}}$ only, an intuition tells us that critical points are candidates for contributing points. Let's say a critical point $\mathbf{p}$ gives a homology generator $\sigma_{\mathbf{p}}$ to $\mathrm{H}_d(\mathcal{M})$. When writing $[T]$ as a sum of homology generators in $\mathrm{H}_d(\mathcal{M})$, such a homology generator $\sigma_{\mathbf{p}}$ may not appear in the sum. Therefore, $\mathbf{p}$ is not a contributing point even though it is a critical point.

Intuitively, we can think of what we do in the topological deformation is pushing T in $\mathcal{M}$ to lower height defined by $h_{\hat{\mathbf{r}}}$ because the higher the height, the larger the integrand in $\int_T \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}}$. The ultimate goal is to push T to infinitely lower height so that the integral decays super exponentially. This coincides with the first observation we obtain from Example 1.8. The difference is that we don't need to (and we cannot) keep T as a torus in $\mathcal{M}$. We can deform T to any torus in the domain of convergence $\mathcal{D}$. Beyond the domain $\mathcal{D}$, singular points from $\mathcal{V}$ appear and T can't keep the shape of a torus. Morse theory tells us that the topology of the level sets $\mathcal{M}_{\leq c}$ of $\mathcal{M}$ defined by $\mathcal{M}_{\leq c} := \{\mathbf{z} \in \mathcal{M} : h_{\hat{\mathbf{r}}}(\mathbf{z}) \leq c\}$ changes only at critical points. Therefore, the deformation of T to lower height will only be possibly get 'stuck' at critical points $\mathbf{p}$. If T is at lower height than $\mathbf{p}$, then $\mathbf{p}$ will not contribute because T will never meet $\mathbf{p}$ in the push-down deformation.

The homology arguments make these deformations easy. Indeed, we do not need to give an explicit deformation of T. It suffices to write the homology class $[T]$ as an integer sum of homology generators for the sake of coefficient extraction because the integral (1.1) depends via the homology class of T in $\mathcal{M}$.

In general, this is a hard question to answer. When $\mathcal{V}$ is smooth and $F(\mathbf{z})$ is a convergent power series, a minimal critical point is a contributing point. However, when $\mathcal{V}$ is not smooth, it is generally not a contributing point. [BP11] shows that minimal critical quadratic cone points often are contributing points, but [BMP24a] later shows a lacuna phenomenon when a minimal critical quadratic cone point is not contributing. This phenomenon happens in even dimensions higher than four.

1.2. Overview of ACSV

In Chapter 1.2.1, we give an easy upper bound on the asymptotics of $a_{\mathbf{r}}$. It is given by a minimizer of $h_{\hat{\mathbf{r}}}$ on the boundary of $\mathcal{D}$. In Chapter 1.2.2, we first give a topological deformation analogous to the univariate expanding-torus argument. Then we introduce a more general topological deformation at the homology level, motivated by (stratified) Morse theory. Indeed, we adopt this modern viewpoint at the homology level throughout the whole paper. In Chapter 1.2.3, we list results given in [PWM24, Chapter 4, 5] for saddle point integrals.

1.2.1. Exponential growth

We let $T(\mathbf{w})$ to denote the torus $\{\mathbf{z} \in \mathbb{C}^d : |z_i| = |w_i|\}$ with the same coordinatewise modulus as $\mathbf{w}$. If $F(\mathbf{z})$ has a convergent power (or Laurent) series $\sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ with domain of convergence $\mathcal{D}$, for any $\mathbf{w} \in \mathcal{D}$, we can express the coefficient $a_{\mathbf{r}}$ using the Cauchy integral formula

$$
\begin{aligned}
|a_{\mathbf{r}}| = \left| \left(\frac{1}{2\pi i}\right)^d \int_{T(\mathbf{w})} \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}} \right| &\leq |w_1|^{-r_1} \cdots |w_d|^{-r_d} \max_{\mathbf{z} \in T(\mathbf{w})} |F(\mathbf{z})| \\
&= |w_1|^{-n\hat{r}_1} \cdots |w_d|^{-n\hat{r}_d} \max_{\mathbf{z} \in T(\mathbf{w})} |F(\mathbf{z})| \\
&= e^{n h_{\hat{\mathbf{r}}}(\mathbf{w})} \max_{\mathbf{z} \in T(\mathbf{w})} |F(\mathbf{z})|
\end{aligned}
$$

Since $T(\mathbf{w})$ is compact and $F(\mathbf{z})$ is continous, the quantity $\max_{\mathbf{z}\in T(\mathbf{w})}|F(\mathbf{z})|$ is finite. Therefore, the only thing matters for the asymptotic behavior of $|a_{\mathbf{r}}| := |a_{n\hat{\mathbf{r}}}|$ is the height $h_{\hat{\mathbf{r}}}(\mathbf{w}) := -\hat{\mathbf{r}} \cdot \mathrm{Relog}(\mathbf{w})$. In other words, we have an upper bound $e^{h_{\hat{\mathbf{r}}}(\mathbf{w})}$ for the exponential growth rate of $|a_{\mathbf{r}}|$ defined by $\limsup_{n\to\infty} |a_{\mathbf{r}}|^{1/n}$. This upper bound works for any $\mathbf{w} \in \overline{\mathcal{D}}$ and thus our goal is to lower the height $h_{\hat{\mathbf{r}}}(\mathbf{w})$ by choosing an appropriate $\mathbf{w}$. In other words, we can deform the original torus T to $T(\mathbf{w})$ such that $h_{\hat{\mathbf{r}}}(\mathbf{w})$ achieves minimum on $\overline{\mathcal{D}}$. The deformation can be explicitly chosen so that we change each coordinate radius of T to $|w_i|$ linearly and it remains to be a shape of torus during deformation.

Instead of finding the minimum of $h_{\hat{\mathbf{r}}}(\mathbf{w})$ on $\overline{\mathcal{D}}$, we often find the minimum of its log-version $h_{\hat{\mathbf{r}}} \circ \exp : \mathbb{R}^d \to \mathbb{R}$ on $\overline{\mathrm{Relog}(\mathcal{D})}$. In particular, $\mathrm{Relog}(\mathcal{D})$ is denoted as B in Chapter 2.4 and it is a component of the complement of $\mathsf{amoeba}(Q)$ in $\mathbb{R}^d$ if $F = P/Q$ is a rational (or meromorphic) function. For now, let's take it for granted that $B := \mathrm{Relog}(\mathcal{D})$ is an open convex subset of $\mathbb{R}^d$ and more details are in Chapter 2.4. The log-version of $h_{\hat{\mathbf{r}}}(\mathbf{w})$ is $h_{\hat{\mathbf{r}}} \circ \exp(\mathbf{x}) := -\hat{\mathbf{r}} \cdot \mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^d$, a linear function. Since B is convex, the minimizer of this linear function must be on the boundary, if exists. There are cases where the minimizer doesn't exist but we leave it for Chapter 2.4.

If the minimizer of $h_{\hat{\mathbf{r}}} \circ \exp$ exists on $\overline{\mathcal{D}}$, then we know that there exists a point $\mathbf{x}_* \in \partial B$ such that $h_{\hat{\mathbf{r}}}$ achieves lowest height on $T(\mathbf{w}_*)$ for $\mathbf{w}_* = \exp(\mathbf{x}_0)$. Since $\mathbf{x}_* \in \partial B$, points on $T(\mathbf{w}_*)$ must be on $\partial\mathcal{D}$. Points on $\partial\mathcal{D} \cap \mathcal{V}$ are called **minimal points**. Minimal points which minimize $h_{\hat{\mathbf{r}}}$ on $\overline{\mathcal{D}}$ often (though not always) provide information for the leading asymptotics of $a_{\mathbf{r}}$.

Example 1.9 (Continuation of Example 1.8)**.** *In Example 1.8, the domain of convergence* $\mathcal{D} := \{z \in \mathbb{C} : |z| < 1\}$ *and* $B := \mathrm{Relog}(\mathcal{D}) = (-\infty, 0)$*. The log-version height function* $h_{\hat{\mathbf{r}}} \circ \exp(x) := -x$ *because the direction* $\hat{\mathbf{r}}$ *in the univariate case is just a vector with one component, or more explicitly, a scalar* 1*. Therefore, the minimizer of* $h_{\hat{\mathbf{r}}} \circ \exp(x)$ *on* $\overline{B}$ *is the point* $x_* = 0$*. This implies that the exponential growth rate* $\limsup_{n\to\infty}(a_n)^{1/n} \le e^0 = 1$*. In particular,* $T(1) \cap \mathcal{V} := \{1\}$ *and so there is one minimal point* $z = 1$*.* □

A big difference on exponential growth between the univariate case and the multivariate case is whether or not the location of minimal points determines exponential growth order. In univariate case, by the first principle of analytical combinatorics [FS09, Chapter IV.1], the location of the singularity with smallest modulus determines exponential growth order. However, there are multivariate examples where the exponential growth is less than $e^{-\hat{\mathbf{r}}\cdot\mathbf{x}_*}$ where $\mathbf{x}_*$ is the minimizer of $h_{\hat{\mathbf{r}}} \circ \exp$ on $\overline{B}$.

Example 1.10 (Ghost Intersection)**.** *[Mel21, Example 5.8] Let $F(x,y) := 1/(2+y-x(1+y)^2)$ be a bivariate function that has a power series domain of convergence $\mathcal{D}$. That is, for $(x,y) \in \mathcal{D}$, we have a convergent power series $F(x,y) = \sum_{r,s\geq 0} a_{r,s}x^r y^s$. Let $\hat{\mathbf{r}} = (1,1)$ be the diagonal direction. We look at the asymptotic behavior of $a_{n\hat{\mathbf{r}}} = a_{n,n}$.*

By some knowledges in Chapter 2.4, $h_{(1,1)} \circ \exp$ achieves its minimum on $\overline{B}$. The set of minimal points are

$$\mathcal{V} \cap \partial\mathcal{D} = \left\{ \left(\frac{2+y}{(1+y)^2}, y \right) : y \in [-2, -\sqrt{3}] \cap [0, \sqrt{3}] \right\}$$

The minimizer of $h_{(1,1)}$ on $\overline{\mathcal{D}}$ are $(1/2, \pm\sqrt{3})$. It gives an upper bound for $\limsup_n a_{n,n}^{1/n} \leq \frac{2}{\sqrt{3}}$. However, the upper bound can be even lowered to 1 since it has a (non-minimal) critical point at infinity (see Chapter 5.2.5) as $y \to \infty$ on $\mathcal{V}$.

This peculiar behavior, as noted by Stephen Melczer in [Mel21, Chapter 5], is due to the Relog *map projecting $\mathcal{V} := \mathcal{V}_Q$ into $\mathsf{amoeba}(Q) \subseteq \mathbb{R}^d$ does not reflect the properties of $\mathcal{V}_Q$ in $\mathbb{C}^d$. In Example 1.10, the two points $(1/2, \sqrt{3})$ and $(1/2, -\sqrt{3})$, though projected to the same point by* Relog*, are actually from two connected components of $\mathcal{V} \cap \mathbb{R}^2$. This is called a ghost intersection.*

Even though we can't enlarge radii of the torus $T(1/2, \sqrt{3})$ further as it will touch the singular variety $\mathcal{V}_Q$, we can try to deform the torus $T(1/2, \sqrt{3})$ so that every point on the deformed one will all be below height $h_{(1,1)}(1/2, \sqrt{3}) = \log(2) - \log(3)/2$. During the deformation, we don't need to keep the shape of the torus. This deformation is given by Morse theory (Chapter 2.3) by a downward vector flow $\mathbf{v}$ such that $dh(\mathbf{v}_\mathbf{p}) < 0$ for all but finitely many points $\mathbf{p} \in \mathbb{C}^d_*$. The downward vector flow $\mathbf{v}$ will push the torus down to lower height until there is some point $\mathbf{p}_* \in \mathcal{V}$ such that the differential of h equals zero. We call the point $\mathbf{p}_*$ a **critical point** *(introduced in Chapter 1.2.2). By Morse* *theory, these points are the only possible places where we can't push the torus further down.* $\square$.

The multivariate complex geometry empowers more flexibility to the deformation. The original Cauchy torus T in (1.1) is a d-chain. The singular variety $\mathcal{V}$ is a $(d-1)$-complex space, or $2d-2$ in real dimensions. When $d \geq 2$, there is enough space in $\mathcal{M} := \mathbb{C}_*^d - \mathcal{V}$ for us to deform T to lower heights than the height given by the minimizer on $\overline{\mathcal{D}}$. When $d = 1$, then $\mathcal{V}$ consists of isolated points and T is a circle. Intuitively, these singularity are like nails on the complex plane and any way to push the circle to larger radius results in stucking at these nails. In ACSV, since singularities in univariate case are isolated, they are always critical points (see Chapter 5.2.2). If we are looking for the coefficient a_n of the power series of F at the origin, then the dominant singularities (i.e. singularities with smallest modulus) will always be a **contributing point** to the asymptotics of a_n. We will define critical points and contributing points rigorously in Chapter 1.2.2. In particular, we will not try to describe an explicit deformation of T as we did in Example 1.8. We are going to use homology arguments and Morse theory will guide us to push the torus T to lower heights.

1.2.2. Topological deformation

A more modern framework in [PWM24] using Morse theory and relative homology makes the topology arguments easier and we do not need to explicitly deform the torus T, though we can do that in some cases, for example, in the case of [BMP24b] where the singular variety is a hyperplane arrangement. We try our best to give some motivations here and introduce basic knowledge on Morse theory and homology in Chapter 2.1, 2.2, and 5.2. More details in the context of ACSV can be found in [PWM24, Appendix B,C,D]. Standard literature includes [Mil63] for classical Morse theory and [GM88] for stratified Morse theory.

This homology-level deformation is motivated by (stratified) Morse theory. Given a direction $\hat{\mathbf{r}}$, we can define the height function $h := h_{\hat{\mathbf{r}}}$ as in Definition 1.7 by $h := -\hat{\mathbf{r}} \cdot \mathrm{Relog}(\mathbf{z})$. We omit the subscript $\hat{\mathbf{r}}$ in $h_{\hat{\mathbf{r}}}$ when there is no abmiguity on the direction $\hat{\mathbf{r}}$. In particular, the height function is real-valued and so we can compare two points on $\mathbb{C}^d$ by their heights. For a space X and a Morse function $h : X \to \mathbb{R}$, we denote $X_{\leq a}$, $X_{<a}$ as the sublevel sets $\{\mathbf{z} \in X : h(\mathbf{z}) \leq a\}$ and $\{\mathbf{z} \in X : h(\mathbf{z}) < a\}$. When $X = \mathcal{V}_*$, the height function $h_{\hat{\mathbf{r}}}$ is a Morse function. When $\mathcal{V}_*$ is smooth, critical points on $\mathcal{V}_*$ are defined to be those points $\mathbf{w} \in \mathcal{V}_*$ such that $dh(\mathbf{w}) = 0$. The critical values are then defined to be the height value $h(\mathbf{w})$ at critical points $\mathbf{w}$. Morse theory tells us that the topology of the sublevel set $X_{\leq a}$ will only change at critical values a.

When $\mathcal{V}$ is smooth

Remark. The topological deformation we do happens on $\mathcal{V}_*$ and $\mathcal{M}$, both as subsets of $\mathbb{C}_*^d$. Indeed, we only need smoothness of $\mathcal{V}_*$ here. The intersection class defined later is also well-defined on $\mathcal{V}_*$ instead of $\mathcal{V}$. In the practice of ACSV, the singular variety $\mathcal{V}$ is often defined by a polynomial Q, and thus it is easier to check smoothness of $\mathcal{V} := \mathcal{V}_Q$ instead of $\mathcal{V}_*$ which is $\mathcal{V}_Q$ minus some unions of algebraic sub-varieties. Therefore, we make the dichotomy based on smoothness of $\mathcal{V}$ though everything is done on $\mathcal{V}_*$ and $\mathcal{M} := \mathbb{C}_*^d - \mathcal{V}_*$. It is also consistent with the dichotomy in most literature in ACSV.

For now, let's assume that $\mathcal{V}$ is smooth. In this case, we can do some analogous work to what we did in Example 1.8. Every point $\mathbf{z}$ on the original Cauchy torus T has the same height $\mathbf{x} := h(\mathbf{z}) \in B$. There is another component B' of $\mathsf{amoeba}(Q)^c$ where $\mathbf{x} \mapsto -\hat{\mathbf{r}} \cdot \mathbf{x}$ is unbounded from below. Here Q is the denominator of the rational function $F = P/Q$ and $\mathcal{V} := \mathcal{V}_Q$. Choose $\mathbf{y} \in B'$ and let $T' = \text{Relog}^{-1}(\mathbf{y}) := \{\mathbf{z} \in \mathbb{C}^d : |z_i| = \exp(y_i)\}$ be the torus in $\text{Relog}^{-1}(B')$. Both T and T' are in $\mathcal{M}$ because their projections under Relog are not in $\mathsf{amoeba}(Q)$. Deforming T to T' needs to cross the variety $\mathcal{V}_*$. In particular, $[T - T'] \in \mathrm{H}_d(\mathcal{M})$ and there is a homology class $\mathbf{INT}[T, T'; \mathcal{V}_*]$ in $\mathrm{H}_d(\mathcal{V}_*)$ called **intersection class** (see Definition 2.9) such that

$$\int_T \omega - \int_{T'} \omega = \int_{\circ \mathbf{INT}[T,T';\mathcal{V}_*]} \omega$$

where $\circ$ is an operator from $\mathrm{H}_{d-1}(\mathcal{V}_*) \to \mathrm{H}_d(\mathcal{M})$ (see Theorem 2.8). We can choose the radii $\mathbf{y}$ of T' in B' so that $-\hat{\mathbf{r}} \cdot \mathbf{y} \to \infty$. Therefore, $\int_{T'} \omega$ decays super-exponentially, i.e. in the order $O(e^{-an})$ for any $a > 0$. Moreover, when F is a rational function, $\int_{T'} \omega = 0$ for all but finitely many multi-indices in the sequence $\{\mathbf{r}\}$ because the denominator of F grows at most polynomially. The multivariate residue theorem (Theorem 2.16) tells us that

$$\int_{\circ\mathbf{INT}[T,T';\mathcal{V}_*]} \omega = (2\pi i) \int_{\mathbf{INT}[T,T';\mathcal{V}_*]} \mathrm{Res}\,(\omega)$$

where $\mathrm{Res}\,(\omega)$ is called the **residue class**, a $(d-1)$-form on $\mathcal{V}_*$, or more precisely, a cohomology class in $\mathrm{H}^{d-1}(\mathcal{V}_*)$. Therefore,

$$a_{n\hat{\mathbf{r}}} = (2\pi i)^{-d} \int_T \omega = (2\pi i)^{1-d} \int_{\mathbf{INT}[T,T';\mathcal{V}_*]} \mathrm{Res}\,(\omega) + O(e^{-an})$$

where $a > 0$ is arbitrarily large.

Example 1.11 (Continuation of Example 1.8)**.** *The component B where* $\mathrm{Relog}(T)$ *lies is* $(-\infty, 0)$*. In pariticular,* $\mathsf{amoeba}(Q) = \{0, \log 2\}$*. We let $B' = (\log 2, \infty)$ and $T' = \mathrm{Relog}^{-1}(y)$ for any $y \in B'$ so that T' is a circle in $\mathbb{C}$ with radius $e^y > 2$. Deformation from T to T' will cross out the singular variety $\mathcal{V} := \mathcal{V}_Q = \{1, 2\}$. **INT**$[T, T'; \mathcal{V}_*] = \{1, 2\}$ and $\circ$**INT**$[T, T'; \mathcal{V}_*]$ can be thought of two small circles centered at 1 and 2 with radius $\epsilon < 0.1$.* $\mathrm{Res}\,(\omega)$ *is 0-form on $\mathcal{V}_*$, and so it is a function on $\mathcal{V}_*$. In particular,* $\mathrm{Res}\,(\omega) = -1$ *and* $\mathrm{Res}\,(\omega) = 2^{-n-1}$*. The multivariate residue theorem then becomes the residue theorem in single variable complex analysis.* $\square$.

The next step is to deform the $(d-1)$-chain $\gamma = \mathbf{INT}[T, T'; \mathcal{V}_*]$ on $\mathcal{V}_*$ so that the integral

$$\int_{\mathbf{INT}[T,T';\mathcal{V}_*]} \mathrm{Res}\,(\omega)$$

becomes a sum of saddle point integrals. We define **smooth critical points** by those points $\mathbf{w} \in \mathcal{V}_*$ such that $dh(\mathbf{w}) = 0$. We say that a critical point is **nondegenerate** if the Hessian matrix of h at $\mathbf{w}$ is nonsingular. We need to deform γ so that within a small neighborhood of each non-degenerate critical point $\mathbf{w}$, γ achieves maximum height at $\mathbf{w}$. It is hard to deform γ explicitly on $\mathcal{V}_*$ but Morse theory gives us enough information on the topology of $\mathcal{V}_*$.

Definition 1.12 (smooth critical point)**.** *A smooth critical point of a rational function $F = P/Q$ in the direction $\hat{\mathbf{r}}$ is a smooth point $\mathbf{w}$ in $\mathcal{V}_* := \mathcal{V}_Q \cap \mathbb{C}_*^d$ such that $dh(\mathbf{w}) = 0$ where h is defined to* *be a smooth mapping of real manifolds from $\mathcal{V}_*$ to $\mathbb{R}$, sending $\mathbf{z}$ to $-\hat{\mathbf{r}} \cdot \mathrm{Relog}(\mathbf{z})$.*

Definition 1.13 (square-free part of a polynomial Q)**.** *The square-free part $\widetilde{Q}$ of a polynomial Q is the product of its distinct irreducible factors over the complex number. If $Q = \widetilde{Q}$, we say that Q is square-free.*

The square-free part $\widetilde{Q}$ can be computed by dividing Q by the greatest common divisor of all of its first order derivatives (see [PWM24, Exercise 8.1]).

Proposition 1.14. *[PWM24, Lemma 7.8] Let $F = P/Q$ and let $\tilde{Q}$ be the square-free part of Q. Suppose that $\mathcal{V}_Q$ is smooth. Then $\mathbf{w}$ is a critical point for $h_{\hat{\mathbf{r}}}$ if and only if it satisfies the* **critical point equations***:*

$$\begin{aligned} \tilde{Q}(\mathbf{w}) &= 0 \\ \hat{r}_k w_1 \tilde{Q}_{z_1}(\mathbf{w}) - \hat{r}_1 w_k \tilde{Q}_{z_k}(\mathbf{w}) &= 0 \qquad 2 \leq k \leq d \end{aligned}$$

Definition 1.15 (level sets of $\mathcal{V}_*$)**.** *Let $c \in \mathbb{R}$. We define $\mathcal{V}_{\leq c}$ to be the level set $\{\mathbf{z} \in \mathcal{V}_* : h(\mathbf{z}) \leq c\}$. We omit the $*$ in the subscript for simplicity.*

If there is no critical value or critical value at infinity (see Definition 5.33) in $[a, b]$, then $\mathcal{V}_{\leq b}$ is homotopy equivalent to $\mathcal{V}_{\leq a}$. [Mil63, Theorem 3.1] is analogous to this result but requires compactness on $h^{-1}[a, b]$ where the domain of h is $\mathcal{V}_*$. The version we used is [BMP22, Theorem 2]. Any chain supported in $\mathcal{V}_{\leq b}$ can then be pushed down to lower heights to be only supported in $\mathcal{V}_{\leq a}$.

The topology for $\mathcal{V}_{\leq c}$ only changes at critical values c. In particular, suppose that there is only one critical value c (not a critical value at infinity) in (a, b) and $h^{-1}[a, b]$ contains exactly one critical point $\mathbf{p}$ with $h(\mathbf{p}) = c$ for $a < c < b$. Here the domain of h is $\mathcal{V}_*$. If $\mathbf{p}$ is nondegenerate, then $\mathcal{M}_{\leq b}$ is homotopy equivalent to $\mathcal{M}_{\leq a}$ with a λ-cell attached along the boundary. Here, λ is the number of negative eigenvalues of the Hessian matrix of h at $\mathbf{p}$. In our choice of $h(\mathbf{z}) := -\hat{\mathbf{r}} \cdot \mathrm{Relog}(\mathbf{z})$, the index is always $d - 1$. The analogous result is [Mil63, Theorem 3.1].

Therefore, the homology $\mathrm{H}_d(\mathcal{V}_{\leq a})$ changes only when a crosses a critical value, assuming no critical value at infinity. We can list critical points of h by their heights, from highest to lowest, as $\mathbf{w}_1, \cdots, \mathbf{w}_m$. Let $c_i := h(\mathbf{w}_1)$ and thus $c_1 \geq \cdots \geq c_m$ where equality is allowed when there are more than one critical point at one critival value. The space $\mathcal{V}_{\leq c_m - \epsilon}$ is homotopy equivalent to $\mathcal{V}_{\leq a}$ for arbitrarilly small $a < c_m$ because there are no more critical points below height c_m. When the sublevel set first crosses the lowest critical value c_m, there is a $(d-1)$-cell attached and so we have one generator (the $(d-1)$-cell) added to the homology. Building it by attachments, we add exactly m generators corresponding to the m critical points. The result of attachment is Theorem 2.27. It says that the homology generators of $\mathrm{H}_{d-1}(\mathcal{V}_*, -\infty)$ are $(d-1)$-cycle $\gamma_{\mathbf{w}_i}$ for each critical point $\mathbf{w}_i$ such that $\gamma_{\mathbf{w}_i}$ attains its maximal height at $\mathbf{w}_i$. The relative homology group $\mathrm{H}_{d-1}(\mathcal{V}_*, -\infty)$ is defined in Chapter 2.1.2. For now, we can interpret this result as

$$[T] := [\gamma'] + \sum_{i=1}^{m} n_i [\gamma_{\mathbf{w}_i}]$$

in $\mathrm{H}_{d-1}(\mathcal{V}_*)$ where γ' is supported on $\mathcal{V}_{\leq c_m - \epsilon}$ for any $\epsilon > 0$. Here n_i are integers to be determined. It is very hard to determine these coefficient except the coefficient for minimal smooth critical points. When $\mathcal{V}$ is smooth, every critical point is smooth. A critical point $\mathbf{w}$ is minimal if $\mathbf{w} \in \partial\mathcal{D}$ where $\mathcal{D}$ is the domain of convergence of $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$. The coefficient n_i for a minimal smooth critical point is ± 1 depending on the orientation of $\gamma_{\mathbf{w}}$.

Since there are no more critical points below height c_m, $\mathcal{V}_{\leq c_m - \epsilon}$ is homotopy equivalent to $\mathcal{V}_{\leq a}$ for any $a < c_m$. We can push γ' down arbitrarily low. The integral of $\mathrm{Res}\,(\omega)$ over γ' decays super-exponentially. Therefore,

$$a_{n\hat{\mathbf{r}}} = (2\pi i)^{-d} \int_T \omega = (2\pi i)^{1-d} \sum_{i=1}^{m} n_i \int_{\gamma_{\mathbf{w}_i}} \mathrm{Res}\,(\omega) + O(e^{an}) \tag{1.4}$$

for arbitrarily large $a > 0$. Integrals $\int_{\gamma_{\mathbf{w}_i}} \mathrm{Res}\,(\omega)$ are saddle point integrals and we can use Fourier-Laplace method to get asymptotics. We will talk about saddle point in Chapter 1.2.3.

When $\mathcal{V}$ is not smooth

We assume that $F = P/Q$ is a rational function with coprime polynomials P and Q. The singular variety of F is then $\mathcal{V} = \mathcal{V}_Q$. In the general case, $\mathcal{V}$ is not smooth. The idea is to use Whitney stratification (Chapter 5.2.1) to decompose $\mathcal{V}$ into unions of smooth manifolds of different dimensions patched in a nice way. That is, we have $\mathcal{V} = \bigcup_{\alpha\in I} \mathcal{S}_\alpha$ where I is a finite index set and $\mathcal{S}_\alpha$ are smooth manifolds of various dimensions. This decomposition satisfies further conditions listed in Definition 5.13 and 5.14. We call each $\mathcal{S}_\alpha$ a stratum in the stratification. Critical points are defined on each stratum by Definition 5.20. In particular, we say that $\mathbf{p}$ is a (stratified) critical point if $dh|_{\mathcal{S}(\mathbf{p})} = 0$ where $\mathcal{S}(\mathbf{p})$ is the stratum on which $\mathbf{p}$ lies. The search of stratified critical points involve some algebraic computations and are listed in procedures (1) to (5) in Chapter 5.2.2.

We directly deform T using (stratified) Morse theory in $\mathcal{M} := \mathbb{C}_*^d - \mathcal{V}$ on which the d-form $\mathbf{z}^{-\mathbf{r}-\mathbf{1}}F(\mathbf{z})\ d\mathbf{z}$ is holomorphic. This is different as we did when $\mathcal{V}$ is smooth; previously we deform T to T' by crossing out $\mathcal{V}$ and we deform the intersection class $\mathbf{INT}[T, T'; \mathcal{V}]$ using Morse theory in $\mathcal{V}$. This time we first use the Morse theory to deform T in $\mathcal{M}$, and then use residue theorem. This is a more general way to do topological deformation and works regardless of smoothness of $\mathcal{V}$.

Going back to (1.1), the integrand $\mathbf{z}^{-\mathbf{r}-\mathbf{1}}F(\mathbf{z})$ is a holomorphic function on $\mathcal{M} := \mathbb{C}_*^d - \mathcal{V}$. Here $\mathbb{C}_* := \mathbb{C} - \{0\}$ so that the denominator in $\mathbf{z}^{-\mathbf{r}-\mathbf{1}}$ will be non-zero. The space $\mathcal{V}$ is the singular variety of $F(\mathbf{z})$ and in particular $\mathcal{V} = \mathcal{V}_Q$ when $F = P/Q$ is a rational (or meromorphic) function with coprime P and Q. If T is homotopy equivalent to another d-chain C in $\mathcal{M}$, then

$$\int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}}F(\mathbf{z})\ d\mathbf{z} = \int_C \mathbf{z}^{-\mathbf{r}-\mathbf{1}}F(\mathbf{z})\ d\mathbf{z}$$

Indeed, the integrand $\omega = \mathbf{z}^{-\mathbf{r}-\mathbf{1}}F(\mathbf{z})\ d\mathbf{z}$ is a d-holomorphic form on $\mathcal{M}$ (see Chapter 2.1.1). By Theorem 2.6, the integral value only depends on the homology class $[T]$ in $\mathrm{H}_d(\mathcal{M})$ and the cohomology class $[\omega]$ in $\mathrm{H}^d(\mathcal{M})$. Here $\mathrm{H}^d(\mathcal{M})$ and $\mathrm{H}_d(\mathcal{M})$ are the d-th singular (co)homology group of $\mathcal{M}$ with complex coefficients. Details on the different types of (co)homology theory and the well-definedness of the integral above are in Chapter 2.1. The topological deformation is done by writing $[T]$ as an integer sum of homology generators $[\sigma_i]$ of $\mathrm{H}_d(\mathcal{M})$. Actually, we often write $[T]$ as an integer sum of homology generators $[\sigma_i]$ of $\mathrm{H}_d(\mathcal{M}, -\infty)$, the relative homology group defined in Chapter 5.2.3. Then $\int_T \omega = \sum_i \int_{\sigma_i} \omega + O(e^{na})$ for any $a < 0$. We sacrifice a super-exponentially decaying error $O(e^{na})$ here.

To describe the generators of $\mathrm{H}_d(\mathcal{M}, -\infty)$, we need to list all stratified critical points of $h_{\hat{\mathbf{r}}}$ on $\mathcal{V}_*$. Let $\mathbf{p}_1, \cdots, \mathbf{p}_m$ be critical points of $h_{\hat{\mathbf{r}}}$, from highest height $h_1 := h(\mathbf{p}_1)$ to lowest $h_m = h(\mathbf{p}_m)$. If all critical points are nondegenerate and no critical value at infinity greater than or equal to h_m, then Theorem 5.28 tells us that generators of $\mathrm{H}_d(\mathcal{M}, -\infty)$ are $\sigma_{j,i} = \gamma_j \times \beta_{j,i}$ where

1. if the stratum $\mathcal{S}(\mathbf{p}_j)$ containing $\mathbf{p}_j$ has complex codimension k_j, then γ_j is a $(d - k_j)$-cycle and $\beta_{j,i}$ are k_j-cycle and homology generators for the normal Morse data at $\mathbf{p}_j$. Here i ranges from 1 to s_j where s_j.

2. $h_{\hat{\mathbf{r}}}$ achieves its maximum on γ_j at $\mathbf{p}_j$.

3. γ_j is of homotopy type $(B^{d-k_j}, \partial B^{d-k_j})$.

The number s_j of generators $\beta_{j,i}$ in the k_j-th homology of the normal Morse data at $\mathbf{p}_j$ depends on the geometric property of $\mathcal{V}$ near $\mathbf{p}_j$. If $\mathbf{p}_j$ is a transverse multiple point (Definition 5.2 or 5.8), then $s_j = 1$. In this case, we can explicitly write $\beta_j := \beta_{j,1}$ as $\Psi^{-1}(T_\epsilon, \mathbf{0})$ where

$$\Psi(\mathbf{z}) = (Q_1(\mathbf{z}), \cdots, Q_{k_j}, z_{\pi_1} - p_{\pi_1}, \cdots, z_{\pi_{d-k_j}} - p_{\pi_{d-k_j}}). \tag{1.5}$$

Here we assume $Q = uQ_1^{m_1} \cdots Q_{k_j}^{m_{k_j}}$ in $\mathcal{O}_{\mathbf{p}_j}$ and $\boldsymbol{\pi} := (\pi_1, \cdots, \pi_{d-k_j}) \subset [d]$ is a sub-index such that $\mathbf{z}_{\boldsymbol{\pi}}$ parametrize the stratum $\mathcal{S}(\mathbf{p})$. The torus T_ϵ is a k_j-torus with suffiiciently small radii ϵ centered at $\mathbf{0} \in \mathbb{C}^{k_j}$.

We can represent $[T]$ as an integer sum of these homology generators $\sigma_{j,i}$ in $\mathrm{H}_d(\mathcal{M}, -\infty)$. That is,

$$[T] = \sum_{j=1}^{m} \sum_{i=1}^{s_j} n_{j,i}[\sigma_{j,i}] \tag{1.6}$$

Assume that all critical points are transverse multiple point. There is only one generator corresponding to a critical point, that is, $s_j = 1$ for all $j = 1, \cdots, m$. We can write

$$[T] = \sum_{j=1}^{m} n_j[\sigma_j] \tag{1.7}$$

where the equality holds in $\mathrm{H}_d(\mathcal{M}, -\infty)$ and $n_j \in \mathbb{Z}$.

The integer coefficient $n_{j,i}$ in equation (1.6) remains to be determined and it is also very hard to determine. If $\mathcal{V}$ is a hyperplane arrangement (Chapter 5.3), then every n_j is $0, 1$, or -1. If $\mathbf{p}_j$ is a minimal transverse point, by [PWM24, Corollary 10.41], n_j is $0, 1$, or -1. There are other cases where n_j for a critical point $\mathbf{p}_j$ can be some other integers. For example, [BMPS18] studies the diagonal coefficients of the power series $F(x, y, z, w) = 1/Q$ where $Q(x, y, z, w) = 1 - (x + y + z + w) + 27xyzw$. In direction $\hat{\mathbf{r}} = (1, 1, 1, 1)$, it has two non-minimal smooth critical point $\mathbf{p}_1$ and $\mathbf{p}_2$. The coefficients n_1 and n_2 for these two critical points are 3. Moreover, it has a minimal critical point $\mathbf{p}_0 = (1/3, 1/3, 1/3, 1/3)$ and this point is not a multiple point by Proposition 6.2 because $\mathsf{hom}(Q, \mathbf{p}_0)$ cannot be factorized into linear factors. Moreover, the coefficients $n_{0,i}$ for $\mathbf{p}_0$ is 0 for all i.

With the decomposition (1.6), we can write the integral

$$\int_T \omega = \sum_{j=1}^{m} \sum_{i=1}^{s_j} n_{j,i} \int_{\sigma_{j,i}} \omega + O(e^{an})$$

as $n \to \infty$, where $a < 0$ is arbitrary. In the case (1.7), the decomposition of the integral becomes

$$\int_T \omega = \sum_{j=1}^{m} n_j \int_{\sigma_j} \omega + O(e^{an}) = \sum_{j=1}^{m} n_j \int_{\gamma_j \times \beta_j} \omega + O(e^{an})$$

By Theorem 2.21(iii), the more general residue theorem, the above quantity becomes

$$\sum_{j=1}^{m} n_j (2\pi i)^{k_j} \int_{\gamma_j} \operatorname{Res}\left(\omega; \mathcal{S}(\mathbf{p}_j) \cap \mathcal{D}_j\right) + O(e^{an})$$

where k_j is the complex codimension of the stratum $\mathcal{S}(\mathbf{p}_j)$ and $\mathcal{D}_j$ is a small neighborhood of $\mathbf{p}_j$ in $\mathbb{C}_*^d$. $\operatorname{Res}\left(\omega; \mathcal{S}(\mathbf{p}_j) \cap \mathcal{D}_j\right)$ is a $(d-k_j)$-form and it is called the iterated residue defined in equation (2.3). Moreover, each integral over γ_j is a saddle point integral. We will see more precisely what we mean by saddle point integral in Chapter 1.2.3.

1.2.3. Saddle point integral

The goal is now to state results for saddle point integrals. We introduce results in [PWM24, Chapter 4 and 5] without proof. To begin with, we first define what we mean by a saddle point integral. Let's start from the univariate case.

Definition 1.16 (Fourier-Laplace integrals)**.** *The integral in the form*

$$\int_{\gamma} A(z) \exp(-\lambda \phi(z)) \ dz$$

is called a Fourier-Laplace integrals. In particular, A is called the amplitude function and ϕ is called the phase function. Both A and ϕ are complex analytic functions in a neighborhood $\mathcal{N} \subseteq \mathbb{C}$ of the origin. We study the asymptotics of the integral as the parameter $\lambda \to \infty$.

Let $F(x,y) = P(x,y)/Q(x,y)$ be a bivariate rational function with a convergent power (Laurent) series $F(x,y) = \sum_{r,s} a_{r,s} x^r y^s$ in $\mathcal{D}$. Suppose the singular variety $\mathcal{V}_Q$ of F is smooth and we are looking for asymptotic behavior of $a_{r,s} := a_{n(\hat{r},\hat{s})}$ as $n \to \infty$ in the direction $(\hat{r},\hat{s})$, then every integral

$$\int_{\gamma_{\mathbf{w}_i}} \operatorname{Res}\left(\omega\right)$$

in equation (1.4) can be put in the form of a Fourier-Laplace integral. Here we let (x_*, y_*) denote the critical point $\mathbf{w}_i$. In particular, since $\mathcal{V}_Q$ is smooth, either $\tilde{Q}_x \neq 0$ or $\tilde{Q}_x \neq 0$ on $\mathcal{V}_Q$ where $\tilde{Q}$ is the square-free part of Q. Therefore, we can assume without loss of generality that $Q_y \neq 0$ on $\mathcal{V}_Q$.

When Q is square-free (i.e. $Q = \tilde{Q}$), by equation (2.2),

$$\mathrm{Res}\,(\omega) = \mathrm{Res}\left(x^{-r-1}y^{-s-1}\frac{P(x,y)}{Q(x,y)}\ dx \wedge dy\right) = \frac{x^{-r-1}y^{-s-1}P(x,y)}{Q_y(x,y)}dx.$$

Here we should think $P(x,y)$ and $Q_y(x,y)$ as a univariate function of the variable x because $y(x)$ is parametrized by x on $\mathcal{V}_Q$ by the implicit function theorem. Therefore, we can write $\int_{\gamma_{\mathbf{w}_i}} \mathrm{Res}\,(\omega)$ as

$$x_*^{-r}y_*^{-s}\int_{\gamma_{(x_*,y_*)}} \exp(-\lambda\phi(x))A(x)\ dx$$

where $A(x) := \dfrac{P(x)}{Q_{z_d}(x,y)xy}$, $\phi(x) := r/s\log x + \log y - r/s\log x_* - \log y_*$, and $\lambda = s$. We think of the variable $y(x)$ as parametrized by x and so A and ϕ are univariate function in variable x. Since $\gamma_{\mathbf{w}_i}$ is a path in a neighborhood of $\mathbf{w}_i$, there is no concern over the branch choice of log and so ϕ is analytic at least over $\gamma_{\mathbf{w}_i}$.

We can write $\phi(x) = 1/\hat{s}[\hat{r}\log x + \log y - \hat{r}\log x_* - \log y_*]$ and $\lambda = n\hat{s}$. It is often a convention that in smooth point asymptotics, we choose the direction vector $(\hat{r},1)$ so that $\Re(\phi(x)) = h(x_*,y_*) - h(x,y)$ and $\lambda = n$. Normalizing the direction vector will not change the location of critical points (but may change the asymptotics by a sign if we normalize the direction by a negative value). Now, assume the direction vector is $(\hat{r},1)$, then $\int_{\gamma_{\mathbf{w}_i}} \mathrm{Res}\,(\omega)$ is equal to

$$x_*^{-r}y_*^{-s}\int_{\gamma_{(x_*,y_*)}} \exp(-n\phi(x))A(x)\ dx \tag{1.8}$$

where $A(x) := \dfrac{P(x)}{Q_{z_d}(x,y)xy}$ and $\phi(x) = \hat{r}\log x + \log y - \hat{r}\log x_* - \log y_*$.

The exponential growth of equation (1.8) is given by $x_*^{-r}y_*^{-s}$ where $r = n\hat{r}$ and $s = n$. The subexponential growth is given by the integral in (1.8). Remember that in Chapter 1.2.2, we say that $\gamma_{(x_*,y_*)}$ is a chain on which $h_{\hat{\mathbf{r}}}$ attains maximum at (x_*,y_*). In other words, over $\gamma_{(x_*,y_*)}$, the real part of the phase function $\phi(x)$ attains minimum value 0 at x_*. This is a univariate saddle point and we have the following theorem that tells us the exact asymptotics of the integral.

Theorem 1.17 (univariate Fourier-Laplace asymptotics)**.** *[PWM24, Theorem 4.1] Let A and ϕ be analytic functions in a neighborhood $\mathcal{N} \subseteq \mathbb{C}$ of the origin. Let*

$$A(z) = \sum_{j=0}^{\infty} b_j z^j, \ \ \phi(z) = \sum_{j=0}^{\infty} c_j z^j$$

be the power series for A and ϕ at the origin, and let $\ell \geq 0$ and $k \geq 2$ be the indices of the least nonvanishing terms in these series, so that $b_\ell, c_k \neq 0$ and $b_j = c_i = 0$ for any $j < \ell$ and $i < k$. Let $\gamma : [-\epsilon, \epsilon] \to \mathbb{C}$ be any smooth curve with $\gamma(0) = 0 \neq \gamma'(0)$ and assume that $Re\{\phi(\gamma(t))\} \geq 0$ with equality only at $t = 0$. Denote

$$\begin{aligned}
\mathcal{I}_+(\lambda) &:= \int_{\gamma|_{[0,\epsilon]}} A(z) \exp(-\lambda\phi(z)) \ dz \\
\mathcal{I}_+(\lambda) &:= \int_{\gamma} A(z) \exp(-\lambda\phi(z)) \ dz \\
C(k, \ell) &:= \frac{\Gamma((1+\ell)/k)}{k}
\end{aligned}$$

where Γ is the Euler gamma function. Then there are asymptotic expansions

$$\begin{aligned}
\mathcal{I}_+(\lambda) &\approx \sum_{j=\ell}^{\infty} a_j C(k, j)(c_k\lambda)^{-(1+j)/k} \\
\mathcal{I}(\lambda) &\approx \sum_{j=\ell}^{\infty} \alpha_j C(k, j)(c_k\lambda)^{-(1+j)/k}
\end{aligned}$$

with the following explicit description.

(i) a_j is a polynomial expression in the values $b_\ell, \cdots, b_j, c_k^{-1}, c_{k+1}, \cdots, c_{k+j-\ell}$ whose first two values are $a_\ell = b_\ell$ and $a_{\ell+1} = b_{\ell+1} - b_\ell \dfrac{2+\ell}{k}\dfrac{c_{k+1}}{c_k}$.

(ii) the choice of k-th root in the expression $(c_k\lambda)^{-(1+j)/k}$ is made by taking the principal root in $x^{-1}(c_k\lambda x^k)^{1/k}$ where $x = \gamma'(0)$.

(iii) the numbers α_j are related to the numbers a_j by

$$\alpha_j = \begin{cases} 2a_j & \textit{if } k \textit{ is even and } j \textit{ is even} \\ 0 & \textit{if } k \textit{ is even and } j \textit{ is odd} \\ (1-\xi^{j+1})a_j & \textit{if } k \textit{ is odd} \end{cases}$$

where

$$\xi = -\exp\left(\frac{i\pi}{k}\ \textit{sgn Im}\{\phi(\gamma'(0))\}\right)$$

□

We omit the proof here because it takes more than twenty pages to develop the univariate theory. Chapter 4.4.1 gives an example of using the above theorem to get the asymptotics of Catalan number.

Let $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$ is a d-variate rational function with convergent series $F(\mathbf{z}) = \sum_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ in $\mathcal{D}$. Assume $\mathcal{V} := \mathcal{V}_Q$ is smooth. Let $\hat{\mathbf{r}}$ be the direction and suppose that the asymptotics of $a_{n\hat{\mathbf{r}}}$ is expressed in equation (1.4). Since $\mathcal{V}_Q$ is smooth, assume without loss of generality that $Q_{z_d} \neq 0$ on $\mathcal{V}_Q$. Normalize $\hat{\mathbf{r}}$ so that $\hat{\mathbf{r}} = (\hat{r}_1, \cdots, \hat{r}_{d-1}, 1)$. Then we can write each $\int_{\gamma_{\mathbf{w}_i}} \operatorname{Res}(\omega)$ in (1.4) as

$$\mathbf{w}^{-\mathbf{r}} \int_{\gamma_{\mathbf{w}}} \exp(-n\phi(\mathbf{z}_{\hat{d}}))A(\mathbf{z}_{\hat{d}})\ d\mathbf{z}_{\hat{d}}$$

where

$$\begin{aligned} \mathbf{z}_{\hat{d}} &:= (z_1, \cdots, z_{d-1}) \\ d\mathbf{z}_{\hat{d}} &:= dz_1 \wedge \cdots \wedge dz_{d-1} \\ A(\mathbf{z}_{\hat{d}}) &:= \frac{P(\mathbf{z})}{Q_{z_d}(\mathbf{z}) \prod_{i=1}^{d} z_i} \\ \phi(\mathbf{z}_{\hat{d}}) &:= \sum_{i=1}^{d-1} \hat{r}_i \log(z_i) + \log(z_d) - \sum_{i=1}^{d-1} \hat{r}_i \log(w_i) - \log(w_d) \end{aligned}$$

and z_d is a function parametrized by $\mathbf{z}_{\hat{d}}$ on $\mathcal{V}_Q$. If $h_{\hat{\mathbf{r}}}(\mathbf{z})$ over $\gamma_{\mathbf{w}}$ attains maximum uniquely at $\mathbf{w}$, then we have the following result.

Theorem 1.18 ($\Re(\phi)$ attains strict minimum)**.** *[PWM24, Theorem 5.2] Let A and ϕ be complex-valued analytic functions on a compact neighborhood $\mathcal{N}$ of the origin in $\mathbb{R}^d$. Suppose that the real part of ϕ is nonnegative on $\mathcal{N}$ and vanishes only at the origin, and that the Hessian matrix $\mathcal{H}$ of ϕ at the origin is nonsinuglar. Then $\mathcal{I}(n) = \int_{\mathcal{N}} A(\mathbf{z}) \exp(-n\phi(z))\, d\mathbf{z}$ has an asymptotic expansion*

$$\mathcal{I}(n) \approx \sum_{\ell \geq 0} c_\ell n^{-d/2-\ell}$$

with leading coefficient

$$c_0 = A(\mathbf{0}) \frac{(2\pi)^{d/2}}{\sqrt{\det \mathcal{H}}}$$

where $\sqrt{\det \mathcal{H}}$ is the product of the principal square roots of the eigenvalues of $\mathcal{H}$. The other coefficient c_ℓ can be calculated by [PWM24, Corollary 5.17]. □

It is possible, however, that $\Re(\phi)$ attains minimum on a positive dimension subset of $\gamma_{\mathbf{w}}$. In this case, we need to use the most general version in [PWM24, Theorem 5.3]. Indeed, when $\mathcal{V}$ is not smooth, one needs to use the most general version for stratified space $\mathcal{V}$. We don't bother giving a full statement of the theorem here. Instead, we will give the modified version of the theorem in the appropriate setting we need. For example, Proposition 4.6 gives a complete description on the asymptotics of the saddle point integral in the setting where amplitude function vanishes to order 1.

1.3. Main Results

The results of this thesis are in two parts. The first part (Chapter 4) is on coefficient asymptotics of multivariate algebraic generating functions. The second part (Chapter 6) is on coefficient asymptotics of pseudo multiple points in the setting of rational functions. These two parts are relatively independent to each other.

1.3.1. Coefficient asymptotics of algebraic generating functions

The complexity hierarchy for ACSV on generating functions starts from the rational functions, then to meromorphic functions, and extends to algebraic functions and D-finite functions. The first main result of this thesis is on coefficient asymptotics of multivariate algebraic generating functions. We give a brief introduction to algebraic generating functions and our results here, with more details given in Chapter 4.

A d-variate algebraic generating function $F(\mathbf{z})$ is encoded by a $(d+1)$-variate real polynomial $P = \sum_{j=0}^{m} P_j(\mathbf{z}) f^j \in \mathbb{R}[\mathbf{z}][f]$. There is a neighborhood $\mathcal{N}$ of $\mathbf{0} \in \mathbb{C}^d$ such that there is an absolutely convergent power series $F(\mathbf{z}) = \sum_{\mathbf{r}\in\mathbb{N}^d} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ satisfying $P(\mathbf{z}, F(\mathbf{z})) = 0$. Such a polynomial P encodes at most m such convergent power series at $\mathbf{0} \in \mathbb{C}^d$. To distinguish the power series F from the others, we need to give extra information other than the polynomial P. We assume henceforth that $P(\mathbf{0}, \cdot)$ has a simple root f_0. Therefore, the two conditions $P(\mathbf{z}, f) = 0$ and $F(\mathbf{0}) = f_0$ specify a unique d-variate algebraic generating function $F(\mathbf{z})$.

To study the asymptotics of $a_{\mathbf{r}}$ in the, we can use the diagonal embedding method (Chapter 3) which embeds the d-variate algebraic generating function $F(\mathbf{z})$ as the i-th elementary diagonal of a $(d+1)$-variate rational generating function $\tilde{F}(f, \mathbf{z})$. Explicitly,

$$a_{\mathbf{r}} = [f^{r_i} \mathbf{z}^{\mathbf{r}}] \tilde{F}(f, \mathbf{z})$$

if z_i divides $F(\mathbf{z})$. This method converts the problem of finding coefficient asymptotics of an algebraic function back to the easier problem of finding those of a rational function.

Motivation: The embedding method involves the Safonov embedding theorem (Theorem 3.5) to embed the algebraic generating functions into a one-more-variable rational function. Moreover, analysis of coefficient asymptotics on the rational function involves residue forms and intersection classes which are not familiar to combinatorists. Therefore, we develop the alternative lifting method to make the analysis more transparent. After all, it only involves saddle point integrals in ACSV.

Let T be a torus centered at the origin in $\mathbb{C}^d$ such that T is in the domain of convergence E of the power series $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$. Then by Cauchy integral formula, we have

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_T \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \, \frac{d\mathbf{z}}{\mathbf{z}}$$

Let $\widetilde{\mathcal{V}}$ denote the variety $\{(\mathbf{z}, f) : P(\mathbf{z}, f) = 0\}$ in $\mathbb{C}^{d+1}$. Let $L : E \to \widetilde{\mathcal{V}}$ be the lifting map $L(\mathbf{z}) := (\mathbf{z}, F(\mathbf{z}))$ and $\pi : \widetilde{\mathcal{V}} \to \mathbb{C}^d$ be the projection map $\pi(\mathbf{z}, f) = \mathbf{z}$. We lift T to $\tilde{T}$ by the lifting map L onto $\widetilde{\mathcal{V}}$. Then we can write

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_{\tilde{T}} \mathbf{z}^{-\mathbf{r}} f \, \frac{d\mathbf{z}}{\mathbf{z}} \tag{1.9}$$

by Proposition 4.1. The story begins with equation (1.9) by deforming $\tilde{T}$ within $\widetilde{\mathcal{V}}$ so that it becomes a saddle point integral. Deforming $\tilde{T}$ within $\widetilde{\mathcal{V}}$ will not change the integral value by Proposition 4.2.

Indeed, we can write equation (1.9) in the form of a saddle point integral

$$\int_{\tilde{T}} A(\mathbf{z}) \exp(-N(\phi(\mathbf{z}))) \, d\mathbf{z}$$

where $A(\mathbf{z}) := f/\mathbf{z}$, $\phi(\mathbf{z}) := \sum_{i=1}^{d} \hat{\mathbf{r}}_i \log z_i$, $N := |\mathbf{r}|$, and $\hat{\mathbf{r}} := \mathbf{r}/N$. To put the integral into a saddle point integral, we need to deform $\tilde{T}$ so that it can contain some critical point of the phase function ϕ. They can be calculated using formulae in (4.5).

Our goal is to deform $\tilde{T}$ to a chain Γ within $\widetilde{\mathcal{V}}$ so that Γ goes through some stationary points (Definition 4.4) where $\Re\{\phi\}$ achieves its minimum on Γ. If there are any stationary points on Γ, we say that Γ is in stationary phase position. Therefore, critical points are candidates for stationary points. We just need to know which of these critical points are stationary points and how to deform $\tilde{T}$ to reach stationary phase position.

The deformation of $\tilde{T}$ can be done by enlarging the first d coordinates so as to remain a torus at every fixed time and varying the f-coordinate so as to remain on $\widetilde{\mathcal{V}}$, until it hits some critical points. This deformation ensures that $\Re\{\phi\}$ is always constant and thus at the end of the deformation, the resulting chain $\tilde{T}'$ is in stationary phase position if it contains finitely many critical points. However, there is no guarantee that in general such a deformation will succeed.

Theorem 4.14 deforms $\tilde{T}$ within $\widetilde{\mathcal{V}}$ in the spirit of the previously introduced deformation. We list somes notations and assumptions first before stating the result.

Notations:

- $\mathrm{discr}(P)$: discriminant polynomial of the polynomial P.
- br: the variety $\{\mathbf{z} \in \mathbb{C}^d : \mathrm{discr}(P)(\mathbf{z}) = 0\}$.
- $\widetilde{\mathsf{br}}$: $\{(\mathbf{z}, f) \in \widetilde{\mathcal{V}} : \partial P/\partial f = 0\}$.
- weakly minimal: $\mathbf{w}$ is weakly minimal for a polynomial G if G does not vanish on $t \cdot T(\mathbf{w})$ for $0 < t < 1$ where $T(\mathbf{w}) := \{\mathbf{z} \in \mathbb{C}^d : |z_i| = |w_i|\}$.
- smooth critical point on a manifold M in direction $\hat{\mathbf{r}}$: a smooth point $\mathbf{z} \in M$ such that $d\phi|_M = 0$ where $\phi(\mathbf{z}) := \sum_{j=1}^{d} \hat{\mathbf{r}}_j \log z_j$.

Standing assumptions:

1. $P \in \mathbb{R}[\mathbf{z}][f]$ and $\widetilde{\mathcal{V}}$ is smooth.
2. All roots of $P(\mathbf{0}, \cdot)$ is simple and $\mathrm{discr}(P)$ is square-free.
3. F is an algebraic power series in a neighborhood of the origin in $\mathbb{C}^d$ defined by the equation $P(\mathbf{z}, F(\mathbf{z})) = 0$ and the initial condition $F(\mathbf{0}) = f_0$

Result of Theorem 4.14: Let $\mathbf{z} := \exp(\mathbf{x}) \in \mathsf{br}$ be a positive real zero of $\mathrm{discr}(P)$. Assume $\mathbf{z}$ is a smooth critical point on br in direction $\hat{\mathbf{r}}$ with $p_m(\mathbf{z}) \neq 0$ and $\mathbf{z}$ weakly minimal for $p_m \cdot \mathrm{discr}(P)$. Let $(\mathbf{z}, f_1) \in \widetilde{\mathsf{br}}$ be a real point. Then

1. the point $(\mathbf{z}, f_1)$ is a critical point on $\widetilde{\mathcal{V}}$ in direction $(\hat{\mathbf{r}}, 0)$.

2. If $(\mathbf{z}, f_1)$ is on the branch defined by f_0, let W denote the set of $\mathbf{y} \in \mathbb{R}^d$ such that there is a complex number $f_{\mathbf{y}}$ with $(\exp(\mathbf{x}+i\mathbf{y}), f_{\mathbf{y}})$ on the branch of $\widetilde{\mathcal{V}}$ defined by f_0 and critical in direction $(\hat{\mathbf{r}}, 0)$. Assume W is finite and for each $\mathbf{y} \in W$, the root of $P(\exp(\mathbf{x}+i\mathbf{y}), \cdot)$ has multiplicity precisely 2 and that the Hessian matrix of $\phi := \sum_{i=1}^{d} \hat{\mathbf{r}}_i \log z_i$ restricted to $\widetilde{\mathcal{V}}$ is nonsingular, then we have an asympotic expansion

$$a_{\mathbf{r}} \approx \exp(-\mathbf{r}\cdot\mathbf{x}) \sum_{\ell=1}^{\infty} \sum_{\mathbf{y}\in W} C_{\mathbf{y},\ell} \exp(-i\mathbf{r}\cdot\mathbf{y}) |\mathbf{r}|^{-d/2-\ell}$$

where the constants $C_{\mathbf{y},\ell}$ are given explicitly in Proposition 4.6. In particular, when $d = 1$ and 2, we have explicit formula for $C_{\mathbf{y},1}$ given in Corollary 4.7.

There is no guarantee that there will be $(\mathbf{z}, f_1)$ on the branch defined by f_0 in general. Therefore, we give sufficient conditions under which the above theorem works. In addition to the previous standing assumptions, we give the following extra assumptions.

Extra assumptions:

1. p_m is a monomial

2. all coefficients of F are nonnegative

3. P is quadratic in f

Result of Theorem 4.16: If the above three extra assumptions hold together with the standing assumptions, then $(\mathbf{z}, f_1)$ is a critical point on the branch defined by f_0 and so we can deform $\tilde{T}$ to $\tilde{T}'$ which is the torus containing the critical point $(\mathbf{z}, f_1)$.

Result of Theorem 4.17: If only the second extra assumption holds together with the standing assumptions, then we don't need the weak minimality condition on $\mathbf{z}$ in Theorem 4.14. Let $\mathbf{x}_*$ be a minimizer of $-\mathbf{x}\cdot\hat{\mathbf{r}}$ on the closure of $\mathcal{D}$ where $\mathcal{D} = \mathrm{Relog}(E)$. Let $\mathbf{z}_* = \exp(\mathbf{x}_*)$. Then $(\mathbf{z}_*, P(\mathbf{z}_*))$ is a critical point in direction $(\hat{\mathbf{r}}, 0)$. Moreover, $\tilde{T}$ can be deformed to a torus $\tilde{T}'$ that contains $(\mathbf{z}_*, F(\mathbf{z}_*))$. Algorithm 1 gives a way to find $\mathbf{z}_*$ and reports failure if p_m or $\nabla\mathrm{discr}(P)$ vanishes at

$\mathbf{z}_*$.

1.3.2. Asymptotics contribution of pseudo multiple points

We have seen in Chapter 1.2.2 that contributions to the coefficient asymptotics can be decomposed into a sum of integrals over generators of $\mathrm{H}_d(\mathcal{M}, -\infty)$. These generators correspond to critical points of $h_{\hat{\mathbf{r}}}$. Assume that all critical points are non-degenerate, then generators of $\mathrm{H}_d(\mathcal{M}, -\infty)$, given by (stratified) Morse theory in Chapter 5.2, are $\sigma_{j,i} = \gamma_j \times \beta_{j,i}$ where γ_j is a $(d-k_j)$-chain over which $h_{\hat{\mathbf{r}}}$ attains maximum at the non-degenerate critical point $\mathbf{p}_j$ and $\beta_{j,i}$ are generators of the k_j-th homology of normal Morse data of $\mathcal{M}$ at $\mathbf{p}_j$. Here k_j is the complex codimension of the stratum $\mathcal{S}(\mathbf{p}_j)$. The local geometry of $\mathcal{V}$ near $\mathbf{p}_j$ determines the normal Morse data at $\mathbf{p}_j$ and thus affect these $\beta_{j,i}$.

When $\mathbf{p}_j$ is a transverse (multiple) point, the rank of the k_j-th homology of normal Morse data of $\mathcal{M}$ at $\mathbf{p}_j$ is one and so there is only one generator $\sigma_j = \gamma_j \times \beta_j$ corresponding to the critical point $\mathbf{p}_j$. Moreover, β_j can be explicitly represented by $\Psi^{-1}(T_\epsilon, \mathbf{0})$ where Ψ is defined in (1.5). When $\mathbf{p}_j$ is a minimal arrangement (multiple) point, then [PWM24, Corollary 10.46] gives a way to turn it into a transverse point. When $\mathbf{p}_j$ is a non-arrangement (multiple) point, [PWM24, Chapter 10.5] gives the surgery method to solve some cases.

Motivation: All these methods have the assumption that $\mathbf{p}_j$ is a multiple point but verifying whether a point $\mathbf{p}_j$ is a multiple point or not is not computationally efficient.

By Definition 5.8, a point $\mathbf{p}$ is a multiple point on $\mathcal{V}_Q$ if and only if Q factors in $\mathcal{O}_{\mathbf{p}}$, the ring of analytic germs at $\mathbf{p}$, and each factor Q_i satisfies $\nabla Q_i(\mathbf{p}) \neq 0$. There is no efficient algorithm to do the factorization in the ring $\mathcal{O}_{\mathbf{p}}$.

Proposition 6.2 says that if $\mathbf{p}$ is a multiple point, then the leading homogeneous part $\mathsf{hom}(Q, \mathbf{p})$ of Q at $\mathbf{p}$, defined in Definition 6.1, can be factorized into linear factors. Fortunately, it is simple to factorize $\mathsf{hom}(Q, \mathbf{p})$ in the polynomial ring $\mathbb{C}[\mathbf{z}]$. Therefore, we define a **pseudo multiple point** to be a point $\mathbf{p}$ satisfying this necessary condition of being a multiple point. That is, $\mathsf{hom}(Q, \mathbf{p})$ can be factorized into linear factors in $\mathbb{C}[\mathbf{z}]$. To be more explicit, we write the factorization of $\mathsf{hom}(Q, \mathbf{p})$

as

$$\mathsf{hom}(Q, \mathbf{p}) = \ell_1^{m_1} \cdots \ell_n^{m_n}$$

where ℓ_i are non-associated elements in the ring $\mathbb{C}[\mathbf{z}]$ such that ℓ_i are linear and $\ell_i(\mathbf{0}) = 0$.

Result I: All pseudo multiple points $\mathbf{p}$ are multiple points when the dimension d is 2 and $m_1 = \cdots = m_n = 1$. Proved in Theorem 6.13

Given a bivariate function $Q(x, y)$, if $\mathsf{hom}(Q, \mathbf{p})$ can be factorized into non-associated linear factors without multiple power in $\mathbb{C}[x, y]$, then Q can be factorized into non-associated analytic germs in ${}_2\mathcal{O}_{\mathbf{p}}$. This result is proved in Chapter 6.2 and is based on Lemma 6.11, a special (and stronger) version of Weierstrass Preparation Theorem for bivariate analytic functions.

Result II: When a minimal critical point happens to be a pseudo multiple point satisfying the following assumptions, we can treat it as a multiple point for the sake of coefficient asymptotics, in the price of the asymptotic error terms listed below.

Let $F(\mathbf{z})$ be a d-variate rational generating function $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$ with coprime polynomials in $\mathbb{C}[\mathbf{z}]$. Assume that $F(\mathbf{z})$ has a convergent power series $F(\mathbf{z}) = \sum_{\mathbf{r} \in \mathbb{N}^d} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ with domain of convergence $\mathcal{D}$.

If the following conditions are satisfied,

Assumption 1.1.

1. *The point* $\mathbf{p}$ *is a pseudo multiple point such that* $\mathsf{hom}(Q)$ *factors into exactly* d *distinct linear factors. That is,*

$$\mathsf{hom}(Q, \mathbf{p}) = \prod_{i=1}^{d} \ell_i^{m_i}(\mathbf{z})$$

   where $\ell_i(\mathbf{p})$ *are non-zero and* $\ell_i(\mathbf{z})/\ell_i(\mathbf{p}) \in \mathbb{R}[\mathbf{z}]$.

2. *gradients of* $\ell_i(\mathbf{z})/\ell_i(\mathbf{p})$ *are linearly independent.*

3. The point $\mathbf{p}$ *is minimal* [1] *for functions* $H(\mathbf{z}) + tR(\mathbf{z})$ *for* $t \in [0,1]$ *where*

$$H(\mathbf{z}) = L_1^{m_1}(\mathbf{z}) \cdots L_d^{m_d}(\mathbf{z}),$$
$$L_i(\mathbf{z}) = \ell_i(\mathbf{z} - \mathbf{p}), \quad R(\mathbf{z}) = Q(\mathbf{z}) - H(\mathbf{z})$$

then we can normalize $F = P/Q$ by dividing both numerator and denominator by $\prod_{i=1}^d \ell_i(\mathbf{p})^{m_i}$. We still use the same symbol P, and Q to denote the normalized numerator and denominator, and the above assumptions are equivalent to the following assumptions.

Assumption 1.2.

1.
$$\mathsf{hom}(Q, \mathbf{p}) = \prod_{i=1}^d \ell_i^{m_i}(\mathbf{z}),$$
a product of d *polynomials* $\ell_1, \cdots \ell_d \in \mathbb{R}[\mathbf{z}]$ *with multiplicities* $m_1, \cdots, m_d$ *such that* $L_i(\mathbf{z}) := \ell_i(\mathbf{z} - \mathbf{p}) = 1 - \mathbf{b}^{(i)}\mathbf{z}$ *and* $\{\mathbf{b}^{(i)}\}$ *are linearly independent in* $\mathbb{R}^d$.

2. Let $H(\mathbf{z}) = L_1(\mathbf{z})^{m_1} \cdots L_d(\mathbf{z})^{m_d}$ *and* $R(\mathbf{z}) := Q(\mathbf{z}) - H(\mathbf{z})$. *Assume that the point* $\mathbf{p}$ *is minimal for* $H + tR$ *as* $t \in [0,1]$.

Let $F_{H+tR} = P/(H + tR)$ and thus $F_H = P/H$. Under Assumption 1.2, Theorem 6.16 implies that there is a $\delta > 0$ such that for any $t < \delta$, as $\mathbf{r} \to \infty$ with $\mathbf{r}$ inside a compact subset of the following cone

$$\left\{ \sum_{i=1}^d a_j \tilde{\mathbf{b}}_{\mathbf{p}}^{(i)} : a_j > 0 \right\} \text{ where } \tilde{\mathbf{b}}_{\mathbf{p}}^{(i)} = (b_1^{(i)} p_1, \cdots, b_d^{(i)} p_d)$$

1. if $m_1 = \cdots = m_d = 1$, then $[\mathbf{z}^{\mathbf{r}}]F_{H+tR}(\mathbf{z}) = [\mathbf{z}^{\mathbf{r}}]F_H(\mathbf{z}) + O(e^{a|\mathbf{r}|})$ for some $a < -\hat{\mathbf{r}} \cdot \mathrm{Relog}(\mathbf{p})$.

2. otherwise, $[\mathbf{z}^{\mathbf{r}}]F_{H+tR}(\mathbf{z}) = [\mathbf{z}^{\mathbf{r}}]F_H(\mathbf{z}) + O(\mathbf{p}^{-\mathbf{r}}|\mathbf{r}|^{|\mathbf{m}|-d-1})$.

Indeed, a pseudo multiple point can arise if we perturb the variety $\mathcal{V}_H$ (a hyperplane arrangement made by hyperplanes L_i) by adding higher order terms $R(\mathbf{z})$ onto $H(\mathbf{z})$. Then the point $\mathbf{p}$ will

[1] minimality for a function G hear explicitly means that G is non-vanishing on $tT(\mathbf{p})$ for $0 \leq t < 1$. In other words, the point $\mathrm{Relog}(\mathbf{p})$ is on the boundary of the component of $\mathsf{amoeba}(G)^c$ containing points $\mathrm{Relog}(\mathbf{z})$ as $\mathbf{z} \to \mathbf{0}$.

be a pseudo multiple point on $\mathcal{V}_{H+tR}$. Our result says that for small enough perturbation, the asymptotics of coefficients will not be affected too much. In particular, when $m_1 = \cdots = m_d = 1$, there is an exponentially small error $O(e^{a|\mathbf{r}|})$ because $[\mathbf{z}^\mathbf{r}]F_H(\mathbf{z})$ is generally of order $O(e^{-\hat{\mathbf{r}}\cdot \mathrm{Relog}(\mathbf{p})|\mathbf{r}|})$. Otherwise, the error term is polynomially small.

The advantage of this result is that all its requirements are easier to be verified than verifying multiple points. Once requirements satisfied, we only need to compute the coefficient of $\mathbf{z}^\mathbf{r}$ in the power series expansion of $F_H(\mathbf{z})$ where $\mathcal{V}_H$ is a hyperplane arrangement (see Chapter 5.1.2) made by d linearly independent hyperplanes $L_1, \cdots, L_d$. In particular, [BMP24b] gives an explicit way to calculate $[\mathbf{z}^\mathbf{r}]F_H(\mathbf{z})$ and we summarize their method in Chapter 5.3.

The disadvantage of this result is that it only works for $t < \delta$. If the quantifier $\delta > 1$, setting $t = 1$ makes $[\mathbf{z}^\mathbf{r}]F(\mathbf{z}) = [\mathbf{z}^\mathbf{r}]F_H(\mathbf{z}) +$ error terms listed above. Unfortunately, we can't quantify how large δ is, and therefore we can only say that asymptotic of coefficients in F_H from the hyperplane arrangement case can sustain with the previous asymptotic error term only after a small (but unquantified) amount of perturbation tR.

1.4. Outline of the Paper

This paper is implicitly divided into three parts. The main results are in Chapter 4 and 6. The first part consisting of Chapter 1 and 2 is background knowledge in ACSV. The second part is on multivariate algebraic generating functions where Chapter 3 gives an introduction to the embedding method and Chapter 4 is on one of the main results, the lifting method. The third part is on pseudo multiple point where Chapter 5 is a preparation chapter for terminologies and results used in Chapter 6. The last chapter is on the main result of pseudo multiple point.

We try to make this paper as self-contained as possible. Therefore, we include many background knowledge and review of literature in Chapter 1, 2, 3, and 5, possibly more than needed for some readers. It is not necessary to go over all of them before one proceeds to main results in Chapter 4 and 6. It is more efficient for readers with some knowledge in analytic combinatorics to go back to these background chapters when there is any question on definitions or general set-up. We summarize the content of each chapter below. There are also more detailed outlines of subchapters in the beginning of each chapter.

Chapter 2 gives technical background for things we used explicitly or implicitly in ACSV. Chapter 2.1 gives a gentle introduction to singular homology and cohomology with complex coefficients. Singular homology is important when we use Morse theory to deform the original Cauchy torus T as an integer sum of homology generators of $\mathcal{M}$. Singular cohomology is used when we define residue forms for forms with higher orders. Chapter 2.2 gives a generalized theory on complex residues. We often need to apply the generalized residue theorem after we localize the integral of $\mathbf{z}^{-\mathbf{r}-\mathbf{1}}F(\mathbf{z})\ d\mathbf{z}$ over a homology generator at a multiple point $\mathbf{p}$. In the same section, we also give an overview of the classical Morse theory that is helpful when the singular variety $\mathcal{V}$ is smooth. In Chapter 2.4, we introduce amoeba, a term originally from tropical geometry, which is essentially a projection of the complex variety by the map Relog onto the real space. The polynomial amoeba theory plays well with Newton polytope. In ACSV, amoeba is a good visualization tool when the dimension $d = 2$ or 3. Amoeba has a lot of good properties and it is often easier to work over this real logrithmic space than the space $\mathbb{C}_*^d$.

Chapter 3 gives an introduction to algebraic generating functions. We review the embedding method in [GMRW22] in Chapter 3.2 and 3.3. We apply the embedding method to some examples in Chapter 3.4.

Chapter 4 is on the main result for multivariate algebraic generating functions. The whole chapter was previously published as [BJP24] on collaboration with Yuliy Baryshnikov and Robin Pemantle. Main result are listed in Chapter 4.2 but readers may need to go back to Chapter 4.1 to get a big picture of the lifting method. Proofs are in Chapter 4.3. We illustrate the lifting method with four combinatorial examples in Chapter 4.4. Chapter 4.5, originally an appendix in [BJP24], is for the determination of the sign in Theorem 4.14.

Chapter 5 has three subchapters. The first subchapter is an introduction to multiple points. The definition of multiple points is independent of ACSV theory and it can be defined either in a geometric way or an algebraic way. To define it algebraically, one need the ring of analytic germs discussed in Chapter 5.1.1. The second subchapter is on stratified Morse theory. Different to the classical Morse theory in Chapter 2.2, we need the stratified version when the singular variety $\mathcal{V}$ of the generating function F is not smooth. This part is based on [PWM24, Appendix D] and [BMP22]. The last part is an exposition on the result of [BMP24b] when the singular variety is a hyperplane arrangement. In particular, in this special case, we can explicitly find homology generators of $\mathrm{H}_d(\mathcal{M}, -\infty)$ without use of the stratified Morse theory. This fundamental case is what we need in Chapter 6.3 where we show that pseudo multiple points can be treated as multiple points under appropriate conditions.

Chapter 6 contains the main result of this paper on pseudo multiple point. The first subchapter is on motivation and definition of pseudo multiple points. The second subchapter shows that every pseudo multiple point on an analytic hypersurface $\mathcal{V} \subset \mathbb{C}^2$ is a multiple point if the leading homogeneous term of Q at that point factors into distinct simple linear factors, where Q locally defines $\mathcal{V}$ near the point. The last subchapter 6.3 gives conditions under which a minimal (critical) pseudo multiple points can be treated as points on hyperplane arrangements (with some errors), for the sake of asymptotic analysis. In particular, depending on the multiplicities of each linear factor in the factorization of the leading homogeneous term, we give different asymptotic error terms.

In particular, here are the road maps leading to the main result on multivariable algebraic generating functions in Chapter 4 and on psuedo multiple points in Chapter 6.

Algebraic GF: Chapter 1 $\longrightarrow$ Chapter 2.4 $\longrightarrow$ Chapter 3 $\longrightarrow$ Chapter 4

Pseudo multiple points: Chapter 1 $\longrightarrow$ Chapter 2 $\longrightarrow$ Chapter 5 $\longrightarrow$ Chapter 6

CHAPTER 2

PRELIMINARIES

This chapter covers tools used in ACSV theory, mostly based on [PWM24, Appendix A, B, C, D, Chapter 6], to keep the whole paper more self-contained. We try to provide a complete yet succinct overview. We also include references to standard literature in these topics and readers shall resort to the standard literature when our descriptions are hard to understand.

2.1. Manifolds, Homology, and Cohomology

In this section, we review some basics on complex manifolds and singular homology and cohomology theory. We list some results that are particularly userful in ACSV.

2.1.1. Complex manifolds

The singular variety we consider in the ACSV setting is mostly either a complex manifold or a union of complex manifolds of different dimensions. We can either think these complex manifolds as smooth real manifolds or as complex manifolds. We give a very brief overview on complex manifolds to go over things we need in ACSV theory. Details on complex manifolds can be found on [Huy05, Chapter 2].

We now write $(z_1, \cdots, z_d) = (x_1 + iy_1, \cdots, x_d + iy_d) \in \mathbb{C}^d$ as $(x_1, \cdots, x_d, y_1, \cdots, y_d) \in \mathbb{R}^{2d}$. For smooth functions f from $\mathbb{C}^d$ identified as $\mathbb{R}^{2d}$ to $\mathbb{C}$, we define the following operators.

$$
\begin{aligned}
\frac{\partial f}{\partial z_i} &:= \frac{1}{2}\left(\frac{\partial f}{\partial x_i} - i\frac{\partial f}{\partial y_i}\right), & \frac{\partial f}{\partial \bar{z}_i} &:= \frac{1}{2}\left(\frac{\partial f}{\partial x_i} + i\frac{\partial f}{\partial y_i}\right) \\
dz_i &:= dx_i + idy_i, & d\bar{z}_i &:= dx_i - idy_i. \\
\partial f &:= \sum_{i=1}^{d} \frac{\partial f}{\partial z_i} dz_i, & \bar{\partial} f &:= \sum_{i=1}^{d} \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i \\
df &:= \partial f + \bar{\partial} f.
\end{aligned}
$$

A function $f : \mathbb{C}^d \to \mathbb{C}$ is called a **holomorphic function** if $\bar{\partial} f = 0$, and a **anti-holomorphic**

function if $\partial f = 0$.

We can extend the operator d to any smooth n-form. Let $I, J \subset [d]$ and let $d\mathbf{z}_I$ denote the wedge product of dz_i for $i \in I$. Similarly, $d\bar{\mathbf{z}}_J$ denote the wedge product of $d\bar{z}_j$ for $j \in J$. For any form $\omega = f d\mathbf{z}_I \wedge d\bar{\mathbf{z}}_J$, define $d\omega = df \wedge d\mathbf{z}_I \wedge d\bar{\mathbf{z}}_J$. For general forms, extend the definition linearly. We say that a n-form ω is a **holomorphic form** if $d\omega$ can be written as a sum of $f_J d\mathbf{z}_I$ with no $d\bar{z}_j$ and f_J is a holomorphic function. The operator d keeps holomorphicity of a form ω. In $\mathbb{C}^d$, a holomorphic form can not be of dimension higher than d. In particular, integrating a holomorphic d-form ω over a boundary of a $d+1$ chain C is zero by Stokes' theorem since

$$\int_{\partial C} \omega = \int_C d\omega = 0.$$

Here $d\omega = 0$ since there is no holomorphic $(d+1)$-form and the operator d keeps holomorphicity. You may already notice that we give two meanings to the symbol d. It represents both the dimension d and the differential operator d. What the symbold d means is clear from the context.

One cornerstone of ACSV theory is the Cauchy Integral Formula. Let D be a polydisk centered at $\mathbf{w} \in \mathbb{C}^d$ with radii $\mathbf{r} \in R^d$, that is $\{\mathbf{z} \in \mathbb{C}^d : |z_i - w_i| < r_i, i = 1, \cdots, d\}$. Let T be a torus centered at $\mathbf{w} \in \mathbb{C}^d$ with radii $\mathbf{r} \in R^d$, that is $\{\mathbf{z} \in \mathbb{C}^d : |z_i - w_i| = r_i, i = 1, \cdots, d\}$.

Proposition 2.1 (Cauchy Integral Formula)**.** *Let f be an analytic function on U where $U \subseteq \mathbb{C}^d$ is an open set. If D and T have the same radii and the same center, and $\overline{D} \subseteq U$, then for any $\mathbf{p} \in D$,*

$$f(\mathbf{p}) = \left(\frac{1}{2\pi i}\right)^d \int_T \frac{f(\mathbf{z})}{(z_1 - p_1) \cdots (z_d - p_d)}\, d\mathbf{z}.$$

2.1.2. Homology and cohomology

In this paper, we will consider singular homology and cohomology with complex coefficients only. Construction of singular homology can be found in any algebraic topology textbook, for example, [Hat02, Chapter 2] and [Lee13, Chapter 18.1]. We will see that the cauchy integral $\int_T F(\mathbf{z})\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\, d\mathbf{z}$ only depends on the homology class of T and the cohomology class of $F(\mathbf{z})\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\, d\mathbf{z}$ in the domain of holomorphy $\mathcal{M} := \mathbb{C}_*^d - \mathcal{V}_*$. Actually, we only care the relative homology in ACSV setting, for the sake of coefficient asymptotics. The above homology class $[T]$ can be replaced by a relative homology class and the asymptotics of the integral are only affected by an exponentially smaller terms. We briefly explain how every part works out.

We denote the p-th singular (co)homology group with complex coefficients of a Hausdorff topological space X by $\mathrm{H}_p(X;\mathbb{C})$ (resp. $\mathrm{H}^p(X;\mathbb{C})$), or simply $\mathrm{H}_p(X)$ (resp. $\mathrm{H}^p(X)$). By universal coefficient theorem (see [May99, Chapter 17.1]), $\mathrm{H}_p(X;\mathbb{C}) \simeq \mathrm{H}_p(X;\mathbb{Z}) \otimes \mathbb{C}$ and $\mathrm{H}^p(X;\mathbb{C}) \simeq \mathrm{Hom}(\mathrm{H}_p(X;\mathbb{Z}), \mathbb{C})$. Therefore, there is no torsion in $\mathrm{H}_p(X;\mathbb{C})$ and $\mathrm{H}_p(X;\mathbb{C})$ is indeed a complex vector space. So is $\mathrm{H}^p(X;\mathbb{C})$. In ACSV setting, the space X is often the singular variety $\mathcal{V}_*$ or the domain of holomorphy $\mathcal{M} := \mathbb{C}_*^d - \mathcal{V}_*$ where the integrand $F(\mathbf{z})\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\, d\mathbf{z}$ is a holomorphic d-form.

We denote the p-th relative singular homology group with complex coefficients of a space X relative to a subspace A by $\mathrm{H}_p(X, A)$. In ACSV, we have the height function $h : \mathcal{M} \to \mathbb{R}$ defined by $h(\mathbf{z}) := -\hat{\mathbf{r}} \cdot \mathrm{Relog}(\mathbf{z})$. The subspace A is chosen to be $\{\mathbf{z} \in \mathcal{M} : h(\mathbf{z}) \leq c\}$ for some constant c. We denote such subspace A by $\mathcal{M}_{\leq c}$. We are mostly interested in the relative homology groups $\mathrm{H}_p(\mathcal{M}, \mathcal{M}_{\leq c})$.

Integral depends on homology and cohomology classes

Proposition 2.2. *Let ω, ω' be closed p-forms on a smooth manifold X and γ, γ' be smooth p-cycles on X. If $[\omega - \omega'] = 0$ in $\mathrm{H}^p(X)$ and $[\gamma - \gamma'] = 0$ in $\mathrm{H}_p(X)$, then $\int_{\gamma'} \omega' = \int_\gamma \omega$.*

Proof: There are two results we need in the proof. [Lee13, Theorem 18.7] shows that the singular homology group $\mathrm{H}_p(X)$ is isomorphic to the smooth homology group $\mathrm{H}_p^\infty(X)$. The de Rham theorem [Lee13, Theorem 18.14] tells us that the de Rham cohomology $\mathrm{H}_{\mathrm{dR}}^p(X)$ is isomorphic to the singular cohomology $\mathrm{H}^p(X)$. Therefore, $\omega - \omega'$ is an exact p-form and $\gamma - \gamma'$ is a smooth p-boundary.

Let $\gamma - \gamma' = \partial\beta$ where β is a smooth $(p+1)$-cycle. $\int_{\gamma-\gamma'} \omega = \int_\beta d\omega = 0$ by Stokes' theorem and closedness of ω. Let $\omega - \omega' = d\eta$ where η is a $(p+1)$-form. $\int_\gamma \omega - \omega' = \int_\gamma d\eta = \int_{\partial\gamma} \eta = 0$ since γ is a cycle. $\square$

Remark. Indeed, de Rham theorem implies the following isomorphism $\ell : \mathrm{H}^p_{\mathrm{dR}}(X) \to \mathrm{H}^p(X)$ defined by

$$\ell[\omega][\alpha] = \int_{\alpha'} \omega$$

where α' is a smooth p-cycle in the same singular homology class as α. Proposition 2.2 shows that the map is well-defined. From now on, for the sake of simplicity, we abuse the notation by writing $\ell[\omega][\alpha]$ as $\int_\alpha \omega$, though rigorously we can't integrate over a singular chain (since we need at least C^1 smoothness).

Actually, the integral value of $\int_\gamma \omega$ in the ACSV setting depends only on the relative homology class of γ if we tolerate an asymptotically exponentially smaller error terms.

Proposition 2.3. *[PWM24, Proposition B.10] Let X be a manifold of dimension n with submanifold A also of dimension n, and let ϕ be a smooth complex-valued function on X satisfying $\mathrm{Re}(\phi) \le \beta$ on A for some $\beta \in \mathbb{R}$. Suppose that $\omega = \omega_\lambda = \exp(\lambda\phi(\mathbf{z}))\eta$ is a closed k-form on X with $k \le n$ where η is a k-form here. If C and C' are k-chains on X with $[C] = [C']$ in $\mathrm{H}_k(X, A)$, then as $\lambda \to \infty$,*

$$\int_C \omega_\lambda = \int_{C'} \omega_\lambda + O(e^{\lambda\beta})$$

Proof: Since $[C] = [C']$, $C - C'$ is relative boundary and thus $C - C' = \partial D + C''$ where $D \in C_{k+1}(X)$ and $C'' \in C_k(A)$. By Stokes' theorem,

$$\int_C \omega_\lambda - \int_{C'} \omega_\lambda = \int_{\partial D} \omega_\lambda + \int_{C''} \omega_\lambda = \int_D d\omega_\lambda + \int_{C''} \omega_\lambda = \int_{C''} \omega_\lambda$$

and $\left|\int_{C''} \omega_\lambda\right| \le \int_{C''} e^{\lambda\beta}|\eta| \le Ke^{\lambda\beta}$. □

In particular, in ACSV, the space X is $\mathcal{M} := \mathbb{C}^d_* - \mathcal{V}_*$ and the function ϕ in the proposition above is chosen to be $-\hat{\mathbf{r}} \cdot \log(\mathbf{z})$ and thus $\mathrm{Re}(\phi)(\mathbf{z}) = h(\mathbf{z}) = -\hat{\mathbf{r}} \cdot \mathrm{Relog}(\mathbf{z})$. The space A is thus $\mathcal{M}_{\le\beta} := \{\mathbf{z} \in \mathcal{M} : h(\mathbf{z}) \le \beta\}$. The integrand $\omega = \omega_\lambda = \mathbf{z}^{-\lambda\cdot\hat{\mathbf{r}}}\eta$ where $\eta = F(\mathbf{z})/\mathbf{z}\ d\mathbf{z}$. The chain C is the initial torus T in the domain of convergence of the power series $F(\mathbf{z})$. Therefore, the corollary below says that the coefficient asymptotics $[\mathbf{z}^{\mathbf{r}}]F(\mathbf{z})$ in direction $\hat{\mathbf{r}} := \mathbf{r}/|\mathbf{r}|$ is equal to

$$\int_{C'} \omega_\lambda$$

with an asymptotic error term $O(e^{\lambda\beta})$ which has an exponential growth order less than β. It is not clear whether or not ω is closed and therefore we are not sure now whether we can apply Proposition 2.3 or not. We therefore state the following corollary fit to our setting. In particular, instead of requiring ω to be closed, we require ω to be holomorphic and top-form (and thus closed).

Corollary 2.4. *[PWM24, Corollary B.23] Let X be a complex manifold of dimension n with submanifold A also of dimension n, and let ϕ be a complex-valued function on X satisfying $\mathrm{Re}(\phi) \leq \beta$ on A for some $\beta \in \mathbb{R}$. Suppose that $\omega = \omega_\lambda = \exp(\lambda\phi(\mathbf{z}))\eta$ is a holomorphic n-form on X where η is a n-form here. If C and C' are n-chains on X with $[C] = [C']$ in $\mathrm{H}_n(X, A)$, then as $\lambda \to \infty$,*

$$\int_C \omega_\lambda = \int_{C'} \omega_\lambda + O(e^{\lambda\beta})$$

Remark. We made a small change to the original statement in [PWM24, Corollary B.23]. We only require ω to be holomorphic, instead of ϕ and η being holomorphic.

Proof: Proof is similar to Proposition 2.3. The operator $d : \omega \to d\omega$ keeps holomorphicity and there is no holomorphic $(n+1)$-form on a complex manifold of complex dimension n. Therefore, the integral $\int_D d\omega_\lambda = 0$ in the proof of Proposition 2.3 is zero. $\square$

Relations to other homologies

Direct computation on singular homology by definition is nearly impossible because the singular chain space of each dimension is infinite-dimensional. There are homology theories easier to be calculated with some additional requirement on the space X. If a space X is homotopic equivalent to a Δ-complex, one can define the simplicial homology [Hat02, Chapter 2.1]. If a space X is homotopic equivalent to a CW complex, one can define the cellular homology [Hat02, Chapter 2.2]. The singular homology (resp. relative homology) is isomorphic to the simplicial homology (resp.

relative homology) when X is a Δ-complex (resp. when A is a subcomplex of X). This is the result of [Hat02, Theorem 2.27]. The singular homology is isomorphic to the cellular homology when X is a CW-complex [Hat02, Theorem 2.35]. For a CW pair (X, A), there is a relative cellular chain complex formed by the groups $\mathrm{H}_i(X^i, X^{i-1} \cup A^i)$, having homology groups isomorphic to the singular relative homology $\mathrm{H}_n(X, A)$ [Hat02, Exercise 2.18].

2.1.3. Embedded complex manifolds

In ACSV, we care mainly about the topology of $\mathcal{V}_*$ and $\mathcal{M} := \mathbb{C}_*^d - \mathcal{V}_*$. They are complex manifolds, or unions of complex manifolds. A direct corollary of [AF59, Theorem 1] shows that a biholomorphically embedded complex manifold of complex dimension p into a closed submanifold of $\mathbb{C}^N$ (Stein manifold) is homotopic equivalent to a CW-complex of real dimension p. Therefore, all its singular homology groups and cohomology groups of order $> p$ vanish.

Theorem 2.5 (Andreotti-Frankel Theorem)**.** *Let X be a complex manifold of complex dimension p. Assume that X can be embedded biholomorphically as a closed submanifold of $\mathbb{C}^N$ for some $N > p$. Then X is homotopic equivalent to a CW-complex of dimension at most p. As a result,* $\mathrm{H}^n(X)$ *and* $\mathrm{H}_n(X)$ *vanish for all $n > p$.*

Before giving a sketch proof for Theorem 2.5, we first note that the requirement of this theorem is very strong. For smooth manifolds of real dimension p, Whitney embedding theorem shows that we can embed them in $\mathbb{R}^{2p}$. However, it is rather hard to embed a complex p-manifold biholomorphically. The extra holomorphic requirement on the embedding map, for example, makes any compact connected complex manifold (unless just a point) not able to be embedded in this way into any complex space. Complex manifolds biholomorphically embedded as a closed submanifold of $\mathbb{C}^N$ are called **Stein manifolds**. This definition, though different from the original definition in [Ste51], is equivalent as shown in [Hör90, Theorem 5.3.9, Theorem 5.1.5] and the fact that $\mathbb{C}^N$ is a Stein manifold. Every smooth affine complex algebraic variety is a Stein manifold. Moreover, every domain of holomorphy is a Stein manifold. Therefore, $\mathrm{H}^n(\mathcal{M})$ and $\mathrm{H}_n(\mathcal{M})$ vanish for $n > d$.

Proof: The statement in [AF59, Theorem 1] does not explicitly contain the result of Theorem

2.5, but the proof of [AF59, Theorem 1] shows exactly that X is homotopy equivalent to a CW-complex of dimension at most p by using Morse theory. Choose a point $\mathbf{z}_0 \in \mathbb{C}^N - X$ and let $f(\mathbf{z}) := |\mathbf{z} - \mathbf{z}_0|$ be the distance function from $\mathbf{z} \in X$ to $\mathbf{z}_0$. Then f is a Morse function and if we choose $\mathbf{z}_0$ properly, all critical points of f on X is nondegenerate. That is, the Hessian matricies of f at critical points are nonsingular. Indeed, [AF59, Lemma] shows that such good choices of $\mathbf{z}_0$ are dense in $\mathbb{C}^N - X$. The proof of [AF59, Theorem 1] then shows that there is no critical points of index $n > p$ on X. Therefore, by Morse lemma, X is homotopy equivalent to a CW-complex of dimension at most p. Since all of our (co)homology groups take coefficients in $\mathbb{C}$, the (co)homology groups are torsion-free. Therefore, the zero rank of the n-th (co)homology group (where $n > p$) implies that this (co)homology group vanishes. □

Remark. A detailed treatment can be found on [Voi03, Chapter 1.2]. [PWM24, Notes of Appendix B] says that the Andreotti-Frankel theorem is true in a much more generality than Theorem 2.5. It is true for all affine complex algebraic varieties, regardless of being smooth or singular [Kar79]. The theorem also applies to the complement of complex affine algebraic variety.

Another thing to notice is that when we integrate $\omega := F(\mathbf{z})\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\, d\mathbf{z}$ over the torus T in the ACSV setting, ω is not only a differentiable form. The form ω is indeed a holomorphic form on $\mathcal{M}$, the domain of holomorphy of ω. It thus make sense to define a subcomplex of the de Rham complex, consisting of all holomorphic forms. We can then have the corresponding (holomorphic) de Rham cohomology $\mathrm{H}^p_{\mathrm{holo}}(X)$. When X is a Stein manifold, $\mathrm{H}^p_{\mathrm{holo}}(X)$ is isomorphic to $\mathrm{H}^p_{\mathrm{dR}}(X)$ by [PWM24, Proposition B.22], and hence isomorphic to the singular cohomology $\mathrm{H}^p(X)$ by de Rham theorem. Therefore, we have the following theorem.

Theorem 2.6. *[PWM24, Theorem B.18] Let ω be a closed p-form holomorphic on an embedded complex manifold $\mathcal{M} \subset \mathbb{C}_n$ (if $p = n$ then ω is always closed). Let C be a singular p-cycle on $\mathcal{M}$. Then $\int_C \omega$ depends on C only via the homology class $[C]$ of C in $\mathrm{H}_p(\mathcal{M})$ and on ω only via the cohomology class $[\omega]$ of ω in $\mathrm{H}^p(\mathcal{M})$.*

Remark. In ACSV, $\mathcal{M} := \mathbb{C}^d_* - \mathcal{V}_*$ is always a Stein manifold. The d-form ω is holomorphic on $\mathcal{M}$ and thus closed. Moreover, three cohomology groups $\mathrm{H}^p(\mathcal{M}), \mathrm{H}^p_{\mathrm{dR}}(\mathcal{M})$, and $\mathrm{H}^p_{\mathrm{holo}}(\mathcal{M})$ are isomorphic. Therefore, singular cocycles, closed forms, and closed holomorphic forms are corresponded.

2.2. Residue Forms

In this section, we will generalize the idea of residues in single variable complex analysis to several complex variables. In ACSV, we often have a rational generating function $F = P/Q$ where P, Q are coprime polynomials. The coefficient asymptotics of the power series of F in the direction $\hat{\mathbf{r}}$ is determined by the integral $\int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, d\mathbf{z}$. What we do in general is to deform the torus T and let the deformed one to wrap around critical points on the singular variety defined by Q. However, though the idea is the same as what one do in single variable complex analysis, the term "wrap around" is difficult to visualize because singular points are no longer isolated when $d \geq 2$. In particular, singular points form the so-called singular variety $\mathcal{V}$.

The residue theory in this section starts from the case when F has poles near which $\mathcal{V}$ is a smooth analytic hypersurface, to the case when $\mathcal{V}$ is locally a union of smooth analytic hypersurfaces intersecting transversely.

Definition 2.7 (analytic hypersurface)**.** *A set $\mathcal{V} \subseteq \mathbb{C}^d$ is an* **analytic hypersurface** *if for any $\mathbf{z} \in \mathcal{V}$ and any sufficiently small neighborhood $\mathcal{D}$ of $\mathbf{z}$ in $\mathbb{C}^d$, there is an analytic function Q on $\mathcal{D}$ such that $\mathcal{V} \cap \mathcal{D} = \mathcal{V}_Q \cap \mathcal{D}$.*

If in addition ∇Q is nonvanishing on $\mathcal{V} \cap D$, then $\mathcal{V}$ is a **smooth analytic hypersurface at $\mathbf{z}$** *and in particular $\mathbf{z}$ is called a* **smooth point***. If $\mathcal{V}$ is an analytic hypersurface and every point on $\mathcal{V}$ is smooth, then $\mathcal{V}$ is a* **smooth analytic hypersurface***.*

If $\mathcal{V}_Q \cap \mathcal{D} = (\mathcal{V}_{Q_1} \cup \cdots \cup \mathcal{V}_{Q_k}) \cap \mathcal{D}$ where (i) Q_j's are analytic on $\mathcal{D}$ and ∇Q_j's are nonvanishing on $\mathcal{V} \cap \mathcal{D}$, and (ii) $\nabla Q_j(\mathbf{z})$'s are linearly independent, then $\mathbf{z}$ is called a **transverse multiple point***. If every point of $\mathcal{V}$ is a transverse multiple point, then $\mathcal{V}$ is a* **transverse analytic hypersurface***.*

Remark. The locally analytic function Q depends on the neighborhood $\mathcal{D}$. By definition, a smooth analytic hypersurface is always a transverse analytic hypersurface. However, the converse is not true. Locally at each point on a transverse analytic hypersurface, $\mathcal{V} \cap \mathcal{D}$ is equal to $\mathcal{V}_Q \cap \mathcal{D}$ where $Q = Q_1^{m_1} \cdots Q_k^{m_k}$. Since $Q_j(\mathbf{z}) = 0$ for all j, it is immediate that $\nabla Q = 0$ and thus $\mathcal{V}$ not smooth unless $k = 1$. The number k of smooth analytic hypersurfaces depends on the location of $\mathbf{z}$.

When $F = P/Q$ is a rational function with coprime polynomials P and Q, the singular variety of F is $\mathcal{V}_Q$, which is an algebraic hypersurface and hence an analytic one. In this case, the singular variety is given globally by the polynomial Q and $\mathcal{V}_Q$ is smooth at $\mathbf{z}$ if and only if $\nabla Q(\mathbf{z}) \neq 0$. We say that $\mathcal{V}_Q$ is smooth if $\nabla \widetilde{Q} \neq 0$ everywhere on $\mathcal{V}_Q$.

In the practice of ACSV, we actually consider the subset $\mathcal{V}_* = \mathcal{V}_Q \cap \mathbb{C}_*^d$ of $\mathcal{V} = \mathcal{V}_Q$, which is also an analytic hypersurface. One of the reason is that the intersection classes are defined on $\mathcal{V}_*$ instead of $\mathcal{V}$.

2.2.1. Tubular neighborhood and intersection classes

Now let's consider the case when we have d complex variables. Let $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$ be a rational generating function with P, Q coprime and $\mathcal{V}_Q$ is smooth hypersurface. Consider the integral $\int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, d\mathbf{z}$. Let ω be the d-form $\mathbf{z}^{-n-1} F(\mathbf{z})\, d\mathbf{z}$. The singular variety of ω is then $\mathcal{V}_Q$ union coordinate axes. The domain of holomorphicity of ω is $\mathcal{M} := \mathbb{C}_*^d - \mathcal{V}_*$. We want to expand the small torus T to a larger one T' on which the integral decays super-exponentially. We can consider this expansion as a cobordism, a $(d+1)$-chain whose boundary is $T - T'$. This $(d+1)$-chain can be made to cross $\mathcal{V}_*$ transversely. The crossing will leave a $(d-1)$-chain γ on the $(d-1)$ complex dimensional manifold $\mathcal{V}_*$. We will call it γ, or more precisely, the homology class $[\gamma]$ in $\mathrm{H}_{d-1}(\mathcal{V}_*)$, the **intersection class** (see Definition 2.9) of T and T' in $\mathcal{M}$. In the trivial example above when $d = 1$, the intersection class γ is just the point $z = 1$. In Chapter 2.2.1, we assume that the analytic hypersurface $\mathcal{V}$ is defined globally by an analytic function $Q(\mathbf{z})$ on $\mathbb{C}^d$ so that $\mathcal{V} = \mathcal{V}_Q$. Moreover, we assume that $\nabla \widetilde{Q}$ is non-vanishing on $\mathcal{V}_Q$ and so $\mathcal{V}_Q$ is smooth. This is the case when the concerning generating function $F = P/Q$ is rational and $\mathcal{V}_Q$ is smooth.

The singular variety $\mathcal{V} = \mathcal{V}_Q$ by assumption is always an embedded complex manifold. For a point $\mathbf{w} \in \mathcal{V}_Q$, we let $N_{\mathbf{w}}\mathcal{V}$ denote the normal space of $\mathcal{V}$ at $\mathbf{w}$. In particular, $N_{\mathbf{w}}\mathcal{V}$ is a one-dimensional complex vector space (hence of real dimension two) since $\mathcal{V}_Q$ is a hypersurface. The total space of the normal bundle to $\mathcal{V}$ is the set $\{(\mathbf{w}, \mathbf{v}) \in \mathcal{V} \times \mathbb{C}^d : \mathbf{v} \in N_{\mathbf{w}}\mathcal{V}\}$. [PWM24, Lemma C.1] shows that there is an open neighborhood of $\mathcal{V}$ that is diffeomorphic to the total space of the normal bundle to $\mathcal{V}$. Moreover, this diffeomorphism ψ maps $\mathbf{w} \in \mathcal{V}$ to $(\mathbf{w}, \mathbf{0}) \in \mathcal{V} \times \mathbb{C}^d$.

For any n-chain γ on $\mathcal{V}_*$, we define a $(n+1)$-chain $\circ\gamma$ on $\mathcal{M} := \mathbb{C}^d_* - \mathcal{V}_*$ by taking the union of small circles in the fibers of normal bundle to $\mathcal{V}$ with centers on γ. The radii of these small circles are chosen to be small enough so that these circles are in the range of the previous diffeomorphism ψ. The radii differ by base points and vary continously. You may wonder what happens if we choose different radii. You will get different $\circ\gamma$ but they are homologous in $\mathrm{H}_{n+1}(\mathcal{M})$. Later we will see that we care things only at the homology level and thus this definition of $\circ\gamma$ is well-defined for our sake. If we take the union of open solid disks formed by these circles, we denote it by $\bullet\gamma$.

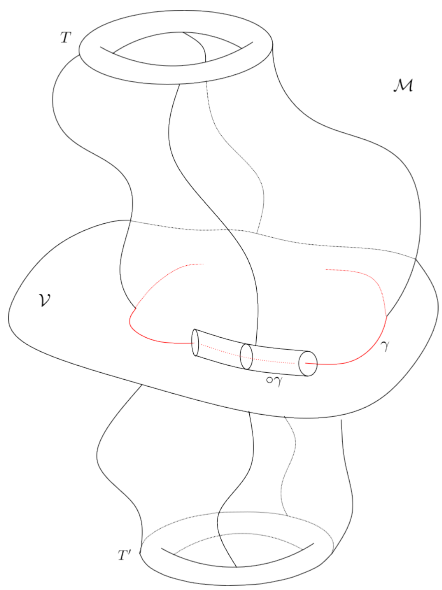

Figure 2.1: The intersection class $\gamma = \mathbf{INT}[T, T'; \mathcal{V}_*]$ of two torus T and T' in $\mathcal{M}$ and the corresponding chain $\circ\gamma$.

Details on these constructions are in [PWM24, Appendix C.1]. It coincides with the definition in [Mat12, Chapter 6] such that

$$\begin{tikzcd} & \gamma \arrow[dl, "s"'] \arrow[dr, "\text{inclusion}"] & \\ \bullet\gamma \arrow[rr, "\psi"] & & \mathbb{C}^d \end{tikzcd}$$

commutes. Here s is the zero section of γ in the normal bundle and ψ is the diffeomorphism aforementioned.

The operator $\circ$ commutes with the boundary operator δ such that $\partial(\circ\gamma) = \circ(\partial\gamma)$. Therefore, we have a map induced by $\circ$ on $\mathrm{H}_n(\mathcal{V}_*)$ to $\mathrm{H}_{n+1}(\mathcal{M})$ where $\mathcal{M} := \mathbb{C}_*^d - \mathcal{V}_*$. This map is also denoted by $\circ$.

Theorem 2.8. *[PWM24, Theorem C.2] Let $\circ : \mathrm{H}_{d-1}(\mathcal{V}_*) \to \mathrm{H}_d(\mathcal{M})$ be the map induced by $\circ$ on homology groups. Assume that $\mathcal{V}$ is a manifold.*

(i) $\circ$ is injective and its image is the kernel of the map ι_* induced by the inclusion $\mathcal{M} \xrightarrow{\iota} \mathbb{C}_*^d$.

(ii) Given $\alpha \in Ker(\iota_*)$, one may compute the inverse $I(\alpha) = \circ^{-1}(\alpha)$ by intersecting $\mathcal{V}_*$ with any $(d+1)$-chain in $\mathbb{C}_*^d$ whose boundary is α and for which the intersection with $\mathcal{V}_*$ is transverse.

Remark. The condition $\alpha \in Ker(\iota_*)$ means that α is null-homologous in $\mathbb{C}_*^d$. Topologically, $\mathbb{C}_*^d$ is homotopy equivalent to a product of d circles. Then $\mathrm{H}_d(\mathbb{C}_*^d)$ has exactly d generators, each of which is a circle around one coordinate axis. The kernel of ι_* then consists of all classes whose representatives α satisfy

$$(2\pi i)^{-d} \int_\alpha 1/\mathbf{z}\ d\mathbf{z} = 0. \tag{2.1}$$

When $d = 1$, the integral on the left hand side of (2.1) measures the number of times α wrapping around the origin.

Definition 2.9 (intersection class)**.** *When C and C' are two d-cycles in $\mathcal{M}$ such that $[C - C']$ is zero in $\mathrm{H}_d(C_*^d)$, we call $I(C - C') := \circ^{-1}(C - C')$ the* **intersection class** *of C and C' with $\mathcal{V}_*$. We denote it by $\boldsymbol{INT}[C, C'; \mathcal{V}_*]$.*

The intersection class of C and C' can be seen as a $(d-1)$-chain but in fact it is an element in the homology group $\mathrm{H}_{d-1}(\mathcal{V}_*)$. In ACSV, C is normally the original small torus T and C' is a larger torus T' on which the Cauchy integral decays super-exponentially.

2.2.2. Residue forms and residue classes

The theory for residue forms in ACSV is fully discussed in [PWM24, Appendix C.2] in a progressive path of generality. One can consider forms with poles on a single smooth analytic hypersurface, or on a union of transversely intersection smooth analytic hypersurfaces. The first category is mostly useful when one consider smooth points in ACSV. The second category is essential in the case when one needs to find the contribution of multiple points to the coefficient asymptotics. In each one of the category, one needs to consider forms with simple poles and forms with higher order poles.

Poles on a single smooth analytic hypersurface

Definition 2.10 (smooth poles)**.** *Fix an analytic hypersurface $\mathcal{V}_*$ contained in $\mathbb{C}_*^d$. A holomorphic d-form on $\mathcal{M} := \mathbb{C}_*^d - \mathcal{V}_*$ has* **a smooth pole of order** *n at $\mathbf{w} \in \mathcal{V}_*$ if there is a small open neighborhood $\mathcal{D}$ of $\mathbf{w}$ in $\mathbb{C}_*^d$ satisfying the following conditions.*

(smoothness) There exists an analytic function $Q_{\mathcal{D}}(\mathbf{z})$ on $\mathcal{D}$ satisfying $\mathcal{V} \cap \mathcal{D} = \mathcal{V}_{Q_{\mathcal{D}}} \cap \mathcal{D}$ and $\nabla Q_{\mathcal{D}}(\mathbf{w}) \neq 0$

(pole) The form $Q_{\mathcal{D}}(\mathbf{z})^n \omega$ extends to a holomorphic form on $\mathcal{D}$ but $Q_{\mathcal{D}}(\mathbf{z})^i \omega$ does not for $0 \leq i < n$.

When $n = 1$, we say that ω has a smooth simple pole at $\mathbf{w}$.

Remark. For an analytic hypersurface $\mathcal{V}$ in $\mathbb{C}^d$, its intersection with $\mathbb{C}_*^d$ is an analytic hypersurface in $\mathbb{C}_*^d$.

In the practice of ACSV, when the generating function F is a rational function P/Q defined by two coprime polynomials, the singular variety $\mathcal{V}$ is an analytic hypersurface globally defined by Q. If $\nabla \tilde{Q} \neq 0$ on $\mathcal{V}_*$ (i.e. $\mathcal{V}_*$ is smooth) and $Q = \tilde{Q}^n$, then the form $\omega = P(\mathbf{z})/Q(\mathbf{z})\, d\mathbf{z}$ has smooth poles of order n on $\mathcal{V}_*$. That is, any point on $\mathcal{V}_*$ is a smooth pole of order n. If $\nabla Q \neq \mathbf{0}$ on $\mathcal{V}_*$, then ω has simple smooth poles on $\mathcal{V}_*$.

Definition 2.11 (smooth poles on $\mathcal{V}_*$)**.** *Let $\mathcal{V}$ be an analytic hypersurface and ω be a holomorphic d-form on $\mathcal{M}$. If ω has smooth poles of the same order n at all points $\mathbf{w} \in \mathcal{V}_*$, then ω is said to* *have smooth poles of order n at on $\mathcal{V}_*$. When $n = 1$, the form ω has smooth simple poles on $\mathcal{V}_*$.*

Proposition 2.12 (residues on smooth poles)**.** *Let ω be a holomorphic d-form on $\mathcal{M}$ and has a smooth simple pole at $\mathbf{w} \in \mathcal{V}_*$. Let $\mathcal{D}$ and $Q_{\mathcal{D}}$ be defined in Definition 2.10. Suppose that ω has a local representation in $\mathcal{M} \cap \mathcal{D}$ as a quotient $P(\mathbf{z})/Q_{\mathcal{D}}(\mathbf{z})$ of two analytic functions on $\mathcal{D}$ and the gradient of $Q_{\mathcal{D}}$ does not vanish on $\mathcal{D}$.*

There exists a $(d-1)$-form θ on $\mathcal{D}$ satisfying $dQ_D \wedge \theta = P\ d\mathbf{z}$. Let $\iota : \mathcal{V} \cap \mathcal{D} \to \mathcal{D}$ be the inclusion map, then θ restricts to a unique $(d-1)$-form $\iota^*\theta$ on $\mathcal{V}_* \cap \mathcal{D}$. We denote $\iota^*\theta$ by $\operatorname{Res}(\omega; \mathcal{V}_* \cap \mathcal{D})$, the **residue form of** *ω* **on** *$\mathcal{V}_* \cap \mathcal{D}$.*

Proof: See [PWM24, Proposition C.6]. □

Remark. We see from the above proposition that residue form is a local construction at a pole $\mathbf{w}$ on $\mathcal{V}$. Moreover, notice that we require $\mathbf{w} \in \mathcal{V}_*$ instead of $\mathcal{V}$. This is proper for the purpose of ACSV and the coordinate formula in Proposition 2.14 also requires non-zero coordinate. In [AY83, Chapter 16], the construction of residue forms works for any complex manifolds X and S of dimension d and $d-1$. Any holomorphic d-form on $X - S$ has a well-defined residue form on S. In our case, we take $X = \mathbb{C}_*^d$, $S = \mathcal{V}_*$, and thus $X - S = \mathcal{M}$.

In the practice of ACSV, when F is a rational generating function, the analytic hypersurface $\mathcal{V}$ is globally defined by a polynomial Q. The form ω is written globally as $P(\mathbf{z})/Q(\mathbf{z})\ d\mathbf{z}$ on $\mathcal{M}$. The requirement of smooth simple poles is equivalent to saying that $\mathcal{V}_Q$ is smooth and Q is square-free. The neighborhood $\mathcal{D}$ can be chosen to be an open neighborhood of $\mathcal{V}_*$. The function $Q_{\mathcal{D}}$ above can be chosen to be $\mathcal{D}$ uniformly across all smooth simple poles on $\mathcal{V}_*$. The residue form can be defined globally on $\mathcal{V}_*$. We denote it by $\operatorname{Res}(\omega)$ as in [PWM24, Proposition C.6].

Example 2.13. *Let $\omega = 1/Q(x,y)\ dxdy$, where $Q(x,y) = 1 - x - y$. The d-form ω has simple smooth poles on $\mathcal{V}_* = \mathcal{V}_Q \cap \mathbb{C}_*^d$. The solutions θ to the equation $dQ \wedge \theta = Pd\mathbf{z}$ are $-dy$ or dx. The restriction of these two answers to $\mathcal{V}_*$ gives equivalent forms because $1 - x - y = 0$ on $\mathcal{V}_*$ and $dx = -dy$ by implicit differentiation. Therefore, we can write* $\operatorname{Res}(\omega) = dx$ *or* $-dy$ *either.* □

In ACSV, we often need to evaluate the Cauchy integral (1.1) for a rational function $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$ where the integrand $\mathbf{z}^{-\mathbf{r}-\mathbf{1}}F(\mathbf{z})\ d\mathbf{z}$ is a d holomorphic form on $\mathcal{M} := \mathbb{C}_*^d - \mathcal{V}$ where $\mathcal{V} = \mathcal{V}_Q$. It is thus handy to have a coordinate formula listed below.

Proposition 2.14 (Coordinate formula for residue forms)**.** *Let ω be a holomorphic d-form on $\mathcal{M}$ having a smooth simple pole at $\mathbf{w} \in \mathcal{V}_* := \mathcal{V} \cap \mathbb{C}_*^d$. Assume that ω and $\mathbf{w}$ satisfy conditions in Proposition 2.12.*

We can always let $\mathcal{D}$ in Proposition 2.12 be small enough so that $\mathcal{D} \subset \mathbb{C}_*^d$ and the partial derivative $\partial Q_{\mathcal{D}}/\partial z_k$ is nonvanishing on $\mathcal{D}$ for some fixed k. For $\mathbf{r} \in \mathbb{Z}^d$, the residue of $\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\omega$ on $\mathcal{V}_* \cap \mathcal{D}$ is

$$\operatorname{Res}\left(\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\omega; \mathcal{V}_* \cap \mathcal{D}\right) = (-1)^k \frac{\mathbf{z}^{-\mathbf{r}-\mathbf{1}}P(\mathbf{z})}{\partial Q_{\mathcal{D}}/\partial z_k(\mathbf{z})}\ d\mathbf{z}_{\hat{k}} \tag{2.2}$$

where $d\mathbf{z}_{\hat{k}} = dz_1 \wedge \cdots \wedge dz_{k-1} \wedge dz_{k+1} \wedge \cdots \wedge dz_d$.

Proof: See the proof for the formula in [PWM24, Proposition C.8]. The original version of the proposition assumes that $\mathcal{V}$ is a smooth analytic hypersurface globally defined by an analytic function Q. It also works locally. Notice that $\mathbf{z}^{-\mathbf{r}-\mathbf{1}}$ is analytic on $\mathcal{D}$. $\square$

Remark. If ω satisfies conditions in Proposition 2.12, then $\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\omega$ also satisfies these conditions because $\mathbf{z}^{-\mathbf{r}-\mathbf{1}}$ is holomorphic in $\mathbb{C}_*^d$. Therefore, $\operatorname{Res}\left(\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\omega\right)$ is well-defined.

Just like what we have in single variable complex analysis, poles can have order greater than one. A good thing is that forms with smooth poles of higher order can often be reduced to forms with smooth simple poles. [AY83, Lemma 17.1] captures this reduction when the variety $\mathcal{V}$ is globally given by an analytic function Q.

Lemma 2.15. *Let $\mathcal{V}_*$ be an analytic hypersurface in $\mathbb{C}_*^d$ defined by an analytic function Q in a neighborhood $\mathcal{D}$ of $\mathcal{V}_*$ in $\mathbb{C}_*^d$ and ∇Q is nonvanishing on $\mathcal{D}$. Let ω be a holomorphic d-form on $\mathcal{M} := \mathbb{C}_*^d - \mathcal{V}$ with representation on $\mathcal{D}$ as the ratio of analytic functions $P(\mathbf{z})/Q(\mathbf{z})^k\ d\mathbf{z}$ for $k \geq 2$,* *then*

$$\omega = \frac{dQ}{Q^k} \wedge \psi + \frac{\theta}{Q^{k-1}}$$

where ψ and θ are holomorphic forms in $\mathcal{D}$.

PROOF: [AY83, Lemma 17.1]. The original proof is for smooth forms, but it also works for holomorphic forms.

Remark. The first term on the right hand side is $d\left(\frac{-\psi}{(k-1)Q^{k-1}}\right)$ and thus ω is cohomologous to another form with poles of order $k-1$ on $\mathcal{V}_*$. A subtle detail here is that we identify the singular cohomology group with the de Rham cohomology group and the holomorphic de Rham cohomology group by results in Chapter 2.1.3. Therefore, when we say ω is cohomologous to θ/Q^{k-1}, we do not specify which cohomology group we refer to, because they are isomorphic to each other.

Applying Lemma 2.15 $k-1$ times, a form with smooth poles of order k is then cohomologous to a form ω' with smooth simple pole. The residue form of ω on $\mathcal{V}_*$ is then defined to be the residue form of ω' on $\mathcal{V}_*$. In other words,

$$[\mathrm{Res}\,(\omega)] = [\mathrm{Res}\,(\omega')] \text{ in } \mathrm{H}^{d-1}(\mathcal{V}_*)$$

Another subtlety here is that $\mathrm{Res}\,(\omega')$ is a form and $[\mathrm{Res}\,(\omega)]$ is a cohomology class. For forms with higher order poles, we define its **residue class**, instead of its **residue form**. This is good to use in ACSV because the integral we care only depends on the homology class and the cohomology class by Theorem 2.6. The generalization of residues to residue forms and residue classes is for the generalization of the resiude theorem.

Theorem 2.16 (Residue theorem)**.** *[PWM24, Theorem C.9, C.12] Suppose $\mathcal{V}_* = \{\mathbf{z} \in \mathbb{C}_*^d : Q(\mathbf{z}) = 0\}$ is defined globally by a function Q that is analytic on a neighborhood of $\mathcal{V}$ and $\nabla Q \neq 0$ on $\mathcal{V}_*$. Let ω be a holomorphic d-form on $\mathcal{M}$ with poles of order k on $\mathcal{V}_*$. If α and β are d-cycles in $\mathcal{M}$*

that are homologous in $\mathbb{C}_*^d$*, then*

$$\int_\alpha \omega - \int_\beta \omega = 2\pi i \int_{\boldsymbol{INT}[\alpha,\beta;\mathcal{V}_*]} \mathrm{Res}\,(\omega)$$

For forms with higher order poles, $\mathrm{Res}\,(\omega)$ *is taken from a representative in* $[\mathrm{Res}\,(\omega')]$ *where* ω' *is the form with smooth simple poles that are cohomologous to* ω*.*

In ACSV, the cycle α is often the small torus T around the origin and the cycle β is a torus T' with larger radius on which $-\hat{\mathbf{r}} \cdot \mathrm{Relog}(\mathbf{z})$ is smaller than any critical points. We will see in Chapter 2.4 on how to choose T'. Morse theory, introduced later, tells us that as long as β is lower than the lowest critical point, it can be pushed down even further. In other words, the radius of β can be made arbitrarily large and thus the integral over it decays super-exponentially. The relative version of this result is more useful.

Theorem 2.17 (Relative residue theorem)**.** *[PWM24, Theorem C.10] Let* $\mathcal{V}_*$ *and* ω *be as in Theorem 2.16. If* Y *is any closed subspace of* $\mathbb{C}_*^d$ *such that* $\mathrm{H}_d(C_*^d, Y)$ *vanishes, and* α *is a* d*-cycle in* $\mathcal{M}$*, then*

$$\int_\alpha \omega = 2\pi i \int_{\boldsymbol{INT}[\alpha,0;\mathcal{V}_*]} \mathrm{Res}\,(\omega) + \int_{C'} \omega$$

for some chain C' *supported on the interior of* Y*. In ACSV,* ω *is often chosen to be* $\mathbf{z}^{-\mathbf{r}}\eta$ *for some holomorphic form* η *on* $\mathcal{M}$ *and if* Y *is the set where* $-\hat{\mathbf{r}} \cdot \mathrm{Relog}(\mathbf{z})$ *is at most* c*, then as* $|\mathbf{r}| \to \infty$*,*

$$\int_\alpha \omega = 2\pi i \int_{\boldsymbol{INT}[\alpha,0;V_*]} \mathrm{Res}\,(\omega) + O\left(e^{|\mathbf{r}|c'}\right)$$

for any $c' > c$*.*

Forms with poles on transverse sheets

In ACSV, we will often have a rational generating function whose denominator is a product of two or more polynomials. Other times, the generating function may not be a rational function but at a point $\mathbf{p} \in \mathcal{V}$, the singular variety $\mathcal{V}$ is locally defined by a product of two or more locally analytic functions. If we consider the d-form $\mathbf{z}^{-\mathbf{r}} F(\mathbf{z})\, d\mathbf{z}$, it then has a pole which lies on the intersection of analytic hypersurfaces. We deal with the case when the intersection is transverse.

Let $\mathcal{V}_*$ be a transverse analytic hypersurface as in Definition 2.7. For any point $\mathbf{w} \in \mathcal{V}_*$. We have $Q_1, \cdots, Q_k$ that locally defines $\mathcal{V}_*$ near $\mathbf{w}$ and each Q_i has non-vanishing gradients near $\mathbf{w}$ on $\mathcal{V}_*$. Given a multi-index $\mathbf{m} \in \mathbb{N}^k$, we write $Q(\mathbf{z})^{\mathbf{m}} = Q_1(\mathbf{z})^{m_1} \cdots Q_k(\mathbf{z})^{m_k}$.

Definition 2.18 (transverse pole)**.** *Suppose that $\mathcal{V}_*$ is a transverse analytic hypersurface in $\mathbb{C}_*^d$ and ω is a d-form on $\mathcal{M} := \mathbb{C}_*^d - \mathcal{V}_*$. Let $\mathbf{w} \in \mathcal{V}_*$ be a transverse multiple point so that there exists a neighborhood $\mathcal{D}$ of $\mathbf{w}$ in $\mathbb{C}_*^d$ such that $\mathcal{D} \cap \mathcal{V}_* = \mathcal{D} \cap (\mathcal{V}_{Q_1} \cup \cdots \cup \mathcal{V}_{Q_k})$ for analytic functions $\{Q_i\}$ on $\mathcal{D}$ whose gradients are non-vanishing on $\mathcal{D}$ and are linearly independent at $\mathbf{w}$. We say that ω has a* **transverse pole** *of order $\mathbf{m}$ at $\mathbf{w} \in \mathcal{V}_*$ if it satisfies all of the following conditions.*

(i) there exists a multi-index $\mathbf{m} = (m_1, \cdots, m_k)$ such that $Q(\mathbf{z})^{\mathbf{m}}\omega$ is holomorphic in $\mathcal{D}$.

(ii) it is not possible to choose another set of $\mathcal{D}, k, Q_1(\mathbf{z}), \cdots, Q_k(\mathbf{z}), P(\mathbf{z})$ such that the previous condition holds with another $\mathbf{m}'$ with one of the coordinate less than that of $\mathbf{m}$.

In particular, when $\mathbf{m} = \mathbf{1} := (1, \cdots, 1)$, we say that ω has a **transverse simple pole** *at $\mathbf{w}$.*

If ω has a smooth simple pole at $\mathbf{w}$, as we previously did, we can compute the residue form of ω on $\mathcal{V}_*$ where $\mathcal{V}_*$ is a smooth analytic hypersurface on which $\mathbf{w}$ lies. If ω has a transverse pole at $\mathbf{w}$, then $\mathbf{w}$ is actually on the intersection of smooth analytic hypersurfaces $\mathcal{V}_{Q_1}, \cdots, \mathcal{V}_{Q_k}$. We denote the intersection by $S := \bigcap_{i=1}^k \mathcal{V}_{Q_i}$ and let $S_{\mathcal{D}} := S \cap \mathcal{D}$. The goal is to define an analogous residue form in this setting: if ω is a holomorphic d-form $\omega = \frac{P(\mathbf{z})}{\prod_{i=1}^k Q_i(\mathbf{z})}\, d\mathbf{z}$ on $\mathcal{M} \cap \mathcal{D}$ where P and Q_i are analytic functions on $\mathcal{D}$, then there exists a solution of $(d-k)$-form θ on $\mathcal{D}$ such that

$$dQ_1 \wedge \cdots \wedge dQ_k \wedge \theta = P d\mathbf{z}. \tag{2.3}$$

Furthermore, all such solutions have the same restriction to $S_{\mathcal{D}}$. We define the restriction of θ to $S_{\mathcal{D}}$ as the **residue form** of ω on $S_{\mathcal{D}}$.

Example 2.19 (Two transverse poles)**.**

(i) *($d = 2$ and $k = 2$) Let $Q(x, y) := (y-1)^2 - (x-1)^2 - (x-1)^3$ and $\mathcal{V} = \mathcal{V}_Q \in \mathbb{C}^2$. The point $\mathbf{1} = (1, 1)$ is a transverse multiple point. There exist a sufficiently small open neighborhood $\mathcal{D} := \{(x, y) : |x-1| = |y-1| < 9/10\}$ and two analytic functions Q_1 and Q_2 in $\mathcal{D}$ defined by*

$$Q_1(x, y) = (y-1) + (x-1)\sqrt{x} \quad , \quad Q_2(x, y) = (y-1) - (x-1)\sqrt{x}$$

such that $\mathcal{V} \cap \mathcal{D} := (\mathcal{V}_{Q_1} \cup \mathcal{V}_{Q_2}) \cap \mathcal{D}$. Both $\mathcal{V}_{Q_1}$ and $\mathcal{V}_{Q_2}$ are smooth analytic hypersurfaces in $\mathcal{D}$. Gradients $\nabla Q_1(\mathbf{1})$ and $\nabla Q_2(\mathbf{1})$ are linearly independent. The set $\mathcal{S}_\mathcal{D}$ is just a singleton set consisting of $\mathbf{1}$.

Let ω be a 2-form defined as $\omega = 1/Q \; dx \wedge dy$ on $\mathcal{M}$. In $\mathcal{D} \cap \mathcal{M}$, it has a local representation $\omega = 1/(Q_1 Q_2) \; dx \wedge dy$. Therefore, ω has a transverse simple pole at $\mathbf{1}$.

(ii) *($d = 3$ and $k = 2$) Let $Q(x, y, z) := (x + 2y + z - 4)(x + y + 2z - 4)(x + y + z - 2)$ and $\mathcal{V} = \mathcal{V}_Q$. Therefore, $\mathcal{V}$ is an algebraic hypersurface. The point $\mathbf{1} = (1, 1, 1)$ is a transverse multiple point. We can choose $\mathcal{D} := \{(x, y, z) : |x-1| = |y-1| = |z-1| < 1/10\}$ and thus $\mathcal{V} \cap \mathcal{D} = (\mathcal{V}_{Q_1} \cup \mathcal{V}_{Q_2}) \cap \mathcal{D}$ where $Q_1(x, y, z) = x + 2y + z - 4$ and $Q_2(x, y, z) = x + y + 2z - 4$. The intersection of $\mathcal{V}_{Q_1}$ and $\mathcal{V}_{Q_2}$ is the complex line $(1, 1, 1) + (-3z, z, z)$ parametrized by $z \in \mathbb{C}$. The set $\mathcal{S}_\mathcal{D}$ is then the open segment of the complex line $(1 - 3z, 1 + z, 1 + z)$ for $|z| < 1/30$.*

Let ω be a 3-form defined on $\mathcal{M}$ by $\omega = (x + 2y + z - 4)^{-1}(x + y + 2z - 4)^{-1}(x + y + z - 2)^{-1} \; dx \wedge dy \wedge dz$. In $\mathcal{D} \cap \mathcal{M}$, it has a local representation $\omega = P/(Q_1 Q_2) \; dx \wedge dy \wedge dz$ where $P(x, y, z) = 1/(x + y + z - 2)$ is analytic on $\mathcal{D}$. Therefore ω has a transverse pole of order $(1, 1)$ at $\mathbf{1}$.

To calculate residues on a transverse pole $\mathbf{w}$, notice that the gradients of Q_i at $\mathbf{w}$ are linearly independent by transversality. By the implicit function theorem, there exist a set of indices $\pi = \{\pi_1, \cdots, \pi_{d-k}\}$ such that $z_{\pi_1}, \cdots, z_{\pi_{d-k}}$ analytically parametrize S near $\mathbf{w}$ in a (possibly smaller than $\mathcal{D}$) open neighborhood in $\mathbb{C}^d_*$. In other words, there exist k analytic functions $\xi_i : \mathcal{D}' \to \mathbb{C}$ for $i \notin \pi$ such that $\mathbf{z} \in S$ if and only if $z_i = \xi_i(\mathbf{z}_\pi)$ for $i \notin \pi$. Here $\mathcal{D}'$ is a sufficiently small neighborhood of the origin in $\mathbb{C}^{d-k}$. Without loss of generality, let's assume that this open neighborhood is $\mathcal{D}$.

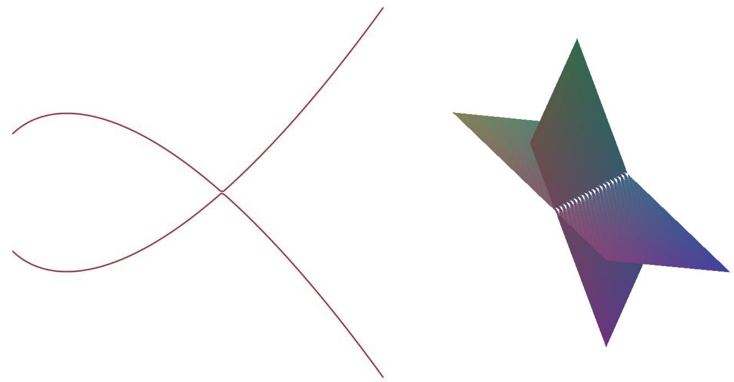

Figure 2.2: Pictures of $\mathcal{V} \cap \mathcal{D} \cap \mathbb{R}^d$ in Example 2.19

Otherwise, replace $\mathcal{D}$ by a smaller open neighborhood. Consider the map $\Psi : \mathcal{D} \to \mathbb{C}^d$ defined by

$$\Psi(\mathbf{z}) := \left(Q_1(\mathbf{z}), \cdots, Q_k(\mathbf{z}), z_{\pi_1} - w_{\pi_1}, \cdots, z_{\pi_{d-k}} - w_{\pi_{d-k}}\right).$$

Definition 2.20 (Augmented lognormal matrix)**.** *For a differentiable function Q, the logrithmic gradient of Q at $\mathbf{z}$ is defined as*

$$\nabla_{\log} Q(\mathbf{z}) := (z_1 Q_{z_1}(\mathbf{z}), \cdots, z_d Q_{z_d}(\mathbf{z})).$$

For each $\mathbf{z} \in S_{\mathcal{D}}$, the **augmented lognormal matrix** *is the following $d \times d$ matrix*

$$\Gamma_\Psi(\mathbf{z}) = \begin{bmatrix} \nabla_{\log} Q_1(\mathbf{z}) \\ \vdots \\ \nabla_{\log} Q_k(\mathbf{z}) \\ z_{\pi_1} e_{\pi_1} \\ \vdots \\ z_{\pi_{d-s}} e_{\pi_{d-s}} \end{bmatrix} \tag{2.4}$$

where e_i is the i-th elementary basis vector in $\mathbb{R}^d$.

We will see the explicit use of this matrix in equation (5.10) in Chapter 5.3.5 when we deal with hyperplane arrangements where all Q_i are linear. One can also see that Γ_Ψ depends on choose Q in $\omega = P/Q\ d\mathbf{z}$ and the factorization of Q into $Q_1, \cdots, Q_k$. In other words, Γ_Ψ is only defined up to a complex multiple. One way of normalization is to let all Q_i have $Q_i(\mathbf{0}) = 1$ (and this is exactly what we adopt in Chapter 6.1).

Theorem 2.21 (Iterated residues)**.** *[PWM24, Theorem C.17, C.18] Suppose that ω is a holomorphic d-form $\omega = P(\mathbf{z})/\prod_{i=1}^k Q_j(\mathbf{z})$ on $\mathcal{M} \cap \mathcal{D}$ where gradients of Q_j are linearly independent inside $\mathcal{D}$. Here P, Q_i are analytic in $\mathcal{D}$. Then we have*

(i) The solution to equation (2.3) *exists and the restriction of such solution θ to $S_\mathcal{D}$ is independent of the particular choice of θ. We denote this restriction as* $\operatorname{Res}(\omega; S_\mathcal{D})$*, and call it the* **iterated residue** *of ω on $S_\mathcal{D}$.*

(ii) In particular, the $(d-k)$-form $\operatorname{Res}(\omega; S_\mathcal{D})$ *is given by the formula*

$$\operatorname{Res}(\omega; S_\mathcal{D}) = \left.\frac{P(\mathbf{z})}{\det J_\Psi(\mathbf{z})}\right|_{z_i = \xi_i(\mathbf{z}_\pi) \text{ for all } i \notin \pi} dz_{\pi_1} \wedge \cdots \wedge dz_{\pi_{d-k}}$$

where J_Ψ is the Jacobian matrix of Ψ. When $k = d$, the formula becomes

$$\frac{P(\mathbf{w})}{\det J_\Psi(\mathbf{w})}$$

where $\{\mathbf{w}\}$ is the 0-dimensional stratum $S_\mathcal{D}$.

(iii) **Residue Theorem** *Let σ be any $(d-k)$-chain on $S_\mathcal{D}$ and $T = \Psi^{-1}(T_\epsilon)$ as we defined previously, then*

$$\frac{1}{(2\pi i)^k}\int_{T\times\sigma} \omega\ d\mathbf{z} = \int_\sigma \operatorname{Res}(\omega; S_\mathcal{D})$$

(iv) **Cauchy Integral** *If $\mathcal{D}$ is chosen so that $\mathcal{D} \subset \mathbb{C}_*^d$, then*

$$\frac{1}{(2\pi i)^k}\int_{T\times\sigma} \mathbf{z}^{-\mathbf{r}-\mathbf{1}}\omega\ d\mathbf{z} = \int_\sigma \left.\frac{\mathbf{z}^{-\mathbf{r}}P(\mathbf{z})}{\det \Gamma_\Psi(\mathbf{z})}\right|_{z_i = \xi_i(\mathbf{z}_\pi) \text{ for all } i \notin \pi} dz_{\pi_1} \wedge \cdots \wedge dz_{\pi_{d-k}}. \tag{2.5}$$

When $d = k$, the above formula becomes

$$\frac{\mathbf{w}^{-\mathbf{r}} P(\mathbf{w})}{\det \Gamma_\Psi(\mathbf{w})}.$$

Remark. When $k \neq d$, we need to compute the saddle point integral (2.5) after we apply the residue theorem. In particular, σ is often a chain on $S_\mathcal{D}$ such that $Re(-\hat{\mathbf{r}} \cdot \log(\mathbf{z}))$ attains unique maximum at $\mathbf{w}$ on σ. Even though Γ_Ψ is not uniquely defined, the quantity $P(\mathbf{z})/\det \Gamma_\Psi(\mathbf{z})$ is indepdent of choices of factorization of Q into Q_i.

Example 2.22 (Continuation of Example 2.19)**.**

(i) The 2-form in Example 2.19 (i) is written as $\omega = 1/(Q_1 Q_2)$ in $\mathcal{M} \cap \mathcal{D}$ and $\mathcal{S}_\mathcal{D} = \{(1,1)\}$. By Theorem 2.21 (ii), $\text{Res}\,(\omega; \mathcal{S}_\mathcal{D})$ *is a constant defined as $1/\det J_\Psi(1,1)$. The Jacobian matrix of Ψ at $(1,1)$ is $[\nabla Q_1(1,1), \nabla Q_2(1,1)]^T$ and its determinant is 2. Therefore,* $\text{Res}\,(\omega; \mathcal{S}_\mathcal{D}) = 1/2$.

(ii) The 3-form in Example 2.19 (ii) is written as $\omega = P/(Q_1 Q_2)$ in $\mathcal{M} \cap \mathcal{D}$ and $\mathcal{S}_\mathcal{D} = \{(1-3z, 1+z, 1+z) : |z| < 1/30\}$. The iterated residue $\text{Res}\,(\omega, \mathcal{S}_\mathcal{D})$ *is a 1-form on $\mathcal{S}_\mathcal{D}$. In particular, coordinate z parametrizes $\mathcal{S}_\mathcal{D}$ and by the same theorem as above,*

$$\text{Res}\,(\omega; \mathcal{S}_\mathcal{D}) = P(4-3z, z, z)/\det J_\Psi(4-3z, z, z)\ dz$$

for $|z-1| < 1/30$. The Jacobian matrix $J_\Psi(4-3z, z, z)$ is

$$\begin{bmatrix} \nabla Q_1 \\ \nabla Q_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, $\text{Res}\,(\omega; \mathcal{S}_\mathcal{D}) = -P(4-3z, z, z)\ dz = -(2-z)^{-1}\ dz.$ □

For general transverse poles with order $\mathbf{m} > \mathbf{1}$, we have a similar Gelfand-Shilov reduction (Lemma 2.15) reducing a higher order pole to a simple one.

Lemma 2.23 (Gelfand-Shilov reduction)**.** *[PWM24, Propostion C.21] Let* $\omega = \dfrac{P(\mathbf{z})}{\prod_{i=1}^{k} Q_i^{m_i}(\mathbf{z})}\ d\mathbf{z}$ *be a holomorphic d-form on* $\mathcal{M} \cap \mathcal{D}$ *where* P, Q_i *are holomorphic functions on* $\mathcal{D}$*, then* ω *is cohomologous in* $\mathcal{M} \cap \mathcal{D}$ *to a holomorphic d-form*

$$\omega' := \frac{P'(\mathbf{z})}{\prod_{i=1}^{k} Q_i(\mathbf{z})}\ d\mathbf{z}$$

with poles of order $\mathbf{1}$ *on* $S_{\mathcal{D}} := \mathcal{V}_{Q_1} \cap \cdots \cap \mathcal{V}_{Q_k} \cap \mathcal{D}$*. In particular,* $\operatorname{Res}(\omega'; S_{\mathcal{D}}) = \operatorname{Res}(\omega; S_{\mathcal{D}})$ *because* $\operatorname{Res}(d\eta; S_{\mathcal{D}}) = 0$ *for any holomorphic d-form* $\eta = P(\mathbf{z})/Q(\mathbf{z})^{\mathbf{m}}$ *on* $\mathcal{M} \cap \mathcal{D}$*.*

Theorem 2.24. *[PWM24, Theorem C.24] Let* $\omega = \dfrac{P(\mathbf{z})}{\prod_{i=1}^{k} Q_i^{m_i}(\mathbf{z})}\ d\mathbf{z}$ *be a holomorphic d-form on* $\mathcal{M} \cap \mathcal{D}$ *where* P, Q_i *are holomorphic in* $\mathcal{D}$*. Shrink* $\mathcal{D}$ *if necessary so that* $\mathcal{D} \subset \mathbb{C}_*^d$*. Then*

$$\operatorname{Res}\left(\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\omega; S_{\mathcal{D}}\right) = \mathbf{z}^{-\mathbf{r}}\mathcal{P}(\mathbf{r}, \mathbf{z})\big|_{z_j = \xi(\mathbf{z}_\pi), j \notin \pi}\ d\mathbf{z}_\pi$$

where $\mathcal{P}(\mathbf{r}, \mathbf{z})$ *is a polynomial in* $\mathbf{r}$ *of degree* $|\mathbf{m}| - k$*. The leading term of* $\mathcal{P}(\mathbf{r}, \mathbf{z})$ *is*

$$\frac{(-1)^{|\mathbf{m}-\mathbf{1}|}}{(\mathbf{m}-\mathbf{1})!} \frac{P(\mathbf{z})}{\det \Gamma_\Psi(\mathbf{z})} (\mathbf{r}\Gamma_\Psi^{-1})^{\mathbf{m}-\mathbf{1}}$$

where $(\mathbf{r}\Gamma_\Psi)^{\mathbf{m}-\mathbf{1}}$ *is defined to be* $\prod_{i=1}^{k} (\mathbf{r}\Gamma_\Psi)_i^{m_i - 1}$.

Alternative definitions for poles and residues

On both Definition 2.10 and 2.18, we first fix an analytic hypersurface $\mathcal{V}_* \subset \mathbb{C}_*^d$ and then define poles for a holomorphic d-form ω in $\mathcal{M} := \mathbb{C}_*^d - \mathcal{V}_*$. Residues of ω are also defined on $\mathcal{V}_*$. In ACSV, when the GF is a rational function P/Q, the form ω can be thought of as $F(\mathbf{z})\ d\mathbf{z}$ or $\mathbf{z}^{-\mathbf{r}-\mathbf{1}}F(\mathbf{z})\ d\mathbf{z}$. Both forms have poles on $\mathcal{V}_Q \cap \mathbb{C}_*^d$ and holomorphic in $\mathcal{M} := \mathbb{C}_*^d - \mathcal{V}_Q$.

One may wonder how to deal with points on coordinate axes. For example, the form $\mathbf{z}^{-\mathbf{r}-\mathbf{1}}F(\mathbf{z})\ d\mathbf{z}$ is not analytic on the z_i-coordinate if $r_i > -1$. The current definitions we have for poles don't apply to points on coordinate axes but we can define them analogously. In short, fix an analytic hypersurface $\mathcal{V}$ in $\mathbb{C}^d$. A holomoprhic d-form in $\mathbb{C}^d - \mathcal{V}$ is said to have transverse poles on $\mathbf{w} \in \mathcal{V}$ if

(i) there exists a neighborhood $\mathcal{D}$ of $\mathbf{w}$ in $\mathbb{C}^d$ such that $\mathcal{D} \cap \mathcal{V} = \mathcal{D} \cap (\mathcal{V}_{Q_1} \cup \cdots \cup \mathcal{V}_{Q_k})$ for analytic functions $\{Q_i\}$ on $\mathcal{D}$ whose gradients are all non-vanishing on $\mathcal{D}$ and linearly independent at $\mathbf{w}$.

(ii) there exists a multi-index $\mathbf{m} = (m_1, \cdots, m_k)$ such that $Q(\mathbf{z})^{\mathbf{m}}\omega$ is holomorphic in $\mathcal{D}$.

(iii) it is not possible to choose another set of $\mathcal{D}, k, Q_1(\mathbf{z}), \cdots, Q_k(\mathbf{z}), P(\mathbf{z})$ such that the previous condition holds with another $\mathbf{m}'$ with one of the coordinate less than that of $\mathbf{m}$.

If $k = 1$, then ω has a smooth pole at $\mathbf{w}$. Compare this definition to Definition 2.18.

The reasons we don't use this general definition are based on our applications to ACSV. When the generating function F is a rational function P/Q, the form ω can be thought of $\mathbf{z}^{-\mathbf{r}-\mathbf{1}}F(\mathbf{z})\ d\mathbf{z}$ as the integrand in the Cauchy integral (1.1). If we only look for poles in $\mathbb{C}_*^d$, then $\mathbf{z}^{-\mathbf{r}-\mathbf{1}}F(\mathbf{z})\ d\mathbf{z}$ has the same poles as $F(\mathbf{z})\ d\mathbf{z}$, which are $\mathcal{V}_Q \cap \mathbb{C}_*^d$. In particular, the smoothness or transversality of these poles depends solely on the polynomial Q. On the other hand, if we consider poles in $\mathbb{C}^d$, then the form $\mathbf{z}^{-\mathbf{r}-\mathbf{1}}F(\mathbf{z})\ d\mathbf{z}$ has poles in $\mathcal{V} := \mathcal{V}_Q \cup \{\mathbf{z} \in \mathbb{C}^d : z_i = 0 \text{ for any } i\}$ while the form $F(\mathbf{z})\ d\mathbf{z}$ has poles in $\mathcal{V}_Q$. In both cases, for any pole $\mathbf{w}$ lying on the coordinate axes, even if $\mathcal{V}_Q$ is smooth, the pole is not a smooth pole. In ACSV, we never need to calculate residues for poles on coordinate axes and therefore we don't need to define poles on coordinate axes.

In a nut shell, considering forms in $\mathbb{C}_*^d$ makes a big convenience because $\mathbf{z}^{-\mathbf{r}-\mathbf{1}}$ is always holomorphic in $\mathbb{C}_*^d$ and thus smoothness or transversality of poles are captured by the polynomial Q solely. Moreover, since poles and residues are defined locally, these two definitions (in $\mathbb{C}_*^d$ and $\mathbb{C}^d$) agree for any pole $\mathbf{w} \in \mathbb{C}_*^d$.

2.3. Classical Morse Theory

2.3.1. Why Morse theory

Let's now go back to the case of forms with smooth poles. Let our generating function $F(\mathbf{z})$ be $P(\mathbf{z})/Q(\mathbf{z})$ such that P, Q are analytic in $\mathbb{C}_*^d$. Then consider the form $\omega := P(\mathbf{z})/Q(\mathbf{z})\ d\mathbf{z}$ which is holomorphic in $\mathcal{M} := \mathbb{C}_*^d - \mathcal{V}_Q$. Let's assume that $\nabla\widetilde{Q}$ is nonvanishing and so $\mathcal{V} := \mathcal{V}_Q$ is a smooth analytic hypersurface. The coefficient asymptotic of the power series $\sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ that converges to $F(\mathbf{z})$ is given by the Cauchy integral (1.1)

$$a_{\mathbf{r}} = \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} \omega \; d\mathbf{z}$$

where T is a sufficiently small torus around the origin in $\mathbb{C}^d$. Let T' be a sufficiently large torus over which the above integral contributes a negligibly smaller term. Then we can compute the intersection class $\sigma := \mathbf{INT}[T, T'; \mathcal{V}_*]$ and the residue class $\mathrm{Res}\,[\omega; \mathcal{V}_*]$. Then

$$a_{\mathbf{r}} = \int_\sigma \mathbf{z}^{-\mathbf{r}-\mathbf{1}} \mathrm{Res}\,[\omega; \mathcal{V}_*] + \int_{T'} \mathbf{z}^{-\mathbf{r}-\mathbf{1}} \omega \; d\mathbf{z}$$

where the second integral contributes exponentially smaller terms than the first integral. The difficulty here is how to deform σ on $\mathcal{V}_*$ so that we can compute the first integral in an easy way (as a saddle point integral). This requires us to know the topological properties of $\mathcal{V}_*$ and Morse theory [Mil63] helps us to see the topology of $\mathcal{V}_*$. Indeed, we have already used it in the proof of Theorem 2.5. In particular, given a real-valued Morse function (defined later) h on $\mathcal{V}_*$, topology of $\{\mathbf{p} \in \mathcal{V}_* := \mathcal{V}_Q \cap \mathbb{C}_*^d : h(\mathbf{p}) \leq c\}$ only changes at critical points of h. For forms with transverse poles, stratified Morse theory should be applied. We postpone the discussion of such application to transverse poles until Chapter 5.2 where we discuss multiple points. Here, we give a rather brief introduction to the classical Morse theory.

2.3.2. Condensed introduction to Morse theory

Let X be a smooth manifold of dimension n and let $h : X \to \mathbb{R}$ be a smooth real-valued function. A point $\mathbf{p} \in X$ is called a **critical point** if the induced map $dh : T_{\mathbf{p}}X \to T_{h(\mathbf{p})}\mathbb{R}$ is zero. In other words, if $(x_1, \cdots, x_n)$ is a local coordinate near $\mathbf{p}$, then $\frac{\partial h}{\partial x_i}(\mathbf{p}) = 0$ for all i. The real number $h(\mathbf{p})$ is called a **critical value**. Using a local cooridinate system $(x_1, \cdots, x_n)$, the Hessian matrix of h

at $\mathbf{p}$ is defined to be the matrix

$$\mathcal{H} := \begin{bmatrix} \frac{\partial^2 h}{\partial x_1 \partial x_1}(\mathbf{p}) & \cdots & \frac{\partial^2 h}{\partial x_1 \partial x_n}(\mathbf{p}) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 h}{\partial x_d \partial x_1}(\mathbf{p}) & \cdots & \frac{\partial^2 h}{\partial x_d \partial x_d}(\mathbf{p}) \end{bmatrix}.$$

We say that a critical point $\mathbf{p}$ is **non-degenerate** if the Hessian matrix of h at $\mathbf{p}$ is non-singular. The singularity does not depend on the particular choice of local coordinate systems. In [Mil63, Chapter 2], the Hessian is defined to be a symmetric bilinear functional on $T_{\mathbf{p}}X$ whose matrix representation is the above matrix. The **index** of h is the number of negative eigenvalues (counting multiplicities) of the Hessian matrix. Indeed, the Hessian matrix $\mathcal{H}$ is real symmetric and thus there exists orthogonal matrix S such that $S\mathcal{H}S^T$ is a diagonal matrix consisting of eigenvalues. If there is no eigenvalue equal to zero, the point is non-degenerate; otherwise, degenerate. If all critical points of h are non-degenerate, then h is called a **Morse function** or **height function**. Given a Morse function h, we denote the set $\{x \in X : h(x) \leq c\}$ by $X_{\leq c}$. There are two important results from Morse theory.

Theorem 2.25. *[Mil63, Theorem 3.1] Let $a < b$ be two real numbers and assume that $h^{-1}[a,b]$ is compact. Further assume that there is no critical value in $[a,b]$. Then $X_{\leq a}$ is diffeomorphic to $X_{\leq b}$. Furthermore, $X_{\leq a}$ is a deformation retraction of $X_{\leq b}$ so that the inclusion $X_{\leq a} \hookrightarrow X_{\leq b}$ is a homotopy equivalence.*

The above theorem works for any smooth real valued function h on a smooth manifold X. The next theorem works only for Morse function h.

Theorem 2.26. *[Mil63, Theorem 3.2] Let h be a Morse function on X and assume that $h^{-1}[a,b]$ is compact. If $h^{-1}[a,b]$ contains exactly one critical non-degenerate point $\mathbf{p}$ with $h(\mathbf{p}) \in (a,b)$, then $X_{\leq b}$ is homotopy equilvalent to $X_{\leq a}$ with a λ-cell attached along its boundary. Here λ is the index of h at $\mathbf{p}$.*

If for any a, $X_{\leq a}$ is compact, then by [Mil63, Theorem 3.5], X is homotopy equivalent to a CW-complex, with one λ-cell for each critical point with index λ. Actually, this topological change happens only inside a small neighborhood of the critical point. Let $a < c < b$ where $h^{-1}[a, b]$ contains exactly one critical non-degenerate point $\mathbf{p}$ with critical value c. Then choose $\epsilon > 0$ small enough and by Theorem 2.25, $X_{\leq c-\epsilon}$ is homotopy equivalent to $X_{\leq a}$. Therefore the pair $(X_{\leq b}, X_{\leq a})$ is homotopy equivalent to $(X_{\leq b}, X_{\leq c-\epsilon})$. There exist $\delta(\epsilon) > 0$ such that $X_{\leq c-\epsilon} \cup B_\delta(\mathbf{p})$ is homotopy equivalent to $X_{\leq b}$. We denote the pair $(X_{\leq c-\epsilon} \cup B_\delta(\mathbf{p}), X_{\leq c-\epsilon})$ by $X^{\mathbf{p},loc}$. By all the above homotopy equivalence, we see that $(X_{\leq b}, X_{\leq a})$ has the same homotopy type of $X^{\mathbf{p},loc}$. By Theorem 2.26, $X^{\mathbf{p},loc}$ has the homotopy type of $X_{\leq c-\epsilon}$ with a λ-cell attached along its boundary.

This local pair $X^{\mathbf{p},loc}$ is handy when we have several critical points with the same critical value. Let $a < c < b$ where c is the only critical value in $[a, b]$. If $h^{-1}[a, b]$ is compact, then

$$(X_{\leq b}, X_{\leq a}) \sim \widetilde{\bigoplus}_{\mathbf{p}:h(\mathbf{p})=c} X^{\mathbf{p},loc}$$

where $\widetilde{\bigoplus}$ means the wedge of spaces, taking disjoint union with the second space $X_{\leq c-\epsilon}$ in each pair identified. The reduced relative homology $\tilde{H}_*(X_{\leq b}, X_{\leq a})$ is then $\bigoplus_{\mathbf{p}:h(\mathbf{p})=c} \tilde{H}_*(X^{\mathbf{p},loc})$ where $\bigoplus$ is the direct sum.

2.3.3. Set-Up in ACSV

In ACSV, as we said in the very beginning of Chapter 2.3, we care the topology of $\mathcal{V}_* := \mathcal{V}_Q \cap \mathbb{C}_*^d$. Since $\mathcal{V}_*$ is a complex manifold of dimension $d-1$, it is thus a smooth manifold of dimension $2d-2$. Now let $X = \mathcal{V}_*$ and let's see how to apply previous assertions on this particular space X. The Morse function we consider in ACSV is $h(\mathbf{z}) := -\hat{\mathbf{r}} \cdot \mathrm{Relog}(\mathbf{z})$ where $\hat{\mathbf{r}}$ is the given direction for the coefficient asymptotic and $\mathrm{Relog}(\mathbf{z})$ is the vector $(\log|z_1|, \cdots, \log|z_d|)$. It is not always true that h is a proper Morse function but we assume that it is for now. Indeed, if we consider $X := \mathcal{V}_*$ as a smooth manifold, we should consider $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ and thus h is a function with $2d$ variables. This particular choice of height function is harmonic because it is the real part of (a branch of) a holomoprhic function $-\hat{\mathbf{r}} \cdot \log \mathbf{z}$. Because h is harmonic, the Morse index at each critical point to be half of the dimension and thus $d-1$.

We can enumerate its critical points by height, from the largest to the smallest, as $\mathbf{p}_1, \cdots, \mathbf{p}_m$ with corresponding critical value $c_1, \cdots, c_m$. Any cycle supported on $X_{<c_m}$ can be pushed to $X_{\leq a}$ where a can be arbitrarily small as long as $a < c_m$ by Theorem 2.25. Therefore, we identify the homotopy type of $X_{\leq a}$ where $a < c_m$ as the same and denote it by $X_{-\infty}$. In particular, if γ is a cycle supported on $X_{<c_m}$, then $\int_\gamma \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\ d\mathbf{z}$ decays super-exponentially, that is $O(e^{-a|\mathbf{r}|})$ for any $a > 0$. We denote the pair $(X, X_{-\infty})$ by $(X, -\infty)$.

By Proposition 2.3, the integral depends on the relative homology classes. We have an explicit description of the relative homology $\mathrm{H}_*(X, -\infty)$ for this particular choice of h.

Theorem 2.27. *[PWM24, Theorem C.38] Assume that $h := -\hat{\mathbf{r}} \cdot \mathrm{Relog}(\mathbf{z})$ is defined on a smooth algebraic hypersurface X and h is a proper Morse function. $\mathrm{H}_k(X, -\infty)$ vanishes for all $k < d-1$. $\mathrm{H}_{d-1}(X, -\infty) \simeq \mathbb{C}^m$ where m is the number of critical points. Generators of $\mathrm{H}_{d-1}(X, -\infty)$ can be chosen to be a $(d-1)$-cycle $\gamma_\mathbf{p}$ for each critical point $\mathbf{p}$ such that $\gamma_\mathbf{p}$ attains its maximal height at $\mathbf{p}$.*

Since the critical points all have index $d-1$, it is not hard to see why the homology at other dimensions vanishes. The cycle $\gamma_\mathbf{p}$ can be smooth by the isomorphism between singular homology and smooth singular homology by [Lee13, Theorem 18.7]. We can even relax the condition of Theorem 2.27 such that h need not be a proper function. Indeed, as [BMP22] shows, as long as there is no critical point at infinity (CPAI), Theorem 2.25, 2.26, and 2.27 are true. CPAI is defined and discussed later in Chapter 5.2.5. With the presence of CPAIs (and even degenerate critical points), we can have a weaker theorem analogous to Theorem 2.27.

Corollary 2.28. *Assume that $h := -\hat{\mathbf{r}} \cdot \mathrm{Relog}(\mathbf{z})$ is defined on a smooth algebraic hypersurface X. Let $\mathbf{p}_1, \cdots, \mathbf{p}_m$ enumerate all critical points by decreasing critical values. Let $\mathbf{p}_i$ be the first degenerate critical point and $\mathbf{p}_j$ be the first CPAI in this enumeration. Let $c := \max\{h(\mathbf{p}_i), h(\mathbf{p}_j)\}$.*

Then $\mathrm{H}_k(X, X_{\leq c+\epsilon})$ vanishes for all $k < d-1$. $\mathrm{H}_{d-1}(X, X_{\leq c+\epsilon}) \simeq \mathbb{C}^k$ where k is the number of critical points with critical values greater than c. Generators of $\mathrm{H}_{d-1}(X, X_{\leq c+\epsilon})$ can be chosen to be a $(d-1)$-cycle $\gamma_\mathbf{p}$ for each critical point $\mathbf{p}$ with $h(\mathbf{p}) > c$ such that $\gamma_\mathbf{p}$ attains its maximal height *at* $\mathbf{p}$.

Indeed, these k generators of $\mathrm{H}_{d-1}(X, X_{\leq c+\epsilon})$ come from the generators of the local homology group $\mathrm{H}_{d-1}(X^{\text{loc}}_{\mathbf{p}_n})$ for $n = 1, \cdots, k$. The rank of each local homology group is one. Morse lemma says that $X^{\text{loc}}_{\mathbf{p}_n}$ is homotopy equivalent to a $(d-1)$-ball attacned modulo boundary on $X_{\leq h(\mathbf{p}_n)-\epsilon}$. The generator can be chosen to be maximized its height at $\mathbf{p}_n$.

2.4. Amoeba

Amoebas are powerful tools in ACSV. They help us visualize the higher dimensional complex space by projecting the space onto a real space of half the dimension. Convex properties of Amoebas tell us an upper bound on the exponential order of the coefficient asymptotics. Amoebas can be understood by their connection with Newton polytopes [GKZ94]. In this paper, we focus on the amoeba of a polynomial, though results can be extended to amoebas for Laurent polynomials in [PWM24, Chapter 6].

Definition 2.29 (amoeba)**.** *Define the log magnitude map* $\mathrm{Relog} : \mathbb{C}_*^d \to \mathbb{R}^d$ *by*

$$\mathrm{Relog}(\mathbf{z}) = (\log|z_1|, \cdots, \log|z_d|).$$

*The **amoeba** of a polynomial $f : \mathbb{C}_*^d \to \mathbb{C}$ is the image of the zero locus of f under the map* Relog. *Explicitly,*

$$\mathsf{amoeba}(f) := \{\mathrm{Relog}(\mathbf{z}) \in \mathbb{R}^d : \mathbf{z} \in \mathbb{C}_*^d \textit{ and } f(\mathbf{z}) = 0\}$$

When $d = 2$ or 3, the amoeba maps from a complex space of real dimension greater than four to a real space of dimension less than three, therefore from a space where visualization is hard to one where it is relatively easy. The amoeba helps us visualize the domain of convergence $\mathcal{D}$ of the power series converging to a rational function $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$. Actually, $\mathrm{Relog}(\mathcal{D})$ is a component of the complement of $\mathsf{amoeba}(Q)$ in $\mathbb{R}^d$ and every components of $\mathsf{amoeba}(Q)^c$ is open and convex.

Proposition 2.30. *All connected components of* $\mathsf{amoeba}(f)^c$ *are open convex subsets of* $\mathbb{R}^d$*, in one-*

to-one correspondence of Laurent series expansions of the rational function $1/f(\mathbf{z})$*. In particular, if* $1/f(\mathbf{z})$ *has a power series expansion (i.e.* $f(\mathbf{0}) \neq 0$*), then there is a component containing* $(-\infty, M)^d$ *for some* M *sufficiently small.*

PROOF: See [GKZ94, Chapter 6 Corollary 1.6]. □

2.4.1. Connection to convex geometry

We introduce some results on amoebas and Newton polytopes. In particular, Newton polytopes of a Laurent polynomial $f(\mathbf{z})$ provides information on the convex properties of $\mathsf{amoeba}(f)$.

Definition 2.31 (Newton polytopes)**.** *Let* f *be a* d*-variate polynomial in* $\mathbb{C}[\mathbf{z}]$*. The support of the polynomial* f *is the (finite) set of exponents of the monomials in* f*. The* ***Newton polytope*** $\mathcal{N}_f$ *of* f *is the convex hull of its support,*

$$\mathcal{N}_f := \mathsf{hull}\{\mathbf{r} \in \mathbb{N}^d : a_{\mathbf{r}}\mathbf{z}^{\mathbf{r}} \textit{ is a monomial of } f\}.$$

Definition 2.32 (order map)**.** *Let* f *be a* d*-variate polynomial. The* ***order map*** $\nu : \mathsf{amoeba}(f)^c \to \mathbb{R}^d$ *sends a real* d*-vector* $\mathbf{x} \in \mathsf{amoeba}(f)^c$ *to another real* d*-vector* $\nu(\mathbf{x})$ *whose* k*-th coordinate is*

$$\nu(\mathbf{x})_k := \frac{1}{2\pi i}\int_{|z_k|=e^{x_k}} \frac{f_{z_k}(e^{x_1}, \cdots, z_k, \cdots, e^{x_d})}{f(e^{x_1}, \cdots, z_k, \cdots, e^{x_d})}\, dz_k$$

where $(e^{x_1}, \cdots, z_k, \cdots, e^{x_d})$ *denotes the vector* $\mathbf{z}$ *with* $z_j = e^{x_j}$ *for* $j \neq k$*.*

Definition 2.33 (various cones)**.** *A* ***convex cone*** *is a subset of* $\mathbb{R}^d$ *that is closed under addition and closed under multiplication by positive scalars. The* ***dual cone*** K^* *of an open convex cone* $K \subseteq \mathbb{R}^d$ *is the set of vectors* $\mathbf{s} \in \mathbb{R}^d$ *such that* $\mathbf{s} \cdot \mathbf{x} \geq 0$ *for all* $\mathbf{x} \in K$*.*

Let B *be a convex set. The* ***recession cone*** *of* B *is the set of vectors* $\mathbf{v} \in \mathbb{R}^d$ *such that* $\mathbf{x} + \mathbf{v} \in B$ *for all* $\mathbf{x} \in B$*. The* ***tangent cone*** *to* B *at* $\mathbf{x} \in B$ *is defined as the set of* $\mathbf{s} \in \mathbb{R}^d$ *such that* $\mathbf{x} + \epsilon\mathbf{s} \in B$ *for all sufficiently small* $\epsilon > 0$*. The* ***normal cone*** *to* B *at* $\mathbf{x} \in B$ *is the set of* $\mathbf{s} \in \mathbb{R}^d$ *such that* $\mathbf{s} \cdot \mathbf{x} \geq \mathbf{s} \cdot \mathbf{b}$ *for all* $\mathbf{b} \in B$*.*

Proposition 2.34. *We have the following properties on* $\mathsf{amoeba}(f)$ *and the Newton polytope* $\mathcal{N}_f$*.*

(i) The image of $\mathsf{amoeba}(f)^c$ *under the order map* ν *is on integer points of the Newton polytope of* f*.*

(ii) $\nu(\mathbf{x}) = \nu(\mathbf{x}')$ *if and only if* $\mathbf{x}$ *and* $\mathbf{x}'$ *belong to the same component of* $\mathsf{amoeba}(f)^c$*.*

(iii) Vertices of the Newton polytope of f *are contained in the image of* $\mathsf{amoeba}(f)^c$ *under the order map* ν*. In particular, vertices of the Newton polytope are in bijection with those connected components of* $\mathsf{amoeba}(f)^c$ *which contain an affine convex cone with non-empty interior.*

(iv) If B *is a component of* $\mathsf{amoeba}(f)^c$ *and* $\mathbf{v} = \nu(\mathbf{x})$ *for some and hence every* $\mathbf{x} \in B$*, then the recession cone of* B *is equal to the normal cone to* $\mathcal{N}_f$ *at* $\mathbf{v}$*. If* B *is bounded, then both are empty.*

PROOF: Proofs for i and iv follow from [FPT00, Proposition 2.4 and 2.6] and the fact that coordinates of $\nu(\mathbf{x})$ are integer-valued by argument principle in complex analysis. The IF part of ii follows from the fact that ν is continous and ν takes value in integer lattice. The ONLY IF part of ii follows from [FPT00, Proposition 2.5]. The proof for the first statement in iii follows from [FPT00, Theorem 2.8]. The second statement in iii follows from ii and iv. □

It is a convention in ACSV that we use B to denote a connected component of $\mathsf{amoeba}(f)^c$ and we use $\mathcal{D}$ to denote $\mathrm{Relog}^{-1}(B)$. In this paper, we mainly study a convergent power series that converges to a rational function $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$ with domain of convergence $\mathcal{D}$. By Proposition 2.30, there is a connected component $B = \mathrm{Relog}(\mathcal{D})$ of $\mathsf{amoeba}(Q)^c$ that contains $(-\infty, M)^d$ for some M. It is much convenient to work with B because of its convexity. For example, the component B often tells us an upper bound for the exponential order of the coefficient asymptotics.

Let $F(\mathbf{z}) = \sum_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ be a convergent Laurent series converging to a rational function $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$ and let $\mathcal{D}$ be the domain of convergence. In this paper, we often have a power series F and thus $\mathcal{D}$ contains the origin. Let $\mathbf{w} \in \mathcal{D}$, then $T(\mathbf{w}) := \{\mathbf{z} \in \mathbb{C}^d : |z_i| = |w_i|\}$ is contained in $\mathcal{D}$. Then by Cauchy integral formula and maximum modulus principle,

$$|a_{\mathbf{r}}| = \left| \left(\frac{1}{2\pi i}\right)^d \int_{T(\mathbf{w})} F(\mathbf{z})\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\ d\mathbf{z} \right| \leq |\mathbf{w}^{-\mathbf{r}}| \max_{\mathbf{z}\in T(\mathbf{w})} |F(\mathbf{z})|$$

Since the torus $T(\mathbf{w})$ is compact and F is continuous over $T(\mathbf{w})$, the term $\max_{\mathbf{z}\in T(\mathbf{w})} |F(\mathbf{z})|$ is finite. We can lower the upper bound by minimizing $|\mathbf{w}^{-\mathbf{r}}|$, or equivalently its logarithm $-\hat{\mathbf{r}}\cdot \operatorname{Relog} \mathbf{w}$. Finding $\inf_{\mathbf{z}\in\mathcal{D}} -\hat{\mathbf{r}} \cdot \operatorname{Relog} \mathbf{z}$ is equivalent to finding $\inf_{\mathbf{x}\in B} -\hat{\mathbf{r}} \cdot \mathbf{x}$ where B is the component in $\mathsf{amoeba}(Q)^c$ corresponding to the Laurent series F. In particular, $B = \operatorname{Relog}(\mathcal{D})$. This is much easier to do because $-\hat{\mathbf{r}} \cdot \mathbf{x}$ is a linear function on a convex set B.

Definition 2.35 (supporting hyperplane)**.** *Let B be a convext set. A* ***supporting hyperplane*** *to B at $\mathbf{x} \in B$ is a hyperplane through $\mathbf{x}$ such that all elements of B lie on one side of the hyperplane. A normal vector $\mathbf{v}$ to the plane is said to be inward-facing if $(\mathbf{y} - \mathbf{x}) \cdot \mathbf{v} \geq 0$ for all $\mathbf{y} \in B$ and outward-facing if $(\mathbf{y} - \mathbf{x}) \cdot \mathbf{v} \leq 0$ for all $\mathbf{y} \in B$.*

Since B is a convex open set, the minimizer $\mathbf{x}_*$ of $-\hat{\mathbf{r}} \cdot \mathbf{x}$, if exists, is at the boundary of B. If $\mathbf{x}_*$ exists, there is then a supporting hyperplane with outward-facing normal $\hat{\mathbf{r}}$ at $\mathbf{x}_*$ to $\overline{B}$ ([PWM24, Theorem 6.44(i)]).

On the other hand, if there is no minimizer of $-\hat{\mathbf{r}} \cdot \mathbf{x}$ on $\overline{B}$, i.e. $-\hat{\mathbf{r}} \cdot \mathbf{x}$ unbounded from below in B, then $a_{\mathbf{r}} = 0$ for all but finitely many terms (see [PWM24, Proposition 6.24]). A useful lemma [PWM24, Corollary 6.29] in ACSV is that there is always a component of $\mathsf{amoeba}(Q)^c$ such that $-\hat{\mathbf{r}} \cdot \mathbf{x}$ is unbounded from below.

Proposition 2.36. *Fix $\hat{\mathbf{r}} \in \mathbb{R}^d$. There is a component B of $\mathsf{amoeba}(f)^c$ on which $\mathbf{x} \mapsto -\hat{\mathbf{r}} \cdot \mathbf{x}$ is unbounded.*

Proof: There is always a vertex $\mathbf{v}$ on $\mathcal{N}_f$ such that $\hat{\mathbf{r}}$ is in the normal cone to $\mathcal{N}_f$ at $\mathbf{v}$. By Proposition 2.34 (iii, iv), there is a component B of $\mathsf{amoeba}(f)^c$ such that $\nu(B) = \mathbf{v}$ and the recession cone of B is equal to the normal cone of $\mathcal{N}_f$ at $\mathbf{v}$. In particular, $\hat{\mathbf{r}}$ is in the recession cone of B. By definition of the recession cone, $-\hat{\mathbf{r}} \cdot \mathbf{x}$ is unbounded from below. □

This result enables us to deform the torus $T(\mathbf{w}) \in \mathcal{D}$ to another torus $T(\mathbf{w}')$ where $\mathbf{w}'$ is in the component where $-\hat{\mathbf{r}} \cdot \mathbf{x}$ is unbounded. Then the integral of $\mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, d\mathbf{z}$ over $T(\mathbf{w}')$ is zero as $\mathbf{r} \to \infty$. The deformation of $T(\mathbf{w})$ to $T(\mathbf{w}')$ cuts the singular variety $\mathcal{V}$ of F by the so-called intersection class. This is the foundation of what we do in Chapter 1.2.2 when $\mathcal{V}$ is smooth.

2.4.2. Connection to minimality

When $-\hat{\mathbf{r}} \cdot \mathbf{x}$ is bounded from below in a component B, then we obtain an upper bound on the exponential order of the coefficient asymptotics of the series $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ in the domain of convergence $\mathcal{D} = \mathrm{Relog}^{-1}(B)$. Suppose that this minimizer is $\mathbf{x}_*$. Since $\mathbf{x}_*$ is in the Relog-space, it is a projection under Relog of all points $\mathbf{z} \in \mathbb{C}_*^d$ such that $|z_i| = e^{x_i}$. Among all these points, there are some points that are singularities of F. Indeed, for any point $\mathbf{x} \in \partial B$, there is some point $\mathbf{w} \in \mathcal{V}$ such that $\mathrm{Relog}(\mathbf{w}) = \mathbf{x}$. We call such points minimal points.

Proposition 2.37. *If $\mathbf{w} \in \partial\mathcal{D}$, then $T(\mathbf{w}) \cap \mathcal{V} \neq \emptyset$. Conversely, if $\mathbf{p} \in \mathcal{D}$ and $\mathbf{w} \in \mathcal{V}$ and $T(\mathbf{z}) \cap \mathcal{V} = \emptyset$ for all $\mathbf{z}$ in the open line segment between $(|p_1|, \cdots, |p_d|)$ and $(|w_1|, \cdots, |w_d|)$, then $\mathbf{w} \in \partial\mathcal{D}$.*

PROOF: Proof is on [PWM24, Proposition 6.33]

Definition 2.38 (minimal points)**.** *Given a convergent Laurent series $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ with domain of convergence $\mathcal{D}$. A point $\mathbf{w} \in \mathbb{C}_*^d$ is called a* ***minimal point*** *if $\mathbf{w} \in \partial\mathcal{D} \cap \mathcal{V}$, or equivalently, $\mathbf{w} \in \mathcal{V}$ and $\mathrm{Relog}(\mathbf{w}) \in \partial B$ where $B = \mathrm{Relog}(\mathcal{D})$. We call $\mathbf{w}$* ***finitely minimal*** *if $T(\mathbf{w}) \cap \mathcal{V}$ is finite, and* ***strictly minimal*** *if $T(\mathbf{w}) \cap \mathcal{V}$ contains only the point $\mathbf{w}$.*

Suppose that $F(\mathbf{z}) = \sum_{\mathbf{r} \in \mathbb{N}^d} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ is a convergent power series with domain of convergence $\mathcal{D}$ and $B = \mathrm{Relog}(\mathcal{D})$. If $\mathbf{x} \in \partial B$, then $\mathbf{y} \in B$ as long as $y_i \le x_i$ for $1 \le i \le d$ with at least one of the inequality being strict. Therefore, a point $\mathbf{w} \in \mathcal{V}$ is minimal for the convergent power series $F(\mathbf{z})$ if and only if $\mathbf{z} \notin \mathcal{V}$ for any $\mathbf{z}$ such that $|z_i| \le |w_i|$ with at least one of the inequality being strict. Actually, there is an easier way to verify minimality by checking on a sequence of tori.

Corollary 2.39. *[PWM24, Corollary 6.36] Let $\mathbf{p} \in \mathcal{D}$. The point $\mathbf{w} \in \mathcal{V}$ is minimal if and only if* *the open line segment*

$$\{t\,\mathrm{Relog}(\mathbf{p}) + (1-t)\,\mathrm{Relog}(\mathbf{w}) : t \in (0,1)\}$$

stays in $B := \mathrm{Relog}(\mathcal{D})$*. If* F *is a convergent power series, then it suffices to show that* $T(t\mathbf{w}) \cap \mathcal{V} = \emptyset$ *for all* $t \in (0,1)$*.*

Remark 2.40. If $\mathcal{V} = \mathcal{V}_Q$ is the variety defined by a polynomial Q, then the condition $T(t\mathbf{w}) \cap \mathcal{V} = \emptyset$ is equivalent to $\mathbf{w}$ being weakly minimal (see Definition 4.12) for Q.

When $F(\mathbf{z})$ is a combinatorial series, we have a much simpler minimality test which only requires testing on positive real points.

Definition 2.41 (combinatorial series). *A Laurent series* $\sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ *is a* ***combinatorial series*** *if* $a_{\mathbf{r}} \geq 0$ *for all but a finite number of* $\mathbf{r} \in \mathbb{Z}^d$*.*

Proposition 2.42 (minimality test for combinatorial series)**.** *Let* $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$ *be a rational gunction with a convergent combinatorial Laurent series in the domain of convergence* $\mathcal{D}$*. Let* $\mathbf{p} \in \mathcal{D}$*. The point* $\mathbf{w} \in \mathcal{V}$ *is minimal if and only if*

$$Q(|p_1|^t |w_1|^{1-t}, \cdots, |p_d|^t |w_d|^{1-t}) \neq 0 \text{ for all } t \in (0,1).$$

If F *is a convergent power series, then* $\mathbf{w} \in \mathcal{V}$ *is minimal if and only if* $Q(t|w_1|, \cdots, t|w_d|) \neq 0$ *for* $t \in (0,1)$*.*

Proof: This is a direct application of the multivariate Pringsheim theorem [PWM24, Propostion 6.38] on Corollary 2.39.

Minimal points have some properties. We define the **logarithmic gradient** $\nabla_{\log} f$ of an analytic function f at the point $\mathbf{z} \in \mathbb{C}_*^d$ by

$$(\nabla_{\log} f)(\mathbf{z}) := (z_1 f_{z_1}, \cdots, z_d f_{z_d})$$

Theorem 2.43. *[PWM24, Theorem 6.44] Let $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$ be a rational function with co-prime polynomials P and Q in $\mathbb{R}[\mathbf{z}]$ with Laurent series expansion in the domain of convergence $\mathcal{D} = \mathrm{Relog}^{-1}(B)$. Then for any minimal point $\mathbf{w}$, there exists $\mathbf{v} \in \mathbb{R}^d$ and $\lambda \in \mathbb{C}_*$ such that $(\nabla_{\log}\tilde{Q})(\mathbf{w}) = \lambda\mathbf{v}$. Assume $\mathbf{v} \neq \mathbf{0}$. The hyperplane through $\mathrm{Relog}(\mathbf{w})$ with normal vector $\mathbf{v}$ is a supporting hyperplane to $\overline{B}$ at $\mathbf{w}$. If $\mathbf{v}$ is outward-facing, then $\mathbf{w}$ is a minimizer of $\mathbf{x} \mapsto -\mathbf{v}\cdot\mathbf{x}$ in $\overline{B}$. Otherwise, it is a maximizer.*

Conversely, let $\hat{\mathbf{r}} \in \mathbb{R}^d - \{\mathbf{0}\}$. If $\mathbf{w}$ is a minimizer of $\mathbf{x} \mapsto -\hat{\mathbf{r}}\cdot\mathbf{x}$ on $\overline{B}$, then the hyperplane through $\mathrm{Relog}(\mathbf{w})$ with normal vector $\hat{\mathbf{r}}$ is a supporting hyperplane to $\overline{B}$ and $\hat{\mathbf{r}}$ is outward-facing.

There are two things to notice in Theorem 2.43. First of all, we use the square-free part $\widetilde{Q}$ instead of Q. Secondly, $\nabla_{\log}(\tilde{Q})(\mathbf{w})$ may not be $\lambda\hat{\mathbf{r}}$ when $\mathbf{w}$ is a minimizer of $-\hat{\mathbf{r}}\cdot\mathbf{x}$ on $\overline{B}$. In Example 1.10, $\hat{\mathbf{r}} = (1,1)$ and the minimizer $\mathbf{w}$ is $(1/2, \pm\sqrt{3})$ but $\nabla_{\log}(\widetilde{Q})(\mathbf{w}) \neq [1,1]$ in $\mathbb{CP}^1$. It is also possible that $\nabla_{\log}(\widetilde{Q})(\mathbf{w}) = \mathbf{0}$ if $\mathbf{w}$ is not a smooth point on $\mathcal{V}_Q$.

Finally, we try to give some explanations on several notions introduced previously in Chapter 1 and 2. We have defined three kinds of points so far: minimal points, critical points, and contributing points. Minimal points are on the boundary of the domain of convergence and the singular variety $\mathcal{V}$. Critical points are those points on $\mathcal{V}$ such that the differential of the height function $h_{\hat{\mathbf{r}}}(\mathbf{z}) = -\hat{\mathbf{r}}\cdot\mathrm{Relog}(\mathbf{z})$ equals zero (when restricted to the stratum on which the critical point lies). Contributing points are a subset of critical points containing those contributing to the asymptotics of $a_{\mathbf{r}}$. In other words, the integer coefficients n_i in equation (1.4) for homology generators at contributing points are non-zero.

Indeed, for a minimal point $\mathbf{w}$, if $(\nabla_{\log}\tilde{Q})(\mathbf{w}) = \lambda\hat{\mathbf{r}}$ for $\lambda \neq 0$, then $\mathbf{w}$ is a smooth critical point. The point $\mathbf{w}$ is smooth because not all partials of $\tilde{Q}$ at $\mathbf{w}$ vanish (otherwise $(\nabla_{\log}\tilde{Q})(\mathbf{w}) = 0$). Since $(\nabla_{\log}\tilde{Q})(\mathbf{w}) = \lambda\hat{\mathbf{r}}$, then the following matrix is rank 1.

$$\begin{bmatrix} (\nabla_{\log}\tilde{Q})(\mathbf{w}) \\ \hat{\mathbf{r}} \end{bmatrix} = \begin{bmatrix} w_1\tilde{Q}_{z_1}(\mathbf{w}) & \cdots & w_d\tilde{Q}_{z_d}(\mathbf{w}) \\ \hat{r}_1 & \cdots & \hat{r}_d \end{bmatrix}$$

which is equilvalent to the smooth critical point equations in Proposition 1.14. The above minimal point $\mathbf{w}$ such that $(\nabla_{\log}\tilde{Q})(\mathbf{w}) = \lambda\hat{\mathbf{r}}$ is contributing if $\hat{\mathbf{r}}$ is outward-facing. In other words, if $\mathbf{w}$ is a minimizer of $-\hat{\mathbf{r}}\cdot\mathbf{x}$ over $\overline{B}$. If we consider a convergent power series, then a minimal smooth critical point is always a contributing point.

CHAPTER 3

ALGEBRAIC GENERATING FUNCTIONS VIA EMBEDDING

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3.1. Introduction

Sometimes, the generating function may not be in the form of a rational function, or even a meromorphic function, on $\mathbb{C}_*^d$. One example will be the generating function for Catalan numbers $F(z) = (1 - \sqrt{1-4z})/(2z)$, with the principal square root, is not meromorphic. On the other hand, though, this univariate function F satisfies a bivariate polynomial relation P such that $P(z, F(z)) = 0$. In the case of Catalan numbers, P will be $P(z, f) = zf^2 - f + 1$. One can easily verify that $F(z) = (1 - \sqrt{1-4z})/(2z)$ satisfies $P(z, F(z)) = 0$. Therefore, it can be easier to work with this bivariate polynomial $P(z, f)$ than to deal with the square root in $F(z)$. This type of d-variate generating function $F(\mathbf{z})$, which satisfies a $(d+1)$-variate polynomial $P(\mathbf{z}, f)$, is called **algebraic generating functions**. One thing to notice is that there is more than one such function $F(\mathbf{z})$ that satisfies the condition $P(\mathbf{z}, F(\mathbf{z})) = 0$. In the previous example, another possible choice (branch) is $F'(z) = (1 + \sqrt{1-4z})/(2z)$. Therefore, we should also specify which branch the generating function is on by stating that $F(0) = 1$. Letting $P(z, F(z)) = 0$, the two conditions $P(z, f) = zf^2 - f + 1$ and $F(0) = 1$ uniquely determine the Catalan generating function $F(z) = (1 - \sqrt{1-4z})/(2z)$.

In this chapter, we are going to discuss one method of working with coefficient asymptotics of algebraic multivariable generating functions. Currently, there are two methods, the **diagonal embedding method** and the **lifting method**. The diagonal embedding method embeds the coefficient array of an algebraic function $F(\mathbf{z})$ as an elementary diagonal of a rational function $\tilde{F}(\mathbf{z}, f)$ in one more variable. One can use ACSV theory on rational functions to extract the elementary diagonal coefficient asymptotics. The advantage and disadvantage both come from this connection. On the one hand, one can easily use the current theory on rational functions to algebraic functions. On the other hand, for people who don't know the ACSV theory, using the enormous toolbox, containing for example residue forms and intersection cycles, is burdensome. The lifting method is more streamlined, not requiring the function F to go up one dimension. It also bypasses the residual forms and intersection cycles in ACSV theory, only involving stationary phase integrals. We will briefly review the diagonal embedding method and the work of [GMRW22]. For the next chapter, we introduce the lifting method, a new method due to [BJP24], which is one of the central parts of this thesis.

Before we proceed, let's formally define what an algebraic generating function means.

Definition 3.1 (Algebraic generating function)**.** *A d-variate generating function $F(\mathbf{z})$ is **algebraic** if there is a polynomial $P(\mathbf{z}, f) \in \mathbb{C}[\mathbf{z}][f] = \sum_{j=0}^{m} p_j(\mathbf{z}) f^j$ such that $P(\mathbf{z}, F(\mathbf{z})) = 0$.*

Definition 3.2 (Minimal polynomial)**.** *A polynomial $P(\mathbf{z}, f) \in \mathbb{C}[\mathbf{z}][f]$ satisfying $P(\mathbf{z}, F(\mathbf{z})) = 0$ is **minimal** if P has the lowest degree in the variable f among all polynomials with $P(\mathbf{z}, F(\mathbf{z})) = 0$.*

Notice that a minimal polynomial $P(\mathbf{z}, f)$ for an algebraic generating function $F(\mathbf{z})$ is not unique. It differs up to a unit in the ring $\mathbb{C}[\mathbf{z}][f]$, or more explicitly, up to a complex multiple.

Example 3.3 (Catalan number)**.** *The generating function for Catalan number $F(z) = \dfrac{1-\sqrt{1-4z}}{2z}$ is algebraic because it satisfies a minimal polynomial $P(z, f) = zf^2 - f + 1$.*

Example 3.4 (Bi-colored Motzkin paths)**.** *A bicolored Motzkin path on the x-y plane starts at the origin and ends at the x-axis, never goes below the x-axis and takes steps $U = (1, 1)$, $D = (1, -1)$, and two colored horizontal steps $H_1 = (1, 0)$ and $H_2 = (1, 0)$. Let $\mathcal{M}^2$ be the set of bicolored Motzkin paths. Define $u(M), d(M), h_1(M), h_2(M)$ to be the number of U, D, H_1, H_2 steps in the bicolored Motzkin path $M \in \mathcal{M}^2$ respectively.*

The generating function $F(x, y) = \sum_{M \in \mathcal{M}^2} x^{d(M)+h_1(M)} y^{u(M)+h_2(M)}$ is counting the number of paths by the total number of D and H_1 steps and the total number of U and H_2 steps. In particular,

$[x^iy^j]F(x,y)$ is the number of such paths with i steps in D and H_1 and j steps in U and H_2. [Eli21, Lemma 2.1] shows that

$$F(x,y) = \frac{1-x-y-\sqrt{(1-x-y)^2-4xy}}{2xy}.$$

$F(x,y)$ is algebraic because it satisfies a minimal polynomial

$$P(x,y,f) = xyf^2 + (x+y-1)f + 1.$$

We have seen previously that there are two functions $F(z)$ satisfying $P(z,F(z)) = 0$ when F is the Catalan number generating function and P is its minimal polynomial. We call each satisfying function $F(z)$ a **branch**. To specify at which branch we look, provided the roots of $P(0,\cdot) = 0$ are simple, we need to specify $F(0)$. If $P(0,\cdot)) = 0$ has non-simple roots, things can become more complicated. In this chapter, we assume that $\frac{\partial P}{\partial f} \neq 0$ at $(\mathbf{0},0)$ so that all branches are simple at the origin.

We have the following literature review in [BJP24] on the development of asympotic analysis on multivariable algebraic generating functions. It is known [Fur67, Saf00] that the coefficient array of any algebraic function appears as a diagonally embedded sub-array of the coefficients of some rational function in one more variable. This may be used to reduce the problem of coefficient extraction for algebraic functions to the same problem for rational functions. The applicability of the embedding result to ACSV was first noticed by Raichev and Wilson in [RW07, RW12] and is exploited in [GMRW22] to compute coefficients asymptotics for a number of algebraic generating functions of combinatorial interest. Prior to this, the best known method for approximating coefficients of algebraic generating functions was the one developed by Bender, Richmond Gao et al. [Ben73, BR83, GR92]. This pioneering work, which extended to the algebraic-logarithmic paradigm, required a number of assumptions such as nonnegativity of coefficients and a quasi-power representation, meaning that for some distinguished variable, say z_d, disaggregating $F(z) = \sum_{n=0}^{\infty} \phi_n(z_1,\dots,z_{d-1})z_d^n$, one has $\phi_n \sim c_n g\lambda^n$ for some functions g and λ. Raichev and Wilson's work represents a considerable step forward in terms of generality and effective computation. The steps of the Raichev-Wilson **diagonal embedding method** are as follows.

1. Use theorems of Furstenburg or Safonov to embed the coefficient array of an arbitrary algebraic function F diagonally in the coefficient array of rational function $\tilde{F}$.

2. Apply the multivariate Cauchy integral formula.

3. Transfer the integral to the integral of a residue form over an intersection cycle in the pole variety.

4. Use Morse theoretic techniques to represent the intersection cycle as a sum of cycles local to critical points of the height function.

5. Evaluate each of these integrals asymptotically via stationary phase integration methods.

The remainder of this chapter explains what is involved in these steps, so that the complexity of this method can be compared to that of the new results presented in Chapter 4.

3.2. Embedding Theorem

Furstenberg [Fur67] showed that the diagonal of the power series of a rational function of two variables is an algebraic univariate function and the converse is also true. That is, for any univariate algebraic power series $F(z) = \sum_{i\geq 0} a_i z^i$, there is a bivariate rational power series $\tilde{F}(z, f) = \sum_{i,j\geq 0} b_{i,j} z^i f^j$ such that

$$\sum_{i\geq 0} a_i z^i = \sum_{i\geq 0} b_{i,i} z^i.$$

[Saf00] then generalized the result for the connection between a d-variate algebraic power series and a $(d+1)$-variate rational power series.

Before we introduce the generalized embedding theorem, let's define everything first. We consider a d-variate power series $F(\mathbf{z})$ such that

$$F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}} \tag{3.1}$$

where

$$\mathbf{r} = (r_1, r_2, \ldots, r_d) \text{ and } r_1 \geq 1, r_i \geq 0 \text{ for all } i \in \{2, \ldots, d\}.$$

In other words, we require that z_1 divides $F(\mathbf{z})$. The method only requires one of the variables in F to divide F. However, without loss of generality, we assume that it's z_1. We also assume that this power series $F(\mathbf{z})$ converges in an open neighborhood of the origin in $\mathbb{C}^d$ and this power series is algebraic. Let $\tilde{F}(\mathbf{z}, f)$ be a $(d+1)$-variate power series such that

$$\tilde{F}(\mathbf{z}, f) = \sum_{(r_0, \mathbf{r})} b_{r_0, \mathbf{r}} \mathbf{z}^{\mathbf{r}} f^{r_0} \tag{3.2}$$

where

$$(r_0, \mathbf{r}) = (r_0, r_1, r_2, \ldots, r_d) \text{ and } r_0 \geq 1, r_i \geq 0 \text{ for all } i \in \{1, \ldots, d\}.$$

Similarly, we assume that $\tilde{F}$ converges in an open neighborhood of the origin in $\mathbb{C}^{d+1}$. We define the i-th **elementary diagonal** of $\tilde{F}$ to be the d-variate power series

$$\sum_{(r_i, \mathbf{r})} b_{r_i, \mathbf{r}} \mathbf{z}^{\mathbf{r}} := \sum_{(r_i, r_1, \cdots, r_d)} b_{r_i, r_1, \cdots, r_d} z_1^{r_1} \cdots z_d^{r_d}.$$

From the assumption on $F(\mathbf{z})$, we know that z_1 divides F and thus $F(\mathbf{0}) = 0$. Let P be the minimal polynomial for the algebraic function F. It is then clear that $P(\mathbf{z}, f(\mathbf{z})) = 0$ and $P(\mathbf{0}, 0) = 0$. Let's denote the ring of convergent power series as $\mathbb{C}\{\mathbf{z}\}$. Sometimes, it's called the ring of analytic germs at the origin, or the ring of germs of holomorphic functions at the origin. In [Saf00], it is denoted as $\mathcal{O}$, as well as in many texts on several complex variables. We are now ready to state the embedding theorem.

Theorem 3.5 (Lemma 2, [Saf00])**.** *Let $F(\mathbf{z})$ be a d-variate algebraic power series in $\mathbb{C}\{\mathbf{z}\}$ in the form of (3.1). Suppose that there is a polynomial P defining F in the form*

$$P(\mathbf{z}, f) = (f - F(\mathbf{z}))^k u(\mathbf{z}, f)$$

in a neighborhood of zero, where k is an integer greater than 1, and $u(\mathbf{z}, f)$ is a unit in $\mathbb{C}\{\mathbf{z}, f\}$.

Then there exists a rational function $\tilde{F}(\mathbf{z}, f)$ in the form of (3.2) such that F is the first elementary diagonal of $\tilde{F}$, or more explicitly $b_{r_1,\mathbf{r}} = a_{\mathbf{r}}$.

PROOF: The proof is on [Saf00]. To make this paper more self-contained, we replicate the proof as follows with more details. Consider the auxiliary rational function

$$A(\mathbf{z}, f) = \frac{1}{k} \frac{f^2 P_f(\mathbf{z}, f)}{P(\mathbf{z}, f)}$$

where P_f is $\frac{\partial}{\partial f} P$. The candidate for $\tilde{F}$ is $\tilde{F}(\mathbf{z}, f) = A(z_1 f, z_2, \cdots, z_d, f)$. Clearly $\tilde{F}$ is rational because A is.

We first show that $\tilde{F}$ converges near the origin in $\mathbb{C}^{d+1}$. The denominator of $\tilde{F}(\mathbf{z}, f)$ is

$$kP(z_1 f, z_2, \cdots, z_d, f) = k(f - F(z_1 f, z_2, \cdots, z_d))^k u(z_1 f, z_2, \cdots, z_d)$$

because of the assumption on P. Since we also require $F(\mathbf{z})$ be in the form of (3.1), the first variable of $F(\mathbf{z})$ divides F. That is, $F(\mathbf{z}) = z_1 F_1(\mathbf{z})$ for some $F_1(\mathbf{z}) \in \mathbb{C}\{\mathbf{z}\}$. Then the above quantity is equal to $k(f - z_1 f F_1(z_1 f, z_2, \cdots, z_d))^k u = k f^k (1 - z_1 F_1)^k u$. We will now on omit the variables in functions like F, F_1 and u just for the sake of simplicity.

The numerator of $\tilde{F}(\mathbf{z}, f)$ is then

$$\begin{aligned}
&f^2 \left(k(f-F)^{k-1} u + (f-F)^k u_f \right) \\
&= f^2 \left(k(f - z_1 f F_1)^{k-1} u + (f - z_1 f F_1)^k u_f \right) \\
&= f^{k+1} \left(k(1 - z_1 F_1)^{k-1} u + f(1 - z_1 F_1)^k u_f \right)
\end{aligned}$$

The f^k in the denominator cancels out with f^k in the numerator. Since $u(\mathbf{z})$ is a unit in $\mathbb{C}\{\mathbf{z}\}$, it's not equal to zero at the origin and thus we can choose a small neighborhood around the origin where $u(z_1 f, z_2, \cdots, z_d, f)$ is not zero. On the other hand, since $F_1(\mathbf{z}) \in \mathbb{C}\{\mathbf{z}\}$, we know that $F_1(z_1 f, z_2, \cdots, z_d) \in \mathbb{C}\{\mathbf{z}, f\}$ because it is a composition of two analytic functions. Thus in a small neighborhood around the origin, we can let $1 - z_1 F_1(z_1 f, z_2, \cdots, z_d)$ not equal to zero. Therefore, at least in a small neighborhood of the origin, the denominator will not have any zeros.

As for the numerator, notice that $u_f \in \mathbb{C}\{\mathbf{z}, f\}$ because u is. Any other part in the numerator is just a composition, a product, or a sum of analytic functions, which continues to be analytic at the origin. Therefore, the numerator is holomorphic in a small neighborhood of the origin in $\mathbb{C}^{d+1}$. Therefore, at least in a small polydisk $\{|z_i| \leq \rho, |f| \leq \rho\}$ of $\mathbb{C}^{d+1}$, the function $\tilde{F}(\mathbf{z}, f)$ is analytic. Moreover, the extra factor f in the numerator indicates that f divides $\tilde{F}(\mathbf{z}, f)$ and thus in the form of (3.2).

Let $\tilde{F}(\mathbf{z}, f) = \sum_{(r_0,\mathbf{r})} b_{r_0,\mathbf{r}} \mathbf{z}^{\mathbf{r}} f^{r_0}$ be the power series expansion of $\tilde{F}$ in the polydisk $\{|z_i| \leq \rho, |f| \leq \rho\}$. Choose $0 < \epsilon < \rho$, then the function $\tilde{F}(z_1/f, z_2, \cdots, z_d, f)$ is analytic for $|f| = \epsilon$, $|z_1| \leq \min\{\rho\epsilon, \rho\}$ and $|z_i| \leq \rho$. By Cauchy's integral formula, one finds the elementary diagonal to be

$$\sum_{(r_1,\mathbf{r})} b_{r_1,\mathbf{r}} \mathbf{z}^{\mathbf{r}} = \frac{1}{2\pi i} \int_{|f|=\epsilon} \tilde{F}(z_1/f, z_2, \cdots, z_d, f) \, \frac{df}{f}. \tag{3.3}$$

On the other hand, the right hand side of (3.3) is

$$\frac{1}{2\pi i} \int_{|f|=\epsilon} A(z_1, z_2, \cdots, z_d, f) \, \frac{df}{f} = \frac{1}{2\pi i k} \int_{|f|=\epsilon} \frac{f P_f(\mathbf{z}, f)}{P(\mathbf{z}, f)} \, df.$$

A breakdown of the integrand gives you

$$\frac{f P_f(\mathbf{z}, f)}{P(\mathbf{z}, f)} = \frac{kf}{f - F} + \frac{f u_f}{u}$$

The second factor is holomorphic near zero and thus if chosen ϵ small enough, it will integrate to zero. The first factor is equal to $k + \frac{kF}{f-F}$. The integral of the scalar k along a closed path is zero. The integral of $\frac{kF}{f-F}$ along the circle $|f| = \epsilon$ is then equal to kF by the residue theorem, given that $F(\mathbf{z})$ is inside the circle. This can be done by shrinking the radius of the polydisk since $F(\mathbf{0}) = 0$.

Therefore, we see that

$$\frac{1}{2\pi ik}\int_{|f|=\epsilon}\frac{fP_f(\mathbf{z},f)}{P(\mathbf{z},f)}\,df = F(\mathbf{z})$$

for $\mathbf{z}$ in a small enough polydisk.

Then shrink the polydisk if needed to get the convergent power series of $F(\mathbf{z})$, and we have

$$\sum_{\mathbf{r}} a_{\mathbf{r}}\mathbf{z}^{\mathbf{r}} = F(\mathbf{z}) = \frac{1}{2\pi ik}\int_{|f|=\epsilon}\frac{fP_f(\mathbf{z},f)}{P(\mathbf{z},f)}\,df = \frac{1}{2\pi i}\int_{|f|=\epsilon}\tilde{F}(z_1/f,z_2,\cdots,z_d,f)\,\frac{df}{f} = \sum_{(r_1,\mathbf{r})} b_{r_1,\mathbf{r}}\mathbf{z}^{\mathbf{r}}$$

□

Let's review the condition that $P(\mathbf{z},f) = (f-F(\mathbf{z}))^k u(\mathbf{z},f)$ near the origin. Since u is a unit, it's not zero near the origin. Therefore, the assumption we put on P is that F is the only branch through the origin with multiplicity k. In other words, there is only one branch (with multiplicity k) $F(\mathbf{z})$ satisfying $P(\mathbf{z},F(\mathbf{z}))=0$ and $F(\mathbf{0})=0$. There are two things to notice. First of all, if $P(\mathbf{z},f)=(f-F(\mathbf{z}))^k u(\mathbf{z},f)$ defines $F(\mathbf{z})$, then $Q(\mathbf{z},f)=(f-F(\mathbf{z}))u(\mathbf{z},f)$ also defines $F(\mathbf{z})$. Indeed, since a minimal polynomial is always of lowest degree, we can just work with $P(\mathbf{z},f)$ in the form of $(f-F(\mathbf{z}))u(\mathbf{z},f)$. This observation yields the following corollary.

Corollary 3.6. *Let $F(\mathbf{z})$ be a d-variate algebraic power series such that z_i divides F in the form $\sum_{\mathbf{r}} a_{\mathbf{r}}\mathbf{z}^{\mathbf{r}}$. Suppose that there is a minimal polynomial $P(\mathbf{z},f)$ defining F and $\frac{\partial}{\partial f}P(\mathbf{0},0)\neq 0$, Then there exists a rational function $\tilde{F}(\mathbf{z},f)$ with power series $\sum_{(r_0,\mathbf{r})} b_{(r_0,\mathbf{r})}\mathbf{z}^{\mathbf{r}}f^{r_0}$ such that F is the i-th elementary diagonal of $\tilde{F}$, or more explicitly $b_{r_i,\mathbf{r}}=a_{\mathbf{r}}$. Explicitly,*

$$\tilde{F}(\mathbf{z},f) = \frac{f^2P_f(z_1,\cdots,z_if,\cdots,z_d,f)}{P(z_1,\cdots,z_if,\cdots,z_d,f)}$$

The second thing to notice is that when a minimal polynomial factors into

$$P(\mathbf{z},f) = (f-F(\mathbf{z}))(f-G(\mathbf{z}))u(\mathbf{z},f)$$

near the origin, there are two branches crossing the origin. Then Corollary 3.6 does not work any more. The problem is that in the proof of Theorem 3.5, the quantity $\frac{1}{2\pi i k}\int_{|f|=\epsilon} \frac{fP_f(\mathbf{z},f)}{P(\mathbf{z},f)}\,df$ will become the sum of these two branches F and G. In another words, the diagonal series of $\tilde{F}$ in (3.3) will be the sum of these two algebraic power series F and G. They get mixed up and so we cannot tell which is which. This issue is dealt with in [Saf00, Lemma 3], resolving multiple branches of $\{P = 0\}$ at the origin.

3.3. Pre-processing

The assumption of Corollary 3.6 requires the algebraic generating function $F(\mathbf{z})$ to satisfy that (C1) one of the variables z_i divides F and (C2) F is the only branch of its minimal polynomial P at the origin (i.e. $\frac{\partial}{\partial f}P(\mathbf{0}, 0) \neq 0$. The first example we want to try would be the Catalan generating function. Sadly, it does not satisfy the first condition. Indeed, Catalan generating function F has constant term $F(0) = 1$. [GMRW22] suggests three methods, **additive substitution**, **multiplicative substitution**, and **monomial substitution** to modify the original power series to satisfy the first condition (C1).

The additive substitution in the univariate algebraic series case is simply subtracting the constant term. For example, $F(z)-1$ satisfies (C1) and its minimal polynomial can be $P(z, f) = z(f+1)^2 - (f+1)+1$. Corollary 3.6 suggests an embedding of $F(z)-1$ into $\frac{f(1-2zf^2-2zf)}{1-(zf^2+2zf+z)}$. For general multivariate algebraic series, we need to subtract a finite number of initial terms F_0 from the power series $F(\mathbf{z})$ to possibly make $F - F_0$ satisfy (C1). A necessary condition for z_i dividing $F - F_0$ is that $F(z_1, \cdots, z_{i-1}, 0, z_{i+1}, \cdots, z_d)$ is a polynomial [GMRW22, Proposition 5]. Notice that $F - F_0$ satisfies (C2) if and only if F does. In other words, manipulating the original power series via the additive substitution will not affect (C2).

The multiplicative substitution is simply multiplying the original power series by z_i. For example, if $F(z)$ is the Catalan generating function, then $zF(z)$ satisfies (C1) and its minimal polynomial will be $P(z, f) = f^2 - f + z$. Again, Corollary 3.6 suggests an embedding of $zF(z)$ into $\frac{f(1-2zf^2-2zf)}{1-(zf^2+2zf+z)}$. The problem, however, is that the multiplicative substitution sometimes violates (C2), even if the original power series satisfies (C2). [GMRW22] gave an example of the ternary tree analog of the shifted Catalan generating function where $P(z, f) = f^3 - zf + z^2$ and $P_f(0, 0) = 0$.

The monomial substitution is used when $F(\mathbf{0}) = 0$ but neither of the variables divide F. This situation is not possible in the univariate case but can happen when we have two variables or more. In two variables, we use $(x, y) \to (x, xy)$ to make a transformed generating function $F'(x, y) = F(x, xy)$. Doing so will not change (C2) because $P_f(x, xy, f)$ has the same value as $P_f(x, y, f)$ at the origin. Here, $P(x, y, f)$ is a minimal polynomial of $F(x, y)$.

In practice, even if a power series already satisfy (C1) and (C2), we may still apply some transformation to make sure the embedding is **combinatorial**. That is, if we write $\tilde{F}(\mathbf{z}, f) = \dfrac{H(\mathbf{z}, f)}{G(\mathbf{z}, f)}$, then $\dfrac{1}{G}$ is a combinatorial series (i.e. all but a finite number of its power series coefficients are nonnegative). A combinatorial series makes it significantly easier to apply ACSV theory. The major reason is due to a multivariate version of Vivanti-Pringsheim theorem which gives a straightforward way to verify minimality [PWM24, Corollary 6.39]. In particular, if we are looking for power series expansion of $\tilde{F}(\mathbf{z}, f)$, then [PWM24, Lemma 6.41] tells us that every root of $G(\mathbf{z}, f)$ with positive coordinates is a minimal point and if $(\mathbf{z}, f)$ is a minimal point, then $(|z_1|, \cdots, |z_d|, |f|)$ is also a minimal point. In addition, if $G(\mathbf{z}, f) = 1 - g(\mathbf{z}, f)$ where g is combinatorial, **aperiodic** (i.e. the exponent of terms appearing in g generates $\mathbb{Z}^{d+1}$), and $g(\mathbf{0}, 0) = 0$, then the only minimal points are those roots of $G(\mathbf{z}, f)$ with positive coordinates.

It is not always guaranteed that the final embedding can be made combinatorial. In the noncombinatorial case, it will be hard to determine minimality. Sometimes, the contributing point may not even be minimal [PWM24, Example 7.3]. Without minimality on contributing points, one needs to use more involved Morse theory. A complete algorithm for finding all contributing critical points for bivariate power series can be found in [PWM24, Chapter 9.3.1]. For bivariate Laurent series, one needs to look at the signed intersection numbers [PWM24, Chapter 9.3.2]. We will not elaborate these techniques here but we do want to stress out the importance of having a combinatorial embedding because it will make things much easier.

Finally, one needs to pay special attention to the relation between indices in the original power series $F(\mathbf{z})$ and the embedding rational function $\tilde{F}'(\mathbf{z}, f)$ of the preprocessed power series $F'(\mathbf{z})$.

3.4. Examples

We replicate two examples of the embedding method given in [GMRW22] with added details. We assume that one is familiar with ACSV on rational series. The first is our running toy example, the Catalan series.

3.4.1. Catalan GF

Let $F(z)$ be Catalan GF. We use the additive substitution to embed $F(z) - 1$ diagonally into $\tilde{F}(z, f) = \dfrac{f(1 - 2zf^2 - 2zf)}{1 - (zf^2 + 2zf + z)}$. Notice that $[z^n]F(z) = [z^n f^n]\tilde{F}(z, f)$. Write $\tilde{F}$ in the form of $H(z, f)/G(z, f)$ where $H(z, f) = f(1 - 2zf^2 - 2zf)$ and $G(z, f) = 1 - (zf^2 + 2zf + z) = 1 - g(z, f)$. First of all, the variety defined by $\{G = 0\}$ is smooth everywhere because $\{G = 0, G_z = 0, G_f = 0\}$ has no solution. Moreover, notice that $g(z, f)$ is combinatorial, aperiodic, and $g(0, 0) = 0$. Therefore, by [PWM24, Lemma 6.41], the only minimal points are roots of $G(z, f)$ with positive coordinates.

Now let's look at the smooth critical point equations given by Proposition 1.14 in the direction $\hat{\mathbf{r}} = (1, 1)$, namely $\{G = 0, zG_z - fG_f = 0\}$. The solution is $z = 1/4, f = 1$. Since the point $\mathbf{w} = (1/4, 1)$ has positive coordinates, it is minimal, or in particular, strictly minimal because there is no other minimal critical points with the same modulus.

By [PWM24, Theorem 9.4] or [Mel21, Theorem 5.2], one can find an asymptotic expansion in the direction $\hat{\mathbf{r}} = (1, 1)$,

$$[z^n f^n]\tilde{F}(z, f) = \mathbf{w}^{-(n,n)} n^{-1/2} \frac{(2\pi)^{-1/2}}{\sqrt{1/2}} \sum_{\ell=0}^{\infty} C_\ell n^{-\ell}$$

where constants C_ℓ can be explicitly calculated from derivatives of $H(z, f)$ and $G(z, f)$ at $\mathbf{w}$ (see [PWM24, Corollary 5.17] for explicit formulas or [Hör83, Theorem 7.7.5]). In our case, $C_0 = 0$. The next constant $C_1 = 1$.

Therefore, we obtain an asymptotic expansion

$$[z^n]F(z) = [z^n f^n]\tilde{F}(z, f) = \frac{4^n}{\sqrt{\pi}} \left[n^{-3/2} + O(n^{-5/2}) \right]$$

whose leading coefficient agrees with the result one obtain via Stirling's formula.

Indeed, one can compute the asymptotics up to any precisions. For example, we use the code provided by [Mel21] and get

$$[z^n]F(z) = [z^n f^n]\tilde{F}(z,f) = \frac{4^n}{\sqrt{\pi}}\left[n^{-3/2} - \frac{9}{8}n^{-5/2} + \frac{145}{128}n^{-7/2} + O(n^{-9/2})\right]$$

3.4.2. Bi-colored Motzkin paths

We give an example on the generating function of bi-colored Motzkin paths in Chapter 4.4.3 using the lifting method. Here we apply the embedding method to its generating function

$$F(x,y) = \frac{1-x-y-\sqrt{(1-x-y)^2-4xy}}{2xy}.$$

Let $\hat{\mathbf{r}} = (\hat{r}, 1-\hat{r})$ and let $\mathbf{r} = (r,s) = N\hat{\mathbf{r}}$. We calculate the asymptotic formula for $a_{\mathbf{r}} = [x^{\hat{r}N}y^{(1-\hat{r})N}]F(x,y)$. The minimal polynomial $P(x,y,f)$ satisfying $P(x,y,F(x,y)) = 0$ is

$$P(x,y,f) = xyf^2 + (x+y-1)f + 1\,.$$

We apply the **multiplicative substitution** and **additive substitution** in Chapter 3.3 together, namely embedding $F(x,xy)-1$ into the first diagonal of a rational function $\tilde{F}$ by Corollary 3.6. In other words, $[x^{\hat{r}N}y^{(1-\hat{r})N}]F(x,y) = [x^N y^{(1-\hat{r})N}](F(x,xy)-1) = [x^N y^{(1-\hat{r})N} f^N]\tilde{F}(f,x,y)$.

The minimal polynomial for $F(x,xy)-1$ is $P(x,xy,f+1)$, that is,

$$P(x,xy,f+1) = (f+1)^2x^2y + (xy+x-1)(f+1) + 1.$$

By Corollary 3.6, we can embed $F(x,xy)$ into the first diagonal of $\tilde{F}$ defined as

$$\tilde{F}(x,y,f) = \frac{f(2x^2yf^3 + 2x^2yf^2 + xyf + xf - 1)}{x^2yf^3 + 2x^2yf^2 + x^2yf + xyf + xf + xy + x - 1}.$$

The singular variety of $\tilde{F}$ defined by its denominator is smooth. Therefore, we apply the smooth critical point equation in Proposition 1.14. There is only one critical point equation in direction $(1, 1-\hat{r}, 1)$, namely the point (x_0, y_0, f_0):

$$x_0 = \frac{(1-\hat{r})\hat{r}^3}{\hat{r}^2 - \hat{r} + 1}, \quad y_0 = \frac{(\hat{r}-1)^2}{\hat{r}^2}, \quad f_0 = \frac{\hat{r}^2 - \hat{r} + 1}{r(1-\hat{r})}.$$

Notice that the denominator of $\tilde{F}$ can be put in the form of $1 - A(\mathbf{z})$ where A is a polynomial with nonnegative coefficients. By [PWM24, Lemma 6.41], roots of $1 - A(\mathbf{z})$ with positive coordinates are minimal. Since $0 < \hat{r} < 1$, the point (x_0, y_0, f_0) has positive coordinates and thus minimal.

We can now apply [PWM24, Theorem 9.12] to say that

$$[x^N y^{(1-\hat{r})N} f^N]\tilde{F}(f, x, y) = x_0^{-N} y_0^{-N(1-\hat{r})} f_0^{-N} N^{-1} \frac{1}{2\pi\sqrt{\det \mathcal{H}}} \sum_{\ell=0}^{\infty} C_\ell(\hat{r}) N^{-\ell}.$$

where $\mathcal{H}$ is a $(d-1) \times (d-1)$ matrix defined in [PWM24, Lemma 8.22]. One can compute the matrix $\mathcal{H}$ as well as the coefficients C_ℓ up to any ℓ needed. We use the SageMath code provided by [Mel21] to compute C_0 and C_1 and get the following asymptotics.

$$\begin{aligned}
[x^{\hat{r}N} y^{(1-\hat{r})N}]F(x, y) &= [x^N y^{(1-\hat{r})N} f^N]\tilde{F}(f, x, y) \\
&= \left(\frac{1}{\hat{r}^{2\hat{r}}\ (1-\hat{r})^{2(1-\hat{r})}} \right)^N \left(\frac{1}{2\pi \hat{r}^2 (\hat{r}-1)^2} N^{-2} + O(N^{-3}) \right).
\end{aligned}$$

CHAPTER 4

ALGEBRAIC GENERATING FUNCTIONS VIA LIFTING

This chapter was previously published as [BJP24] Baryshnikov, Y., Jin, K. and Pemantle, R. Coefficient Asymptotics of Algebraic Multivariable Generating Functions. *La Matematica* **3**, 293–336 (2024). The article is reproduced here with permission from Springer Nature. The first section of the original published paper is moved to Chapter 3.1. The appendix of the original published paper is listed below as the final section 4.5.

This chapter gives an alternative method for algebraic generating functions that avoids some of the complexities of the embedding method. The primary reason for doing so is not to streamline the computation, although we do provide specialized formulas for stationary phase asymptotics that simplify the most common computations arising in coefficient extraction for algebraic series. The chief motive for developing the alternative **lifting method** is transparency. Its derivation relies only on stationary phase methods, and does not use residue forms or intersection cycles. It also avoids the use of a black-box embedding in Step 1 of the diagonal embedding approach. The lifting method is therefore considerably more accessible to the analytic combinatorics community.

Note: In this chapter, we consider the direction $\hat{\mathbf{r}} \in \mathbb{RP}^{d-1}$ and we write $\hat{\mathbf{r}} = [\hat{r}_1 : \cdots : \hat{r}_d]$ as an equivalence class. We use $\hat{\mathbf{r}}$ satisfying $|\hat{\mathbf{r}}| = 1$ as a canonical representative of the equivalence class $[\hat{\mathbf{r}}] \in \mathbb{RP}^{d-1}$.

4.1. Integral Representations

4.1.1. Notation

To specify an algebraic generating function, one requires a defining polynomial along with a choice of solution near the origin. Some global notation is as follows. Fix an integer $d \geq 1$. The coordinates of $\mathbb{C}^{d+1}$ will be denoted $z_1, \ldots, z_d, f$. The f-coordinate plays a different role from the others. Accordingly, we let $\pi : \mathbb{C}^{d+1} \to \mathbb{C}^d$ denote projection to the first d coordinates, and we make the roles of the z variables visually easier to distinguish from f by denoting $(z_1, \ldots, z_d)$ by $\mathbf{z}$ and

$(z_1, \ldots, z_d, f)$ by $(\mathbf{z}, f)$. Names of objects in $\mathbb{C}^{d+1}$ will typically have a tilde, while names of their projections to $\mathbb{C}^d$ will drop the tilde.

We assume henceforth that $P = \sum_{j=0}^{m} p_j(\mathbf{z}) f^j$ is a real polynomial function on $\mathbb{C}^{d+1}$, written as a polynomial in $\mathbb{R}[\mathbf{z}][f]$. Suppose there is a neighborhood $\mathcal{N}$ of the origin in $\mathbb{C}^d$ on which there is an absolutely convergent power series $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ satisfying $P(\mathbf{z}, F(\mathbf{z})) = 0$. Because there are at most m locally analytic solutions $f + f(\mathbf{z})$ to $P(\mathbf{z}, F) = 0$, one of these may be uniquely specified by naming P together with the values of F at a set of points of cardinality at most m. We assume throughout that $P(\mathbf{0}, \cdot)$ has a simple root at a real value f_0, and therefore that the two conditions $P(\mathbf{z}, F) = 0$ and $F(\mathbf{0}) = f_0$ specify a unique d-variable algebraic generating function F that is absolutely convergent in a neighborhood of the origin. Assume without loss of generality, that P is irreducible and square-free; for if it is not, then replacing P by its square-free part (the generator of the radical of the ideal generated by P) defines the same solutions, and some irreducible factor defines F.

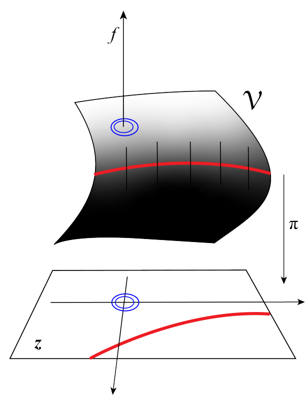


Figure 4.1: The variety $\widetilde{\mathcal{V}}$, projection π, tori T and $\tilde{T}$, branch locus br and vertical tangent locus $\widetilde{\mathsf{br}}$

Let $\widetilde{\mathcal{V}}$ denote the variety $\{(\mathbf{z}, f) : P(\mathbf{z}, f) = 0\}$ in $\mathbb{C}^{d+1}$. The map L defined by

$$L(\mathbf{z}) := (\mathbf{z}, F(\mathbf{z})) \tag{4.1}$$

on the domain of convergence E of F is inverted by π, that is, $\pi \circ L$ is the identity. It is smooth on some neighborhood of the origin because a power series is smooth on the interior of its domain of absolute convergence. Therefore, it is a diffeomorphism from such a neighborhood onto its lifting into $\widetilde{\mathcal{V}}$. Letting T be any torus within the domain of absolute convergence of the series for F, we denote by $\tilde{T}$ the lifting of T into $\widetilde{\mathcal{V}}$. Figure 4.1 illustrates these definitions (those in red will be defined later).

The usual methodology of stationary phase integration is to move the contour of integration, T, into a position where it passes through a stationary phase point where the gradient of the "large term" $\mathbf{z}^{-\mathbf{r}}$ vanishes. Univariate functions with branch points typically require a customized contour, for example one that hugs a slit (a segment or ray whose removal get rid of the branching) at a distance going to zero. A multivariate version of such a contour is not obvious. One case, namely a nonintegral power of a polynomial, was handled in [Gre18]. Our method is distinct from the branch contour method of [Gre18] and the diagonal method of [RW07] as well as the earlier quasi-power methods. Our approach is to transfer the integral "upstairs" to $\widetilde{\mathcal{V}}$, where everything becomes smooth.

4.1.2. Integration upstairs

Proposition 4.1. *The coefficients $a_{\mathbf{r}}$ in the Laurent expansion of F are given by*

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_{\tilde{T}} \mathbf{z}^{-\mathbf{r}} f \frac{d\mathbf{z}}{\mathbf{z}} \tag{4.2}$$

where $\mathbf{z}^{-\mathbf{r}}$ denotes $\prod_{j=1}^d z_j^{r_j}$ and $d\mathbf{z}/\mathbf{z}$ denotes the logarithmic volume form $dz_1 \wedge \cdots \wedge dz_d / \prod_{j=1}^d z_j$.

PROOF: Pulling back via π induces a map π^* on forms where $\pi^*(d\mathbf{z}/\mathbf{z})$ is still $d\mathbf{z}/\mathbf{z}$ in global coordinates and $\pi^* F|_T = F \circ \pi|_{\tilde{T}} = f$. Hence,

$$\pi^* F(\mathbf{z}) \mathbf{z}^{-\mathbf{r}} \frac{d\mathbf{z}}{\mathbf{z}} = \mathbf{z}^{-\mathbf{r}} f \, \frac{d\mathbf{z}}{\mathbf{z}} \, .$$

Functoriality then implies that the RHS of (4.2) is equal to the RHS of (1.1), proving the proposition. □

Whereas F may be defined only in a small domain, not extendable around branchpoints or through poles, the form on the RHS of (4.2), which we denote $\eta := \mathbf{z}^{-\mathbf{r}} f d\mathbf{z}/\mathbf{z}$, is well defined and holomorphic on all of $\mathbb{C}_*^{d+1}$. Here, $\mathbb{C}_* := \mathbb{C} \setminus \{0\}$ denotes the set of nonzero complex numbers[2]. We cannot deform $\tilde{T}$ freely through $\mathbb{C}^{d+1}$ without altering the integral, but we can deform it within $\widetilde{\mathcal{V}}$, or any other complex d-manifold in which it happens to reside. More formally, if $\gamma_1 : \mathcal{C} \to \widetilde{\mathcal{V}}$ and $\gamma_2 : \mathcal{C} \to \widetilde{\mathcal{V}}$ are chains over the same cell complex $\mathcal{C}$, we say that γ_1 and γ_2 are **homotopic within** $\widetilde{\mathcal{V}}$ if there is a continuous map $H : \mathcal{C} \times [0,1] \to \widetilde{\mathcal{V}}$ with $H(\cdot, 0) = \gamma_1$ and $H(\cdot, 1) = \gamma_2$. The map H is said to be a **homotopy** from γ_1 to γ_2. The homotopy H is a map from the cell complex $\mathcal{C} \times [0,1]$ to the space $\widetilde{\mathcal{V}}$, so it is a $(d+1)$-chain and we can integrate $(d+1)$-forms over it. Because the torus $\tilde{T}$ has no boundary, the boundary of the chain H is given by $\partial H = \tilde{T} \times \partial[0,1] = \tilde{T}' - \tilde{T}$.

Proposition 4.2. *If $\tilde{T}$ and $\tilde{T}'$ are homotopic within $\widetilde{\mathcal{V}}$ then $\int_{\tilde{T}} \eta = \int_{\tilde{T}'} \eta$.*

Proof: Let $H : \tilde{T} \times [0,1] \to \widetilde{\mathcal{V}}$ be a homotopy from $\tilde{T}$ to $\tilde{T}'$ in $\widetilde{\mathcal{V}}$. From Stokes' Theorem (see for example [Lee03, Theorem 14.20]) we see that

$$\int_H d\eta = \int_{\partial H} \eta = \int_{\tilde{T}'} \eta - \int_{\tilde{T}} \eta \,.$$

The integral of the holomorphic $(d+1)$-form $d\eta$ vanishes over any chain supported in a complex d-manifold. Because H is a homotopy within $\widetilde{\mathcal{V}}$, the integral of the left-hand side vanishes, proving the proposition. □

4.1.3. Stationary phase integration

The integrand in (4.2) is easy to evaluate asymptotically when it has the form of a **stationary phase** integral. It is a little easier to see why (4.2) is a stationary phase integral if we let $N := |\mathbf{r}| = \sum_{j=1}^d r_j$, let $\hat{\mathbf{r}} := N^{-1}\mathbf{r}$, and write the integrand as

$$I(N) = I(N; A, \phi) := A(\mathbf{z}) \exp(-N\phi(\mathbf{z}))\, d\mathbf{z} \tag{4.3}$$

where $A(\mathbf{z}) := f / \prod_{j=1}^d z_j$ and $\phi(\mathbf{z}) := \sum r_j \log z_j$. In an integral of such a form, the term $A(\mathbf{z})\, d\mathbf{z}$ is called the **amplitude** and ϕ is called the **phase**. We have used N rather than the traditional

[2] We use a subscript of $*$ instead of the more conventional superscript in order to avoid a double superscript.

λ for the parameter that goes to infinity to remind us that $N\hat{\mathbf{r}}$ is always an integer vector so $\exp(-N\phi(\mathbf{z})) = \mathbf{z}^{-\mathbf{r}}$ and the branching of the logarithm does not matter. The following defines critical points at smooth points of algebraic varieties. There is a more general definition of stratified critical points that need not concern us here.

Define the **logarithmic gradient** of an analytic function g to be the vector whose coordinates are the partial derivatives in logarithmic coordinates:

$$\nabla_{\log} g := \left(z_j \frac{\partial g}{\partial z_j} \right)_{1 \le j \le d} . \tag{4.4}$$

Definition 4.3 (critical points and directions)**.**

(i) A ***(smooth) critical point*** *for a function ϕ on an algebraic variety $\mathcal{M}$ is a point $\mathbf{p}$ which is a smooth point of $\mathcal{M}$ and satisfies $d\phi|_{\mathcal{M}}(\mathbf{p}) = 0$.*

(ii) A ***critical point in direction*** $\hat{\mathbf{r}}$ *on an algebraic variety $\mathcal{M}$ is a smooth critical point for the function $\phi_{\hat{\mathbf{r}}}$ where $\phi_{\hat{\mathbf{r}}}(\mathbf{z}) := \sum_{j=1}^{d} \hat{r}_j \log z_j$.*

Smooth critical points on the surface $\{P = 0\}$ in direction $[\mathbf{r}]$ satisfy $\nabla_{\log} P(\mathbf{z}) = [\mathbf{r} : 0]$, projectively. This may be captured by the $d + 1$ **critical point equations** (see [PWM24, Equation (7.8)] or [Mel21, Page 11]):

$$\begin{aligned} P(\mathbf{z}) &= 0 \\ \frac{\partial P}{\partial f}(\mathbf{z}) &= 0 \\ r_j z_1 \frac{\partial P}{\partial z_1}(\mathbf{z}) - r_1 z_j \frac{\partial P}{\partial z_j}(\mathbf{z}) &= 0 \qquad 2 \le j \le d \, . \end{aligned} \tag{4.5}$$

Generically, this defines a finite set and is easily computed by a computer algebra system. When $\widetilde{\mathcal{V}}$ is smooth, which will usually be the case, these equations precisely define the set of critical points[3].

[3]Where $\widetilde{\mathcal{V}}$ is not smooth, i.e., where ∇P vanishes, the last $d - 1$ equations (4.5) are trivially satisfied and one requires further equations for criticality; this will not concern us, as we assume smoothness.

Definition 4.4 (stationary phase)**.** *Suppose a contour Γ contains finitely many critical points for ϕ. Let* `contrib` *denote the subset of these at which the real part $\Re\{\phi\}$ achieves its minimum on Γ. Points of* `contrib` *are called* ***stationary phase points*** *for ϕ on Γ, and if* `contrib` *is nonempty, Γ is said to be in* ***stationary phase position****.*

We remark that being a critical point on Γ is in principle a weaker condition than being a critical point on a variety $\mathcal{M}$ that Γ lies in, however it is the same wherever the real tangent space to Γ has the same span over $\mathbb{C}$ as the tangent space to $\mathcal{M}$; this will always be the case for our contours. Existence of stationary phase points is what makes an integral of the form $\int_\Gamma Ae^{-N\phi}$ easy to evaluate asymptotically. The precise nature of Γ is not relevant, only the orientation of Γ, along with the fact that $\mathbf{p}$ is a critical point at which $\Re\{\phi\}$ is minimized on Γ. Off-the-shelf stationary phase computations at this level of generality can be found in [PWM24, Lemma 5.15], [MW19, Chapter 7.1], or [PV19, Theorem 4.2]. We find it useful to state coordinate-free hypotheses when possible, while giving the resulting formulae in coordinates. For example in Proposition 4.6, the input data are a phase function ϕ on a complex d-manifold and a holomorphic d-form η for the amplitude, while the formula for the integral uses a coordinate representation $A(\mathbf{z})d\mathbf{z}$ for η.

All the examples in this paper have an expansion (1.3) in which all terms with $\ell = 0$ vanish. We therefore find it convenient to state an explicit formula for the leading term C_1, in the special case that the amplitude $\eta = \mathbf{z}^{-\mathbf{r}} f \, d\mathbf{z}/\mathbf{z}$ vanishes to order precisely 1 at the stationary phase point. We base our formulae on some useful reductions for this case that can be found in the Appendix to [MW19].

Definition 4.5. *The notation $\sqrt{\det M}$ denotes the product of the principal square roots of the eigenvalues of the matrix M. The notation $(\det M)^{1/2}$ leaves open which choice of square root is intended.*

We begin with the case where `contrib` is a singleton $\{\mathbf{p}\}$. It is well known that the leading term of a stationary phase integral is inversely proportional to a curvature invariant at $\mathbf{p}$, which is given in coordinates by the determinant of the Hessian matrix of the phase function. These formulae make more sense when one takes into account the way such a determinant transforms under changes of variable. If the Jacobian is J then at a point where the gradient vanishes, the Hessian matrix H transforms to $J^T H J$; as the amplitude $A(\mathbf{z})d\mathbf{z}$ transforms to $\det(J)A(\mathbf{z}')d\mathbf{z}'$, this means that

$$\frac{A}{\sqrt{\det(H)}} \text{ is independent of the choice of coordinates.} \tag{4.6}$$

Proposition 4.6 (stationary phase formula and case where amplitude vanishes to order 1)**.** *Let η be a holomorphic d-form on a complex d-manifold $\mathcal{M}$ and let ϕ be a holomorphic function on $\mathcal{M}$. Let $\mathbf{p}$ be a point of $\mathcal{M}$ at which $d\phi$ vanishes. Fix a coordinate system $z_1, \ldots, z_d$ on a neighborhood of $\mathbf{p}$ in $\mathcal{M}$ and suppose*

(i) the form η is represented by $A(\mathbf{z})d\mathbf{z}$;

(ii) the function ϕ has a nondegenerate Hessian matrix H at $\mathbf{p}$, which condition is invariant under coordinate changes[4] *by (4.6).*

Define quantities

- *$g(\mathbf{z}) := \phi(\mathbf{z}) - \frac{1}{2}(\mathbf{z}-\mathbf{p})^T H(\mathbf{z}-\mathbf{p})$, in other words, ϕ with its leading (quadratic) term subtracted off;*
- *a second order differential operator $\mathbb{H} := \sum_{i,j=1}^{d} -(H^{-1})_{ij} \frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j}$.*

Let $\mathcal{N}$ be a neighborhood of the origin in $\mathbb{R}^d$ and let $\Gamma : (\mathcal{N}, \mathbf{0}) \to (\mathcal{M}, \mathbf{p})$ be a compact, smooth, real d-chain supported on a set $|\Gamma|$ on which $\Re\{\phi\}$ is uniquely minimized at an interior point, $\mathbf{p}$. Then the integral

$$I(N) := \int_\Gamma \eta \exp(-N\phi) \tag{4.7}$$

[4]Technically, a coordinate system is a map $\Psi(\mathcal{M}, \mathbf{p}) \to (\mathbb{C}^d, \mathbf{0})$ and we should refer to the Hessian matrix of $\phi \circ \Psi^{-1}$ at $\mathbf{0}$, however continue to use "$\phi(\mathbf{p})$ in coordinates", "$A(\mathbf{p})$ in coordinates" and so forth instead of $\phi \circ \Psi^{-1}(\mathbf{0})$, $(\Psi^{-1})^* A(\mathbf{0})$ and so forth because most readers can more easily read the former.

has an asymptotic series expansion

$$I(N) \approx e^{-N\phi(\mathbf{p})} \left(\frac{2\pi}{N}\right)^{d/2} \sum_{\ell=0}^{\infty} C_\ell N^{-\ell} \tag{4.8}$$

for some constants C_ℓ that can be computed from the partial derivatives of ϕ and A at $\mathbf{p}$. Specifically,

$$C_\ell = (\det H)^{-1/2} \sum_{j=0}^{2\ell} (-1)^\ell \frac{\mathbb{H}^{\ell+j}(A \cdot g^j)}{2^{\ell+j} j!(\ell+j)!}(\mathbf{p}) \,. \tag{4.9}$$

If $A(\mathbf{p}) = 0$ and $dA(\mathbf{p}) \neq \mathbf{0}$, then $C_0 = 0$ and the leading term is given by

$$C_1 = -\frac{1}{2(\det H)^{1/2}} \left[\mathbb{H}(A)(\mathbf{p}) + \frac{1}{4}\mathbb{H}^2(A \cdot g)(\mathbf{p})\right] . \tag{4.10}$$

The square root in (4.10) *should be chosen as follows. In the coordinate system that represents $\eta = A\,d\mathbf{z}$, the chain $\Gamma : \mathbb{R}^d \to \mathcal{M}$ pulls back to a chain $\gamma : \mathbb{R}^d \to \mathbb{C}^d$; let $J = d\mathbf{z}/d\mathbf{t}$ denote the Jacobian matrix of γ; choose the $\frac{1}{2}$ power in* (4.10) *to be* $\det J/\sqrt{\det J^T H J}$, *which is a choice for* $\det H^{1/2}$ *by* (4.6).

Proof: Lemma A.3 and Theorem 5.5 of [MW19] compute a general asymptotic series expansion for power series coefficients, first writing this as $(2\pi i)^{-d} \int \tilde{u}(\mathbf{z}) \exp(\tilde{g}(\mathbf{z}))d\mathbf{z}$ and then evaluating this integral asymptotically. The form of their series is as given in (4.8) – (4.9).

When $A(\mathbf{p}) = 0$, C_0 vanishes. When $dA(\mathbf{p}) \neq 0$, C_1 does not vanish. Plugging in ϕ for $\tilde{g}$ and A for $\tilde{u}$, their result is that our $I(N)$ is asymptotic to

$$L_1(A,\phi)\; N^{-1} \left(\frac{2\pi}{N}\right)^{d/2} (\det H)^{-1/2}$$

where $L_1(A,\phi)$ is given by several formulas, of which we use the second displayed equation in their Lemma A.3 due to our assumption that A vanishes to order precisely 1. Equation (4.10) is then a direct statement of their result.

To check the choice of sign, first note that $\det J / \sqrt{\det J^T H J}$ is indeed a square root of $\det H$. Pulling back the integral $I(N) = \int_\Gamma \eta \exp(-N\phi)$ to $\int_\gamma A(\mathbf{z}(\mathbf{t}))J \, d\mathbf{t} \, \exp(-N\phi \circ \Gamma^{-1})$ and applying formula (5.4) of [PWM24] shows that this sign choice evaluates the integral (see also [PWM24, Equation (5.6)]). $\square$

Chapter 4.4 computes a number of examples, mostly taken from [GMRW22], where the majority have $d = 2$ (bivariate algebraic generating functions). It is helpful to pre-compute (4.10) for $d = 1, 2$: not only does this show the degree of simplification but it gives users an off-the-shelf formula that does not require them to program a differential operator in their computer algebra platform. Chapter 4.4.2 contains some further symbolic algebra techniques for obtaining simplified representations for algebraic quantities such C_1. When $d = 1$, the Hessian matrix H reduces to the scalar quantity $V := \phi''(\mathbf{p})$, while the operator $\mathbb{H}$ is V^{-1} times the second derivative operator. This leads to a rather compact formula. While the explicit formula for $d = 2$ is somewhat messier, we will see in Chapter 4.4.2 that the formula can simplify drastically when some of the partial derivatives vanish.

Corollary 4.7. *When $d = 1$, the formula* (4.10) *reduces to the following expresion.*

$$C_1 = -\frac{1}{2} \cdot V^{-1/2} \left[-\frac{1}{V} A''(\mathbf{p}) + \frac{1}{V^2} \phi'''(\mathbf{p}) A'(\mathbf{p}) \right] \tag{4.11}$$

When $d = 2$, the formula (4.10) *reduces to the following expression, where again H denotes $\phi_{xx}\phi_{yy} - \phi_{xy}^2$ and all partial derivatives are evaluated at $\mathbf{p} = (p_1, p_2)$.*

$$\begin{aligned} C_1 \quad = \quad & -\frac{1}{2} H^{-1/2} \ \times \\ & \Big[H^{-1} \Big(-A_{xx}\phi_{yy} + 2A_{xy}\phi_{xy} - A_{yy}\phi_{xx} \Big) \\ & + H^{-2} \Big(A_x \phi_{yy}^2 \phi_{xxx} - A_y \phi_{xxx} \phi_{xy} \phi_{yy} \\ & - 3A_x \phi_{xxy} \phi_{xy} \phi_{yy} + (A_x \phi_{xyy} + A_y \phi_{xxy})(\phi_{xx}\phi_{yy} + 2\phi_{xy}^2) \\ & - \ 3A_y \phi_{xyy} \phi_{xy} \phi_{xx} - A_x \phi_{yyy} \phi_{xy} \phi_{xx} + A_y \phi_{xx}^2 \phi_{yyy} \Big) \Big] \end{aligned} \tag{4.12}$$

The square root in both cases is chosen as in Proposition 4.6.

Proposition 4.6 and Corollary 4.7 extend easily to allow `contrib` to be a finite set of cardinality greater than 1. The following generalization can be found in [PWM24, Theorem 5.3; see also Theorem 9.25].

Corollary 4.8. *Proposition 4.6 and Corollary 4.7 continue to hold if the hypothesis of a single critical point* $\mathbf{p}$ *at which* $\Re\{\phi\}$ *is uniquely minimized is replaced by the hypothesis that there are finitely many critical points* $\mathbf{p}^{(1)}, \ldots, \mathbf{p}^{(m)}$ *of* ϕ *on* Γ *whose common value of* $\Re\{\phi\}$ *attains the minimum on* Γ*, and the conclusion* (4.8) *is altered replacing the right-hand side by a sum of the same quantity with* $\mathbf{p}$ *replaced by* $\mathbf{p}^{(j)}$*, for* $j = 1, \ldots, m$*.* □

4.1.4. The lifting method

Putting together Propositions 4.1, 4.2 and the computation in Proposition 4.6 we obtain the following plan for computing coefficient asymptotics in a direction $[\mathbf{r}]$ for the series F.

(i) Compute the set of critical points on $\widetilde{\mathcal{V}}$.

(ii) Deform $\tilde{T}$ to a contour $\tilde{T}'$ in stationary phase position, so that there is a nonempty finite set `contrib` of stationary phase points for ϕ on $\tilde{T}'$ at which $\Re\{\phi\}$ takes the value $\inf_{\mathbf{z}\in\tilde{T}'} \Re\{\phi(\mathbf{z})\}$.

(iii) Use standard stationary phase estimates to asymptotically evaluate $\int_{\tilde{T}'} A(\mathbf{z}) \exp(-N\phi(\mathbf{z}))\, d\mathbf{z}$.

The first and last of these steps require viturally no work, the first being accomplished by (4.5) and Proposition 4.18 and the last by Proposition 4.6 and Corollary 4.8. Already, with these two steps accomplished, one can get a pretty good idea of the asymptotics of $a_{\mathbf{r}}$. Each critical point $\mathbf{p}$ leads to an asymptotic series of the form (4.8); according to Corollary 4.8, summing these over the correct set `contrib` of critical points for ϕ on $\widetilde{\mathcal{V}}$, with the correct orientations, will give an asymptotic series for $a_{\mathbf{r}}$, provided the torus $\tilde{T}$ may be deformed into stationary phase position.

Often it is easy to determine by inspection which among finitely many choices for `contrib` yield the correct asymptotics. There are also cases where the homotopy mapping taking $\tilde{T}$ to an appropriate chain $\tilde{T}'$ is geometrically obvious. For the remainder of the cases, we develop a number of sufficient conditions for determining `contrib`. While rigorous numerical homotopy procedures exist that may be used to investigate deformations of $\tilde{T}$ into stationary phase position, these are cumbersome and few researchers possess or are familiar with the software. Lemma 4.23 gives a method to compute `contrib` without resorting to numerical homotopies. The remainder of this section contains a prelimiary discussion of the properties of the covering map π, along with various assumptions that hold in many applications and greatly simplify the identification of `contrib`.

One way to deform $\tilde{T}$ into stationary phase position is to enlarge the first d coordinates so as to remain a torus at every fixed time, while varying f so that $(\mathbf{z}, f)$ remains in $\widetilde{\mathcal{V}}$. By the nature of ϕ, this ensures that $\Re\{\phi\}$ is constant on the resulting torus, which means that at the end of the homotopy, the final contour $\tilde{T}'$ will be in stationary phase position, provided the set of critical points on $\tilde{T}'$ is finite and nonempty. This type of deformation can be described as a homotopy of T, lifted via π^{-1} to $\widetilde{\mathcal{V}}$. It is easier keep track of deformations of T in $\mathbb{C}^d$ than deformations of $\tilde{T}$ in $\widetilde{\mathcal{V}}$. Therefore, we have two reasons to pursue deformations that can be described as liftings via π^{-1} of homotopies in $\mathbb{C}^d$.

Definition 4.9. *The* ***discriminant*** $\mathrm{discr}(P) = \mathrm{discr}(P, f)$ *of P with respect to f is the minimal polynomial in $\mathbb{C}[\mathbf{z}]$ that vanishes if and only if $P(\mathbf{z}, \cdot)$ does not have distinct roots*[5]*. Let $\mathtt{br} \subseteq \mathbb{C}^d$ denote the branching locus, that is, the algebraic hypersurface defined by* $\mathrm{discr}(P)$*. The* ***vertical tangent locus*** $\widetilde{\mathtt{br}}$ *is the set of points $(\mathbf{z}, f) \in \widetilde{\mathcal{V}}$ such that $\partial P/\partial f = 0$. Under the assumption that $\widetilde{\mathcal{V}}$ is smooth, this is precisely the set of points of $\widetilde{\mathcal{V}}$ whose tangent space is vertical. A lower star, such as $\mathtt{br}_*, \widetilde{\mathtt{br}}_*, \widetilde{\mathcal{V}}_*$, refers respectively to not allowing a zero in the first d coordinates.*

Proposition 4.10. *The image of $\widetilde{\mathtt{br}}$ under π is $\mathtt{br}$. The polynomial* $\mathrm{discr}(P)$ *may be computed by eliminating f from the ideal generated by P and $\partial P/\partial f$.*

PROOF: The first statement, namely that vertical tangents occur precisely where roots coalesce, is well known. The second follows from the fact that projection to the $\mathbf{z}$-plane corresponds to eliminating f. $\square$

[5] One needs to be careful how one categorizes those $\mathbf{z}$ for which one of the roots goes to infinity, but that won't be relevant for us.

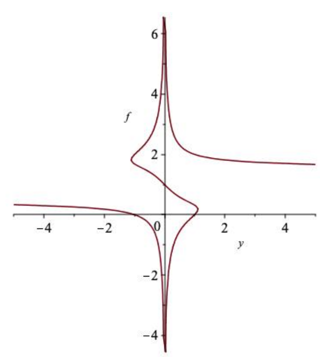
4.2. Main Results

Theorems 4.14 and 4.16 as well as Corollary 4.15 will assume the following properties of P and F:

- P is a smooth real polynomial on $\mathbb{C}^{d+1}$ whose zero set is denoted $\widetilde{\mathcal{V}}$;
- All roots of $P(\mathbf{0}, \cdot)$ are simple and $\mathrm{discr}(P)$ is squarefree;
- F is an algebraic power series in a neighborhood of the origin in $\mathbb{C}^d$ defined by the equation $P(\mathbf{z}, F(\mathbf{z})) = 0$ and the initial condition $F(\mathbf{0}) = f_0$;

Definition 4.11 (rank)**.** *Fix* $\mathbf{z} \in \mathbb{R}^d_+$ *and let* $\xi_1 \leq \cdots \leq \xi_k$ *denote the finite real roots of* $P(\mathbf{z}, \cdot)$ *listed, with repetition for multiple roots, in increasing order. By convention we consider* P *to have* ℓ *real roots at* $-\infty$ *if for all* $M > 0$*, all vectors* $\mathbf{v}$ *of positive real numbers, and sufficiently small* $\varepsilon = \varepsilon(M) > 0$*, the univariate polynomial* $P(\mathbf{z} + \varepsilon\mathbf{v}, \cdot)$ *has* ℓ *negative real roots less than* $-M$*. If* $f = \xi_j$ *is a simple root, the* ***rank*** *of* f*, denoted* $\mathrm{rk}(f; \mathbf{z})$ *is defined to be* $\ell + j$*. When* $\mathbf{z} = \mathbf{0}$ *we omit it from the notation, thus,* $\mathrm{rk}(f) := \mathrm{rk}(f; \mathbf{0})$*.*

Example. *Let* $P(y, z, f) = 1 - f + z[((f-1)^2 - 1) + y(f-1)^5]$*. The unique algebraic function* F *solving* $P(y, z, F(y, z)) = 0$ *in a neighborhood of the origin in the* yz*-plane has value 1 at the origin. In Chapter 4.4 we show that* $\mathrm{rk}(1) = 2$ *be establishing that* $P(\varepsilon, \varepsilon, \cdot)$ *has precisely three real roots for small positive* ε*, with one going to* ∞ *and one to* $-\infty$ *as* $\varepsilon \downarrow 0$*. The sketch to the right shows the real zero set of* $P(y, y, f)$*.*



Definition 4.12 (minimal point)**.** *Let* T *be the centered torus containing a point* $\mathbf{z} \in \mathbb{C}^d_*$*. Say that* $\mathbf{z}$ *is* ***weakly minimal*** *for a polynomial* $G : \mathbb{C}^d \to \mathbb{C}$ *if* G *is nonvanishing on the torus* $t \cdot T$ *for all* $0 < t < 1$*. Weak minimality is implied by the usual notion of* **minimality** *in ACSV [Mel21, Definition 3.9], namely that* $G(\mathbf{w}) \neq 0$ *for every* $\mathbf{w}$ *satisfying* $|w_j| \leq |z_j|$ *for all* j*, with at least one of the inequalties being strict.*

For the next definition, all we need is a real polynomial P, a real number f_0 such that $P(\mathbf{0}, f_0) = 0$, and the assumption that all roots of $P(\mathbf{0}, \cdot)$ are simple.

Definition 4.13. *Let $(\mathbf{z}, f_1)$ be a point of $\widetilde{\mathcal{V}}$ and let T' denote the torus through $\mathbf{z}$. Say that $(\mathbf{z}, f_1)$ is* ***on the branch defined by*** *f_0 if the map $H : T' \times [0, 1] \to \mathbb{C}^d$ by $H(\mathbf{w}, t) = t\mathbf{w}$ lifts to a continuous map $\tilde{H} : T' \times [0, 1] \to \widetilde{\mathcal{V}}$ with $\tilde{H}(\mathbf{w}, 0) = f_0$ for every $\mathbf{w} \in T'$ and $\tilde{H}(\mathbf{z}, 1) = f_1$.*

The following is our first main result and will be proved in Chapter 4.3. Recall that exp denotes coordinatewise exponentiation on complex vectors and that p_m denotes the leading coefficient of P as a polynomial in f.

Theorem 4.14. *Let $\mathbf{z} := \exp(\mathbf{x}) \in \mathtt{br}$ be a positive real zero of $\operatorname{discr}(P)$. Assume $\mathbf{z}$ is a smooth, critical point for $\operatorname{discr}(P)$ in direction $[\mathbf{r}]$, with $p_m(\mathbf{z}) \neq 0$ and $\mathbf{z}$ weakly minimal for $p_m \cdot \operatorname{discr}(P)$. Let $(\mathbf{z}, f_1) \in \widetilde{\mathtt{br}}$ be a real point of the vertical tangent locus.*

Conclusion 1:

> *The point $(\mathbf{z}, f_1)$ is critical for P in direction $[\mathbf{r} : 0]$. It is on the branch defined by f_0 if and only if $f_1 = \xi_{\mathrm{rk}(f_0)}$ where $\xi_1 \geq \xi_2 \geq \ldots$ enumerates the real roots of $P(\mathbf{z}, \cdot)$ in decreasing order with multiplicities.*

Assuming that $(\mathbf{z}, f_1)$ is on the branch defined by f_0, suppose the set of $\mathbf{y}$ such that there is a complex number $f_\mathbf{y}$ with $(\exp(\mathbf{x} + i\mathbf{y}), f_\mathbf{y})$ is on the branch of $\widetilde{\mathcal{V}}$ defined by f_0 and critical in direction $[\mathbf{r}]$ is finite and denote this set by W. Finally, assume for each $\mathbf{y} \in W$, the root $f_\mathbf{y}$ of $P(\exp(\mathbf{x} + i\mathbf{y}), \cdot)$ has multiplicity precisely 2 and that the Hessian matrix of $\phi_\mathbf{r} := \sum_{j=1}^d r_j \log z_j$ restricted to $\widetilde{\mathcal{V}}$ is nonsingular.

Conclusion 2: *There is an asymptotic expansion*

$$a_\mathbf{r} \approx \exp(-\mathbf{r} \cdot \mathbf{x}) \sum_{\ell=1}^{\infty} \sum_{\mathbf{y} \in W} C_{\mathbf{y},\ell} \exp(-i\mathbf{r} \cdot \mathbf{y}) |\mathbf{r}|^{-d/2-\ell} \tag{4.13}$$

where the constants $C_{\mathbf{y},\ell}$ are the constants C_ℓ determined in Proposition 4.6 with $\mathbf{p} = \exp(\mathbf{x} + i\mathbf{y})$, as well as $\phi(\mathbf{z}) = \hat{\mathbf{r}} \cdot \log \mathbf{z}$ and $\eta = f dz_1 \cdots dz_d / \prod_{j=1}^d z_j$. The expansion of (4.13) will be nonzero for some $\ell \geq 1$ and uniform as $[\mathbf{r}]$ varies over compact neighborhoods where the hypotheses hold.

We amplify on the most common form of the final formula, which occurs in the case that $|W| = 1$. This corresponds, more or less, to the aperiodic case.

Corollary 4.15 (computational form)**.** *Suppose that W in Theorem 4.14 contains exactly one point $\mathbf{y}$ and denote $\mathbf{p} := \exp(\mathbf{x} + i\mathbf{y})$. Let $\eta := A(\mathbf{z})d\mathbf{z} := f d\mathbf{z} / \prod_{j=1}^d z_j$ and let $\phi(\mathbf{z}) := \sum_j \hat{r}_j \log z_j$. Fix $k \le d$ with $\partial P / \partial z_k$ nonvanishing at $(\mathbf{p}, f_1)$. Then*

$$a_{\mathbf{r}} \sim C|\mathbf{r}|^{-d/2-1}\mathbf{p}^{-\mathbf{r}} \tag{4.14}$$

where C is the constant $C_{\mathbf{0},1}$ from (4.13) and is determined as follows.

Using the analytic implicit function theorem, reparametrize $\widetilde{\mathcal{V}}$ near $\mathbf{p}$ by $\{z_j : j \neq k\}$ and f. Let H denote the Hessian of ϕ in the new coordinates, let $\tilde{\phi}$ represent ϕ in the new coordinates with the quadratic term subtracted off, and let

$$\tilde{A}\, dV_k := A \frac{dz_k}{df}\, dV_k = A \frac{\partial P/\partial f}{\partial P/\partial z_k}\, dV_k \tag{4.15}$$

denote the form η in the new coordinates, where $dV_k := df \wedge dz_1 \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_d$ and the hat denotes absence of a dz_k term. Let $\mathbb{H}$ denote the second order differential operator $\mathbb{H} := \sum_{i,j=1}^{d} -(H^{-1})_{ij} \frac{\partial^2}{\partial z_i' \partial z_j'}$ where z_j' denotes z_j if $j \neq k$ and f if $j = k$. Then

$$C = \left(\frac{1}{2\pi i}\right)^d (2\pi)^{d/2} C_1$$

with C_1 given by (4.10) with $\tilde{A}$ in place of A:

$$C_1 = -\frac{1}{2}(\det H)^{-1/2}\left[\mathbb{H}(\tilde{A})(\mathbf{p}) + \frac{1}{4}\mathbb{H}^2(\tilde{A}\cdot g)(\mathbf{p})\right]. \tag{4.10}$$

The $\frac{1}{2}$ power should be taken as $i^d / \prod_{j=1}^d \sqrt{-N_j}$, where $\{N_j : 1 \le j \le d\}$ are the eigenvalues of H, and $\sqrt{}$ denotes the principal square root, namely the one in the right half-plane. Finally, one must multiply by -1 if the branch of F at the origin is the larger of the two coalescing at $(\mathbf{p}, f_1)$.

Proof: This is a straightforward application of Proposition 4.6 except for the choice of square root. The square root is determined by the oriented tangent plane to the chain of integration near $\mathbf{p}$. In log coordinates, $d\phi$ is positive definite on the real tangent space (by the strong convexity assumption), hence negative definite on the imaginary tangent space. In the original coordinates, at any real point such as $(\mathbf{p}, f_1)$, the imaginary log tangent space maps to the imaginary tangent space. Therefore, the chain Γ parametrized by $(f + it_1, p_1 + it_2, \ldots, p_{k-1} + it_k, p_{k+1} + it_{k+1}, \ldots, p_d + it_d)$ has a strict minimum of $\Re\{\phi\}$ at $\mathbf{p}$. The Jacobian determinant of the parametrization is i^d and the Hessian is the negative of H, therefore the integral over Γ with this parametrization has constant C_1 determined by (4.10) with $(\det H)^{1/2}$ taken to be $i^d / \sqrt{\prod_{j=1}^d (-N_j)}$.

To go from the integral over Γ to the integral over $\tilde{T}'$, observe first that the local homology group has rank 1 at a quadratically nondegenerate critical point, hence $\tilde{T}'$ is homologous either to Γ or $-\Gamma$. The computation of the sign depends on Lemma 4.37, which is placed in a separate section at the end of the chapter so as not to interrupt the flow here. Letting $\mathbf{T}_\varepsilon$ denote the original small torus $\tilde{T}$ and $\mathbf{T}_*$ denote $\tilde{T}'$, Lemma 4.37 states that the orientation of $\mathbf{T}_*$ is positive with respect to $df \wedge \eta$ if and only if f_0 is the lesser of the two roots eventually coalescing at $(\mathbf{p}, f_1)$. The orientation of Γ with respect to $df \wedge \eta$ is positive, hence the integral computed by this parametrization requires a sign flip to compute the integral over $\tilde{T}'$ if and only if f_0 is the greater of the two coalescing roots. $\square$

Simplifying conditions

Theorem 4.14 does not guarantee the existence of such a pair $(\mathbf{z}, f_1)$ satisfying the hypotheses. In fact a number of further hypotheses hold in many examples that help to assure this. We take as our base of examples the twenty examples analyzed in [GMRW22]. Our simplifying assumptions are catalogued in Table 4.1 along with which examples from [GMRW22] satisfy them. The first is set off because it is a standing hypothesis, repeated here so as to display which examples satisfy it. Four of these do not satisfy our standing hypothesis that P should be smooth; twelve of the twenty satisfy all our simplifying assumptions. All eight of the examples in [GMRW22] that don't satisfy our simplifying hypotheses are among the ten for which no final asymptotic formula given.

1	P is smooth	16 out 20 examples (all except 1, 5, 6, 20)
2	p_m is a monomial	13 out of 20 examples (all except 1, 2, 3, 5, 6, 11, 12)
3	all coefficients of F are nonnegative	20 out of 20 examples
4	P is quadratic in f	18 out of 20 examples (all except 6, 20)

Table 4.1: Simplifying conditions

The following results will be proved in the next section. The first requires all the simplifying hypotheses. It is easier to use because it guarantees the procedure already described in Theorem 4.14 will produce the desired asymptotic formula. The second is more general, requiring only one extra assumption, namely nonnegativity of coefficients. It also guarantees one can find a critical point from which one can automatically produce a valid asymptotic formula, however it involves the introduction of a slightly more complicated algorithm.

Theorem 4.16. *In addition to the standing hypotheses, assume all three remaining hypotheses in Table 4.1. Suppose there is a weakly minimal critical point* $(\mathbf{w}, f)$ *of* P *in direction* $[\mathbf{r} : 0]$*. Define* $z_j := |w_j|$*. Then the point* $\mathbf{z}$ *is a minimal critical point of* $p_m \cdot \operatorname{discr}(P)$ *in direction* $[\mathbf{r}]$*, where the factor* $\operatorname{discr}(P)$ *is the one that vanishes; the fiber* $\pi^{-1}(\mathbf{z})$ *contains a single point* $(\mathbf{z}, f_1)$*; this point is a minimal critical point for* P *in direction* $[\mathbf{r} : 0]$*, and is on the branch determined by* f_0*. Consequently, the conclusions of Theorem 4.14 hold.*

Our last main result is the most general, guaranteeing results without any of the simplifying assumptions in Table 4.1 other than smoothness of P and nonnegativity of coefficients. Let F be an algebraic power series defined by a polynomial $P(\mathbf{z}, f)$ and convergent in a neighborhood of the origin in $\mathbb{C}^d$. A standard result from the theory of several complex variables, e.g., [Hör90, Theorem 2.4.3] describes the open domain of convergence E of F as the set

$$\{\mathbf{z} : \exists\, \mathbf{x} \in D \text{ s.t. } |z_j| \leq e^{x_j} \text{ for all } 1 \leq j \leq d\}$$

where the logarithmic domain of convergence D is an open convex set in $\mathbb{R}^d$ closed under decrease in any coordinate.

Theorem 4.17. *Suppose that f_0 is a simple root of the smooth polynomial $P(\mathbf{z}, \cdot)$, that $\operatorname{discr}(P, f)$ is squarefree, and that the algebraic power series F defined by $P(\mathbf{z}, F(\mathbf{z})) = 0$ and $F(\mathbf{0}) = f_0$ converges in a neighborhood of the origin and has nonnegative coefficients. Fix a nonnegative real unit vector $\hat{\mathbf{r}}$, suppose that $\mathbf{x} \cdot \hat{\mathbf{r}}$ achieves a maximum on $\overline{D}$ uniquely at a point $\mathbf{x}_*$, and denote $\mathbf{z}_* := \exp(\mathbf{x}_*)$. Assume that p_m and the gradient of $\operatorname{discr}(P)$ are nonvanishing at $\mathbf{z}_*$. Then*

(i) $\nabla_{\log} P(\mathbf{z}_*, F(\mathbf{z}_*)) = [\hat{\mathbf{r}}, 0]$.

(ii) The lift $\tilde{T}$ of a small torus in $\mathbb{C}_*^d$ may be deformed within $\widetilde{\mathcal{V}}$ to a lift of a torus $\tilde{T}'$ that contains $(\mathbf{z}_*, F(\mathbf{z}_*))$ and satisfies the requirements to be in stationary phase position for direction $\hat{\mathbf{r}}$ with positive orientation, expect possibly for the requirement of containing finitely many critical points.

(iii) The points $\mathbf{x}_*$ and $\mathbf{z}_*$ can be determined by Algorithm 1 below, which will report failure if p_m or $\nabla \operatorname{discr}(P)$ vanishes at $\mathbf{z}_*$.

4.3. Proofs and Effective Procedures

4.3.1. Proof of Theorem 4.14

The critical point equations (4.5) define a (generically) finite subset of $\mathbb{C}^{d+1}$ and may be rewritten as $P(\mathbf{z}) = 0$ together with $\nabla_{\log} P(\mathbf{z}) = [\mathbf{r} : 0]$ in $\mathbb{CP}^d$. The next proposition shows that critical points for P in direction $[\mathbf{r} : 0]$ upstairs correspond to critical points for $\operatorname{discr}(P)$ in direction $[\mathbf{r}]$ downstairs. This establishes the first part of Conclusion 1 of Theorem 4.14.

Lemma 4.18. *Let $(\mathbf{z}, f)$ be a critical point in direction $[\mathbf{r} : 0]$ for P. Then $\mathbf{z}$ is a critical point in direction $[\mathbf{r}]$ for $\operatorname{discr}(P)$. Conversely, if $\mathbf{z}$ is a critical point in direction $[\mathbf{r}]$ for $\operatorname{discr}(P)$ and $(\mathbf{z}, f) \in \widetilde{\mathtt{br}}$, then $(\mathbf{z}, f)$ is critical for P in direction $[\mathbf{r} : 0]$.*

Proof: Assume $(\mathbf{z}, f) \in \widetilde{\mathtt{br}}$. The tangent space $T_{(\mathbf{z},f)}(\widetilde{\mathcal{V}})$ is a d-dimensional linear space containing the elementary basis vector in the f-direction. Consequently, it is mapped by π to a $(d-1)$-

dimensional subspace of $T_{\mathbf{z}}(\mathbb{C}^d)$. Because $\widetilde{\mathcal{V}} \supseteq \widetilde{\mathtt{br}}$, $T_{(\mathbf{z},f)}(\widetilde{\mathcal{V}}) \supseteq T_{\mathbf{z}}(\widetilde{\mathtt{br}})$, hence

$$\pi(T_{(\mathbf{z},f)}(\widetilde{\mathcal{V}})) \supseteq \pi(T_{(\mathbf{z},f)}(\widetilde{\mathtt{br}})) = T_{\mathbf{z}}(\pi(\widetilde{\mathtt{br}})) = T_{\mathbf{z}}(\mathtt{br})\,.$$

As the first and last linear spaces both have dimension $d-1$, they must coincide. By hypothesis, $T_{(\mathbf{z},f)}(\widetilde{\mathcal{V}})$ consists of all vectors orthogonal to $[z_1 r_1 : \cdots : z_d r_d : 0]$. Hence $T_{\mathbf{z}}(\mathtt{br})$ consists of all vectors orthogonal to $[z_1 r_1 : \cdots : z_d r_d]$. This establishes the conclusion in both directions. □

The restriction of π to $\widetilde{\mathcal{V}}$ is an m-to-1 covering map except over points of two kinds: the branch locus $\mathtt{br}$ and the pole variety $\mathtt{pole}$, defined by the vanishing of the leading coefficient p_m. On the pole variety the degree of $P(\mathbf{z},\cdot)$ is less than m, corresponding to one or more roots at infinity. The following proposition states a well known property of algebraic branched coverings; see, e.g., [Hat02, Chapter 1.3] for further definitions involving covering spaces.

Proposition 4.19. *Let $\mathcal{A} := \mathbb{C}^d_* \setminus (\mathtt{pole} \cup \mathtt{br})$ and denote $\tilde{\mathcal{A}} := \pi^{-1}(\mathcal{A})$. Then $\pi : \tilde{\mathcal{A}} \to \mathcal{A}$ is an m to 1 covering of $\mathcal{A}$.* □

Covering spaces are useful because homotopies on the base space lift uniquely to homotopies on the covering space.

Definition 4.20 (lifting notation)**.**

(i) Given a path $\gamma : [0,t] \to \mathbb{C}^d$ for some $t > 0$, if there exists a unique lifting of γ to $\widetilde{\mathcal{V}}$ with value $(\mathbf{0}, f_0)$ at $t = 0$, we denote this lifting by $\tilde{\gamma}$.

(ii) Conversely, given a path $\tilde{\gamma} : [0,t] \to \widetilde{\mathcal{V}}$, we let γ denote $\pi \circ \tilde{\gamma}$.

*(iii) A path $\gamma : [0,t] \to \mathbb{R}^d$ is called **admissible** if $\gamma(0) = \mathbf{0}$, the path γ is increasing in the coordinatewise partial order, that $\gamma[0,t] \subseteq E$, and that and a neighborhood $\tilde{\gamma}[0,t]$ in $\widetilde{\mathcal{V}}$ is diffeomorphic to a neighborhood of $\gamma[0,t]$ in $\mathbb{C}^d$ via π in one direction and L in the other.*

Proposition 4.21. *Suppose $\gamma : [a,b] \to \mathcal{A}$ is a continuous path in $\mathcal{A}$ and f_0 is a complex number* *with $(\gamma(a), f_0 \in \widetilde{\mathcal{V}})$. Then there is a unique continuous lifting $\tilde{\gamma}$ of γ to $\widetilde{\mathcal{V}}$ such that $\tilde{\gamma}(a) = (\gamma(a), f_0)$. If γ is positive real and increasing in the coordinatewise partial order, then γ is admissible.*

PROOF: Unique lifting is a restatement of the homotopy lifting property [Hat02, Proposition 1.30]. For any $s \in [a, b]$ there is a neighborhood of $\tilde{\gamma}(s)$ in $\widetilde{\mathcal{V}}$ where the partial derivativein $\partial P/\partial f$ does not vanish, hence π induces a diffeomorphism from this neighborhood to a neighborhood of $\gamma(s)$ in $\mathbb{C}^d$. On the union of these, π is a local diffeomorphism, but in fact if we choose product neighborhoods, π is one to one because γ is increasing. $\square$

Lemma 4.22. *Let T be a torus in the closure of the domain of convergence of F. Let $H : T \times [0, 1] \to \mathbb{C}^d$ be a homotopy from T to a chain T' such that $H(\mathbf{z}, t) \in \mathcal{A}$ when $t < 1$ and $H(\mathbf{z}, 1) \notin \texttt{pole}$. Then there is a unique lifting of H to a homotopy $\tilde{H} : T \times [0, 1] \to \widetilde{\mathcal{V}}$ such that $\tilde{H}(\mathbf{z}, 0)$ is the chain $\tilde{T}$. The lifted chain is smooth for every $t < 1$ and continuous at $t = 1$. Consequently,*

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_{\tilde{T}'} \mathbf{z}^{-\mathbf{r}} f \frac{d\mathbf{z}}{\mathbf{z}} \,. \tag{4.16}$$

PROOF: For any $t < 1$, existence and uniqueness of $\tilde{H}|_{[0,t]}$ follow from Proposition 4.21. When $H(\mathbf{z}, 1) \in \mathcal{A}$, define $\tilde{H}(\mathbf{z}, 1) = \tilde{\gamma}(1)$ where $\tilde{\gamma}$ is the unique lifting of the path $H(\mathbf{z}, t) : 0 \leq t \leq 1$. When $H(\mathbf{z}, 1) \in \texttt{br}$, define $\tilde{H}(\mathbf{z}, 1) := \lim_{t \uparrow 1} \tilde{H}(\mathbf{z}, t)$. The limit exists because there is no pole; approaching the branch point, the lift within any branch approaches a limit. Proposition 4.2, along with the identity (4.2) then establishes (4.16). $\square$

Continuing with the proof of Theorem 4.14, let T be the torus containing $\mathbf{z}$. For every $t < 1$, the torus $t \cdot T$ is in the domain of convergence of F because the only possible singularities of an algebraic function are at poles and branchpoints, and the open polydisk on whose boundary T lies has no poles or branchpoints by the assumption that $\mathbf{z}$ is minimal for $p_m \cdot \mathrm{discr}(P)$. Applying Lemma 4.22 establishes that $(\mathbf{z}, f)$ is on the branch defined by f_0 for some f satisfying $P(\mathbf{z}, f) = 0$. To finish proving Conclusion 1 we need to see that choosing $f = \xi_{\mathrm{rk}(f_0)}$ yields the correct branch. This is accomplished by the following lemma, which in one variable goes by the name of the Algebraic Continuation Algorithm [FS09, Proposition VII.8]; the multivariate argument is nearly identical;

see [Cha02] for a prior mention and [Mel21, Example 2.16] for a development in the case where the coefficients of F are known to be nonnegative.

Lemma 4.23. *Suppose the line segment $\gamma := \{t\mathbf{v} : 0 < t < 1\}$ lifts to a path avoiding $\widetilde{\texttt{br}}$, satisfying $\tilde{\gamma}(0) = f_0$ and $\tilde{\gamma}(1) = f_1$. Then, listing the positive real roots of $P(\mathbf{v}, \cdot)$ in decreasing order with multiplicities, f_1 will occur at index* $\mathrm{rk}(f_0)$.

Proof: The homotopy lifting property guarantees a unique lifting because the path γ remains in $\mathcal{A}$ up to time 1 and is not in `pole` at time 1. Without coalescing, real roots of a continuously varying family of real univariate polynomials cannot become complex nor can complex roots become real. Therefore the real roots of $P(t\mathbf{v}, \cdot)$ along γ remain in the same order. By hypothesis they remain finite. Coalescing can occur at the end, but counting with multiplicity preserves the order. By continuity and by definition of how infinite roots are handled, for sufficiently small $\varepsilon > 0$, $\tilde{\gamma}(\varepsilon)$ will be the real root of $P(\varepsilon\mathbf{v}, \cdot)$ of index $\mathrm{rk}(f_0)$, hence this will persist up to time $\tilde{\gamma}(1)$. By continuity, this persists at 1 if one counts multiplicities. □

Proof of remaining conclusion in Theorem 4.14: By definition, the fact that $(\mathbf{z}, f_1)$ is on the branch defined by f_0 means that the lift $\tilde{T}$ of a small torus is homotopic to a torus $\tilde{T}'$ through $(\mathbf{z}, f_1)$. By Lemma 4.22, we have the formula (4.16) for $a_{\mathbf{r}}$. Having assumed that $\tilde{T}'$ passes through finitely many critical points for ϕ on $\widetilde{\mathcal{V}}$, we see that $\tilde{T}'$ is in stationary phase position for ϕ. Proposition 4.6 then implies Conclusion 2, provided we verify that $A(\mathbf{z}, f_1) = 0$. But the form η necessarily vanishes at the point $\exp(\mathbf{x} + i\mathbf{y})$ because $dz_1 \wedge \cdots \wedge dz_d$ vanishes on $\widetilde{\mathcal{V}}$ wherever $\partial P / \partial f = 0$. Hence, each coefficient $C_{\mathbf{y},0}$ from [PWM24, Theorem 5.3] will vanish, leaving only terms with $\ell \geq 1$.

To argue that the sum in (4.13) is not identically zero, the form of [MW19, (5.2)] implies that for each $\mathbf{y} \in W$, at least one of the coefficients $C_{\mathbf{y},\ell}$ is nonzero. The quantities $e^{-i\mathbf{r}\cdot\mathbf{y}}$ are linearly independent over $\mathbb{C}$ as functions of $\mathbf{r}$, hence nonvanishing of $C_{\mathbf{y},\ell}$ for a single pair $(\mathbf{y}, \ell)$ implies nonvanishing of the double sum. The set `contrib` varies continuously wherever the hypotheses of the theorem hold, implying uniformity of the estimate (4.8) and finishing the proof of Theorem 4.14. □

4.3.2. Proof of Theorem 4.16

In proving Theorem 4.16, we also take care of one detail not stated in the theorem, namely how one checks weak minimality of $\mathbf{w}$ under the nonnegativity assumption. Chapter 4.3.3 deals with the more difficult algorithm checking minimality in the general case.

Begin by recalling some concepts about polynomial amoebas, e.g., from [GKZ08, Mik04]. Define the coordinatewise log magnitude map Relog by

$$\mathrm{Relog}(\mathbf{z}) := (\log|z_1|, \ldots, \log|z_d|) .$$

The **amoeba** of a polynomial function $g : \mathbb{C}^d \to \mathbb{C}$ is the image of the zero set of g under the Relog. The amoeba is a closed set whose complement is divided into finitely many open convex connected components. If $g(\mathbf{0}) \neq 0$ then there is a unique component G_0 of the complement of $\mathsf{amoeba}(g)$ and a real number M such that G_0 contains $(-\infty, M]^d$. Such a number M can be effectively computed. The torus T in (1.1) can be taken to be the product of circles with radius e^M.

Fix a critical point $\mathbf{z}$ in direction $[\mathbf{r}]$ on the variety $\mathcal{V}$ defined by g. A more general definition of minimality is that $\mathbf{x} = \mathrm{Relog}\, \mathbf{z}$ lie on the boundary of $\mathsf{amoeba}(g)$, and for an ordinary power series, specifically on ∂G_0. When $\mathbf{z} \in \mathcal{V}$, the condition $\mathbf{x} \in \partial G_0$ is equivalent to $\mathcal{D}(\mathbf{z})$ being in the domain of convergence of the power series for $1/g$, where $\mathcal{D} = \mathcal{D}(\mathbf{z})$ is the open polydisk $\{\mathbf{w} : |w_j| < |z_j|, 1 \leq j \leq d\}$. For functions F not necessarily rational but represented by power series absolutely convergent in some domain $\mathcal{D}$ containing the origin (necessarily the union of tori), minimality again generalizes to the condition that $\mathbf{z} \in \partial\mathcal{D}$.

When all coefficients of a power series are known to be nonnegative, checking minimality is particularly easy. This is the case for the power series F in all our examples, but unfortunately this does not help as much as one might think because the minimality testing we need to do is for $g = \mathrm{discr}(P)$ or $g = p_m \cdot \mathrm{discr}(P)$, in neither case leading to a series likely known to have nonnegative coefficients. We therefore also require a condition to ensure minimality upstairs and downstairs are the same, which will be seen below to follow from the second and fourth conditions in Table 4.1.

Proof of Theorem 4.16: The multivariate version of Pringsheim's Theorem (see, e.g., [PWM24, Proposition 6.38]) says that if the power series coefficients of a function F are known to be nonnegative, and if $\mathbf{z} = \exp(\mathbf{x} + i\mathbf{y})$ and $\mathbf{x} \in \partial G_0$, then the positive real point $|\mathbf{z}| := (|z_1|, |z_2|, \ldots, |z_d|)$ is singular for F. For algebraic functions, this means $\mathbf{z}$ is a branchpoint or a pole.

Let $\mathbf{z}$ and $\mathbf{w}$ be as in the statement of the theorem. It is shown in [Mel21, Proposition 5.5.4] that weak minimality of $\mathbf{w}$ along with nonnegative coefficients implies minimality of $\mathbf{w}$ (hence $\mathbf{z}$) for F. Letting T' denote the torus in $\mathbb{C}_*^d$ through $\pi(\mathbf{z})$, this implies that the lifting of the homotopy $t \cdot T', 0 < t < 1$ does not intersect $\widetilde{\texttt{br}}$. Under the assumption that P is quadratic in f, this is equivalent to the homotopy $t \cdot T', 0 < t < 1$ avoiding $\texttt{br}$. Because there are only two branches, we know that $\mathbf{p} \in \texttt{br}$ and $(\mathbf{p}, f) \in \widetilde{\mathcal{V}}$ imply $(\mathbf{p}, f) \in \widetilde{\texttt{br}}$. From the assumption that p_m is a monomial, we conclude that $\texttt{pole}$ is empty in $\mathbb{C}_*^d$, hence $\pi(\mathbf{z})$ is minimal for $p_n \cdot \operatorname{discr}(P)$; in fact it is minimal in direction $[\mathbf{r}]$ by Lemma 4.18. Again, because there are only two branches, $\pi^{-1}(\mathbf{z}) \cap \widetilde{\mathcal{V}}$ contains a single point $(\mathbf{z}, f_1)$, and at this point the two solutions with different initial conditions merge. Hence this solution is on the branch determined by either initial condition. □

4.3.3. Verification of minimal points

When P is not quadratic, the correspondence between the branch locus and the vertical tangent locus may not be complete. In this case minimality upstairs and downstairs need not coincide and one would need nonnegativity of coefficients of $1/g$ as well as of F, in order to test minimality both upstairs and downstairs. There is no reason to expect $1/g$ to have nonnegative coefficients.

Minimality is effectively testable regardless of any nonnegativity condition, because it is a **real semi-algebraic condition**, that is, it is defined by real algebraic equalities and inequalities once $\mathbb{C}^d$ is represented as $\mathbb{R}^d \oplus i\mathbb{R}^d$. Making full use of Theorems 4.14 and 4.16 however, requires a particular algorithm to test minimality rather than an existence result for such an algorithm. We briefly describe one presented by Melczer, referring the reader to [Mel21, Chapter 7.1.3] for details.

We let $g_{\Re}$ and $g_{\Im}$ denote the unique polynomials in the $2d$ real variables $x_1, \ldots x_d$ and $y_1, \ldots y_d$ such that $g(\mathbf{x} + i\mathbf{y}) = g_{\Re}(\mathbf{x}, \mathbf{y}) + i g_{\Im}(\mathbf{x}, \mathbf{y})$. Augment the original d complex critical point equations for g

(d, not $d+1$ as we are dealing with g rather than P) to $d+1$ equations via a complex parameter N describing the ratio between $\nabla_{\log} g(\mathbf{p})$ and $\hat{\mathbf{r}}$. These $d+1$ critical point equations expand to $2d+2$ equations in $\mathbf{x}$ and $\mathbf{y}$ and the real and imaginary parts of N, involving $g_{\Re}$, $g_{\Im}$ and their partial first derivatives in place of g and its derivatives. Thus far these still describe all critical points of g. Next one write equation for another real solution on a linearly shrunken torus via the $d+2$ equations $(x_j')^2 + (y_j')^2 = t(x_j^2 + y_j^2)$ for $1 \le j \le d$ and $g_{\Re}(\mathbf{x}', \mathbf{y}') = g_{\Im}(\mathbf{x}', \mathbf{y}') = 0$.

This gives $3d+4$ equations in $4d+3$ variables, whose real solutions give pairs of critical points on similar tori. It is shown that minimal critical points for g in direction $\hat{\mathbf{r}}$ correspond precisely to real solutions $(\mathbf{x}, \mathbf{y})$ of the critical point equations for which there are no real solutions $(\mathbf{x}', \mathbf{y}', t)$ to the pair equations with $t \in (0, 1)$. This criterion has alternating quantifiers ranging over infinite sets so is not yet algorithmic. However, introducing a new parameter ν representing the common argument in each coordinate of a hypothetical smaller solution in a pair, and d more equations to force this to be the argument, one obtains a set of $4d+4$ equations in $4d+4$ variables whose real solutions with $t \in (0, 1)$ correspond to refutations of minimality for $(\mathbf{x}, \mathbf{y})$. This results in an algorithm which will either determine some nondegeneracy hypotheses have not been met, or will find all real solutions, eliminate refuted solutions, and correctly produce a set of minimal critical points in direction $\hat{\mathbf{r}}$.

We summarize in a proposition.

Proposition 4.24 ([Mel21, Chapter 7.1.3])**.** *Under mild nondegenearcy assumptions, testing for minimality can be accomplished by searching for real solutions to $4d+4$ polynomial equations in $4d+4$ variables, constrained to an interval. Such real semi-algebraic equation solving is effective and is carried out in pseudo-code in [Mel21, Algorithm 3].* □

4.3.4. Proof of Theorem 4.17

Preliminaries: what singularities are encountered by the lift?

Throughout this section we consider $P, \mathbf{z}_*, \hat{\mathbf{r}}, f_0$ and F to be fixed and to satisfy the hypotheses of Theorem 4.17, with D the open logarithmic domain of convergence and $E = (\mathrm{Relog}^{-1}(D))_{\ll}$ the open domain of converegence in $\mathbb{C}^d$.

Lemma 4.25. *For $\mathbf{z} \in E$, the lifting $L(\mathbf{z}) = (\mathbf{z}, F(\mathbf{z}))$ of* (4.1) *maps into $\widetilde{\mathcal{V}} \setminus \widetilde{\mathtt{br}}$.*

PROOF: Any point of E has a neighborhood in which ∇F is well defined and bounded. Fixing $\mathbf{z} \in E$, we need to show that $F(\mathbf{z}) \notin \widetilde{\mathtt{br}}$. Assume for contradiction that $(\mathbf{z}, F(\mathbf{z})) \in \widetilde{\mathtt{br}}$. Then there is a nonzero tangent vector $\mathbf{v} \in T_{\mathbf{z}}(\mathbb{C}^d)$ which fails to be in the projection $\pi_*[T_{(\mathbf{z},F(\mathbf{z}))}(\mathcal{V})]$ of the tangent space to $\widetilde{\mathcal{V}}$. For this vector $\mathbf{v}$, $(d/dt)_{t=0} F(\mathbf{z} + t\mathbf{v})$ cannot have a well defined finite value. By contradiction, we see that $F(\mathbf{z}) \notin \widetilde{\mathtt{br}}$. □

Theorems 4.14 and 4.16 deal with the case where a critical point can be found on the boundary of the amoeba of $p_m \mathrm{discr}(P)$. The reason Theorem 4.17 is harder to prove is that without the simplifying assumptions, we will not know in advance whether a critical point can be found in direction $\hat{\mathbf{r}}$ that is minimal for $p_m \mathrm{discr}(P)$. Nonnegativity of the coefficients gives us a means to look beyond the boundary of the amoeba of $p_m \mathrm{discr}(P)$, but we first have to make precise how one follows the branch F when $\mathbf{z}$ crosses the pole or branch locus. This part of the proof will culminate in Theorem 4.30 below.

Lemma 4.26. *Let $\tilde{\gamma} : [0, t] \to \widetilde{\mathcal{V}} \cap \mathbb{R}_+^{d+1}$ be a continuous increasing map with $\tilde{\gamma}(0) = (\mathbf{0}, f_0)$ and suppose $\tilde{\gamma}[0, t]$ is disjoint from $\widetilde{\mathtt{br}}$. Then γ is admissible.*

PROOF: We first establish that $\gamma[0, t] \subseteq E$. Suppose for contradiction that $\gamma(s) \notin E$ for some $s < t$ and let s_* be the infimum of such s. We know $s_* > 0$ because F has positive radius of convergence. Let $\mathcal{D}_s$ be the centered open polydisk with $\gamma(s)$ on its boundary. We know that F is analytic on $\mathcal{D}_s$ because $\mathcal{D}_s = \bigcup_{s' < s} \mathcal{D}_{s'}$ and each $\mathcal{D}_{s'} \subseteq E$ due to Pringsheim's theorem again. We know that F is not analytic on any neighborhood of $\overline{\mathcal{D}_s}$ because $\gamma(s') \notin \overline{E}$ for $s' > s$. We conclude that F has a singularity on $\partial \mathcal{D}_s$. Because the coefficients of F are nonnegative, the multivariate Pringsheim theorem implies that F is singular at the positive real point $\gamma(s)$.

For $u < s$, taking the limit from below,

$$\tilde{\gamma}(s) = \left(\gamma(s), \lim_{u \uparrow s} F(\gamma(u)) \right) .$$

Because F is algebraic and singular but not a pole at $\gamma(s)$, it must be a branch singularity. This implies that $(\gamma(s), F(\gamma(s))) \in \widetilde{\mathtt{br}}$, contradicting the hypotheses of the lemma.

We conclude that $\gamma[0,t) \subseteq E$. To finish showing $\gamma[0,t] \subseteq E$ we need to show that $\gamma(t) \in E$. If not, then again F must have a singularity on $\overline{\mathcal{D}_t}$ which must again occur at the positive real point $\gamma(t)$. Again this is not a pole, hence must be a branchpoint, contradicting the hypothesis that $\tilde{\gamma}(t) \notin \widetilde{\mathtt{br}}$ and finishing the proof that $\gamma[0,t] \subseteq E$.

The rest of admissibility is easy. By Lemma 4.25, for every $s \in [0,t]$, $(\partial/\partial f)P(\gamma(s), F(\gamma(s)) \neq 0$. Hence, by the analytic implicit function theorem, in some neighborhood in $\widetilde{\mathcal{V}}$ of each such point, the relation $P(\mathbf{z}, f) = 0$ defines a unique locally analytic function agreeing with F. Putting these neighborhoods together gives a neighborhood of $\tilde{\gamma}[0,s]$ diffeomorphic via π to a neighborhood of $\gamma[0,s]$, completing the proof of the lemma. $\square$

In order to have an effective procedure for determining critical points on the boundary of E, we will need to look from the point of view from γ rather than $\tilde{\gamma}$. The key here is to extend the lifting beyond a point in $\mathtt{pole} \cup \mathtt{br}$ where the lifting has no pole or branch.

Lemma 4.27. *Fix real numbers $0 < t_1 < t$ and a continuous path $\gamma : [0,t] \to \mathbb{R}_+^d$ with $\gamma(0) = \mathbf{0}$. Suppose that for $s \in [0,t]$ not equal to t_1 we know that $\gamma(s) \notin \mathtt{br} \cup \mathtt{pole}$, while $\gamma(t_1) \in \mathtt{br} \setminus \mathtt{pole}$. Suppose also that a finite limit $f_1 := \lim_{s \uparrow t_1} F(\gamma(s))$ exists and that $(\gamma(t_1), f_1) \notin \widetilde{\mathtt{br}}$. Then γ is admissible.*

Proof: By Lemma 4.26 we know that $\gamma_{[0,t_1]}$ is admissible, hence $\gamma_{[0,t']}$ is admissible for some $t' \in [t_1, t]$. Denote $f_1 := F(\gamma(t_1))$. Recall the set $\mathcal{A} = \mathbb{C}_*^d \setminus (\mathtt{pole} \cup \mathtt{br})$ from Proposition 4.19. The path $\gamma_{[t',t]}$ lies in $\mathcal{A}$ so by Proposition 4.21 this portion of the path is admissible as well. Again, in a possibly smaller product neighborhood of $|\tilde{\gamma}|$ in $\widetilde{\mathcal{V}}$ the projection π is one-to-one, establishing admissibility of the whole path γ. $\square$

The same argument yields:

Corollary 4.28. *Let $\gamma : [0,t] \to \mathbb{R}_+^d$ be a path with $\gamma(0) = (\mathbf{0}, f_0)$ such that $\gamma_{[0,t']}$ admissible and*

$\gamma_{[t',t]} \in \mathcal{A}$, *Then* γ *is admissible.* $\square$

This leads to the following recursive procedure.

Procedure 4.29. *Given any increasing path* $\gamma : [0,1] \to \mathbb{R}^d_+$, *suppose that* $t_1 < \cdots < t_k$ *enumerates values* $s \in [0,t)$ *such that* $\gamma(s) \in \texttt{pole} \cup \texttt{br}$. *Define* $\tilde{\gamma}$ *recursively as follows.*

- *On* $[0,t_1)$, *define* $\tilde{\gamma}$ *by Proposition 4.21.*
- *If* $\lim_{s\uparrow t_1} \tilde{\gamma}(s)$ *fails to exist or is in* $\widetilde{\texttt{br}}$ *then* $\tilde{\gamma}$ *is undefined on* $[0,t]$.
- *Otherwise, define* $\tilde{\gamma}(t_1)$ *to be this limit. By Lemma 4.26,* $\gamma_{[0,t_1]}$ *is admissible, hence* $\gamma_{[0,t_1+\varepsilon]}$ *is admissible for some* $\varepsilon > 0$. *This defines* $\tilde{\gamma}$ *up to* $t_1 + \varepsilon$. *Letting* f_1 *be the* f*-coordinate of* $\tilde{\gamma}(t_1 + \varepsilon)$, *continue recursively.*
- *Assuming* $\tilde{\gamma}$ *has been defined without failure on* $[0,t_j)$ *for some* $j < k$, *either fail at* t_j *or extend to* $[0, t_j + \varepsilon]$. *Finally, if* γ *is defined on* $[0,t)$, *extend to* t *if* $\lim_{s\uparrow t} \tilde{\gamma}(s)$ *exists, regardless of whether it is in* $\widetilde{\texttt{br}}$ *and fail otherwise.*

We now have all the ingredients in the preliminary result we have been working toward.

Theorem 4.30. *Independently of the choice of* ε *at each step, the above procedure fails, producing a witness of a pole or merger in the branch defined by* f, *or succeeds and produces a unique extension of* $\tilde{\gamma}$ *to* $[0,t]$. *In case of success,* $\gamma[0,t]$ *is admissible,* $\gamma[0,t) \subseteq E$, *and* $\tilde{\gamma}(s) = (\gamma(s), F(\gamma(s)))$ *for all* $s \in [0,t]$. *Furthermore, in the case of success, the homotopy* $H(\mathbf{y}, s) = \exp(i\mathbf{y})\gamma(s)$, *when restricted to* $[\varepsilon, t]$ *for sufficiently small* ε, *produces a deformation between the torus* $\tilde{T}$ *and the centered torus* $\tilde{T}'$ *through* $F(\gamma(t))$.

Proof: The base case is Proposition 4.21. Each extension for $j < k$ is Lemma 4.27, with the final extension from $[0,t)$ to $[0,t]$ automatic. Admissibility at each step and at the end is Lemma 4.27 again. Uniqueness follows from the construction of the extensions as analytic functions on overlapping neighborhoods and contractibility of a sufficiently small neighborhood of the image of γ defined thus far, from which it also follows that the f-coordinate of $\tilde{\gamma}$ agrees with F. If $\gamma[0,t) \notin E$, the procedure is guaranteed to produce a fail by the contrapositive of Lemma 4.27. Finally, the desired homotopy follows from the homotopy lifting theorem. □

Proving Theorem 4.17

Suppose $\mathbf{z}$ is a critical point in diredtion $\hat{\mathbf{r}}$ and γ is a path from $\mathbf{0}$ to $\mathbf{z}$. We have theoretical description of conditions under which γ may be lifted to $\widetilde{\mathcal{V}}$. We could then imitate our previous use of Lemma 4.22 to represent $a_{\mathbf{r}}$ as a stationary phase integral. Two significant pieces remain. One is to show that the values of F at the points $t_1, \ldots, t_k$ can be computed algorithmically, resulting in Algorithm 1. The second is to find an *a priori* argument that an exhaustive search through all critical points in direction $\hat{\mathbf{r}}$ will turn up a real point $\mathbf{z}$ for which the Procedure 4.29 applied to a suitable path, such as $\gamma(t) = t\mathbf{z}, 0 \leq t \leq 1$, will produce a successful outcome. This is the harder half because the minimality condition that $\gamma[0,1)$ projects under the log magnitude map to a component of the complement of $\mathsf{amoeba}(p_n \mathrm{discr}(P))^c$ has been replaced by the condition that $\gamma[0,1)$ remain in E, the domain of convergence, F, about which we know considerably less.

The theorem tells us the candidate critical point: it is the exponential $\mathbf{z}_*$ of the positive real point $\mathbf{x}_*$ maximizing the dot product with $\hat{\mathbf{r}}$ on ∂D. At this point of ∂D there is a support hyperplane normal to $\hat{\mathbf{r}}$. We need to show that $\nabla_{\log} P(\mathbf{z}_*, F(\mathbf{z}_*)) = [\hat{\mathbf{r}} : 0]$, or projecting to $\mathbb{C}^d$ and pulling back to the log space, that

$$\nabla(\mathrm{discr} \circ \exp)(\mathbf{x}_*) = [\hat{\mathbf{r}}] \, . \tag{4.17}$$

In other words, the normal to the discriminant (in log coordinates) should be parallel to the normal to the domain of convergence. If this is true, then even if we can't compute $\mathbf{z}_*$ directly, we will be able to verify when our search through the finite set of critical points arrives there.

Proof of part (i)

By multivariate Pringsheim, $\mathbf{z}_*$ is a singularity. By assumption, it is not a pole, therefore $(\mathbf{z}_*, f_1) \in \widetilde{\mathtt{br}}$, where $f_1 := F(\mathbf{z}_*)$. By assumption, the gradient of $\mathrm{discr}(P)$ does not vanish at $\mathbf{z}_*$, therefore $\mathbf{z}_*$ is a smooth point of $\widetilde{\mathcal{V}}$. Let $\hat{\mathbf{r}}'$ denote the lognormal to $\widetilde{\mathcal{V}}$ at $(\mathbf{z}_*, f_1)$. This is a good time to pull back via the exponential map $e : \mathbb{R}^d \to \mathbb{R}^d_+ \subseteq \mathbb{C}^d_*$. Thus $\widetilde{\mathcal{V}}_{\mathbb{R}} := \widetilde{\mathcal{V}} \cap \mathbb{R}^d$ which contains $\mathbf{z}_*$ pulls back to $e^*\widetilde{\mathcal{V}}_{\mathbb{R}}$ which contains $\mathbf{x}_*$ and the tangent space of $\widetilde{\mathcal{V}}_{\mathbb{R}}$ pulls back to the linear space normal to $\hat{\mathbf{r}}$. In a sufficiently small neighborhood $\mathcal{N}$ of $\mathbf{x}_*$, the set $e^*\widetilde{\mathcal{V}}_{\mathbb{R}}$ is represented as a graph over (and tangent to) the linear space W orthgonal to $\hat{\mathbf{r}}'$. In other words,

$$e^*\widetilde{\mathcal{V}}_{\mathbb{R}} = \{\mathbf{x}_* + \mathbf{v} + g(\mathbf{v}) : \mathbf{v} \in W\}$$

where $|g(\mathbf{v})| = O(|\mathbf{v}|^2)$. Because of the hypothesis that $p_m(\mathbf{z}_*) \neq 0$, we may also choose $\mathcal{N}$ small enough that there are no zeros of $p_m \circ \exp$ in $\mathcal{N}$. By the assumption that $\mathbf{z}_* \in \partial E$ together with Theorem 4.30, we know that the path $\gamma : [0,1) \to \widetilde{\mathcal{V}}$ defined by $\gamma(t) = t\mathbf{z}_*$ remains within E, hence $e^*\gamma$ remains within D. We now perturb $e^*\gamma$ in $[1-\varepsilon, 1]$ for some small $\varepsilon > 0$ to be determined.

If $\hat{\mathbf{r}}' = \hat{\mathbf{r}}$ we are done, so assume not. Then we can pick $\mathbf{v} \in W$ with $\mathbf{v} \cdot \hat{\mathbf{r}} > 0$. Assume first, for simplicity, that the line segment from $e^*\gamma(1-\varepsilon)$ to $\mathbf{x}_*$ is not parallel to $\mathbf{v}$. Then, letting β be a path tangent to $\mathbf{v}$ in $e^*\widetilde{\mathcal{V}}_{\mathbb{R}}$, the line segment from $e^*\gamma(1-\varepsilon)$ to $\beta(\delta)$ is disjoint from $e^*\widetilde{\mathcal{V}}_{\mathbb{R}}$ except at the endpoint as long as δ is sufficiently small. Also, when δ is sufficiently small, $\hat{\mathbf{r}} \cdot \beta(\delta) > \hat{\mathbf{r}} \cdot \mathbf{x}_*$ and the line segment from $e^*\gamma(1-\varepsilon)$ to $\beta(\delta)$ is still increasing coordinatewise.

Applying the exponential map, we obtain a path α defined by concatenating the path $\gamma|_{[0,1-\varepsilon]}$ with the exponential of the line segment from $e^*\gamma(1-\varepsilon)$ to $\beta(\delta)$. When ε and δ are sufficiently small, the arguments $t_1, \ldots, t_k$ where α intersects $\texttt{pole} \cup \texttt{br}$ and the values $\alpha(t_j), 1 \leq j \leq k$ are the same as for the original path γ. Therefore, by Theorem 4.30, $\alpha[0,1)$ is admissible and $\alpha(1) = \exp(\beta(1)) \in \partial E$. By construction $\hat{\mathbf{r}} \cdot \log \alpha(1) > \hat{\mathbf{r}} \cdot \mathbf{x}_*$, implying that the maximum of $\hat{\mathbf{r}} \cdot \mathbf{x}$ on ∂D is not achieved at $\mathbf{x}_*$. This contradiction reveals that $[\hat{\mathbf{r}}] \neq [\hat{\mathbf{r}}']$ is impossible, at least in the case that the line segment from $e^*\gamma(1-\varepsilon)$ is not parallel to $\mathbf{v}$.

In the unlikely event that this line segment was parallel to $\mathbf{v}$, one can simply perturb γ slightly to destroy this parallelism. This can be done because the set of paths admissible on the closed interval $[0, 1-\varepsilon]$ is is an open set, so any replacing γ by a sufficiently small generic perturbation will suffice to make the last ε of the path avoid $e^*\widetilde{\mathcal{V}}_{\mathbb{R}}$ except at the endpoint $\beta(\delta)$, which the first $1-\varepsilon$ segment remains admissible. This finishes the proof by cvontradiction that $[\hat{\mathbf{r}}'] = [\hat{\mathbf{r}}]$ and establishes part (i)

of Theorem 4.17.

Remainder of proof

For (ii), let T' denote the centered torus through $\mathbf{z}_*$. Arguing analogously to the proof of Proposition 4.19, we see that for every $t < 1$ the map $(\mathbf{u}, s) \mapsto s\mathbf{u}$ defines a homotopy from $\varepsilon T'$ to $(1-t)T'$; this lifts to a homotopy defined by $(\mathbf{u}, s) \mapsto (s\mathbf{u}, F(s\mathbf{u}))$ taking $\tilde{T} := F(\varepsilon T')$ to a lift of $(1-t)T'$ into the branch determined by f_0. By smoothness for $t < 1$ and continuity at $t = 1$ (we assumed $p_m(\mathbf{z}_*) \neq 0$), we conclude again that the continuous extension of the lifting to $t = 1$ is a contour $\tilde{T}'$ through $(\mathbf{z}_*, F(\mathbf{z}_*))$. By Part (i), we know that $(\mathbf{z}_*, f_1)$ is a stationary phase point for $h_{\hat{\mathbf{r}}}$ on $\widetilde{\mathcal{V}}$, hence $\tilde{T}'$ satisfies the definition of stationary phase position for direction $\hat{\mathbf{r}}$, modulo the assumption that are only finitely many critical points for $h_{\hat{\mathbf{r}}}$ on $\tilde{T}'$. This establishes (ii).

For part (iii), we first describe some automatically computable facts about Puiseux series that can be used to keep track of the ordering of real roots. Suppose $\tilde{P}$ is a bivariate polynomial over the complex numbers; we have in mind later to use $\tilde{P}(t, f) = P(\gamma(t), f)$ where $\gamma(t) = t\mathbf{z}_*$. Consider the algebraic functions g solving $\tilde{P}(t, g(t)) = 0$ in a neighborhood of the value t_0. At points (t_0, x_0) where $\tilde{P} = 0$ and $\partial\tilde{P}/\partial t \neq 0$, there is a unique power series expansion $g(t) = \sum_n a_n (t - t_0)^n$ solving $\tilde{P}(t, g(t)) = 0$ and $g(t_0) = x_0$. Where $\tilde{P} = 0$ and $\partial\tilde{P}/\partial t = 0$, there are instead **finite Puiseux series** $g(t) = \sum_{e \in \mathcal{E}} a_e (t - t_0)^e$ where e takes values in a set $\mathcal{E}$ of nonnegative rational numbers with bounded denominators. One may also consider nonfinite Puiseux series, $(t - t_0)^{-\alpha} g$ where α is a positive rational number and g is a finite Puiseux series. The following results are presented in in Chapter 8.3 of [BK86]. They do not discuss there how one can tell which Puiseux series are real, however this follows from the recursive nature of the computation of coefficients. Finding the roots of real univariate polynomials, and reducing the defining polynomial, the branches will be real as long as all roots selected in the process are real.

Lemma 4.31. *Suppose $\tilde{P}$ has degree d in the variable f and let $k \leq d$ be the degree of $\tilde{P}(t_0, \cdot)$. Then there are k finite Puiseux series solutions $g_1, \ldots, g_k$ in a neighborhood of t_0 satisfying $\tilde{P}(t, g_l(t)) = 0, l = 1, \ldots, k$. These may be computed automatically to any number of terms (equivalently, for any $M > 0$, all terms with $e < M$ may be computed), and listed with multiplicities. The number of* *series with constant term x_* is the multiplicity of x_* as a root of $\tilde{P}(t_0, \cdot)$. It is possible to compute which series are real in some interval to the right (respectively to the left) of t_0. The lexicographic order on coefficients induces an ordering of the germs of these real functions.* $\square$

We remark that there are $d-k$ infinite Puiseux series solutions. We also remark that the capability to sort Puiseux solutions into real and non-real is needed for tracking real roots because series coefficients being real for some finite number of exponents is never sufficient to imply the series is real. One may also sort infinite solutions by whether they are real to the left or right, and if so, whether they are positive or negative. One may do this either by computing infinite Puiseaux expansions or by enumerating zeros of $x^d \tilde{P}(t, 1/x)$ where d is the x-degree of $\tilde{P}(t_0, \cdot)$. The upshot is that it is algorithmically decidable (and has a reasonably quick implementation) how many real roots there are on each side of t_0 and how many real roots at infinity there are, with what sign, on each side of t_0. We now apply this, setting $\tilde{P}(t, f)$ equal to $P(\gamma(t), f)$, the defining polynomial for the graph of P over γ.

Definition 4.32 (gain)**.** *For $(\gamma(t_0), x_0) \in \pi^{-1}(\mathtt{br}) \setminus \widetilde{\mathtt{br}}$, define $N_+(t_0, x_0)$ to be the number of finite Puiseux series solutions g at t_0 with $g(t_0) < x_0$ that are real and positive to the right of t_0, plus the number of infinite Puiseux series solutions that are real to the right of t_0 and go to $-\infty$ at t_0^+. Informally, N_+ measures the number of real to the right solutions below x_0. Define $N_-(t_0, x_0)$ similarly, but replacing "right of t_0" by "left of t_0". Denote $\delta(t_0, x_0) := N_+(t_0, x_0) - N_-(t_0, x_0)$ and call this the* ***gain*** *across t_0 at x_0. Informally, the gain is the number of new real roots. When $(\gamma(t_0), x_0) \in \widetilde{\mathtt{br}}$, the gain remains undefined.*

Suppose P, f_0 and F obey the standing assumptions at the beginning of Chapter 4.2. Fix $\mathbf{z}_* \in \mathbb{R}_+^d$ in the closure of the domain of convergence of F and suppose there are finitely many values $0 < t_1 < \cdots < t_k < 1$ for which $t_j \mathbf{z}_* \in \mathtt{pole} \cup \mathtt{br}$. For convenience, denote $t_0 := 0$ and $t_{k+1} := 1$. For $0 \le j \le k$ let $k(j)$ denote the common number of real roots of $F(t\mathbf{z}_*)$ for any $t \in (t_j, t_{j+1})$ and let $\xi_1(t) < \cdots < \xi_{k(t)}(t)$ enumerate these in increasing order. There will be precisely $k(j-1)$ real-to-the-left Puiseux series at t_j. The following lemma describes how to compute $F(t_j \mathbf{z}_*)$ by giving its index in an increasing list of real-to-the-left Puiseux expansions at t_j. The index is computed by computing how it changes at each t_j. The value of this lemma is that it reduces the homotopy continuation of the branch defined by f_0 to computations involving a finite set of Puiseux expansions at a finite, computable set of algebraic numbers $\{t_j : 1 \leq j \leq k\}$. The lemma is immediate from Lemma 4.31 and the definitions of rank and gain.

Lemma 4.33 (tracking real roots)**.**

(i) The $k(j-1)$ distinct finite real roots of $P(t\mathbf{z}_*, \cdot)$ for $t \in (t_{j-1}, t_j)$ correspond in order with the $k(j-1)$ real-to-the-left, possibly infinite Puiseux series at t_j solving $P(t\mathbf{z}_*, g(t)) = 0$.

(ii) This correspondence may be computed by listing the values at t_0 of the real-to-the-left Puiseux series at t_0 in order of their germs to the left. This computes $F(t_j\mathbf{z}_*)$ as the constant term of the real-to-the-left series whose germ is number rk_{j-1} in the increasing order on left-germs at t_j.

(iii) Including the series not real to the left but eliminating those not real to the right produces a new ordered set of roots which are limits from the right of the $k(j+1)$ distinct real roots of $F(t\mathbf{z}_*, \cdot)$ on (t_j, t_{j+1}). (iv) The rank of F changes across t_j via

$$\mathrm{rk}_j = \mathrm{rk}_{j-1} + \delta(F(t_j\mathbf{z}_*), t_j\mathbf{z}_*) \,. \tag{4.18}$$

□

We now describe an algorithm and show that it accomplishes the task in part (iii) of Theorem 4.17. The algorithm will succeed in all cases where the hypotheses of the theorem are satisfied and possibly on some occasions when the hypotheses fail. Notation for the algorithm is as follows. We use the term **left-rank** (respectively **right-rank**) to denote the germ-rank of a given real-to-the-left (respectively real-to-the-right) Puiseux series solution. Thus $\mathrm{rk}(f)$ denotes the right-rank of the simple root f of $P(\mathbf{0}, \cdot)$. Our pseudo-code convention is that RETURN always breaks the main loop, while BREAK (`VAR`) breaks the loop over the variable `VAR` and any loops within this.

Having established there is a nonnegative real point $(\mathbf{z}_*, f_1)$ to which we can deform the original torus so as to be in stationary phase position, all we need to do is to test each possible such pair to see if it is the right one. We know that $\mathbf{z}_*$ will be a critical point for $\mathrm{discr}(P)$ in direction $\hat{\mathbf{r}}$ so we begin by enumerating the set S of nonnegative real solutions to the critical point equations for $\mathrm{discr}(P)$ in direction $\hat{\mathbf{r}}$.

For each $\mathbf{z} \in S$, we track the order of $F(t\mathbf{z})$ among the real roots of $P(t\mathbf{z}, \cdot)$. For fixed $\mathbf{z}$, this is done by tracking the rank of $F(t\mathbf{z})$ as t increases through $(0, 1)$. Perturbing the line segment from the origin to $\mathbf{z}$ generically if necessary to start at a nearby point, we can assume without loss of generality that the open line segment is not contained in $\mathtt{pole} \cup \mathtt{br}$ and therefore intersects $\mathtt{pole} \cup \mathtt{br}$ in finitely many points $t_j \mathbf{z}_*$ with $0 < t_1 < \cdots < t_k < 1$. We know we have found $\mathbf{z}_*$ when F can be extended across each t_j without blowing up or coalescing with another root, and when the branch coalesces with a single other branch at $t = 1$. By Lemma 4.33, we can compute rank of $F(t\mathbf{z}_*)$ for $t \in (0, 1)$. To compute it at $t = 1$ we use steps (i) and (ii) of Lemma 4.33 one last time.

Having described an algorithm to search for $\mathbf{z}_*$ by searching critical points of the discriminant variety and tracking the index of the real root $F(t\mathbf{z}_*)$ as $t \uparrow 1$, we have proved all but the last part of (iii) of Theorem 4.17. For this last piece, note that when precisely two solutions to $P(t\mathbf{z}, g(t\mathbf{z}))$, call them $F_1(t\mathbf{z})$ and $F_2(t\mathbf{z})$ coalesce at $t = 1$, the t-derivatives $(d/dt)F_j(t\mathbf{z})$ are of opposite sign, with magnitudes going to infinity. Because the coefficients of F are nonnegative, we know that all derivatives in t of $F(t\mathbf{z})$ are positive as $t \uparrow \infty$, hence F is always the bottom of the two coalescing branches. By Lemma 4.37, the orientation is positive with respect to the reference form $df \wedge \eta$, which completes the proof of the theorem. $\square$

Algorithm 1: Algorithm to compute $(\mathbf{z}_*, f_1)$ and the orientation.

Input: P, $\hat{\mathbf{r}}$, and a finite simple root f_0 of $P(\mathbf{0}, \cdot)$ determining the solution F;

Output: $(\mathbf{z}_*, f_1)$ and σ, where

- $\mathbf{z}_*$ is on the boundary of the domain of convergence of F
- $(\mathbf{z}_*, f_1)$ is a critical point for P in direction $[\hat{\mathbf{r}} : 0]$
- The torus $\tilde{T}$ is homotopic to a contour in stationary phase position through $(\mathbf{z}_*, f_1)$.

`// Puiseux series pre-computation`

- Compute all real-to-the-right Puiseux series solving $P(t\mathbf{v}, g(t)) = 0$ for any positive real vector $\mathbf{v}$.
- If two coincide RETURN FAIL
- Order the series $\xi_1(t) < \cdots < \xi_k(t)$ by right-germ order.

`// Initialize` ρ `and` S `for the loop in the` $\mathbf{z}$ `variable`

- Set $\rho := \mathrm{rk}(f_0)$
- Set S to the list of positive real critical points of $\mathrm{discr}(P)$ in direction $\hat{\mathbf{r}}$.
- If S is empty or infinite then RETURN FAIL

`// MAIN LOOP`

- For $\mathbf{z} \in S$ do:
  - If $p_m(\mathbf{z}) = 0$ then BREAK
  - Set $\mathbf{x} := 0$
  - WHILE line segment $\overline{\mathbf{x}\,\mathbf{z}} \subseteq \texttt{pole} \cup \texttt{br}$ or $\mathbf{x} \in \texttt{pole}$ or $\mathbf{x} \notin G_0$:
    Choose a new small positive rational $\mathbf{x}$
  - If $k > 0$ then do (root tracking):
    Solve for t such that $\mathbf{x} + t(\mathbf{z} - \mathbf{x}) \in \texttt{pole} \cup \texttt{br}$
    Let $0 < t_1 < \cdots < t_k < 1$ enumerate the solutions
    Initialize $s := \rho$, $j := 1$
    While $j \leq k$ do root tracking j-loop:
      Order the series $\xi_1(t_j) < \cdots < \xi_k(t_j)$ by left-germ order.
      Set f equal to the real root of left-rank ρ
      If the root of rank ρ goes to $\pm\infty$ then BREAK $(\mathbf{z})$
      If the root of rank ρ coalesces then BREAK $(\mathbf{z})$
      Set $\rho := \rho + \delta(f, t_j\mathbf{z})$ and $j := j + 1$
    `//` j `loop is done and` ρ `is now set to` rk_k`, whether or not` k `was positive`
  - If precisely two finite roots with indices ρ and $\rho + 1$ merge at $\mathbf{z}$ then RETURN $(\mathbf{z}, \rho)$
  - If the root of left-rank ρ does not coalesce and remains finite then BREAK
  - RETURN FAIL `// because` $\mathbf{z} \in \partial E$ `but is a pole or at least a triple root`
- RETURN FAIL `// because no z worked, so some hypothesis was violated.`

Remarks 4.34.

1. If $p_m(\mathbf{z}_*) = 0$ then a hypothesis of the theorem is violated and the algorithm discovers it by ruling out the correct $\mathbf{z}_*$ and failing when it does not find another. One could remove this BREAK and let the algorithm continue as long as the branch of F determined by f_0 does not blow up. We do not do this because part (i) of the theorem relies on $p_m(\mathbf{z}_*) \neq 0$ and if this fails, we don't have a proof that $(\mathbf{z}_*, f_1)$ will be critical for P in direction $[\hat{\mathbf{r}} : 0]$. Therefore, if the BREAK is removed, one must check before returning a non-fail that a critical point in direction $[\hat{\mathbf{r}} : 0]$ has been found.

2. When $\mathbf{z}_* \in \partial E$ is a pole, then the asymptotics of $\{a_{\mathbf{r}}\}$ are determined by rational ACSV are are not given by the formulas in Chapter 4.1.3.

3. When $\mathbf{z}_* \in \partial E$ is a singular point of $\mathtt{br}$, meaning that $\nabla \mathrm{discr}(P)$ vanishes there, then, even if $(\mathbf{z}_*, f_1)$ is critical in direction $\hat{\mathbf{r}}$ and the contour can be moved into stationary phase position, the resulting stationary phase integral will have a degenerate Hessian, violating part (ii) of Proposition 4.6, and the formulas of Chapter 4.1.3 will again fail to compute the integral.

4. Summarizing, the hypothesis that p_m and $\nabla \mathrm{discr}(P)$ are nonvanishing at $\mathbf{z}_*$ is sharp in the sense that the conclusions (not just the algorithm) may fail in its absence either because the critical point $(\mathbf{z}_*, f_1)$ does not exist, or because the asymptotics are something entirely different than is described by the formulas in Chapter 4.1.3.

4.4. Examples

Before presenting examples to exhibit the main results, we give a couple of examples showing that algebraic functions are not always determined by integrals through branchpoints.

Example 4.35. *In one variable, let* $F(x) = (1-x)^{-3/2}$*. The defining polynomial is* $P(x,f) := (1-x)^3 f^2 - 1 = 0$*, with* $\mathrm{discr}(P) = 4(1-x)^3$*. The unique branchpoint* $x = 1$ *is also a pole:* $\mathtt{br} = \mathtt{pole} = \{1\}$*, so integration through this point is not possible. Not coincidentally, the hypothesis that* $\mathrm{discr}(P)$ *be squarefree is also violated.*

Example 4.36. *The generating function*

$$\beta(x, y, z) = \frac{1}{\sqrt{(1 - x - y)^2 - 4xy} - z}$$

generates certain hypergeometric sums arising when counting solutions to Ulam's problem [HS22]. The discriminant locus is where the quantity under the radical vanishes. The locus of vanishing is the parabola inscribed in the positive x-y-quadrant and tangent at $(0, 1)$ and $(1, 0)$; see Figure 4.2. The pole locus is where the quantity under the radical is equal to z. For fixed $z \in (0, 1)$ this vanishes on an arc in the positive quadrant hitting the axes at $1 - \sqrt{z}$. The minimal points of $\texttt{pole} \cup \texttt{br}$ *are the ones on this arc, where β has a pole, not a branchpoint. The coefficient asymptotics there are governed by the usual ACSV smooth point formula given, for example, in [PW13, Theorem 9.2.7].*

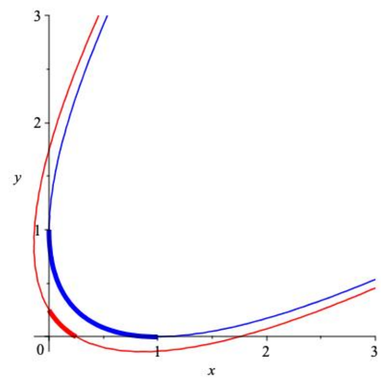


Figure 4.2: pole variety (red) and branching locus (blue)

The remainder of the section gives examples of the application of Theorems 4.14 and 4.16 and Corollary 4.15.

4.4.1. Toy example: Catalan GF

In this example $d = 1$ so the generating function F is a univariate algebraic function, for which well known methods such as the transfer theorems of Flajolet and Odlyzko [FO90] could be applied (see also far earlier works). This example illustrates our methods in the simplest case.

Let $F(z) = (1 - \sqrt{1 - 4z})/(2z) = \sum_{n=0}^{\infty} a_n z^n$ be the generating function for the Catalan numbers

$a_n := \binom{2n}{n}/(2n+1)$. A minimal polynomial representing F via $P(z, F(z)) = 0$ and $F(0) = 1$ is

$$P(z, f) := zf^2 - f + 1\,.$$

The discriminant of P is $1-4z$ and the pole polynomial p_m is z. A graph of F in the $\mathbb{R}\times\mathbb{R}$ subspace of $\mathbb{C}^2$ is shown on the left of Figure 4.3.

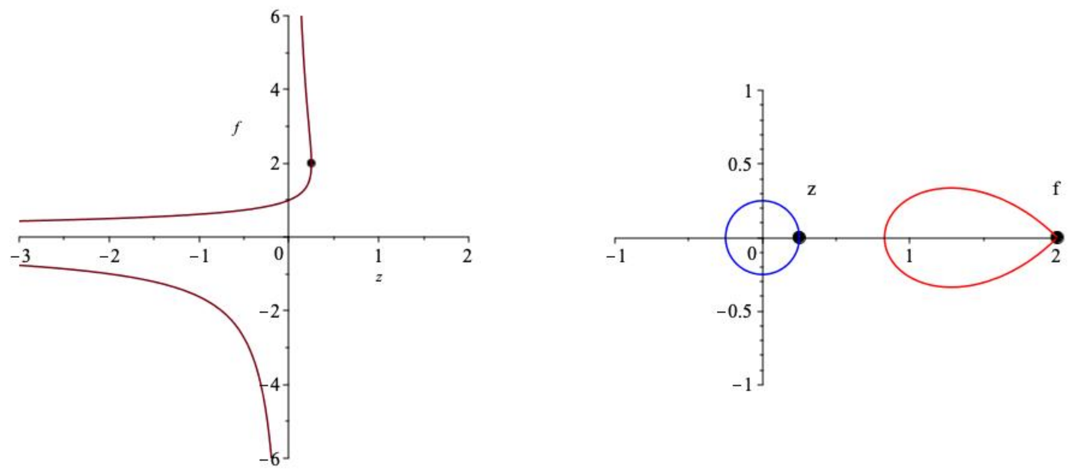


Figure 4.3: Left: $\mathbb{R}\times\mathbb{R}$ graph of the Catalan GF; Right: coordinates in $\mathbb{C}$ of the lifted torus

Here, $m = 2$ but because the origin is a zero of p_m, there is only one function F with a finite value at the origin. This branch has $F(0) = 1$, while the other possible branch, $(1+\sqrt{1-4z})/(2z)$, has a pole at zero. The discriminant of P with respect to f is $1 - 4z$, whose amoeba is the singleton set $\{\log(1/4)\}$. Thus a circle of radius ε about the origin, call it T, may be expanded without hitting `br` or `pole` until it has radius $1/4$. We call this expanded circle T'. Lifting the homotopy from T to T' into the algebraic curve $P(z, f) = 0$ yields a homotopy

$$Y_t(\theta) := (te^{i\theta}, F(te^{i\theta}))$$

between circles $\tilde{T}$ and $\tilde{T}'$ within $\widetilde{\mathcal{V}}$.

We are now in a position to apply Theorem 4.14. This toy example is intended to explain the working parts of the theorem, hence instead of actually applying the theorem we will follow the proof of the theorem to derive the asymptotic expansion from Proposition 4.6 and Corollary 4.7. We begin by determining the oriented stationary phase integral defined by the contour $\tilde{T}'$.

For z positive and sufficiently small, the branch of f that blows up at $z = 0$ takes positive real values, hence the branch defining F will be the lower of two branches coalescing at 2 when $z = 1/4$. Because $F(z)$ is the smaller real root of $P(z, f) = 0$ for small positive real z, Lemma 4.37 implies that $\tilde{T}'$ is positively oriented with respect to $d(if)$. To see the significance of this, look on the right of Figure 4.3, where the z and f coordinates of the circle $\tilde{T}' \subseteq \widetilde{\mathcal{V}}$ are shown. The projection of $\tilde{T}'$ to the f-coordinate is nondifferentiable at $z = 1/4$ because the square root in $f = (1 \pm \sqrt{1-4z})/(2z)$ is the principal root (the one lying in the right half plane); mapping the blue circle by $1 - 4z$ gives a circle tangent to the imaginary axis and lying in the closed right half-plane, whence taking the principal square root produces a discontinuity in the argument resulting in the nondifferentiability in the figure.

The phase $\phi(z, f) = \log|z|$ on the circle $\tilde{T}'$ is constant. We can further deform $\tilde{T}'$ so that the f-coordinate adheres locally to the line $\Re(f) = 2$, and so that the minimum of $h_{\hat{\mathbf{r}}}$ is achieved strictly at $(1/4, 2)$. This illustrates that the imaginary direction is always a direction along which the height, in this case $\log|z|$, will have a local minimum.

Contours with phase minimized at $(1/4, 2)$ are of course not unique. We could deform the f-coordinate to a circle $f = 2e^{i\theta}$ and the z-coordinate to $z = (f-1)/f^2$, for example, however in this case it is simplest to choose the chain Γ obtained from deforming a small arc on the right of the circle to lie on the segment $f = 2 + it, -\varepsilon \le t \le \varepsilon$. Along with $z = (f-1)/f^2$, this defines a parametrization of a curve, along which the derivatives in Corollary 4.7 are computed with minimal effort. Reparametrizing everything in terms of f, we get $dz = (2-f)df/f^3$ and $(f/z)dz = [(2-f)/(f-1)]df$ and (4.16) becomes

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\int_\Gamma z^{-n} f \frac{dz}{z} \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{2-f}{f-1} e^{-n\phi(f)}\, df \end{aligned}$$

where $\phi(f) = \log z = \log((f-1)/f^2)$ has power series expansion around $f = 2$ given by

$$\phi(f) = -\log 4 - \frac{1}{4}(f-2)^2 + \frac{1}{4}(f-2)^3 + O(f-2)^4 \,.$$

To fit into the set-up of Proposition 4.6, we need to recenter the critical point $z = 1/4, f = 2$ at the origin. So by a shift of the coordinate, we get

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\int_{\Gamma'} \frac{-f}{f+1} e^{-n\phi(f+2)}\, df & (4.19)\\ &= \frac{1}{2\pi i} e^{n\log(4)} \int_{\Gamma'} \frac{-f}{f+1} e^{-n\Phi(f)}\, df & (4.20)\end{aligned}$$

where $\Phi(f) = \phi(f+2) + \log(4)$ has power series expansion at the origin given by

$$\Phi(f) = -\frac{1}{4}f^2 + \frac{1}{4}f^3 + O(f)^4 \,.$$

Now we pull back the integral $\int_{\Gamma'} \frac{-f}{f+1} e^{-n\Phi(f)}\, df$ to $\mathbb{R}$. We obtain

$$\int_{\Gamma'} \frac{-f}{f+1} e^{-n\Phi(f)}\, df \approx \int_{\epsilon}^{\epsilon} \frac{-it}{1+it} e^{-n\Phi(it)}\, i\, dt$$

where $\Phi(it) = \log(\frac{1+it}{(2+it)^2}) + \log(4)$. Here, we use $\approx$ and ϵ because the portion of Γ away from the critical point contributes exponentially small value to the integral. We can recognize the amplitude function $A(t) = \frac{-it}{1+it}$ and the phase function $\Phi(it)$.

In Corollary 4.7, $V = 1/2, A'(0) = -i, A''(0) = -2, \Phi'''(0) = -\frac{3i}{2}$. By Corollary 4.15, the choice of $V^{1/2}$ should be the positive one. By (4.11), the leading constant C_1 for the integral is $\sqrt{2}$. Therefore, Proposition 4.6 gives

$$\begin{aligned} a_n &\approx \frac{1}{2\pi} e^{n\log(4)} \int_{\epsilon}^{\epsilon} \frac{-it}{1+it} e^{-n\Phi(it)}\, dt\\ &\approx \frac{1}{2\pi} 4^n \left(\frac{2\pi}{n}\right)^{1/2} \left(\sqrt{2}\, n^{-1}\right)\end{aligned}$$

$$= \frac{4^n}{\sqrt{\pi}} n^{-3/2}$$

To sum up, we have derived asymptotics for Catalan numbers by constructing an explicit lifting of the univariate Cauchy integral to an integral within the algebraic curve $zf^2 - f + 1 = 0$ that witnesses the Catalan generating function as algebraic.

Theorem 4.14 automates most of this procedure. Having checked that $(1/4, 2)$ is a minimal point, we apply the theorem to (4.16) using Proposition 4.6 and its Corollary 4.7 for the computation. Plugging in $A(f) = \frac{-f}{f+1}$ and $\Phi(f) = \log((f+1)/(f+2)^2) + \log(4)$ immediately yields $a_{\mathbf{r}} = \pm 4^n n^{-3/2}/\sqrt{\pi}$. The sign (should it be in doubt) is determined by Lemma 4.37 to be positive because the generating function $F(z) = (1 - \sqrt{1-4z})/(2z)$ is the lower of two branches in a neighbrhood of the origin.

4.4.2. Assembly trees

In this example $d = 2$ and F is the generating function for the number of assembly trees of the complete bipartite graph, counted by the sizes of the left and right vertex sets [BV13]. This generating function is given by

$$F(x, y) = 1 - \sqrt{(1-x)^2 + (1-y)^2 - 1}$$

satisfying an obvious polynomial equation

$$P(x, y, f) := f^2 - 2f - x^2 - y^2 + 2x + 2y = 0\,.$$

The discriminant is given by

$$\mathrm{discr}(P) = (x-1)^2 + (y-1)^2 - 1\,.$$

The branch locus $\mathtt{br}$ where this vanishes is the circle of radius 1 centered at $(1, 1)$. The defining variety $\widetilde{\mathcal{V}}$ is the hyperboloid $\{P = 0\}$ and the vertical tangent locus in $\widetilde{\mathcal{V}}$, which projects to $\widetilde{\mathtt{br}}$, is the intersection of the hyperboloid with the horizontal plane $\{(x, y, f) : f = 1\}$. The branch locus is depicted on the left of Figure 4.4, with $\widetilde{\mathcal{V}}$ and $\widetilde{\texttt{br}}$ shown on the right.

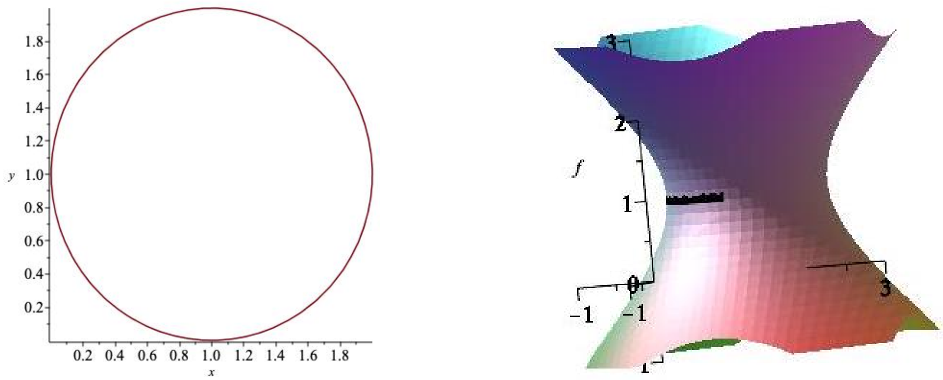


Figure 4.4: Left: the branch locus $\texttt{br}$; Right: $\widetilde{\mathcal{V}}$, with $\widetilde{\texttt{br}}$ shown in black

Given a direction $\hat{\mathbf{r}} = (\hat{r}, 1 - \hat{r})$, there is a corresponding minimal point (x, y) on the branch locus, specifically on the quarter-circle arc joining $(1, 0)$ to $(0, 1)$. This point is given by

$$x(\hat{\mathbf{r}}) = \frac{1}{2} + \hat{r} - \frac{\sqrt{1 + 4\hat{r} - 4\hat{r}^2}}{2} \qquad ; \qquad y(\hat{\mathbf{r}}) = \frac{3}{2} - \hat{r} - \frac{\sqrt{1 + 4\hat{r} - 4\hat{r}^2}}{2}\,.$$

The point $(x(\hat{\mathbf{r}}), y(\hat{\mathbf{r}}))$ lifts to a unique point $(x(\hat{\mathbf{r}}), y(\hat{\mathbf{r}}), 1)$ in the hyperboloid where the two solutions of $P(x, y, f)$ coincide. It follows from (4.13) that, uniformly when r/s and s/r are bounded,

$$a_{r,s} \sim C(r+s)^{-2} e^{-r \log x(\hat{\mathbf{r}}) - s \log y(\hat{\mathbf{r}})}$$

where $\hat{r} := r/N$, $N := r + s$, and C is given by computing the stationary phase integral (4.16) on a contour passing through $\mathbf{p} := (x(\hat{\mathbf{r}}), y(\hat{\mathbf{r}}), 1)$.

Parametrizing $\widetilde{\mathcal{V}}$ near $\mathbf{p}$ by x and f (we could have chosen f and any $ax + by$ other than the one orthogonal to $\widetilde{\mathcal{V}}$) we rewrite $f\,dx\,dy$ as $f J\,dx\,df$ where

$$J = \frac{\partial y}{\partial f} = -\frac{\partial P/\partial f}{\partial P/\partial y}\,.$$

The exponential factor $e^{-r \log x(\hat{\mathbf{r}}) \,-\, s \log y(\hat{\mathbf{r}})} = \left(x(\hat{\mathbf{r}})^{-\hat{r}} y(\hat{\mathbf{r}})^{\hat{r}-1} \right)^N$ comes from the stationary phase integral $\int A(x, f) e^{-N\phi(x,f)} dx\, df$ by Corollary 4.15 where $\phi(x, f) = \hat{r} \log x + (1 - \hat{r}) \log(y(x, f))$

and

$$A(x,f) := \frac{f \cdot J}{x \cdot y(x,f)} = \frac{f(1-f)}{x\, y(x,f)(1-y(x,f))}\,.$$

Here, $y = 1 - \sqrt{1 + (1-f)^2 - (1-x)^2}$, taking the principal root which is well defined near $(x(\hat{\mathbf{r}}), y(\hat{\mathbf{r}}), 1)$.

One can use either Proposition 4.6 or Corollary 4.7 to calculate the constant C_1. In practice, we use Corollary 4.7. This avoids computing $\mathbb{H}^2(A \cdot f^2)$, which is not only messy but wasteful, computing out to four partial derivatives when Corollary 4.7 shows that only third partial derivatives are required. Also, many of the partial derivatives in (4.12) arise only in products with other partial derivatives, meaning that the vanishing of some partial derivatives allow us to avoid the computation of many more. In this case, for example, vanishing partial derivatives lead to the following simple expression for C_1, all partial derivatives being evaluated at $x = x(\hat{\mathbf{r}}), f = 1$.

$$C_1 = \frac{A_{ff}\phi_{xx}}{2}\ (\phi_{xx}\phi_{ff})^{-\frac{3}{2}} \tag{4.21}$$

Having reduced the computation to the evaluation of partial derivatives of algebraic expressions, we illustrate how computer algebra systems handling polynomial computations via Gröbner bases can be harnessed to differentiate algebraic functions. Note that built-in differentiation operators in computer algebra systems such as Sage and Maple do not handle radicals well. Compare (4.22), for example, with the expression taking up a full line in line 45 of the online worksheet attached to [GMRW22].

The idea is that if the arguments to a function $B(x_1, \ldots, x_k)$ are algebraic expressions in other variables $y_{i,j}, 1 \le i \le k, 1 \le j \le m_i$, then implicit differentiation can be used to compute derivatives of B as rational functions of all the variables involved. One can then clear denominators and eliminate the y variables to obtain an algebraic representation for the derivatives of B. To illustrate: in the present example, we need to compute $A_{ff}(x,f)$; originally A was represented as a rational function of x and y; reparametrizing by x and f requires substituting an algebraic expression in x

and f for y, via $P(x, y, f) = 0$. We then use the identity

$$\frac{\partial A(x, y(x, f), f)}{\partial f} = \frac{\partial A(x, y, f)}{\partial f} + \frac{\partial A(x, y, f)}{\partial y} \frac{\partial y(x, f)}{\partial f} .$$

Implicitly differentiating P, this becomes

$$\frac{\partial A(x, y(x, f), f)}{\partial f} = \frac{\partial A(x, y, f)}{\partial f} - \frac{\partial A(x, y, f)}{\partial y} \frac{\partial P/\partial f}{\partial P/\partial y} .$$

Computing terms in (4.21) in this way represents A_{ff}, ϕ_{xx} and ϕ_{ff} as rational functions of x, f and y. A Gröbner basis computation using P to eliminate y, and using the critical point equations to eliminate f and x, one obtains a polynomial satisfied by r and A_{ff}, another satisfied by r and ϕ_{xx} and a third satisfied by r and ϕ_{ff}. These polynomials are quadratic, leading to the following solutions by radicals:

$$\begin{aligned} A_{ff}(x(\hat{\mathbf{r}}), 1) &= \frac{(-\hat{r} - 1)\sqrt{-4\hat{r}^2 + 4\hat{r} + 1} + 2\hat{r}^2 - 3\hat{r} - 1}{4\hat{r}^3(\hat{r} - 1)^2} \\ \phi_{ff}(x(\hat{\mathbf{r}}), 1) &= \frac{1 + \sqrt{-4\hat{r}^2 + 4\hat{r} + 1}}{4\hat{r}(\hat{r} - 1)} \\ \phi_{xx}(x(\hat{\mathbf{r}}), 1) &= \frac{(4\hat{r}^2 - 2\hat{r} - 1)\sqrt{-4\hat{r}^2 + 4\hat{r} + 1} + 4\hat{r}^2 - 4\hat{r} - 1}{16\hat{r}^3(\hat{r} - 1)^2} \end{aligned}$$

These three polynomials, along with the polynomial relation between $A_{ff}, \phi_{xx}, \phi_{ff}$ and C_1 obtained from squaring (4.21) and clearing denominators gives an elimination polynomial $\left(\hat{r}^2 - \hat{r} - \frac{1}{4}\right) C_1^4 + 4$ satisfied by C_1 and $\hat{r}$, yielding the radical expression

$$C_1 = -\frac{2}{(1 + 4\hat{r} - 4\hat{r}^2)^{1/4}} . \tag{4.22}$$

Putting this all together, using $\phi(x(\hat{\mathbf{r}}), 1) = \hat{r}\log(x(\hat{r})) + (1 - \hat{r})\log(y(\hat{r}))$ along with equation (4.16) and Proposition 4.6, yields

$$a_{\mathbf{r}} \approx \frac{-C_1}{2\pi} \frac{\left(x(\hat{\mathbf{r}})^{-\hat{r}} y(\hat{\mathbf{r}})^{\hat{r}-1}\right)^N}{N^2}.$$

Expanding,

$$a_{\mathbf{r}} \approx \left(\frac{1}{2} + \hat{r} - \frac{\sqrt{1+4\hat{r}-4\hat{r}^2}}{2}\right)^{-N\hat{r}} \left(\frac{3}{2} - \hat{r} - \frac{\sqrt{1+4\hat{r}-4\hat{r}^2}}{2}\right)^{N(\hat{r}-1)} \left(\frac{1}{(1+4\hat{r}-4\hat{r}^2)^{1/4}\pi} N^{-2}\right)$$

where $N = |\mathbf{r}|$ and $\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|}$. For example, when $\hat{r} = 1/2$ and N is even,

$$a_{N/2,N/2} \approx 3.4142^N (0.2677\, N^{-2}),$$

agreeing with the value given by the considerably more complicated expression in [GMRW22, online attachment].

4.4.3. Bi-colored Motzkin paths

A bicolored Motzkin path on the x-y plane starts at the origin and ends at the x-axis, never goes below the x-axis and takes steps $U = (1,1)$, $D = (1,-1)$, and two colored horizontal steps $H_1 = (1,0)$ and $H_2 = (1,0)$. Let $\mathcal{M}^2$ be the set of bicolored Motzkin paths. Define $u(M), d(M), h_1(M), h_2(M)$ to be the number of U, D, H_1, H_2 steps in the bicolored Motzkin path $M \in \mathcal{M}^2$ respectively. The generating function $F(x,y) = \sum_{M \in \mathcal{M}^2} x^{d(M)+h_1(M)} y^{u(M)+h_2(M)}$ is counting the number of paths by the total number of D and H_1 steps and the total number of U and H_2 steps. In particular, $[x^i y^j]F(x,y)$ is the number of such paths with i steps in D and H_1 and j steps in U and H_2. [Eli21, Lemma 2.1] shows that

$$F(x,y) = \frac{1 - x - y - \sqrt{(1-x-y)^2 - 4xy}}{2xy}.$$

Let $\hat{\mathbf{r}} = (\hat{r}, 1-\hat{r})$ and let $\mathbf{r} = (r,s) = N\hat{\mathbf{r}}$. We calculate the asymptotic formula for $a_{\mathbf{r}} = [x^{\hat{r}N} y^{(1-\hat{r})N}]F(x,y)$. The minimal polynomial $P(x,y,f)$ satisfying $P(x,y,F(x,y)) = 0$ is

$$P(x,y,f) = xyf^2 + (x+y-1)f + 1.$$

Notice that P satisfies all four assumptions in Table 4.1.

The discriminant is given by

$$\mathrm{discr}(P) = (1 - x - y)^2 - 4xy\,.$$

Given $\hat{\mathbf{r}} = (\hat{r}, 1 - \hat{r})$ with $0 < \hat{r} < 1$, there is one minimal critical point on the branch locus, given by

$$x(\hat{\mathbf{r}}) = \hat{r}^2, y(\hat{\mathbf{r}}) = \hat{r}^2 - 2\hat{r} + 1.$$

The point $(x(\hat{\mathbf{r}}), y(\hat{\mathbf{r}}))$ lifts to a unique point $\mathbf{p} = (x(\hat{\mathbf{r}}), y(\hat{\mathbf{r}}), f(\hat{\mathbf{r}}))$ where $f(\hat{\mathbf{r}}) = -\dfrac{\hat{r}^2 - \hat{r} + 1}{(\hat{r} - 1)\hat{r}}$. The branch defining F is the lower of two branches, the other being at $+\infty$. Hence, the rank of $f_0 = F(0,0) = 1$ is 2. Therefore, when we apply Corollary 4.15, we don't need to flip the sign.

Next, we parametrize near $\mathbf{p}$ by x and f coordinates. The Jacobian is $J = \partial y/\partial f$ is $-P_f/P_y$, the amplitude function is $A(x, f) = fJ/(x \cdot y(x, f))$, and the phase function is $\phi(x, f) = \hat{r}\log(x) + (1 - \hat{r})\log(y(x, f))$. By Proposition 4.6, equation (4.16) becomes

$$\begin{aligned}
a_{\mathbf{r}} &\approx \left(\frac{1}{2\pi i}\right)^2 e^{-N\phi(x(\hat{\mathbf{r}}), f(\hat{\mathbf{r}}))} \left(\frac{2\pi}{N}\right) C_1 N^{-1} \\
&= \left(\frac{1}{2\pi i}\right)^2 \left(x(\hat{\mathbf{r}})^{-N\hat{r}} y(\hat{\mathbf{r}})^{-N(1-\hat{r})}\right) \left(\frac{2\pi}{N}\right) C_1 N^{-1} \\
&= -\frac{1}{2\pi}\left(\hat{r}^{-2N\hat{r}}(\hat{r} - 1)^{2N(\hat{r}-1)}\right) C_1 N^{-2}\,,
\end{aligned}$$

where the constant C_1 is computed by Corollary 4.7. None of partial derivatives of A and ϕ involved in (4.12) of Corollary 4.7 vanishes at the critical point. However, when pieced together, they yield a simple form for the constant C_1:

$$C_1 = -\frac{1}{2(\det H)^{1/2}} \frac{2}{\hat{r}^2(\hat{r} - 1)}\,.$$

Furthermore, $\det H = (1 - \hat{r})^2$, with Corollary 4.15 specifying that the choice $(\det H)^{1/2}$ is to be interpreted as $-\sqrt{\det H}$, in other words, the negative real root. Therefore, $C_1 = -1/(\hat{r}^2(\hat{r} - 1)^2)$

and

$$a_{\mathbf{r}} \approx \left(\frac{1}{\hat{r}^{2\hat{r}}\ (1-\hat{r})^{2(1-\hat{r})}}\right)^N \frac{1}{2\pi\hat{r}^2(\hat{r}-1)^2} N^{-2} \,.$$

4.4.4. 0-2-5 trees

The usual definition of a binary tree is a rooted tree in which each vertex has either zero or two children. The number of binary trees with n nodes is the nth Catalan number, due to the recursion satisfied by binary trees, as follows. If we allow the empty tree, a binary tree is either empty, or a root with a left and right subtree. Thus, the generating function $f(z) := \sum_{n=0}^{\infty} a_n z^n$ for the numbers a_n of n-vertex binary trees is equal to $1 + zf(z)^2$.

One can generalize this to allow the number of children to be either zero or a member of a given subset E of the positive integers. Such trees can be counted by the number of nodes having m children, for each $m \in E$. When E is a finite set, this leads to an algebraic generating function in finitely many variables. The numbers n_j of vertices having j children in such a tree, if non-empty, satisfy the linear relation:

$$N := n_0 + \sum_{j \in E} n_j = 1 + \sum_{j \in E} j n_j \,.$$

For this reason, the number of independent statistics by which these trees can be counted is $|E|$ rather than $|E| + 1$; for example, one might keep track of N and all but one of the n_j.

This subsection analyzes one example, $E = \{2, 5\}$, in other words, trees where every vertex has zero, two or five children. Let a_{ij} count 0-2-5 trees T with $N(T) = j$ and $n_5(T) = i$. Let $F(y,z) = \sum_{i,j \geq 0} a_{ij} y^i z^j$. The relation $F(y,z) = 1 + z[((F(y,z)-1)^2 + 1) + y(F(y,z)-1)^5]$ follows from the recursive description of a 0-2-5 tree as either being empty or consisting of a root and either zero or two trees (counted by 1 or $(F-1)^2$, respectively), or five subtrees (counted by $(F-1)^5$ with an extra factor of y to keep track of the addition of a 5-child vertex). The total number $N(T)$ of a nonempty such tree T is equal to $1 + 2n_2(T) + 5n_5(T)$.

When counting by N and n_5, there is a periodicity because $N - 1 - 5n_5 = 2n_2$ implies $N + n_5 \equiv$

$1 \bmod 2$. Therefore, the generating function has nonconstant terms only of odd total degree[6]. In classical generating function singularity analyses, periodicity is reflected by summing asymptotic series around more than one critical point; the coordinates of the critical points differ by factors that are roots of unity; see, e.g., [Sta97] for a discussion of the univariate case. The same holds for the present analysis. Because $F-1$ is an odd function, critical points come in pairs (x, y) and $(-x, -y)$, whose asymptotics series exactly cancel in even degrees (as we know they must, to produce zero) and are equal in odd degrees.

Let $P(y, z, f) = 1 - f + z[((f-1)^2 - 1) + y(f-1)^5]$. Then $P(y, z, F) = 0$ and $F(0, 0) = 1$ uniquely specifies F as an algebraic function, analytic in a neighborhood of the origin in $\mathbb{C}^2$. We chose the class of 0-2-5 trees to analyze rather than, say, 0-2-3 trees, in order to show that nothing changes when F cannot be expressed via radicals. The term with the highest degree in f is the f^5 term, whose coefficient is xy. Therefore, the pole variety `pole` is the union of two coordinate axes, and the expanding torus will not run into it. Using computer algebra, we verify that P defines a smooth variety and compute the branch locus `br` defined by the discriminant

$$\mathrm{discr}(P) = 3125y^2z^5 - 2250yz^4 + 108z^5 + 1600yz^2 - 27z^3 - 256y\,.$$

As expected, this is an odd function, meaning that $\mathrm{discr}(P)(-y, -z) = -\mathrm{discr}(P)(y, z)$, and leading to an antipodal symmetry among roots (y, z) of the discriminant. However, the polynomial P has degree 5 in f, violating the last assumption in Table 4.1 and requiring extra care. To carry out the branch determination in Conclusion 1 of Theorem 4.14 we determine the rank of $F(0, 0) = 1$ within the ordering of real roots of $P(0, 0, \cdot)$. A little extra work is needed in this case because the coefficients of $P(0, 0, \cdot)$ vanish beyond degree 1, so four of the five roots are at infinity. Computing Puiseux expansions, we see that the roots of $P(\varepsilon, \varepsilon, \cdot)$ occur at roughly $1+\varepsilon, \pm\varepsilon^{-1/2}$ and $c \pm i\varepsilon^{-1/2}$. We see that three are real, and that $F(0)$ is the limit of the middle one of the three.

[6]Indeed, this can be detected directly from the annihilating polynomial P. In this case, for example, letting $F = G - 1$, the defining polynomial is $G = z(1 + G^2 + yG^5)$; the support (the exponents of monomials) is $\{(0, 0, 1), (1, 0, 0), (1, 0, 2), (1, 1, 5)\}$, which is contained in the odd sublattice of $\mathbb{Z}^3$.

Critical points in the direction $n_5/N = \hat{r}/(1-\hat{r})$ correspond to the equation $Q := \hat{r} z\, P_z - (1-\hat{r}) y\, P_y = 0$. Computing a Gröbner basis for the ideal generated Q, P and $\partial P/\partial f$ gives the generators $[\left(4\,\hat{r}^2 - 8\,\hat{r} + 4\right) z^2 + 12\,\hat{r}^2 + 4\,\hat{r} - 1, \left(216\,\hat{r}^3 - 108\,\hat{r}^2 + 18\,\hat{r} - 1\right) y - \left(8\,\hat{r}^3 - 4\,\hat{r}^2 - 4\,\hat{r}\right) z, (2\,\hat{r} - 2)\, z + (2\,\hat{r} + 1)\, f - 2\,\hat{r} - 1]$, in other words, precisely two points (y_0, z_0, f_0) and $(-y_0, -z_0, 2 - f_0)$. Projected to the y-z plane, both of these lie on the same centered torus. We verify, using the techniques described in Chapter 4.3.3, that the positive point is a minimal point for $\mathrm{discr}(P)$ in direction $n_5/N = \hat{r}/(1-\hat{r})$, hence both points are. Computer algebra output of this verification is omitted.

To complete the homotopy continuation, we check that the critical point is on the same branch of F as is $(0,0,1)$. It suffices to check this for any single $\hat{r}$ in the feasible interval $(0, 1/6)$, as long as we also check that roots do not coalesce further for any value of $\hat{r}$ in this interval, which follows from checking that along the subset of the curve in the (y,z)-plane defined by $\mathrm{discr}(P)$ parametrized by $0 < r < 1/6$, the polynomial $F(x(r), y(r), \cdot)$ always has precisely one doubled root, never more. Computer algebra output is again omitted. Now we set $\hat{r} = 1/11$, obtaining the Gröbner basis $[400\, z^2 - 65, 125y + 520\, z, -13 - 20\, z + 13\, f]$. At the positive real point (y_0, z_0) on this curve, the defining polynomial factors into

$$\left(26\, f^3 + \left(4\sqrt{65} - 78\right) f^2 + \left(108 - 8\sqrt{65}\right) f + 9\sqrt{65} - 56\right) \left(-13\, f + 13 + \sqrt{65}\right)^2$$

Then f_0 is the root of the second polynomial, roughly 1.62. This root is doubled and greater than the third real root, which is roughly -0.28, coming from the first polynomial. Therefore the branch of F containing the initial condition $(0,0,1)$ is the lower of two branches passing through (y_0, z_0, f_0). Lifting a homotopy from T to the torus T' through (y_0, z_0, f_0) yields a homotopy of $\tilde{T}$ to $\tilde{T}'$; here, $\tilde{T}'$ is a torus in the pre-image $\pi^{-1} T'$ with positive orientation, passing through (y_0, z_0, f_0), and the height $h_{\hat{\mathbf{r}}}$ with $\hat{\mathbf{r}} = (1/11, 10/11)$ is uniquely maximized on $\tilde{T}'$ at (y_0, x_0, f_0).

The upshot of this is that we have verified that the coefficients $a_{\mathbf{r}}$ of the generating function F for 0-2-5-trees counted by total nodes and outdegree-5 nodes can be estimated by the integral

$$\left(\frac{1}{2\pi i}\right)^2 \int_{\tilde{T}'} \exp(-N\phi) f\, dz\, dy \tag{4.23}$$

over a positively oriented $\tilde{T}' \subseteq \widetilde{\mathcal{V}}$ that passes through the points (y_0, z_0, f_0) and $(-y_0, -z_0, 2 - f_0)$, at which it is stationary for the phase function $\phi = \hat{r}\log y + (1-\hat{r})\log z$. We may compute the result for the positive point only and double it to estimate all a_{rs} with $r+s$ odd, the even coefficients being zero.

Given a direction $\hat{\mathbf{r}} = (\hat{r}, 1-\hat{r})$, we calculate the asymptotics of $a_{\mathbf{r}}$ where $\mathbf{r} = N\hat{\mathbf{r}}$, $N\hat{r} \in \mathbb{N}$, and $N(1-\hat{r}) \in \mathbb{N}$. To make $a_{\mathbf{r}}$ have combinatorial meaning, we restrict $0 < \hat{r} < 1/6$. There are two antipodal minimal critical points on the branch locus defined by the discriminant, the positive one of which is

$$y_0(\hat{\mathbf{r}}) = -\frac{2(2\hat{r}+1)\hat{r}\sqrt{-12\hat{r}^2 - 4\hat{r} + 1}}{(6\hat{r}-1)^3} \qquad z_0(\hat{\mathbf{r}}) = -\frac{\sqrt{-12\hat{r}^2 - 4\hat{r} + 1}}{2(\hat{r}-1)}\,.$$

This lifts uniquely to the vertical tangent locus at (y_0, z_0, f_0); we will not need an explicit expression for f_0; moreover, we omit the argument $\hat{\mathbf{r}}$ unless comparing expressions for different directions. The exponential growth rate is $\left(y_0^{-\hat{r}} z_0^{\hat{r}-1}\right)^N$.

By Proposition 4.6, the next thing left to calculate in the integral (4.23) is the constant term C_1. We show the C_1 for the integral (4.23) at the critical point (y_0, z_0, f_0). That of the integral at the other critical point is the same. We parametrize near (y_0, z_0, f_0) using the z- and f-coordinates. The Jacobian $J := \partial y/\partial f$ at (y_0, z_0, f_0) is $-P_f/P_y$ evaluated at the point. The amplitude function A is $fJ/(yz)$ and the phase function ϕ is $\hat{r}\log(y) + (1-\hat{r})\log(z)$. Using the techniques of implicit differentiation introduced in Chapter 4.4.2, we can calculate every partial derivatives of A and ϕ needed in Corollary 4.7. Unlike Chapter 4.4.2, none of these partial derivatives vanishes. The good news is that the calculation shows that these partial derivatives at the critical point do not depend on the critical point they are evaluated at. For example, at the critical point, $\phi_{zf} = -8(\hat{r}-1)^2/(6\hat{r}-1)$, which doesn't involve any y, z or f in which we need to plug y_0, z_0 and f_0. All partial derivatives in (4.12) for these two critical points are the same and so the constants C_1 are the same. In particular,

$$C_1 = \frac{2(1-\hat{r})}{\hat{r}\sqrt{1-6\hat{r}}\sqrt{1+2\hat{r}}(\det H)^{1/2}}$$

where $\det H = \dfrac{4(1-\hat{r})^3(1+2\hat{r})}{(1-6\hat{r})\hat{r}}$. The square root on $\det H$ is chosen to be $-\sqrt{\det H}$ by Corollary 4.15. Therefore,

$$C_1 = -\frac{1}{\sqrt{\hat{r}}\sqrt{1-\hat{r}}(1+2\hat{r})}\,.$$

Combining everything together, by Proposition 4.6, the integral 4.23 at the critical point (y_0, z_0, f_0) is

$$\begin{aligned}\left(\frac{1}{2\pi i}\right)^2 \int_{\tilde{T}'} \exp(-N\phi) f\,dz\,dy \quad &\approx \quad \left(\frac{1}{2\pi i}\right)^2 \left(y_0^{-\hat{r}} z_0^{\hat{r}-1}\right)^N \frac{2\pi}{N} C_1 N^{-1} \\ &= \quad \frac{\left(y_0^{-\hat{r}} z_0^{\hat{r}-1}\right)^N}{2\pi\sqrt{\hat{r}}\sqrt{1-\hat{r}}(1+2\hat{r})} N^{-2}\end{aligned}$$

When N is even the parity constraint implies $a_{\mathbf{r}} = 0$. When N is odd, the two critical points contribute equally to the asymptotics, leading to

$$a_{\mathbf{r}} = \frac{\left(y_0^{-\hat{r}} z_0^{\hat{r}-1}\right)^N}{\pi\sqrt{\hat{r}}\sqrt{1-\hat{r}}(1+2\hat{r})} N^{-2}\,.$$

For example, when $\hat{\mathbf{r}} = (1/11, 10/11)$, one obtains $a_{N/11,10N/11} \approx \frac{2.1792^N}{0.3397\pi} N^{-2}$ for N an odd multiple of 11. When $\hat{r} = 0$, we are counting full binary trees. The number of full binary trees with N nodes is the $((N-1)/2)$-th Catalan number. We can see that the exponential growth rate of $a_{\mathbf{r}}$ here agrees with that in Chapter 4.4.1 in this case, both yielding $(2+o(1))^N$.

4.5. Orientation

We recall those hypotheses from Theorem 4.14 that will be needed for sign determination. The first two follow from the standing hypotheses preceding the theorem.

real The defining polynomial $P = P(\mathbf{z}, f)$ is real.

branches All roots of the restriction of P to the F axis are simple (i.e. $\operatorname{discr}(\mathbf{0}) \neq 0$), and one of them defines the generating function F we consider.

boundary The polydisk of convergence of F has radii $(e^{p_1}, \ldots, e^{p_d})$, and the corresponding point $\mathbf{p} = (p_1, \ldots, p_d)$ lies on the boundary of the component of the complement to the amoeba of the discriminant discr (and is inside the same component of the complement to the amoeba of the leading coefficient of P, as image of the small torus).

simple The vertical fiber $\{\mathbf{z} = \mathbf{z}_*\}$ has a simple tangency to the variety $\{P = 0\}$ at the real point $(\mathbf{z}_*, f_*)$; here $\mathbf{z}_* = (e^{p_1}, \ldots, e^{p_d})$ and $f_* = \lim_{\mathbf{z}\to\mathbf{z}_*} f_j(\mathbf{z})$.

convex The logarithmic Gauss map $\mathtt{br} \to \mathbb{CP}^{d-1}$ is strongly convex, meaning quadratically nondegenerate at $\mathbf{z}_*$ (here $\mathtt{br} = \{\mathrm{discr} = 0\}$).

If $(\mathbf{z}_*, f_*) \in \widetilde{\mathcal{V}}$ is a critical point of the phase $|\mathbf{z}^{\mathbf{r}}|$, it follows that near that critical point, the defining polynomial P can be represented as

$$P = \hat{\mathbf{r}} \cdot (\mathbf{z} - \mathbf{z}_*) + q(\mathbf{z} - \mathbf{z}_*, f - f_*) \tag{4.24}$$

where $q = Q + R_3$, where Q is a real quadratic form, R_k, here and throughout, denotes a remainder term vanishes to order k in $f - f_*$ and $\mathbf{z} - \mathbf{z}_*$.

Lemma 4.37. *Assume the five preceding hypotheses. Let F be an algebraic function solving $P(\mathbf{z}, F(\mathbf{z})) = 0$ and let $\hat{\mathbf{r}}$ be the logarithmic gradient of the discriminant* $\mathrm{discr}(P, f)$ *at $\mathbf{z}_*$. Then*

1. *The lift of a small d-torus $\mathbf{T}_\epsilon$ around the origin to the branch of $\widetilde{\mathcal{V}}$ defining $f = f_j$ is homologous (can be in fact deformed within the space of embeddings into $\widetilde{\mathcal{V}}$) to a d-torus $\mathbf{T}_* \in \widetilde{\mathcal{V}}$ passing through $\mathbf{z}_*$ so that the restriction of the phase function $|\mathbf{z}^{\hat{\mathbf{r}}}|$ to $\mathbf{T}_*$ attains its global maximum at $(\mathbf{z}_*, f_*)$ and is a Morse function there, meaning that $h_{\hat{\mathbf{r}}}$ restricted to $\widetilde{\mathcal{V}}$ has a nondegenerate hessian matrix at $(\mathbf{z}_*, f_*)$.*

2. *Let η be a $(d-1)$ holomorphic form defined in a vicinity of $\mathbf{z}_*$, such that $d\phi \wedge \eta = \omega$, where $\omega := \prod_k \frac{1}{2\pi i}\frac{dz_k}{z_k}$. Then $df \wedge \eta$ defines the orientation on $\mathbf{T}_*$ (near $(\mathbf{z}_*, f_*)$) consistent with the orientation inherited from the the orientation of $\mathbf{T}_\epsilon$ if the real branch of f approaches f_* from below, and with opposite orientation if the real branch of f approaches f_* from above.*

Remark. The second claim of the proposition seems to depend unexpectedly on the orientation of the f axis; we remark that the integrated form includes a factor $\partial q/\partial f$, which flips sign together with df, leaving the final integral invariant with respect to such flips.

PROOF: To prove the first claim, we can consider a radii increasing homotopy of the tori in the $\mathbf{z}$-space to the torus whose radii are given by $\exp(\mathbf{p})$. During this expansion, up to the last point, all the tori are disjoint from `br`, hence, over the domain of this deformation, the projection $\widetilde{\mathcal{V}}_* \to \mathbb{C}^d_*$ is a covering, and the homotopy can be lifted to $\widetilde{\mathcal{V}}$, producing at time $t = 1$ the torus $\tilde{T}'$. For $\varepsilon > 0$ sufficiently small, we will need to analyze the time-$(1-\varepsilon)$ torus, which we denote by $\mathbf{T}_-$.

We switch to local exponential coordinate chart centered at $\mathbf{p}_*$: we chose the real at $\mathbf{p}_*$ branch of the logarithm, and denote all variables in log space by upper case letters, except for $\mathbf{p}$. Thus, $z_k = \exp(p_k + Z_k)$, with $Z_k = X_k + iY_k$. We do the same with the f coordinate: $f = f_* \exp(G + iH)$.

One can always perform a real translation and linear volume preserving transformation on $\mathbf{Z}$ so that in the new coordinates $(W_1, \ldots, W_d)$ we have $W_1 = \hat{\mathbf{r}} \cdot \mathbf{Z}$. Further, we can choose the W_1 axis and the deformation of tori so that it ends with a short segment on the negative half-axis W_1.

All these transformations together result in the local description of the variety $\mathcal{V}$ in the new coordinates $(F, W_1, \mathbf{W})$ (here $\mathbf{W} = (W_2, \ldots, W_d)$) as

$$\mathcal{V} = \{W_1 = \phi(F, \mathbf{W})\}, \tag{4.25}$$

with ϕ vanishing at the origin to order 2.

The *[simple]* condition implies that we can expand the quadratic part ϕ_2 of ϕ as (the reason for the signs will become clearer later):

$$\phi_2 = -aF^2 + 2F\mathbf{b} \cdot \mathbf{W} - q(\mathbf{W}) \,.$$

Here, $\mathbf{b}$ is a real covector, and q is a real quadratic form.

By the *[branches]* condition, two real branches of f merge at $\mathbf{z}_*$ when $\mathbf{z}$ follows the homotopy from the origin to $e^{\mathbf{p}_*}$, implying that for $\epsilon > 0$ there are real solutions to $-\epsilon = -aF^2 + R_3$, obtained by setting $\mathbf{W} = 0$ in (4.25), so that $a > 0$.

As the logarithmic coordinate change acts in each coordinate independently, the discriminant can be computed in logarithmic coordinates. This can be accomplished by eliminating F from an equation that tracks sufficiently many terms to give us the leading (quadratic) expression for W_1 in terms of the other variables:

$$0 = \frac{\partial \phi}{\partial F} = -2aF + 2\mathbf{b} \cdot \mathbf{W} + R_2; W_1 = \phi(F, \mathbf{W}) .$$

This results in

$$W_1 = \frac{(\mathbf{b} \cdot \mathbf{W})^2}{a} - q(\mathbf{W}) + R_3$$

as the local equation defining the discriminant variety $\mathtt{br}$.

The real part of $\mathtt{br}$ projects under the log mapping to the contour of the amoeba of $\mathrm{discr}(P)$. The *[convex]* condition implies that the contour near the origin is smooth, coincides with the boundary of the amoeba, and is *quadratically convex*, hence the quadratic form $\frac{(\mathbf{b} \cdot \mathbf{W})^2}{a} - q(\mathbf{W})$ is negative definite.

After these preliminaries, we can look at the lift of the torus $\mathbf{T}_-$, i.e., the intersection of the preimage of the torus in $\mathbb{C}^d$ under the projection along F with the variety $\widetilde{\mathcal{V}}$. In our local log coordinates, where we denote the real and imaginary parts as $W_k = U_k + iV_k, k = 2, \ldots, d$, this preimage corresponds to setting $W_1 = -\epsilon + iV_1, W_k = iV_k, k \geq 2$, and $F = G + iH$.

Expanding the terms above, we arrive at

$$-\epsilon + iV_1 = -a(G^2 - H^2) - 2aiGH + 2G\mathbf{b} \cdot i\mathbf{V} + 2iH\mathbf{b} \cdot i\mathbf{V} + q(\mathbf{V}) + R_3$$

where the indices of $\mathbf{V}$ run from 2 to d, that is, $\mathbf{V} = (V_2, \ldots, V_d)$. The real part of this equation is

$$aG^2 = \epsilon + aH^2 - 2H\mathbf{b} \cdot \mathbf{V} + q(\mathbf{V}) = \epsilon + a(H - \mathbf{b} \cdot \mathbf{V}/a)^2 - (\mathbf{b} \cdot \mathbf{V})^2/a + q(\mathbf{V}) + R_3.$$

The quadratic form on right hand side above is positive definite because $(\mathbf{b}\cdot\mathbf{V})^2/a - q(\mathbf{V})$ is negative definite. Hence, for small $\varepsilon > 0$, the equation above defines a hypersurface diffeomorphic to a two-sheeted hyperboloid in the $(G, H, \mathbf{V})$ space. The sign of G on the branch below the merge point, is negative; above, positive. Projection of this surface to $H, \mathbf{V}$ gives coordinates on the sheet, so that G becomes a function of $H, \mathbf{V}$.

Now, one can also express

$$V_1 = -2aGH + 2G\mathbf{b}\cdot\mathbf{V} + R_3\,.$$

This shows that the chain $\mathbf{T}_-$ can be locally coordinatized by $H, \mathbf{V}$.

In our new coordinates the real part of the phase is U_1; the torus $\mathbf{T}_-$ is situated at the level set of the phase. Outside a vicinity of $(\mathbf{z}_*, f_*)$ the gradient of the phase ϕ is non-vanishing, and, by compactness, one can deform the chain there to the zero sublevel set of $\Re\{\phi\}$. *Within* the vicinity of the critical point, one can use the coordinatization by $H, \mathbf{V}$ to deform the chain to the d-space spanned by $H, \mathbf{V}$: along that subspace,

$$U_1 = Re(\phi(F, \mathbf{W})) = aH^2 + q(\mathbf{V}) + R_3$$

showing U_1 to be Morse and positive definite.

To compute the orientation, we represent $\omega = (2\pi i)^{-d} dW_1 \wedge \cdots \wedge dW_d$ in local logarithmic coordinates. On the variety $\widetilde{\mathcal{V}}$ one has, using (4.25)

$$\omega = \frac{1}{(2\pi i)^d}\frac{\partial\phi}{\partial F} dF \wedge d\mathbf{W}$$

where $d\mathbf{W} := dW_2 \wedge \cdots \wedge dW_d$.

Coordinatizing the variety $\widetilde{\mathcal{V}}$ locally by $F, \mathbf{W}$, we obtain

$$\omega = \frac{-2}{(2\pi i)^d}(aF - \mathbf{b}\cdot\mathbf{W}) dF \wedge d\mathbf{W}\,.$$

At the point where $W_1 = -\epsilon$ and $\mathbf{W} = \mathbf{0}$, this reduces to

$$\omega = \frac{-2}{(2\pi)^d} \, (aG) \, dH \wedge d\mathbf{V} \, .$$

Thus ω is a *positive* multiple of $dF \wedge d\mathbf{W}$ on $\mathbf{T}_-$ on the branch where $G < 0$, i.e. $f < f_*$, and *negative* where $f > f_*$. Equivalently, the orientation is given by $dH \wedge dY_2 \wedge \ldots \wedge dY_d$ on the lower branch of f, and is opposite that on the upper branch. $\square$

CHAPTER 5

MULTIPLE POINTS

In this chapter, we shift focus from algebraic generating functions back to rational generating functions. This chapter reviews known material from [PW04], [BMP22], [BMP24b], and [PWM24], in order to understand the new results in Chapter 6. In the most general settings, rational functions are in the form of $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$. We assume without loss of generality that $P(\mathbf{z})$ and $Q(\mathbf{z})$ are coprime in $\mathbb{C}[\mathbf{z}]$. $F(\mathbf{z})$ is assumed to have a convergent power (or Laurent) series expansion $F(\mathbf{z}) = \sum_{\mathbf{r}\in\mathbb{Z}^d} a_{\mathbf{r}}\mathbf{z}^{\mathbf{r}}$ in some domain $\mathcal{D} = \mathrm{ReLog}^{-1}(B) \in \mathbb{C}^d$ for a component B in $\mathsf{amoeba}(Q)^c$. To get the coefficients $a_{\mathbf{r}}$, the first tool used here is the Cauchy integral formula (1.1), that is,

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z}) d\mathbf{z},$$

for a torus $T = \{\mathbf{z} \in \mathbb{C}^d : |\mathbf{z}| = \exp(\mathbf{x})\}$ where $\mathbf{x} \in B$. We then deform the torus T so that we can represent $a_{\mathbf{r}}$ as a finite sum of saddle point integrals around critical points. The deformation requires us to look at the geometry of the singular variety of $F(\mathbf{z})$. Since $P(\mathbf{z})$ and $Q(\mathbf{z})$ are coprime, the singular variety $\mathcal{V}$ of $F(\mathbf{z})$ is precisely the variety $\mathcal{V}_Q$ defined by $\{\mathbf{z} \in \mathbb{C}^d : Q(\mathbf{z}) = 0\}$. In other words, the d-form $\omega = \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z}) d\mathbf{z}$ in the above integral is analytic in $\mathcal{M} = \mathbb{C}^d_* - \mathcal{V}$.

When $\mathcal{V}$ is a smooth variety, we think it as a smooth manifold. If the deformation of T to T' ever needs to cross $\mathcal{V}_*$, then the integral of $\int_T \omega$ is equal to $\int_T' \omega$ plus an integral of a $(d-1)$-form $\mathrm{Res}\,(\omega)$ over a $(d-1)$-chain $\mathbf{INT}[T, T'; \mathcal{V}_*]$ on $\mathcal{V}_*$ (see Chapter 1.2.2).

When $\mathcal{V}$ is not smooth at some point, $\mathcal{V}$ may look very differently from a manifold locally at that point. For example, $Q(x, y) = (1-x)(1-y)$ defines $\mathcal{V}$ to be a union of two lines intersecting at $(1, 1)$. Locally at any other point than $(1, 1)$, $\mathcal{V}$ still looks like a 1-dimensional manifold (a line). However, at $(1, 1)$, $\mathcal{V}$ is not locally a manifold of any dimension. Another example is $Q(x, y, z) = z^2 - x^2 - y^2$, and $\mathcal{V}_Q$ looks like an hourglass shape with two cones touching at $(0, 0, 0)$ and thus not smooth at $(0, 0, 0)$. We call the first type of points multiple points and the second type of points cone points.

[BP11] addresses the problem when there is an isolated quadratic point. The main idea is to borrow results from hyperbolic polynomials and construct vector field to deform the integral chain. The original paper is 80 pages long and a short summary can be found in [PWM24, Chapter 11]. We give an introduction to multiple points in this chapter and pave the way for next chapter to study pseudo multiple points.

5.1. Introduction

We assume that $\mathcal{V}$ is an analytic hypersurface (see Definition 2.7) in $\mathbb{C}^d$. By definition, for any $\mathbf{p} \in \mathcal{V}$, there is an open neighborhood $\mathcal{D}$ of $\mathbf{p}$ in $\mathbb{C}^d$ and an analytic function Q on $\mathcal{D}$ such that $\mathcal{V} \cap \mathcal{D} = \mathcal{V}_Q \cap \mathcal{D}$. The analytic function Q depends on $\mathcal{D}$ and it is only guaranteed to be analytic inside $\mathcal{D}$. A point $\mathbf{p} \in \mathcal{V}$ is a smooth point if we can choose Q so that $\nabla Q(\mathbf{p}) \neq 0$. For ACSV on rational generating functions, the singular variety $\mathcal{V}$ is globally defined by the denominator Q in the generating function $F = P/Q$. Therefore, we give an explicit characterization on what it means for $\mathcal{V} = \mathcal{V}_Q$ being smooth.

Definition 5.1 (smooth variety defined by Q globally)**.** *The variety $\mathcal{V}_Q$ defined by $Q \in \mathbb{C}[\mathbf{z}]$ is smooth if every point on $\mathcal{V}_Q$ is a smooth point. That is, Q and $\nabla \widetilde{Q}$ never vanish together. Here $\widetilde{Q}$ is the* **square-free part** *$\widetilde{Q}$ of Q, the product of distinct irreducible factors of Q in the ring $\mathbb{C}[\mathbf{z}]$.*

Since $\widetilde{Q}$ is a polynomial, $\nabla \widetilde{Q}$ is continuous. Therefore, if $\mathbf{p}$ is a smooth point, then there is a neighborhood of $\mathbf{p}$ in which every point on $\mathcal{V}_Q$ is a smooth point. By implicit function theorem, this implies that $\mathcal{V}_Q$ is a complex manifold of dimension $d-1$ and hence a smooth manifold of real dimension $2d-2$ in a neighborhood of a smooth point $\mathbf{p}$. Conversely, if $\nabla \widetilde{Q}(\mathbf{p}) = 0$, then $\mathcal{V}$ is not a smooth manifold near $\mathbf{p}$. The proof for the converse statement can be found on [PWM24, Lemma 7.6].

We are now ready to define what a **multiple point** is on an analytic hypersurface $\mathcal{V}$. Geometrically, a point $\mathbf{p} \in \mathcal{V}$ is a multiple point if locally near $\mathbf{p}$, the variety $\mathcal{V}$ can be decomposed to a union of complex manifolds.

Definition 5.2 (multiple point, geometric version)**.** *A point* $\mathbf{p} \in \mathcal{V}$ *is a* **multiple point** *if there exist complex manifold* $\mathcal{V}_1, \cdots, \mathcal{V}_s$ *such that* $\mathcal{V} \cap U = (\mathcal{V}_1 \cap U) \cup \cdots \cup (\mathcal{V}_s \cap U)$ *for every sufficiently small neighborhood* U *of* $\mathbf{p}$ *in* $\mathbb{C}^d$*.*

Example 5.3. *[PWM24, Example 12.25] There is a magical biased coin such that one can let the head appears in a flipping with probability* $2/3$ *in the first* N *flips. Afterwards, the probability of a head goes down to* $1/3$*. One can freely choose what number* N *should be. One is told to get* r *heads and* s *tails in the total* $r + s$ *flips. Each choice of* $N = n \leq r + s$ *has a probability of achieving the requirement. On average, what is the number of winning choices for* N*? Explicitly, what is* $\sum_{n=0}^{r+s} \mathbb{P}(r \text{ heads and } s \text{ tails} | N = n)$*?*

Let a_{rs} *be the answer. Then the generating function for* a_{rs} *is*

$$F(x, y) = \frac{P(x, y)}{Q(x, y)} = \frac{1}{(1 - \frac{1}{3}x - \frac{2}{3}y)(1 - \frac{2}{3}x - \frac{1}{3}y)}$$

Notice that $(1, 1)$ *is a multiple point for* $\mathcal{V}_Q$ *here, where* $\mathcal{V}_1 = \{(x, y) \in \mathbb{C}^2 : 1 - \frac{1}{3}x - \frac{2}{3}y = 0\}$ *and* $\mathcal{V}_2 = \{(x, y) \in \mathbb{C}^2 : 1 - \frac{2}{3}x - \frac{1}{3}y = 0\}$

This is the case where the singular variety is an hyperplane arrangement. Chapter 5.3 reviews methods developed in [BMP24b] for this case. In particular, applying equation (5.9) *with* $\boldsymbol{\sigma} = (1, 1)$ *and* $\mathbf{r} = (r, s)$*, we know that* a_{rs} *is* 3 *as* $r, s \to \infty$ *and* $r/(r + s)$ *is in a compact subinterval of* $(1/3, 2/3)$*. Therefore, on average, there are three winning choices when* r *and* s *are sufficiently large.* $\square$

5.1.1. Ring of analytic germs at a point

Besides the geometric viewpoint, there is also an algebraic way to define a multiple point. To begin with, we introduce $\mathcal{O}_{\mathbf{p}}$, the **ring of convergent power series at a point** $\mathbf{p} \in \mathbb{C}^d$. In literature, it is also named **the ring of germs of analytic functions at p** and sometimes one uses the notation ${}_d\mathcal{O}_{\mathbf{p}}$ to emphasize the dimension d for $\mathbf{p} \in \mathbb{C}^d$. If $\mathbf{p} = \mathbf{0}$, we omit the subscript $\mathbf{p}$ and write ${}_d\mathcal{O} = \mathbb{C}\{\mathbf{z}\}$. In general, ${}_d\mathcal{O}_p = \mathbb{C}\{(\mathbf{z} - \mathbf{p})\}$, the ring of convergent power series at $\mathbf{p} \in \mathbb{C}^d$.

Definition 5.4 (Germs of functions). *Let $\mathbf{p} \in \mathbb{C}^d$. Let U, V be two neighborhoods of $\mathbf{p}$ and we define an equivalence relation between two functions $f : U \to \mathbb{C}$ and $g : V \to \mathbb{C}$ that they are equivalent if and only if there is a small neighborhood $\mathbf{p} \in W \subseteq U \cap V$ and $f_{|W} = g_{|W}$. We call this equivalence class $[f]$ a germ of functions at $\mathbf{p}$.*

Definition 5.5 (Ring of convergent power series at $\mathbf{p}$). *We consider the above equivalence relations only on locally analytic functions near $\mathbf{p}$. Each equivalence class is called a germ of analytic functions at $\mathbf{p}$. All germs of analytic functions at $\mathbf{p}$ form a ring. By local analyticity, all representatives in one equivalence class have the same convergent power series representation. Therefore, we can also consider it as the ring of convergent power series at $\mathbf{p}$.*

It should be noted that a representative of a germ of analytic functions at $\mathbf{p}$ need not to be entire. We only require local analyticity. For example, the function $\log(1 + z)$ is a representative of a germ of analytic functions at $z = 0$ but it is not entire.

Lemma 5.6 (${}_1\mathcal{O}_p$ is a PID). *Every ideal I in ${}_1\mathcal{O}_p$ is generated by a polynomial $(z - p)^n$ for some non-negative integer n.*

Remark. For $d \geq 2$, ${}_d\mathcal{O}_p$ is not a PID.

Lemma 5.7 (${}_d\mathcal{O}_p$ is a UFD). *Every element $f \in {}_d\mathcal{O}_p$ can be uniquely factorized into a product of irreducible factors (up to multiplication by a unit and reordering of factors).*

Algebraic viewpoint of multiple points

After introducing these algebraic rings, we are now ready to give another characterization of a multiple point $\mathbf{p}$ on an analytic hypersurface $\mathcal{V}$. Since ${}_d\mathcal{O}_{\mathbf{p}}$ is a UFD, for any $Q \in {}_d\mathcal{O}_{\mathbf{p}}$, we have the following unique (up to units) factorization of Q into a product of non-associated irreducible elements Q_i and a unit u in ${}_d\mathcal{O}_{\mathbf{p}}$

$$Q(\mathbf{z}) = u(\mathbf{z})Q_1^{m_1}(\mathbf{z}) \cdots Q_s^{m_s}(\mathbf{z}) \tag{5.1}$$

where $m_i \in \mathbb{N}_{>0}$. Moreover, since u is a unit in ${}_d\mathcal{O}_{\mathbf{p}}$ we have $u(\mathbf{p}) \neq 0$. Since factors Q_i are irreducibles and thus non-units, we have $Q_i(\mathbf{p}) = 0$. The factorization is unique up to unit u. We say that Q_i and Q_j are **associated** if there is a unit w in $\mathcal{O}_{\mathbf{p}}$ such that $wQ_i = Q_j$. In other words, the variety defined by Q_i near $\mathbf{p}$ is the same as that by Q_j.

Definition 5.8 (multiple points, algebraic version)**.** *A point* $\mathbf{p} \in \mathcal{V}$ *is a* **multiple point** *if* $Q(\mathbf{z})$ *has the factorization (5.1) and* $\nabla Q_i(\mathbf{p}) \neq 0$ *for all* $i = 1, \cdots, s$. *Here* Q *is an analytic function on an open neighorbood* $\mathcal{D}$ *of* $\mathbf{p}$ *in* $\mathbb{C}^d$ *such that* $\mathcal{V} \cap \mathcal{D} = \mathcal{V}_Q \cap \mathcal{D}$.

Remark. In the ring $\mathcal{O}_{\mathbf{p}}$, non-zero gradient implies irreducibility: Let $f \in \mathcal{O}_{\mathbf{p}}$ and suppose that f is reducible and thus $f = gh$. Since $g(\mathbf{p}) = h(\mathbf{p}) = 0$, product rules imply that $\nabla f(\mathbf{p}) = 0$.

The condition $\nabla Q_i(\mathbf{p}) \neq 0$ is to ensure that locally near $\mathbf{p}$, $\mathcal{V}_{Q_i}$ is a complex manifold by implicit function theorem. One can compare the definition of multiple points to that of smooth points. In particular, a smooth point is the simplest case of a multiple point, or in fact, a transverse multiple point. One can also compare the above definition to Definition 5.2 to see that they are equivalent. In particular, to see how Definition 5.2 implies Definition 5.8, one needs to realize the fact that $\mathcal{V}_i$ is complex manifold of codimension one and thus locally at $\mathbf{p}$, it is biholomorphical to a $(d-1)$-slice in $\mathbb{C}^d$. Therefore, there is an analytic function Q_i in a small neighborhood U of $\mathbf{p}$ in $\mathbb{C}^d$ such that $\mathcal{V}_i \cap U = \mathcal{V}_{Q_i} \cap U$. Since $\mathcal{V}_i$ is a complex manifold near $\mathbf{p}$, $\nabla Q_i(\mathbf{p}) \neq 0$ by [PWM24, Lemma 7.6]. We will change back and forth between the algebraic viewpoint and the geometric viewpoint.

Sometimes, we don't need to factorize in the local ring ${}_d\mathcal{O}_{\mathbf{p}}$. In the case of Q being a polynomial, it is much easier to factorize Q in the polynomial ring $\mathbb{C}[\mathbf{z} - \mathbf{p}]$ into

$$Q(\mathbf{z}) = aQ_1(\mathbf{z})^{m_1} \cdots Q_s(\mathbf{z})^{m_s}$$

where $0 \neq a \in \mathbb{C}$ and Q_i are irreducible polynomials in $\mathbb{C}[\mathbf{z} - \mathbf{p}]$. If $Q_i(\mathbf{p}) = 0$ implies that $\nabla Q_i(\mathbf{p}) \neq 0$, then $\mathbf{p}$ is a multiple point. If in addition, $\{\nabla Q_i(\mathbf{p}) : \text{for } i \text{ such that } Q_i(\mathbf{p}) = 0\}$ forms a linearly independent set, we call $\mathbf{p}$ a transverse multiple point.

On the other hand, given an irreducible polynomial $Q \in \mathbb{C}[\mathbf{z}-\mathbf{p}]$ with $Q(\mathbf{p}) = \nabla Q(\mathbf{p}) = 0$, it is hard to say whether $\mathbf{p}$ is a multiple point on $\mathcal{V}_Q$. For example, $Q(x, y) = x^2 - y^2 + x^3$ is irreducible in $\mathbb{C}[x, y]$ and $Q(0, 0) = \nabla Q(0, 0) = 0$, but Q factors in $\mathcal{O}$ into $Q(x, y) = (y - x\sqrt{1 + x})(y + x\sqrt{1 + x})$ where both factors are analytic near $(0, 0)$ and have non-zero gradient there. It is also possible that $\mathbf{p}$ is not a multiple point. For example, when $Q(x, y, z) = z^2 - x^2 - y^2$, the point $\mathbf{0}$ is a not a multiple point. Therefore, when determining multiple points, one should finally resort to factorization in the ring of analytic germs.

5.1.2. Classification of multiple points

In Chapter 5.1.1, we introduce the ring of analytic germs at a point $\mathbf{p}$ and give an algebraic viewpoint of the a multiple point. We now have the algebraic definition of a multiple point (Definition 5.8), and the geometric definition of a multiple point (Definition 5.8). In this section, we classify multiple points by their geometry and simultaneously give the algebraic way to recognize them.

linear v.s. non-linear

When the analytic hypersurface $\mathcal{V}$ is a finite union of hyperplanes. We call $\mathcal{V}$ a **hyperplane arrangement**. Suppose that Q is a polynomial in $\mathbb{C}[\mathbf{z}]$. The vairety $\mathcal{V}_Q$ is a hyperplane arrangement if and only if Q factors in $\mathbb{C}[\mathbf{z}]$ into $Q(\mathbf{z}) = L_1(\mathbf{z})^{m_1} \cdots L_n(\mathbf{z})^{m_n}$ where L_i are linear polynomials and m_i are positive integers.

Example 5.9. *Let* $Q(x, y) = (1 - \frac{2x}{3} - \frac{y}{3})(1 - \frac{x}{3} - \frac{2y}{3})$. *The variety* $\mathcal{V}_Q$ *is a hyperplane arrangement with two hyperplanes defined by* $L_1(x, y) = 1 - \frac{2x}{3} - \frac{y}{3}$ *and* $L_2(x, y) = 1 - \frac{x}{3} - \frac{2y}{3}$

transverse v.s. non-transverse

In Definition 5.2, a point $\mathbf{p}$ is a multiple point on $\mathcal{V}$ if locally near $\mathbf{p}$, the variety $\mathcal{V}$ is a union of s complex manifolds. If in addition these s complex manifolds intersect transversely at $\mathbf{p}$, we say that $\mathbf{p}$ is a **transverse multiple point**. Algebraically, let $\mathcal{D}$ be an open neighborhood of $\mathbf{p}$ in $\mathbb{C}^d$ and let Q be an analytic function in $\mathcal{D}$ such that $\mathcal{V} \cap \mathcal{D} = \mathcal{V}_Q \cap \mathcal{D}$. Since $\mathbf{p}$ is a multiple point, the function Q factors in $\mathcal{O}_{\mathbf{p}}$ into $Q(\mathbf{z}) = u(\mathbf{z})Q_1^{m_1}(\mathbf{z}) \cdots Q_s^{m_s}(\mathbf{z})$ as in equation (5.1). If there is a factorization

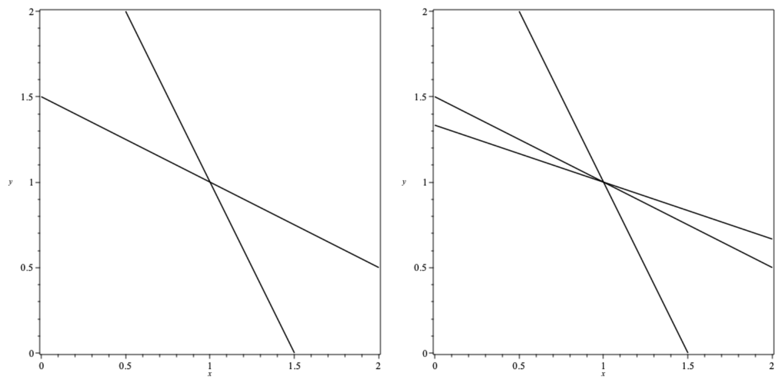


Figure 5.1: The real varieties $\mathcal{V}_Q \cap \mathbb{R}^2$ in Example 5.9 and 5.10.

such that $\{\nabla Q_i(\mathbf{p})\}_{i=1}^s$ forms a linearly independent set, then $\mathbf{p}$ is a transverse multiple point. For example, Example 5.9 is a transverse point.

Example 5.10. *Let* $Q(x,y) = (1-\frac{2x}{3}-\frac{y}{3})(1-\frac{x}{3}-\frac{2y}{3})(1-\frac{x}{4}-\frac{3y}{4})$. *The variety* $\mathcal{V}_Q$ *is a hyperplane arrangement with three hyperplanes and* $\mathbf{p} = (1,1)$ *is not transverse.*

arrangement v.s. non-arrangement

Suppose that $\mathbf{p}$ is a multiple point by Definition 5.2 and thus $\mathcal{V}$ is locally a union of s complex manifolds $\mathcal{V}_1, \cdots \mathcal{V}_s$. For any subset A of $\{1, ..., s\}$, let $\mathcal{V}_A = \cap_{i\in A}\mathcal{V}_i$ and $L_A = \cap_{i\in A} L_i$ where L_i are tangent planes of $\mathcal{V}_i$ at $\mathbf{p}$. Define the intersection lattice of $\{\mathcal{V}_i\}$ by the set of all intersections $\mathcal{V}_A$ indexed by A. It has a lattice structure. Define the intersection lattice of $\{L_i\}$ in the analogous way. If for any sufficiently small neighborhood U of $\mathbf{p}$ in $\mathbb{C}^d$, the intersection lattice of $\{\mathcal{V}_i\}$ within U is the same as the intersection lattice of L_A, we call $\mathbf{p}$ an **arrangement point**.

Algebraically, assume that $\mathbf{p}$ is a multiple point and $Q(\mathbf{z})$ factors in $\mathcal{O}_\mathbf{p}$ into

$$u(\mathbf{z})Q_1^{m_1}(\mathbf{z})\cdots Q_s^{m_s}(\mathbf{z})$$

for non-associated irreducibles Q_i and a unit u. We say that $\mathbf{p}$ is an arrangement point if for any subset $A \subset [s]$, the codimension of $\mathcal{V}_A$ is the same as the dimension of the vector space spanned by $\nabla Q_i := \nabla L_i$ for $i \in A$.

Example 5.10 is an arrangement point. Non-arrangement points occur when there are tangential intersections. That is, $\mathcal{V}_i$ and $\mathcal{V}_A$ share the same tangent plane at $\mathbf{p}$ for some A and $i \notin A$. Here are two example of non-arrangement points in $\mathbb{C}^2$ and $\mathbb{C}^3$, respectively.

Example 5.11. *Let $Q(x, y) = (2x + 2y - xy - 3)(1 - xy)$ and $\mathbf{p} = (1, 1)$. Since the gradients of the first factor and the second factor at $\mathbf{p}$ is both $(1, 1)$, the point $\mathbf{p}$ is not an arrangement point. Indeed, the curve defined by $2x + 2y - xy - 3 = 0$ intersects the curve defined by $1 - xy = 0$ tangentially at $\mathbf{p} = (1, 1)$.*

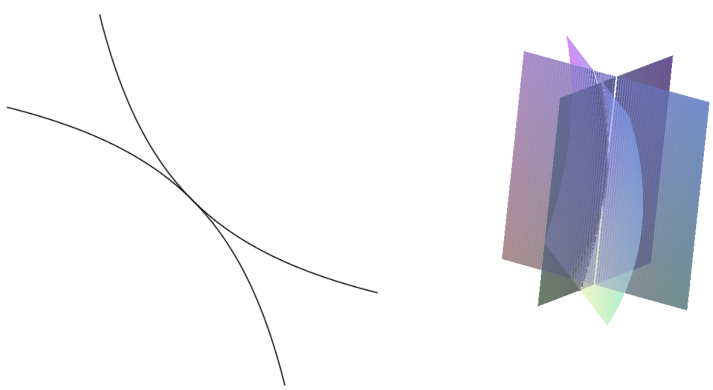


Figure 5.2: The real varieties $\mathcal{V}_Q \cap \mathbb{R}^2$ in Example 5.11 and $\mathcal{V}_Q \cap \mathbb{R}^3$ in Example 5.12.

Example 5.12. *Let $Q(x, y, z) = xy(x + y - z^2)$ and $\mathbf{p} = (0, 0, 0)$. Let $Q_1 = x, Q_2 = y$, and $Q_3 = x + y - z^2$. The three varieties $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ intersect transversely with each other, but $\mathcal{V}_3$ has a tangential intersection with $\mathcal{V}_1 \cap \mathcal{V}_2$. ALgebraically, let $A = \{1, 2, 3\}$ and gradients of Q_1, Q_2, Q_3 at $\mathbf{p}$ span a vector space of dimension 2. However, $\mathcal{V}_A := \{0\}$ has codimension 3. Therefore $\mathbf{p}$ is not an arrangement point.*

Complete Intersection Point

Let $\mathbf{p}$ be a multiple point such that $\mathcal{V} \cap U = (\mathcal{V}_1 \cap U) \cup \cdots (\mathcal{V}_s \cap U)$ for any sufficiently small neighborhoods U of $\mathbf{p}$. If $\bigcap_{i=1}^{s} \mathcal{V}_i \cap U = \{\mathbf{p}\}$ for any sufficiently small U, we call $\mathbf{p}$ a **complete intersection point**. Algebraically, suppose that Q factors into non-associated irreducibles in $\mathcal{O}_\mathbf{p}$ as $Q(\mathbf{z}) = u(\mathbf{z})Q_1^{m_1}(\mathbf{z}) \cdots Q_s^{m_s}(\mathbf{z})$. If for any sufficiently small neighborhood U of $\mathbf{p}$ and any $\mathbf{z} \in U$,

Name	Geometric Definition $\mathcal{V} \cap U = (\mathcal{V}_1 \cap U) \cup \cdots (\mathcal{V}_s \cap U)$ for	Algebraic Definition $Q(\mathbf{z}) = u(\mathbf{z}) Q_1^{m_1}(\mathbf{z}) \cdots Q_s^{m_s}(\mathbf{z})$ in $\mathcal{O}_{\mathbf{p}}$ where
Multiple point	complex manifolds $\mathcal{V}_i$	$\nabla Q_i(\mathbf{p}) \neq 0$
Hyperplane arrangement	complex hyperplanes $\mathcal{V}_i$	$Q(\mathbf{z}) = L_1^{m_1}(\mathbf{z}) \cdots L_s^{m_s}(\mathbf{z})$ in $\mathbb{C}[\mathbf{z}]$ and L_i are linear polynomials
Arrangement point	intersection lattice of $\{\mathcal{V}_i\}$ is the same as that of $\{L_i\}$	for $A \subset [s]$, the codimension of $\mathcal{V}_A$ is the same as the dimension of the vector space spanned by $\{\nabla Q_i, i \in A\}$
Transverse point	complex manifolds $\mathcal{V}_i$ intersecting transversely at $\mathbf{p}$	$\{\nabla Q_i(\mathbf{p})\}$ are linearly independent.
Complete intersection point	complex manifolds $\mathcal{V}_i$ whose intersection is exactly one point $\mathbf{p}$ inside U	for $\mathbf{z} \in U$, we have $Q_1(\mathbf{z}) = \cdots = Q_s(\mathbf{z}) = 0 \Longleftrightarrow \mathbf{z} = \mathbf{p}$.

Table 5.1: Summary of multiple points

$Q_1(\mathbf{z}) = \cdots = Q_s(\mathbf{z}) = 0 \Longleftrightarrow \mathbf{z} = \mathbf{p}$, then $\mathbf{p}$ is a complete intersection point. A complete intersection point can be transverse or not, arrangement or not, and it can arise from hyperplane arrangement or not.

Summary

These classifications are not disjoint. For example, an arrangement point can be transverse or not as seen in Example 5.9 and Example 5.10. In particular, transverse points are a subset of arrangement points. The hyperplane arrangement case is the simplest one to analyze, regardless of transversality. Currently, the most general theorem [PWM24, Theorem Corollary 10.46] exists for a arrangement point $\mathbf{p}$ when it is a minimal critical point and the torus with the radius $(|p_1|, \cdots, |p_d|)$ has finitely many critical points. For non-arrangement points, see [PWM24, Chapter 10.5].

We summarize the taxonomy in Table 5.1 with a translation between geometry and algebra.

It is not immediate to see why a hyperplane arrangement point is an arrangement point from the algebraic viewpoint. You may argue that the factorization of $Q(\mathbf{z}) = L_1^{m_1}(\mathbf{z}) \cdots L_s^{m_s}(\mathbf{z})$ happens in the ring $\mathbb{C}[\mathbf{z}]$, which is not $\mathcal{O}_{\mathbf{p}}$. However, non-zero gradient implies irreducibility in the ring $\mathcal{O}_{\mathbf{p}}$. None of the L_i has zero gradient unless it is a constant. Without loss of generality, assume that $\mathbf{p}$ is a point on the first q hyperplanes but not the rest. In other words, $L_1(\mathbf{p}) = \cdots = L_q(\mathbf{p}) = 0$ and $L_{q+1}(\mathbf{p}) \neq 0, \cdots, L_s(\mathbf{p}) \neq 0$. Then $Q(\mathbf{z}) = u(\mathbf{z})L_1^{m_1}(\mathbf{z}) \cdots L_q^{m_q}(\mathbf{z})$ where $u(\mathbf{z}) = L_{q+1}^{m_{q+1}}(\mathbf{z}) \cdots L_s^{m_s}(\mathbf{z})$. Since $u(\mathbf{p}) \neq 0$, $u(\mathbf{z})$ is a unit in $\mathcal{O}_{\mathbf{p}}$. Since each $\nabla L_i(\mathbf{p})$ is non-zero, they are irreducible in $\mathcal{O}_{\mathbf{p}}$. We see that factorization of Q into linear polynomials in $\mathbb{C}[\mathbf{z}]$ actually induces its factorization in $\mathcal{O}_{\mathbf{p}}$ for any $\mathbf{p} \in \mathcal{V}_Q$. We further note that a hyperplane arrangement point $\mathbf{p}$ is transverse if $\{\nabla L_i(\mathbf{p})\}_i$ are linearly independent for those i such that $L_i(\mathbf{p}) = 0$.

5.1.3. Organization

In Chapter 5.2, the goal is to describe the homology generators for $\mathrm{H}_d(\mathcal{M}, -\infty)$ when $\mathcal{M} = \mathbb{C}_*^d - \mathcal{V}$. This section reviews known Morse theoretical facts about $\mathcal{M}$ presented in [PWM24, Chapter 10 and Appendix D]. When $\mathcal{V}$ is a smooth manifold, these homology generators are given by the classical Morse theory. When the singular variety $\mathcal{V}$ is no longer a smooth manifold, we need to use Whitney stratification (Chapter 5.2.1) to decompose it into unions of smooth manifolds of different dimensions and we call them strata. We characterize critical point equations on each stratum in Chapter 5.2.2. In Chapter 5.2.3 and 5.2.4, we discuss how stratified Morse theory rebuilds the topology of $\mathcal{M}$ by attaching certain Morse data at each critical point. In Chapter 5.2.5, we introduce the critical point at infinity (CPAI) as their presences are obstructions to us applying stratifed Morse theory. In particular, Theorem 5.28 and Corollary 5.30 tell us that when all critical points are multiple points, the original Cauchy torus T is homologous to an integer sum of $\sigma_j = \gamma_j \times \beta_j$ where γ_j is a chain on which the height $h_{\hat{\mathbf{r}}}$ achieves its maximum at the critical point $\mathbf{p}_j$ and β_j is a small torus around those complex manifolds $\mathcal{V}_i$ of the multiple point $\mathbf{p}_j$.

In Chapter 5.3, we review the work of [BMP24b], improving on the result of Chapter 5.2 by giving explicit homology decomposition for $\mathrm{H}_d(\mathcal{M})$ when the singular variety $\mathcal{V}$ is a hyperplane arrangement. This section is also a foundation for what we do in Chapter 6. In Chapter 5.3.1, we explicitly give the stratification and critical points. In Chapter 5.3.2, we introduce the imaginary fibers and linking tori. These two concepts are important to see the explicit construction of homological equivalence. In particular, the linking torus at a crticial point $\mathbf{p}_j$ correspond to the homology generator σ_j in Corollary 5.30 in Chapter 5.2.4. Chapter 5.3.3 and 5.3.4 show that the initial Cauchy torus is homologous to an integer sum of linking tori at each critical point, where the coefficients are in $\{-1, 0, +1\}$. Chapter 5.3.5 gives an explicit formula for integrating the generating function over these linking tori.

5.2. Stratified Morse Theory

For a rational generating function $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$ where P, Q coprime, the singular variety of F is not in general a manifold. We introduce Whitney stratification in this section to find critical points. In particular, this stratification gives multiple strata of different dimensions. On each stratum $\mathcal{S}$, we define a stratified critical point on $\mathcal{S}$ to be the point where $dh|_{\mathcal{S}} = 0$. Remember that when $\mathcal{V}_Q$ is a smooth manifold, a critical point is a point where $dh = 0$ on $\mathcal{V}_Q$. In other words, $\nabla h(\mathbf{z})$ is perpendicular to the tangent space of $\mathcal{V}_Q$ at $\mathbf{z}$. That is, $\nabla h(\mathbf{z})$ is a multiple of $\nabla Q(\mathbf{z})$. The computation for stratified critical points is similar but more involved, requiring the tools of computational algebraic geometry. Whitney stratification also gives a local product structure on each stratum $\mathcal{S}$. For any $\mathbf{p} \in \mathcal{S}$, the neighborhood of $\mathbf{p}$ in the stratified space is locally homeomorphic to $B \times \mathtt{N}$ where B is a ball and $\mathtt{N}$ is called the normal slice. For example, when the variety is a union of two hyperplanes intersecting at a complex line in $\mathbb{R}^3$, the intersecting line forms a stratum of dimension one. Let's call it $\mathcal{S}_\ell$. Then for any point $\mathbf{p} \in \mathcal{S}_\ell$, the neighborhood of $\mathbf{p}$ in the variety is homeomorphic to $B \times \mathtt{N}$ where B is the 1-dimensional ball and $\mathtt{N}$ is the "**X**"-shaped topological space.

Stratified Morse theory works well with the local product structure and it tells us all homology generator of $\mathrm{H}_d(\mathcal{M}, -\infty)$. In particular, for a stratified critical point $\mathbf{p}$ on stratum $\mathcal{S}$, we have a cycle $\gamma_{\mathbf{p}}$ in $\mathcal{S}$ on which the height function h attains maximum at $\mathbf{p}$. If $\mathcal{S}$ is of complex codimension k, we take $\beta_{\mathbf{p}}^{(i)}$ from the k-th homology group of the normal Morse data at $\mathbf{p}$. Then $\gamma_{\mathbf{p}} \times \beta_{\mathbf{p}}^{(i)}$ is a generator of $\mathrm{H}_d(\mathcal{M}, -\infty)$. We will define the normal Morse data in this section. The content below is mostly based on [PWM24, Chapter 7.3, Chapter 8, Appendix D] and [GM88].

5.2.1. Whitney stratification

Though $\mathcal{V}_Q$ is not in general a smooth manifold, we can decompose it into a union of smooth manifolds of different dimensions. For example, when $Q(x, y) = (1 - x)(1 - y)$, $\mathcal{V}_Q$ is the union of two lines $1-x$ and $1-y$. $\mathcal{V}_Q$ is not a smooth manifold because of the singular point $(1,1)$. Instead, we can decompose $\mathcal{V}_Q$ into three strata. The 0-dimensional $\mathcal{S}_0$ is the singular set $\{(1,1)\}$. The other two 2-dimensional (1-complex-dimensional) $\mathcal{S}_1$ and $\mathcal{S}_2$ are $\mathcal{V}_{1-x}$ and $\mathcal{V}_{1-y}$ except the point $(1,1)$.

In general, we use Gröbner basis to detect singular set. Let $\tilde{Q}$ be the square-free part of Q (i.e. the product of distinct irreducible factors of Q). Then $\mathcal{V}_{\tilde{Q}} = \mathcal{V}_Q$. The singular set W_1 of $\mathcal{V}_Q$ is the set of $\mathbf{z}$ such that

$$\tilde{Q}(\mathbf{z}) = \nabla\tilde{Q}(\mathbf{z}) = 0.$$

These points can be found by computing the Gröbner basis of the ideal

$$I = \langle \tilde{Q}, \tilde{Q}_{z_1}, \cdots, \tilde{Q}_{z_d} \rangle.$$

If the Gröbner basis is 1, then there is no point on $\mathcal{V}_Q$ such that $\nabla\tilde{Q} = 0$. Otherwise, W_1 is not empty and it is an algebraic set defined by some polynomials and W_1 has dimension less than $\mathcal{V}_Q$. We can use the same method to compute singular set W_2 of W_1. Iteratively, we have a nested sequence of algebraic sets

$$\emptyset = W_s \subsetneq \cdots \subsetneq W_1 \subsetneq W_0 = \mathcal{V}_Q.$$

Each set difference $W_i \backslash W_{i+1}$ is a smooth manifold. This kind of decomposition is not what we need. We want to refine such a decomposition to get a **Whitney stratification**.

In particular, we want to find a Whitney stratification of $\mathcal{V}_Q$ that not only make $\mathcal{V}_Q$ a union of smooth manifolds, but also make sure that manifolds of different dimensions fit nicely. We gave the definition of Whiteney stratification below.

Definition 5.13 (I-decomposition)**.** *[PWM24, Equation (D.1.1)] An I-decomposition of a space $X \subset \mathbb{R}^n$ is a finite disjoint union $\bigcup_{\alpha \in I} \mathcal{S}_\alpha$ of smooth manifolds of various dimensions, indexed by a partially ordered set I, such that for every $\alpha, \beta \in I$,*

$$\mathcal{S}_\alpha \cap \overline{\mathcal{S}_\beta} \neq \emptyset \iff \mathcal{S}_\alpha \subset \overline{\mathcal{S}_\beta} \iff \alpha \leq \beta.$$

Definition 5.14 (Whitney statification)**.** *[PWM24, Definition D.3] Let Z be a closed subset of $\mathbb{R}^n$. A* **Whitney stratification** *of Z is an I-decomposition of Z satisfying the* **Whitney condition** *as follows.*

(Whitney condition) For any $\alpha, \beta \in I$ with $\alpha < \beta$ and any two sequences $\{x_i \in \mathcal{S}_\beta\}$ and $\{y_i \in \mathcal{S}_\alpha\}$ both converging to some $y \in \mathcal{S}_\alpha$, if the lines $\ell_i = \overline{x_i y_i}$ converge to a line ℓ and the tangent planes $T_{x_i}(\mathcal{S}_\beta)$ converge to a plane T, then $\ell \subseteq T$.

If a Whitney stratification of $\mathbb{Z}$ exists, we call Z a **Whitney stratified space***.*

Remark. In the literature, the above Whitney condition is called the second Whitney condition. The first Whitney condition requires $T_{y_i}(\mathcal{S}_\alpha) \subseteq T$ under the same assumption of the second Whitney condition. The second Whitney condition implies the first and so we omit the first condition.

Example 5.15. *Let $Q(x,y,z) = (x^3 - y^2)(x - y - z)$ defined in $\mathbb{R}^3$. We can perform a Whitney stratification on $\mathcal{V}_Q \subset \mathbb{R}^3$. The singular part W_1 of $W_0 := \mathcal{V}_Q$ is then the union of the z-axis with the one-dimensional curve $\mathcal{V}_{x^3-y^2} \cap \mathcal{V}_{x-y-z}$. The singular part W_2 of W_1 is then the singleton set $\{(0,0,0)\}$. Then we let $\mathcal{S}_0 := W_2$, $\mathcal{S}_1 := W_1 \backslash W_2$, and $\mathcal{S}_2 := W_0 \backslash W_1$. These three strata form a Whitney stratification for $\mathcal{V}_Q$.*

We can even refine these Whitney stratification. In particular, we can split $\mathcal{S}_2$ into two strata of dimension 2, namely $\mathcal{S}_{2,1} := \{x - y - z = 0 \text{ and } x^3 - y^2 \neq 0\}$ and $\mathcal{S}_{2,2} := \{x - y - z \neq 0 \text{ and } x^3 - y^2 = 0\} \backslash \{x = y = 0\}$. Similarly, we can split $\mathcal{S}_1$ into two strata of dimension 1, namely $\mathcal{S}_{1,1} := \{x - y - z = 0 \text{ and } x^3 - y^2 = 0\} \backslash \{(0,0,0)\}$ and $\mathcal{S}_{1,2} := \{x = y = 0\} \backslash \{(0,0,0)\}$. Then $\{\mathcal{S}_0, \mathcal{S}_{1,1}, \mathcal{S}_{1,2}, \mathcal{S}_{2,1}, \mathcal{S}_{2,2}\}$ is also a Whitney stratification of $\mathcal{V}_Q$. □

Example 5.16 (Whitney umbrella)**.** *Let $f(x,y,z) = x^2 + y^2 z$ defined in $\mathbb{R}^3$. The variety $\mathcal{V}_f$ is called* **Whitney umbrella***. One Gröbner basis for the ideal $I = \langle f, f_x, f_y, f_z \rangle$ is $[y^2, x, yz]$. Therefore, the singular set W_1 of $\mathcal{V}_f$ is the z-axis, or more precisely, the non-positive part of the z-axis. Since*

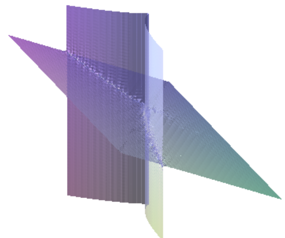


Figure 5.3: The zero locus of $Q(x,y,z) = (x^3 - y^2)(x - y - z)$.

W_1 is smooth, there is no more singular set of W_1. We have the following decomposition

$$\emptyset = W_2 \subsetneq W_1 \subsetneq W_0 = \mathcal{V}_f$$

which gives two strata $\mathcal{S}_0 = W_0 \backslash W_1$ and $\mathcal{S}_1 = W_1$.

It is true that both strata are smooth manifolds. However they do not satisfy the first Whitney condition. In particular, let $p_i = (0, y_i, 0)$ on $\mathcal{S}_0$ and $q_i = (0, 0, z_i)$ on $\mathcal{S}_1$ where $y_i, z_i \to 0$. Then $T_{p_i}(\mathcal{S}_0)$ converges to a plane $T = \{z = 0\}$ and $T_{q_i}(\mathcal{S}_1) = \operatorname{span}\{(0,0,1)\}$. We can see that $T_{q_i}(\mathcal{S}_1)$ is not in T because they are orthogonal to each other.

To get a Whitney stratification, we need to refine the above stratification. In particular, the problem of failing Whitney conditions is the origin in $\mathcal{S}_1$. We can seperate the origin from $\mathcal{S}_1$ and form another stratum consisting of only the origin.

$$\mathcal{S}_0 = W_0 \backslash W_1, \mathcal{S}_1 = W_1 \backslash \{(0,0,0)\}, \mathcal{S}_2 = \{(0,0,0)\}.$$

$\square$

Example 5.17 (Whitney cusp). *[PWM24, Exercise D.6] Let $Q = x^2 - y^3 - z^2y^2$ and let $\mathcal{V}_Q$ be the corresponding real variety. $\mathcal{V}_Q$ is called* **Whitney cusp**. *One Gröbner basis for the ideal $I = \langle Q, Q_x, Q_y, Q_z \rangle$ is $[x, y^3, y^2z, 2yz^2 + 3y^2]$. The singular set W_1 of $\mathcal{V}_Q$ is again the z-axis.*

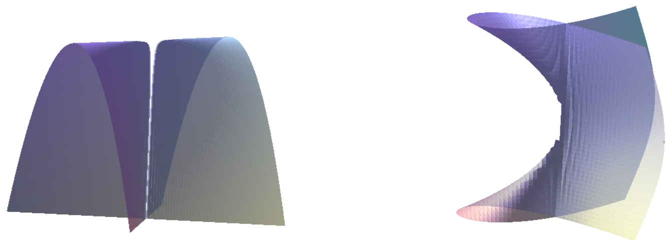


Figure 5.4: The Whitney umbrella (left) and the Whitney cusp (right).

Therefore, we have the naive decomposition again

$$\emptyset = W_2 \subsetneq W_1 \subsetneq W_0 = \mathcal{V}_Q$$

and it gives two strata $\mathcal{S}_0 = W_0 \backslash W_1$ *and* $\mathcal{S}_1 = W_1$*.*

This time the two strata violates the second Whitney condition. The culprit is still the origin in $\mathcal{S}_1$*. In particular, let* $p_i = (0, -z_i^2, z_i)$ *on* $\mathcal{S}_0$ *and* $q_i = (0, 0, z_i)$ *on* $\mathcal{S}_1$ *where* $z_i > 0$ *and* $z_i \to 0$*. Then the line* $\ell_i = \overline{p_i q_i}$ *converges to the line* ℓ *spanned by the vector* $(0, 1, 0)$*. On the other hand,* $T_{p_i}(\mathcal{S}_0)$ *converges to the plane* T *orthogonal to the vector* $(0, 1, 0)$*. Therefore, the second Whitney condition fails.*

The remedy is again to seperate the origin from $\mathcal{S}_1$ *and thus make a substratum* $\mathcal{S}_2$ *of* $\mathcal{S}_1$ *such that* $\mathcal{S}_2 = \{(0, 0, 0)\}$*.* □

The above examples make it appear that Whitney stratification is an art and not an science. Fortunately, canonical Whitney stratification exists. There are nested algebraic sets

$$\emptyset = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_m = \mathcal{V}. \tag{5.2}$$

The set of all connected components in $\mathcal{F}_i \backslash \mathcal{F}_{i-1}$ form a Whitney stratification. Algorithms for computing Whitney stratification exist [Ran98, MR91]. Newer algorithms using Gröbner basis computation [DJ21, HN23]. Indeed, [HN23] provides a Macaulay2 package *WhitneyStratifications* where the input is the ideal defining the variety and the output is a list of prime ideals by dimensions; each prime ideal correspond to a stratum. Usually, in ACSV setting, stratifications are visually available and do not require any computations. For example, in the case of hyperplane arrangement (Chapter 5.3), stratifications are given explicitly by the intersection lattice of hyperplanes. If a multiple point is an arrangement point, then locally the intersection lattice also induces a Whitney stratification.

One important thing of Whitney stratified spaces is that each stratum will have a local product structure. This local product structure implies that on each point of a stratum, the neighborhood of the point looks the same.

Theorem 5.18 (local product structure)**.** *[PWM24, Theorem D.9] Let* $\mathbf{p}$ *be a point in a* k*-dimensional stratum* $\mathcal{S}$ *of a stratified space* Z*. There is a topological space* $\mathtt{N}$*, called the* **normal slice***, depending only on* $\mathcal{S}$ *and not the choice of* $\mathbf{p} \in \mathcal{S}$*, such that some neighborhood of* $\mathbf{p}$ *in* Z *is homeomorphic to* $B^k \times \mathtt{N}$*, where* B^k *is a* k*-dimensional ball.*

Remark. The local product structure is witnessed by diffeomorphisms [PWM24, Proposition D.14] when Z is

- a smooth algebraic hypersurface.
- a simplex or the complexification of a simplex.
- a hyperplane arrangement.
- the product of two spaces on which the local product is induced by a diffeomorphism.

When the homeomorphism is a diffeomorphism, it is easier to prove the stratified Morse lemma. In general, one needs Thom's Isotopy Theorem.

In ACSV, we care both the singular variety $\mathcal{V}$ and its complement in $\mathbb{C}_*^d$. We introduce the concept of the stratification of a pair.

Definition 5.19 (stratification of a pair). *[PWM24, Definition D.10] If $Y \subseteq X$ are closed subsets of real space then a* **stratification of the pair** (X, Y) *is defined to be a stratification of X such that intersecting each stratum with Y gives a stratification of Y and intersecting each stratum with $X \backslash Y$ gives a stratification of $X \backslash Y$.*

[PWM24, Proposition D.11] gives a stratification of $\mathbb{C}_*^d$. In particular, if $\mathcal{V}$ is a complex algebraic variety in $\mathbb{C}_*^d$ with stratification $\{\mathcal{S}_\alpha : \alpha \in I\}$ then adding the stratum $\mathcal{M} = \mathbb{C}_*^d - \mathcal{V}$ produces a stratification of the pair $(\mathbb{C}_*^d, \mathcal{V})$.

5.2.2. Stratified critical points

We define the height function $h_{\mathbf{r}}(\mathbf{z}) := -\mathbf{r} \cdot \mathrm{Relog}(\mathbf{z})$ as usual. When the variety $\mathcal{V}$ is smooth, we define a critical point on V to be the point where the differential dh vanishes. Similarly, we define a **stratified critical point** as follows.

Definition 5.20 (stratified critical point). *Let X be a stratified space and* $\mathbf{p}$ *is a point in a unique stratum $\mathcal{S}$. The point* $\mathbf{p}$ *is a* **stratified critical point** *of the height function $h_{\mathbf{r}}$ on the stratifed space X if $dh|_{\mathcal{S}}(\mathbf{p}) = 0$. In other words, dh is zero on the tangent plane of $\mathcal{S}$ at* $\mathbf{p}$.

The computation of all the stratified critical points of $\mathcal{V}$ is straightforward in computer algebra systems. The following steps are detailed in [PWM24, Chapter 8].

1. Compute a Whitney stratification for the variety $\mathcal{V}$ and obtain a nested algebraic set in equation (5.2).

2. Start from $i = 1$. For each $I_i = I(\mathcal{F}_i)$, do a prime ideal decomposition to get $I_i = P_{i,1} \cap \cdots \cap P_{i,s_i}$.

3. For each $P_{i,j}$, find the codimension c of $\mathcal{V}(P_{i,j})$ and its generators $p_1, \cdots, p_s$. Find $\mathbf{z}$ such that (i.) $M(\mathbf{z}, \mathbf{r})$ in equation (5.3) has rank c, (ii.) $p_1(\mathbf{z}), \cdots, p_s(\mathbf{z}) = 0$, and (iii.) $z_i \neq 0, \forall i$.

4. In addition, remove points $\mathbf{z}$ in Step 3 such that $g(\mathbf{z}) = 0$ for all g in a generating set of I_{i-1}.

5. Assign points $\mathbf{z}$ to the set of critical points. Go to Step 3 for each prime ideal in the decomposition of I_i. After that, Go to Step 2 for the next index i.

$$M(\mathbf{z},\mathbf{r}) := \begin{bmatrix} \nabla p_1(\mathbf{z}) \\ \vdots \\ \nabla p_s(\mathbf{z}) \\ \nabla \phi \end{bmatrix} = \begin{bmatrix} & \nabla p_1(\mathbf{z}) & \\ & \vdots & \\ & \nabla p_s(\mathbf{z}) & \\ -r_1/z_1 & \cdots & -r_d/z_d \end{bmatrix} \tag{5.3}$$

All computations can be done algebraically. In particular, when the generating function is a rational function in $\mathbb{Q}(\mathbf{z})$, we only need to solve a system of polynomial equalities and inequalities in $\mathbb{Q}[\mathbf{z},\mathbf{r}]$. Modern computer algebra systems can do these calculation.

Recall that $\mathcal{S}$ is an algebraic variety minus some sub-varieties (see Example 5.15, 5.16, and 5.17). Therefore the closure $\overline{\mathcal{S}}$ is an algebraic object and is defined by some polynomial ideal. In practice, we canonically choose $\overline{\mathcal{S}}$ to be an irreducible algebriac variety, thus given by some prime ideal $P_{i,j}$ in Step 2. It is possible that stratified critical points calculated for $\mathcal{S}$ do not belong to $\mathcal{S}$, but belong to a sub-variety of $\overline{\mathcal{S}}$. This kind of critical points is excluded in Step 4 as critical points on $\mathcal{S}$ because they have already been counted as critical points on sub-strata of $\mathcal{S}$.

Definition 5.21 (Generic direction). *A direction* $\hat{\mathbf{r}}$ *is* **generic** *if stratified critical points of* $h_{\hat{\mathbf{r}}}$ *on* $\overline{\mathcal{S}}$ *are always on* $\mathcal{S}$.

5.2.3. Attachments by building

Now we are going to explore the topology of $\mathcal{M} := \mathbb{C}_*^d - \mathcal{V}$. Indeed, the integrand $\omega := F(\mathbf{z})/\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\, d\mathbf{z}$ in the Cauchy integral $[\mathbf{z}^{\mathbf{r}}]F(\mathbf{z}) = (2\pi i)^{-d} \int_T \omega$ is a d-form, and $\mathcal{M}$ is the domain of holomorphy where ω is a holomorphic form. Just like in the one-variable complex analysis, when we deform the small torus T in $\mathcal{M}$, the integral value will not change. Furthermore, if we replace T with another representative in its homology class in $\mathrm{H}_d(\mathcal{M})$, the integral value will not change by Stokes' theorem and $d\omega = 0$. We can then replace $[T]$ with a sum of generators $[C_i]$ of $\mathrm{H}_d(\mathcal{M})$. For each $[C_i]$ in $\mathrm{H}_d(\mathcal{M})$, it may be possible to choose a representative cycle C_i and we can evaluate the integral $\int_{C_i} \omega$ as a saddle point integral. All of these sounds very abstract. In this section we are going to have a closer look. We care more about the relative homology group than the homology group. We need the stratified Morse theory to guide us through the process and recognize generators of the relative homology group.

To begin with Morse theory, we need to define what a **Morse function** is in a stratified space.

Definition 5.22 (stratified Morse function)**.** *$h : X \to \mathbb{R}$ is a Morse function on a stratified space X if*

(1) $h|_{\mathcal{S}}$ is a Morse function on each stratum $\mathcal{S}$.

(2) whenever $\mathbf{p}$ is a critical point for h on a stratum $\mathcal{S}_\alpha$ and a sequence of points $\mathbf{p}_i \to \mathbf{p}$ on another stratum $\mathcal{S}_\beta$ with $\alpha < \beta$, then either $T_{\mathbf{p}_i}(\mathcal{S}_\beta) \to T_{\mathbf{p}}(\mathcal{S}_\alpha)$ or the limit tangent plane of $\mathbf{p}_i$ contains a vector v such that $dh(\mathbf{p})(v) \neq 0$.

Remark. The first condition means that every critical point $\mathbf{p}$ on $\mathcal{S}$ is nondegenerate, i.e. the Hessian of $h|_{\mathcal{S}}$ at $\mathbf{p}$ is nonsingular. In the second condition, $\alpha < \beta$ is defined as in Definition 5.13.

The condition is more precisely described in [Pig79, Chapter 3]: if $\alpha < \beta$, for any $(p, H) \in \tau(\mathcal{S}_\beta, \mathcal{S}_\alpha)$, the linear map $dh(\mathbf{p}) : \mathbb{R}^n \to \mathbb{R}$ does not vanish on H. Here n is the dimension of the ambient space of X. [Pig79, Chapter 2] defines $\tau(\mathcal{S}) := \{(\mathbf{p}, T) \in \mathbb{R}^n \times G^{n,r} : \mathbf{p} \in \mathcal{S},\ T$ is in $G^{n,r}$ corresponding to the tangent plane of $\mathcal{S}$ in $\mathbf{p}\}$. Here r is the dimension of stratum $\mathcal{S}$, and $G^{n,r}$ is the Grassmannian of r dimensional subspaces of a n dimensional vector space. Then $\overline{\tau(\mathcal{S})}$ is the closure of $\tau(\mathcal{S})$ in $\mathbb{R}^n \times G^{n,r}$. For two strata $\mathcal{S}_\alpha$ with dimension s and $\mathcal{S}_\beta$ with dimension r, if $\alpha < \beta$, then $\tau(\mathcal{S}_\beta, \mathcal{S}_\alpha)$ is defined to be the set $\mathcal{S}_\alpha \times G^{n,r} \cap \overline{\tau(\mathcal{S}_\beta)}$. Intuitively, we can think of $\mathbf{p}$ as a point on the boundary of $\mathcal{S}_\beta$ and imagine the r dimensional tangent plane of $\mathcal{S}_\beta$ at $\mathbf{p}$ even though the stratum $\mathcal{S}_\alpha$ to which $\mathbf{p}$ belongs is only s dimensional. A more vivid figure can be found in [GM88, page 13].

If critical points of a Morse function have distinct critical values, then it is called a Morse function with distinct critical values. If a Morse function is proper, it is called a proper Morse function. Most treatment in Morse theory assumes that a Morse function is proper.

Let X be a stratified space with proper Morse function h. Let $X_{\leq c}$ be the set $\{\mathbf{z} \in X : h(\mathbf{z}) \leq c\}$. [GM88, Theorem (SMT Part A) page 6] tells us that the topology of $X_{\leq c}$ only changes when c is a critical value. In particular, we use the narrative in [PWM24, Theorem D.13]:

Theorem 5.23 (Stratified Morse Theorem Part A, proper)**.** *[PWM24, Theorem D.13] Let $X \subset \mathbb{C}_*^d$ be a stratified space with proper Morse function h and let $a < b$ be real numbers such that the interval $[a, b]$ contains no critical values of h. The inclusion $X_{\leq a} \hookrightarrow X_{\leq b}$ is a homotopy equivalence.*

The above theorem, however sadly, only works when the Morse function is proper. Moreover, if we stratify $\mathcal{V}$, then it tells us nothing about its complement $\mathcal{M}$ in $\mathbb{C}_*^d$. In the setting of ACSV, h is defined as $-\mathbf{r} \cdot \operatorname{Relog}(\mathbf{z})$ and will fail to be proper. It is more appropriate to introduce a Morse theorem that works in our setting.

Theorem 5.24 (Stratified Morse Theorem Part A)**.** *[BMP22, Theorem 1(i)] Given a Whitney stratification $\{\mathcal{S}_\alpha\}$ of the pair $(\mathbb{C}_*^d, \mathcal{V}_*)$, if there is no critival value (including critical value at infinity) in the interval $[a, b]$, then $\mathcal{S}_{\leq a} \hookrightarrow \mathcal{S}_{\leq b}$ is a homotopy equivalence for any stratum $\mathcal{S}$. In particular, $\mathcal{M}_{\leq a}$ is homotopy equivalent to $\mathcal{M}_{\leq b}$.*

Remark. See Definition 5.19 for Whitney stratification of the pair $(\mathbb{C}_*^d, \mathcal{V}_*)$. In particular, $\mathcal{M}$ is itself a stratum. We haven't defined critical value at infinity (CVAI) yet. For the definition, see [PWM24, Definition 7.43]. The lack of CVAI is a replacement of the condition that h is proper. The new condition guarantees that when we push down a chain in $\mathcal{M}_{\leq b}$ to $\mathcal{M}_{\leq a}$ by a gradient flow, we will not stuck at places infinitely far away.

In [BMP22, Definition 2], critical points at infinity and ordinary critical points are all called stationary critical points at infinity (SPAI). If a critical point has any finite height, it is then called heighted stationary critical points at infinity (H-SPAI).

To save reader's time to go into referenced literature, we briefly summarize how the topology of $\mathcal{M}_{\leq c}$ changes when c crosses critical values of the Morse function h. We are mostly interested in the homology group of $\mathcal{M}$ because

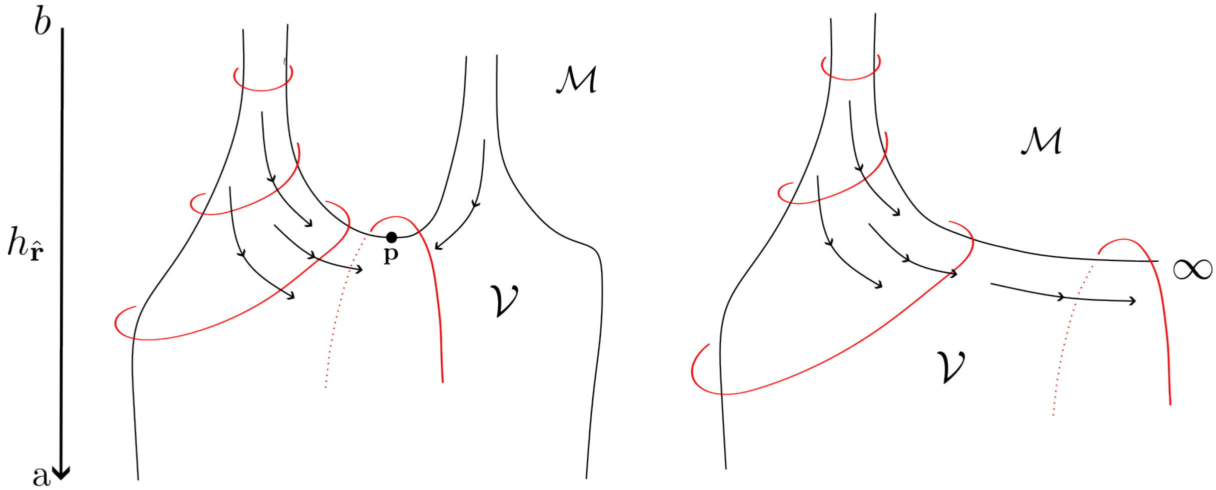


Figure 5.5: (left) A critical point $\mathbf{p}$ with height between a and b. A chain (red) in $\mathcal{M}$ following the gradient flow stucks at $\mathbf{p}$. (right) A critical point at infinity with height between a and b. A chain in $\mathcal{M}$ following the gradient flow stucks at places infinitely far away.

(1) deforming T within $\mathcal{M}$ will not change the integral $\int_T \omega$

(2) only things matters to the integral above is the homology element, rather than the actual cycle.

Assumption: From now on, we assume that we have finite number of critical values for h and there is no critical value at infinity.

We can list them from the smallest to the largest by $c_1 < c_2 < \cdots < c_m$ and each critical value may have multiple critical points with the same value. Let a be any number less than c_1. It does not matter which a we choose here because $\mathcal{M}_{\leq a}$ and $\mathcal{M}_{\leq a'}$ have the same homotopy type by Theorem 5.24 as long as $a, a' < c_1$. Now we try to reconstruct $\mathcal{M}$ from $\mathcal{M}_{\leq a}$ by attaching some topological stuff at each critical point.

Let ϵ be less than half of the minimum of $c_i - c_{i-1}$ for $i = 2, \cdots, m$. Let $\mathcal{M}_{c_i} := \mathcal{M}_{\leq c_i}, \mathcal{M}_{c_i+} := \mathcal{M}_{\leq c_i+\epsilon}$, and $\mathcal{M}_{c_i-} := \mathcal{M}_{\leq c_i-\epsilon}$. Then $\mathcal{M}_{c_1-}$ has the same homotopy type as $\mathcal{M}_{\leq a}$. $\mathcal{M}_{c_m+}$ has the same homotopy type as $\mathcal{M}$. $\mathcal{M}_{c_i+}$ has the same homotopy type as $\mathcal{M}_{c_{i+1}-}$. All of these are guaranteed by Theorem 5.24.

For each stratified critical point $\mathbf{p}$, there is a δ depending on ϵ and let $B(\mathbf{p})$ be the intersection of

$\mathcal{M}$ with the δ-ball centered at $\mathbf{p}$. If $h(\mathbf{p}) = c_i$, we let

$$\mathcal{M}^{\mathbf{p},\mathrm{loc}} := (\mathcal{M}_{c_i-} \cup B(\mathbf{p}), \mathcal{M}_{c_i-})$$

Some part of $B(\mathbf{p})$ will be below height $c_i - \epsilon$.

[BMP22, Theorem 1(ii)] says that $\mathcal{M}_{c_i+}$ is homotopy equivalent to $\mathcal{M}_{c_i-}$ union a sufficiently small neighborhoods in $\mathcal{M}$ of critical points at height c_i. We can then identify the **attachment pair** at critical value c_i as

$$(\mathcal{M}_{c_i+}, \mathcal{M}_{c_i-}) = \left(\mathcal{M}_{c_i-} \cup \bigcup_{j=1}^{n} B(\mathbf{p}_j), \mathcal{M}_{c_i-}\right)$$

where $\mathbf{p}_j$ are critical points with $h(\mathbf{p}_j) = c_i$. By shrinking ϵ if necessary, we can assume that $B(\mathbf{p}_j)$ are disjoint to each other and so for any k,

$$\mathrm{H}_k(\mathcal{M}_{c_i+}, \mathcal{M}_{c_i-}) = \bigoplus_{j=1}^{n} \mathrm{H}_k(\mathcal{M}^{\mathbf{p},\mathrm{loc}}).$$

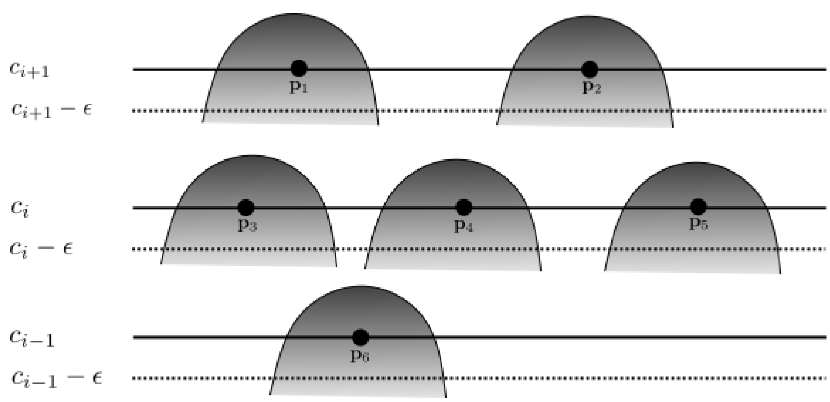


Figure 5.6: Building $\mathcal{M}$ by attachment pairs at critical values $c_1 < \cdots < c_m$. The figure shows that there are two, three, and one critical points at height c_{i+1}, c_i, and c_{i-1} respectively. Bumps around the critical points represent $B(\mathbf{p})$ in the attachment pair at critical values.

By [PWM24, Theorem D.23(ii)], all of these spaces above has homotopy type of CW complexes of dimension at most d. Therefore, homology groups H_k with $k > d$ vanish. Let $G(\mathbf{p})$ be the image in $\mathrm{H}_d(\mathcal{M}_{c_i+}, \mathcal{M}_{c_i-})$ of projecting absolute cycles supported on $\mathcal{M}_{c_i-} \cup B(p_j)$ to relative cycles on

$(\mathcal{M}_{c_i+}, \mathcal{M}_{c_i-})$. Let $G(c_i) := \bigoplus_{h(\mathbf{p})=c_i} G(\mathbf{p})$. Therefore, homology classes in $G(c_i) \subset \mathrm{H}_d(\mathcal{M}_{c_i+}, \mathcal{M}_{c_i-})$ have representative as absolute cycles in $\mathcal{M}_{c_i-} \cup \bigcup_{h(\mathbf{p})=c_i} B(\mathbf{p})$, which is homotopy equivalent to $\mathcal{M}_{c_i+}$. Continue the building by attachments iteratively from c_1 to c_m, we get

$$\mathrm{H}_d(\mathcal{M}) \cong \mathrm{H}_d(\mathcal{M}_{\leq a}) \oplus \bigoplus_{\mathbf{p}:\text{critical}} G(\mathbf{p}) \tag{5.4}$$

By Theorem 5.24, $\mathcal{M}_{\leq a}$ has the same homotopy type with $\mathcal{M}_{\leq a'}$ as long as $a, a' < c_1$. Therefore it does not matter which a we choose and we recognize all these homotopy equivalent spaces as $\mathcal{M}_{-\infty}$. We define $\mathrm{H}_d(\mathcal{M}, -\infty)$ as the d-th homology group of the pair $(\mathcal{M}, \mathcal{M}_{-\infty})$. $\mathrm{H}_d(\mathcal{M}, -\infty)$ is isomorphic to $\mathrm{H}_d(\mathcal{M}, \mathcal{M}_{\leq a})$ for any $a < c_1$. Therefore,

$$\mathrm{H}_d(\mathcal{M}, -\infty) \cong \bigoplus_{\mathbf{p}:\text{critical}} G(\mathbf{p}) \tag{5.5}$$

5.2.4. Homology generators

In the previous section, we see that the $\mathrm{H}_d(\mathcal{M}, -\infty)$ is a direct sum of $G(\mathbf{p})$ where each homology class in $G(\mathbf{p})$ is represented by an absolute cycle supported on an arbitrarily small neighborhood of $\mathbf{p}$. In this section, we describe what generators of each $G(\mathbf{p})$ are. In this section, we continue the previous assumption that there is no critical value at infinity.

Let's begin the story with a theorem in [GM88]. The theorem has more strict conditions than our running assumption but it basically captures the essence; it requires the stratified space to be compact.

Theorem 5.25. *[GM88, Theorem (SMT Part B)] Morse data measuring the change in the topological type of $X_{\leq c}$ as c crosses the critical value v of the critical point p is the product of the normal Morse data at p and the tangential Morse data at p.*

We explain terms in Theorem 5.25 one by one. Let $\mathbf{p}$ be a critical point of h with critical value c on a stratum $\mathcal{S}$ of complex codimension k.

Definition 5.26 (tangential Morse data)**.** *The* **tangential Morse data** *of* $\mathcal{M}$ *at* $\mathbf{p}$ *is the homotopy type of the pair* $(B^{d-k}, \partial B^{d-k})$ *consisting of a ball of codimension* k *modulo its boundary.*

Indeed, since h is the real part of $-\mathbf{r} \cdot \log(\mathbf{z})$, h is harmonic and the Morse index of h is half of the dimension. If $\mathcal{S}$ is of complex codimension k, then it is of complex dimension $d-k$, or real dimension $2(d-k)$. The Morse index is then $d-k$, indicating that there is a $d-k$ dimensions on $\mathcal{S}$ where h goes downward and another $d-k$ dimensions where h goes upward (like a saddle point). Unlike the classical Morse theory where Morse index determines the Morse data, in the stratified case, Morse index only determines the tangential Morse data. We need to know the topology not only of the stratum $\mathcal{S}$, but also the topology of the space around the stratum $\mathcal{S}$.

In our setting, we define the normal plane $N_{\mathbf{p}}(\mathcal{S})$ of $\mathcal{S}$ at $\mathbf{p}$ to be the complex orthogonal complement of the tangent plane $T_{\mathbf{p}}(\mathcal{S})$. We define the **normal slice** $\mathtt{N}(\mathcal{M})$ at $\mathbf{p}$ as the intersection of $\mathcal{M}$ with an arbitrarily small disk D in $N_{\mathbf{p}}(\mathcal{S})$ centered at $\mathbf{p}$. The **normal link** $\mathcal{L}(\mathcal{M})$ at $\mathbf{p}$ is $\mathcal{M} \cap \partial D$. In particular, $\mathbf{p}$ is not in $\mathtt{N}(\mathcal{M})$ and $\mathtt{N}(\mathcal{M})$ retracts to $\mathcal{L}(\mathcal{M})$. Therefore, normal link and normal slice are homotopy equivalent.

Definition 5.27 (normal Morse data)**.** *The* **normal Morse data** *of* $\mathcal{M}$ *at* $\mathbf{p}$ *is the homotopy type of the pair*

$$\big(\mathtt{N}(\mathcal{M}) \cap h^{-1}([c-\epsilon, c+\epsilon]), \mathtt{N}(\mathcal{M}) \cap h^{-1}(c-\epsilon)\big)$$

The homotopy does not change as long as D in $\mathtt{N}(\mathcal{M})$ and ϵ are sufficiently small [GM88].

We take the product of tangential Morse data and normal Morse data in the category of topological pairs,

$$(A, B) \times (C, D) = (A \times C, (A \times D) \cup (B \times C)),$$

and this is the **Morse data** of $\mathcal{M}$ at $\mathbf{p}$. The homology of a product is given by the Künneth formula,

$$\mathrm{H}_k(U \times V) = \bigoplus_{i=0}^{k} \mathrm{H}_i(U) \times \mathrm{H}_{k-i}(V).$$

Homotopy type of $\mathcal{M}^{\mathbf{p},\text{loc}}$ is the same as the homotopy type of the normal Morse data [GM88]. In particular, $G(\mathbf{p})$ in equation (5.4) is the d-th homology of the Morse data of $\mathcal{M}$ at $\mathbf{p}$. If $\mathbf{p}$ is on a stratum of complex codimension k, then $G(\mathbf{p})$ is the product of the $(d-k)$-th homology of the tangential data and the k-th homology of the normal data. The tangential data is of homotopy type $(B^{n-k}, \partial B^{n-k})$ and there is only one generator in its $(d-k)$-th homology. Choosing a relative cycle γ_j as its generator, we can let h achieves its maximum on this cycle γ_j exactly at $\mathbf{p}$. The k-th homology of the normal Morse data instead may have multiple generators. The next theorem summarizes the story.

Theorem 5.28 (Homology generators for $\mathrm{H}_d(\mathcal{M}, -\infty)$)**.** *[PWM24, Theorem 7.35] Fix $\hat{\mathbf{r}}$ and assume that there is no critical values at infinity. Let $\mathbf{p}_1, \cdots, \mathbf{p}_m$ enumerate the stratified critical points of $\mathcal{V}_*$ in decreasing order of the height function $h_{\hat{\mathbf{r}}}$, where the stratum containing $\mathbf{p}_j$ has complex codimension k_j. If all critical points are quadratic nondegenerate (i.e. Hessian of $h_{\hat{\mathbf{r}}}$ doesn't vanish), then there are cycles $\gamma_1, \cdots, \gamma_m$ on $\mathcal{V}_*$ along with basis $\beta_{j,1}, \cdots, \beta_{j,s_j}$ for the k_j-th homology of the normal Morse data, with the following properties.*

(a) $h_{\hat{\mathbf{r}}}$ achieves its maximum on γ_j at $\mathbf{p}_j$;

(b) γ_j is of homotopy type $(B^{d-k_j}, \partial B^{d-k_j})$;

(c) A basis for the integer homology group $\mathrm{H}_d(\mathcal{M}, -\infty)$ can be formed by cycles $\sigma_{j,i} = \gamma_j \times \beta_{j,i}$. For fixed j, $\sigma_{j,i}$ form a basis for $G(\mathbf{p}_j)$.

Example 5.29. *Let $\mathcal{V}$ be the space consisting of two hyperplanes defined by linear polynomials H_1 and H_2 in $\mathbb{C}^3$ intersecting at a complex line L. A Whitney stratification for $\mathcal{V}$ is*

$$\mathcal{S}_0 = L, \mathcal{S}_1 = \mathcal{V}_{H_1} \backslash L, \mathcal{S}_2 = \mathcal{V}_{H_2} \backslash L.$$

Assume that $\mathbf{p}$ is a nondegenerate critical point on stratum $\mathcal{S}_0$. In particular, $\mathbf{p}$ is a multiple point by Definition 5.2.

The complex codimension of $\mathcal{S}_0$ is 2 and so the tangential Morse data of $\mathcal{M} := \mathbb{C}^d_* - \mathcal{V}$ at $\mathbf{p}$ is

$(B^1, \partial B^1)$. The normal plane of $\mathcal{S}_0$ at $\mathbf{p}$ is a complex 2-space. The normal slice $\mathtt{N}(\mathcal{M})$ is the complement of two intersecting complex lines in a small disk in the complex 2-plane. The normal slice is then homotopy equivalent to a 2-torus. This 2-torus can be made arbitrarily close to $\mathbf{p}$ and there the second component in Definition 5.27 becomes $\emptyset$. The first component in the same definition is the 2-torus. Therefore, the normal Morse data is of homotopy type $(S^1 \times S^1, \emptyset)$, or exactly $S^1 \times S^1$. The Morse data is then $(S^1 \times S^1 \times B^1, S^1 \times S^1 \times \partial B^1)$, which is homotopy equivalent to a 3-torus.

The γ in Theorem 5.28 is a 1-cycle on $\mathcal{S}_0$ that maximize $h_{\hat{\mathbf{r}}}$ at $\mathbf{p}$. The second homology group of the normal Morse data (i.e. $\mathrm{H}_2(S^1 \times S^1)$) only has one generator, that is, the 2-torus itself.

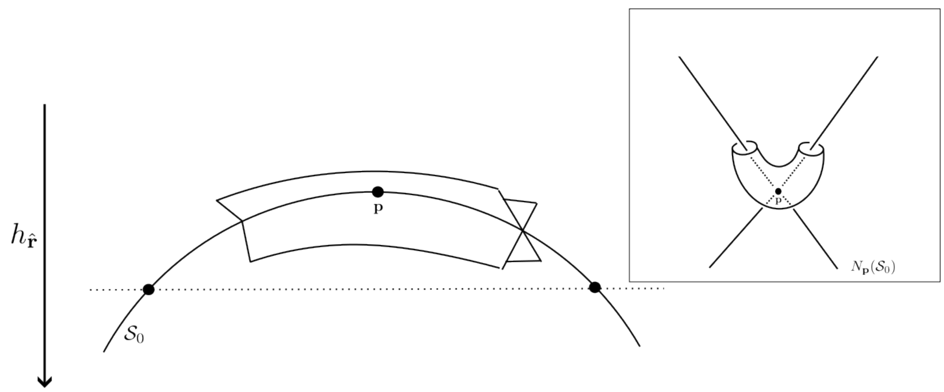


Figure 5.7: The tangential (left) and normal (right) Morse data in Example 5.29. The tangential data is curve above the dotted line with two endpoints identified. $N_{\mathbf{p}}(\mathcal{S}_0)$ is the normal space (of real dimension 4) of $\mathcal{S}_0$ at $\mathbf{p}$. The two solid lines are complex lines intersecting at $\mathbf{p}$.

Indeed, when $\mathbf{p}$ is a transverse multiple point, we can locally parametrize the stratum on which $\mathbf{p}$ lies and there is an explicit description of the generator of the homology group of the normal Morse data. By Definition 5.8, if $\mathbf{p}$ is a multiple point, then there are irreducible analytic germs $Q_1, \cdots Q_s$ in the local ring $\mathcal{O}_{\mathbf{p}}$ with $\nabla Q_i(\mathbf{p}) \neq 0$ such that $Q(\mathbf{z}) = u(\mathbf{z}) Q_1^{m_1} \cdots Q_s^{m_s}(\mathbf{z})$ in $\mathcal{O}_{\mathbf{p}}$. Geometrically, $\mathbf{p}$ is on a stratum $\mathcal{S}$ which in a small neighborhood of $\mathbf{p}$, is the intersection of $\mathcal{V}_{Q_1}, \cdots, \mathcal{V}_{Q_s}$. If in addition $\mathbf{p}$ is a transverse multiple point, then $\nabla Q_i(\mathbf{p})$ are linearly independent and $\mathcal{S}$ has codimension s. By implicit function theorem, there exists $d - s$ coordinates $\boldsymbol{\pi} := \{\pi_1, \cdots, \pi_{d-s}\}$ that locally parametrize $\mathcal{S}$ near $\mathbf{p}$. Consider the map

$$\Phi(\mathbf{z}) = \big(Q_1(\mathbf{z}), \cdots, Q_s(\mathbf{z}), z_{\pi_1} - p_{\pi_1}, \cdots, z_{\pi_{d-s}} - p_{\pi_{d-s}}\big)$$

which is bi-analytic in a small neighborhood of $\mathbf{p}$ in $\mathbb{C}^d$ that it takes a neighborhood of $\mathbf{p}$ in $\mathcal{S}$ to a neighborhood of $\mathbf{0}$ in $\{\mathbf{0}\} \times \mathbb{C}^{d-s}$. The normal plane of $\mathcal{S}$ at $\mathbf{p}$ is the complex s-space formed by the span of $\nabla Q_1(\mathbf{p}), \cdots, \nabla Q_s(\mathbf{p})$. Let $T_\epsilon \subset \mathbb{C}^s \times \{\mathbf{0}\}$ be the product of s circles of radius ϵ in the first s coordinates and $\{0\}$ in the last $d-s$ coordinates. Then $\beta_{\mathbf{p}} := \Phi^{-1}(T_\epsilon)$ is the generator of the s-th homology group of the normal Morse data of $\mathcal{M}$ at $\mathbf{p}$.

Corollary 5.30 (homology generators in transverse multiple points)**.** *For each* $\mathbf{p}_j$ *in Theorem 5.28, if* $\mathbf{p}_j$ *is a transverse multiple point, then there is only one generator for* $G(\mathbf{p}_j)$*, that is* $\sigma_j := \gamma_j \times \beta_j$ *where* β_j *is* $\Phi^{-1}(T_\epsilon)$ *as defined above.*

Corollary 5.31. *The original Cauchy torus* $T := \{|z_i| = \epsilon, i = 1, \cdots, d\}$ *can be decomposed into an integer sum of* $\sigma_{j,i}$ *in* $\mathrm{H}_d(\mathcal{M}, -\infty)$*. In particular, if* $[T] = \sum_{j=1}^{m}\sum_{i=1}^{s_j} \kappa_{j,i}[\sigma_{j,i}]$*, then* $\int_T \omega = \sum_{j=1}^{m}\sum_{i=1}^{s_j} \kappa_{j,i} \int_{\sigma_{j,i}} \omega + O(e^{a|\mathbf{r}|})$ *for any* $a < 0$*.*

Proof: The existence of decomposition of T is given by Theorem 5.28. When $\mathbf{p}_j$ is a transverse multiple point, $s_j = 1$ and σ_j is given in Corollary 5.30. The extra term $O(e^{an})$ is due to the fact that we integrate ω in the relative homology group (see Chapter 5.3.3). □

Finally it should be noted that $\kappa_{j,i}$ is hard to be found by geometric methods. One can use some numerical methods [Mel21, Chapter 9.3.2] to compute the coefficient $\kappa_{j,i}$ because precision up to one decimal digit is enough to find the integer value $\kappa_{j,i}$. The easiest case that one can find these coefficients is the hyperplane arrangement in Chapter 5.3. In particular, there is only one generator for each $G(\mathbf{p})$, which is $\Phi^{-1}(T_\epsilon)$, and we can explicitly write it as a linking torus (Definition 5.44) up to orientation. The coefficient κ_j is then either 0, 1, or -1. There is a computable way to recognize critical points with $\kappa_j = 0$ (Definition 5.41). For other critical points with coefficients $\kappa_j = 1$ or -1, explicit formulas for $\int_{\sigma_j} \omega$ are given in Chapter 5.3.5 and one does not need to take explicit care on the sign of κ_j.

5.2.5. Critical point at infinity

This section is mainly as an appendix to the stratified Morse theory discussed previously, during which we assume no critical point at infinity (CPAI) but we have never seriously define what CPAI is and never talk about what will happen if CPAIs do arise. A major condition for Morse theory is compactness. When the Morse function is proper, $h^{-1}([a,b])$ is compact. When the Morse function is not proper, [BMP22] suggests that no occurrence of CAPI with height values in $[a,b]$ is sufficient in the setting of ACSV where $h = -\hat{\mathbf{r}} \cdot \mathrm{Relog}(\mathbf{z})$. In [BMP22], they call CPAI stationary points at infinity (SPAI). Indeed, SPAI is more general than CAPI. It also involves ordinary critical points whose alias in [BMP22] is affine critical points. We adopt notation and definition in [PWM24, Chapter 7.5] since the book is newer and two of its authors also wrote [BMP22].

Definition 5.32 (Critical points at infinity (CPAI)). *Define a binary relation $\mathcal{R}$ on $\mathbb{C}_*^d \times \mathbb{CP}^{d-1}$ such that $(\mathbf{z}, \hat{\mathbf{r}}) \in \mathcal{R}$ if $\mathbf{z}$ is a stratified critical point for $h_{\hat{\mathbf{r}}}$. Let $\overline{\mathcal{R}}$ be the closure of $\mathcal{R}$ in $\mathbb{CP}^d \times \mathbb{CP}^{d-1}$. A point $\mathbf{z}_* \in \mathbb{CP}^d \backslash \mathbb{C}_*^d$ is a* **critical point at infinity** *in the direction $\hat{\mathbf{r}}_*$ if $(\mathbf{z}_*, \hat{\mathbf{r}}_*) \in \overline{\mathcal{R}}$.*

Remark. We abuse the notation in $\mathbb{CP}^d \backslash \mathbb{C}_*^d$ in which we identify $\mathbb{C}_*^d$ as a subset in $\mathbb{CP}^d$ consisting of $[z_1 : \cdots : z_{d+1}]$ such that $z_i \neq 0$ for $i = 1, \cdots, d+1$. Therefore, a CAPI is either on some coordinate plane in $\mathbb{C}^d$ (when $z_i = 0$ for any $i \in \{1, \cdots, d\}$ and $z_{d+1} \neq 0$), or infinitely far away (when $z_{d+1} = 0$).

Definition 5.33 (Critical values at infinity (CVAI)). *Define a ternary relation $\mathcal{C}$ on $\mathbb{C}_*^d \times \mathbb{CP}^{d-1} \times \mathbb{R}$ such that $(\mathbf{z}, \hat{\mathbf{r}}, \eta) \in \mathcal{C}$ if $(\mathbf{z}, \hat{\mathbf{r}}) \in \mathbb{R}$ and $h_{\hat{\mathbf{r}}}(\mathbf{z}) = \eta$. Let $\overline{\mathcal{C}}$ be the closure of $\mathcal{C}$ in $\mathbb{CP}^d \times \mathbb{CP}^{d-1} \times \mathbb{R}$. A value η_* is a* **critical value at infinity** *if $(\mathbf{z}_*, \hat{\mathbf{r}}_*, \eta_*) \in \overline{\mathcal{C}}$ and $\mathbf{z}_* \notin \mathbb{C}_*^d$.*

Remark. In [BMP22], $z_* \notin \mathbb{C}_*^d$ with $(z_*, \hat{\mathbf{r}}_*, \eta) \in \overline{\mathcal{C}}$ is called heighted stationary critical points (H-SPAI). The closure of $\mathcal{C}$ is taken in $\mathbb{C}_*^d \times \mathbb{CP}^{d-1} \times \mathbb{R}$ instead of $\mathbb{C}_*^d \times \mathbb{CP}^{d-1} \times \mathbb{RP}$. Therefore, it excludes CPAI with height at negative infinity. Indeed, only these H-SPAIs will matter to the topology of $(\mathcal{M}, \mathcal{M}_{-\infty})$.

Let's list critical values (including CVAI) of $h_{\hat{\mathbf{r}}}$ from the least to the largest, $c_1 < c_2 < \cdots < c_m$.

It may be possible that more than one critical points (including CPAI) share the same critical value. Let c_i be the largest CVAI, then Theorem 5.28 works for everything above the height c_i. In particular, we have the following theorem and it is a direct result of [BMP22, Theorem 2(ii)] and Theorem 5.28. Let a be a real number in (c_i, c_{i+1}). Define $\mathcal{M}_{<c_{i+1}}$ to be $\mathcal{M}_{\leq a}$. The definition is well-defined because $\mathcal{M}_{\leq a}$ is homotopy equivalent to $\mathcal{M}_{\leq a'}$ for any $a, a' \in (c_i, c_{i+1})$.

Theorem 5.34 (Homology generators for $\mathrm{H}_d(\mathcal{M}, \mathcal{M}_{<c_{i+1}})$)**.** *Fix $\hat{\mathbf{r}}$ and assume that there is no CVAI larger than or equal to c_{i+1}. Let $\mathbf{p}_1, \cdots, \mathbf{p}_m$ enumerate the stratified critical points with $h_{\hat{\mathbf{r}}}(\mathbf{p}_i) \geq c_{i+1}$ of $\mathcal{V}_*$ in decreasing order of the height function $h_{\hat{\mathbf{r}}}$, where the stratum containing $\mathbf{p}_j$ has complex codimension k_j. If all critical points are quadratic nondegenerate (i.e. Hessian of $h_{\hat{\mathbf{r}}}$ doesn't vanish), then there are cycles $\gamma_1, \cdots, \gamma_m$ on $\mathcal{V}_*$ along with basis $\beta_{j,1}, \cdots, \beta_{j,s_j}$ for the k_j-th homology of the normal Morse data, with the following properties.*

(a) $h_{\hat{\mathbf{r}}}$ achieves its maximum on γ_j at $\mathbf{p}_j$;

(b) γ_j is of homotopy type $(B^{d-k_j}, \partial B^{d-k_j})$;

(c) A basis for the integer homology group $\mathrm{H}_d(\mathcal{M}, \mathcal{M}_{<c_{i+1}})$ can be formed by cycles $\sigma_{j,i} = \gamma_j \times \beta_{j,i}$. For fixed j, $\sigma_{j,i}$ form a basis for $G(\mathbf{p}_j)$.

Intuitively, the original Cauchy torus $T := \{\mathbf{z} : |z_i| = \epsilon\}$ is at height $-\hat{\mathbf{r}} \cdot (\log(\epsilon), \cdots, \log(\epsilon)) = -\sum_{i=1}^d r_i \log(\epsilon)$. Since ϵ is small, the torus T can be made at arbitrary height higher than the largest critical value. Then Theorem 5.34 says that we can push down T to lower height. Each time we meet with an affine critical value (i.e. critical value but not CVAI), T decomposes into an integer sum of generators in local homology groups $G(\mathbf{p}_)$ of critical points at this height and a cycle C that is supported at lower heights. We continue pushing down the cycle C until we meet with a CVAI at which we don't know how to proceed because our version of Morse theory stops working. This is the case of Theorem 5.34. If there is no CVAI, then we can push further below until we pass the lowest critical value. Beyond this level, topology doesn't change anymore. This is the case of Theorem 5.28. We continue the story of multiple points in the next section where we will talk about the easiest case hyperplane arrangements. We can explicitly deform T, pushing it lower and lower.

Existence of CVAI is not yet well understood. However we have a few helpful facts known. First of all, [Gil22] shows that there are sufficient conditions implying that the set of $\hat{\mathbf{r}}$ in which a CVAI exists has codimension at least one. [BMP22] shows that CVAI are effectively computable. CVAI does not appear for generic direction $\hat{\mathbf{r}}$ when $\mathcal{V}$ is a hyperplane arrangement.

5.3. Hyperplane Arrangement

In this section, we are going to review [BMP24b], where they study generating functions in the form of

$$F(\mathbf{z}) = F(z_1, \cdots, z_d) = \frac{G(\mathbf{z})}{H(\mathbf{z})}$$

where $G(\mathbf{z})$ is an entire function and

$$H(\mathbf{z}) = \prod_{j=1}^{m} L_j(\mathbf{z})^{p_j}$$

for integers $p_j \geq 1$ and real linear functions

$$L_j(\mathbf{z}) = 1 - \mathbf{b}^{(j)} \cdot \mathbf{z} \tag{5.6}$$

for real d-vector $\mathbf{b}^{(j)}$. We assume that G and H are coprime; in this case, G does not identically vanish on the zero set of one of the L_j. The singular set of F is the variety defined by H, that is,

$$\mathcal{V} = \mathcal{V}_H = \{\mathbf{z} \in \mathbb{C}^d : H(\mathbf{z}) = 0\}.$$

In particular, $\mathcal{V}$ defines a hyperplane arrangement.

This particular type of generating functions is an important building block when we study pseudo multiple points in Chapter 6. A pseudo multiple point $\mathbf{p}$ for $F = G/Q$ has the property that the leading homogeneous term of the power series expansion of $Q(\mathbf{z})$ at $\mathbf{p}$ is a product of linear polynomials, exactly in the form of $H(\mathbf{z})$ above. This is the big motivation that we review [BMP24b] here.

We will see in Chapter 6 that the proof finally boils down to the case of hyperplane arrangements.

Let's give out some terms before we say their result. We define a **flat** to be an intersection of hyperplanes. If we denote $\mathcal{V}(L_{k_1}, \cdots, L_{k_s}) = \{\mathbf{z} \in \mathbb{C} : L_{k_1}(\mathbf{z}) = \cdots = L_{k_s} = 0\}$, then for any subset $\{k_1, \cdots, k_s\} \subseteq \{1, \cdots, m\}$, $\mathcal{V}(L_{k_1}, \cdots, L_{k_s})$ is a flat. We can see that we can put a lattice structure on flats by giving them a partial order defined by inclusion. Two different subsets of indices can give the same flat. For any index subset $T \subset [m]$, we let $\overline{T}$ to be the set of all indices i such that $\mathcal{V}_{L_i}$ contains the flat defined by T. If $T = \overline{T}$ whenever the flat defined by T is non-empty, we say that the hyperplane arrangement is **transverse**. In [BMP24b], this property is also called **simple**. In fact, they define simple arrangements to be as follows.

Definition 5.35 ([BMP24b], simple arrangements)**.** *The singular set $\mathcal{V}$, or rational function $F(\mathbf{z})$ is said to be* **simple** *if for any subset $\{k_1, \cdots, k_s\} \subseteq [m]$ of indices such that the flat $\mathcal{V}_{k_1}, \cdots, \mathcal{V}_{k_s}$ is nonempty, the coefficient vectors $\mathbf{b}^{(k_1)}, \cdots, \mathbf{b}^{(k_s)}$ are linearly independent.*

Indeed, a hyperplane arrangement is **transverse** if and only if it is **simple**. We use these two notions interchangablely in this paper and [BMP24b] uses the notion of simple arrangements. The analysis will be done to simple arrangements only. For non-simple arrangements, there is a way to decompose F into a sum of simple arrangements [BMP24b, Chapter 5], and we briefly explain how it can be done in the end of this subsection.

5.3.1. Stratification and critical points

Stratification

Whether the singular variety is simple or not, we can give an explicit stratification in the hyperplane case. Let $\mathcal{A}$ to be the collection of maximal sets $\overline{T}$ for each $T \subseteq [m]$ whenever the flat defined by T is non-empty. For each $S \in \mathcal{A}$, we define a stratum $\mathcal{S}_S$ to be the flat defined by S with all subflats removed.

$$\mathcal{S}_S = \mathcal{V}_S \backslash \bigcup_{S \subset T} \mathcal{V}_T$$

where $\mathcal{V}_T = \mathcal{V}(L_i, i \in T)$. For example, a stratum defined by a set of singleton $\{i\}$ is the hyperplane $\mathcal{V}_{L_i}$ after removing all intersections of $\mathcal{V}_{L_i}$ with $\mathcal{V}_{L_j}$. The closure of each stratum $\mathcal{S}_S$ is the flat $\mathcal{V}_S$. The dimension of $\mathcal{S}_S$ is the dimension of $\mathcal{V}_S$ as an algebraic set. It is also the dimension of $\mathcal{S}_S$ as a complex manifold. The maximum number of linearly independent vectors in $\{\mathbf{b}^{(j)}, j \in S\}$ is the codimension. For a simple arrangements, the codimension of $\mathcal{S}_S$ will always be $|S|$ because $\{\mathbf{b}^{(j)}, j \in S\}$ is a linearly independent set. Therefore, the dimension of $\mathcal{S}_S$ will always be $d - |S|$ in a simple arrangement.

Critical points

For multiple points, we find critical points stratum by stratum. By definition, a critical point σ on a stratum $\mathcal{S}$ is a point where $dh_{\hat{\mathbf{r}}}|_{\mathcal{S}} = 0$. Here $h_{\hat{\mathbf{r}}}$ is the height function defined by $h_{\hat{\mathbf{r}}} = -\mathbf{r} \cdot \mathrm{Relog}(\mathbf{z})$ and is the real part of the function $\phi = -\mathbf{r} \cdot \log(\mathbf{z})$. The function ϕ is locally analytic and by Cauchy-Riemann equations, $dh_{\hat{\mathbf{r}}}|_{\mathcal{S}} = 0$ if and only if $d\phi|_{\mathcal{S}} = 0$. That is, $\nabla\phi(\sigma)$ is normal to the tangent plane $T_\sigma(\mathcal{S})$. In other words, $\nabla\phi(\sigma)$ is in the normal space of $\mathcal{S}$ at σ.

For each stratum $\mathcal{S}_S$ in the hyperplane arrangement, take a linearly independent set $\{\mathbf{b}^{(i)}, i \in S'\}$ where $S' \subseteq S$ and $\mathcal{V}_S = \mathcal{V}_{S'}$. If the hyperplane arrangement is simple, then $S' = S$. The span of gradients of L_i for $i \in S'$ is the normal space of $\mathcal{S}_S$ at σ. Suppose that $S' = \{k_1, \cdots, k_s\}$. Then we can essentially capture all critical points on $\mathcal{S}_S$ by finding points $\mathbf{z} \in \mathcal{V}_S$ such that the following matrix is rank deficient.

$$M(\mathbf{z}) := \begin{bmatrix} \nabla L_{k_1} \\ \vdots \\ \nabla L_{k_s} \\ \nabla\phi \end{bmatrix} = \begin{bmatrix} & \mathbf{b}^{(k_1)} & \\ & \vdots & \\ & \mathbf{b}^{(k_s)} & \\ r_1/z_1 & \cdots & r_d/z_d \end{bmatrix} \tag{5.7}$$

In particular, when we have a complete intersection point σ (i.e. $|S'| = d$), the matrix $M(\sigma)$ is a $(d+1) \times d$ matrix. It is always full rank. Instead, in this case, σ is always a critical point because $\nabla\phi(\sigma)$ is always in the normal space of $\{\sigma\}$ because we have d linearly independent $\mathbf{b}^{(i)}$ in this case.

For $s < d$, we can reorder variables in $\mathbf{z}$ and L_{k_i} so that M contains pivots in the first s diagonal entries. Let M_j denote the $(s+1)\times(s+1)$ matrix constructed from the first s columns of M together with the $(s+j)$-th column. The set of critical point equations on the stratum $\mathcal{S}_S$ is characterized by

$$L_{k_1}(\mathbf{z}) = \cdots = L_{k_s}(\mathbf{z}) = \det M_1(\mathbf{z}) = \cdots = \det M_{d-s}(\mathbf{z}) = 0. \tag{5.8}$$

Equation 5.8 does not guarantee that a critical point satisfying these equations is indeed on the stratum $\mathcal{S}_S$. Instead, it only guarantees that the point is on $\mathcal{V}_S$. It is possible that the point is on a subflat $\mathcal{V}_T$ of $\mathcal{V}_S$, and so is not on $\mathcal{S}_S$ because we remove all subflats from $\mathcal{V}_S$ to get $\mathcal{S}_S$. Then this point is a critical point for both the flat $\mathcal{V}_T$ and $\mathcal{V}_S$. When such things happen, we say that the direction $\hat{\mathbf{r}}$ is **non-generic**. We adopt [BMP24b, Definition 3.4] as follows.

Definition 5.36. *The direction $\hat{\mathbf{r}}$ is* **generic** *if no critical point of $h_{\hat{\mathbf{r}}}$ is a critical point of two distinct flats.*

This notion of generic directions coincides with the more general definition of generic directions in Definition 5.21 where $\overline{\mathcal{S}}$ there is a flat here, and $\mathcal{S}$ is a flat minus all its sub-flats.

Definition 5.37. *The* **critical set** *Ω of F in the direction $\hat{\mathbf{r}}$ is the union of all critical points of $h_{\hat{\mathbf{r}}}$ on strata $\mathcal{S}_S$ for $S \in \mathcal{A}$.*

If a critical point σ of $h_{\hat{\mathbf{r}}}$ satisfying (5.8) is not on $\mathcal{S}_S$, we still call it a critical point of the $\mathcal{S}_S$. We denote the unique stratum of lowest dimension where σ lies as $\mathcal{S}(\sigma)$. A direction $\hat{\mathbf{r}}$ is generic if for each $S \in \mathcal{A}$, any solution σ for (5.8) satisfies that $\mathcal{S}(\sigma) = \mathcal{S}_S$.

In our initial setting, L_i is a real polynomial. Therefore, $\mathbf{b}^{(i)}$ is a real vector. This gives the following result.

Lemma 5.38. *[BMP24b, Lemma 3.3] If σ is a critical point of $h_{\hat{\mathbf{r}}}$, then $\sigma \in \mathbb{R}^d$.*

This result is powerful. It tells us the number of critical points for a stratum in each orthant. Let $\mathcal{V}_{S,\mathbb{R}} = \mathcal{V}_S \cap \mathbb{R}^d$ to be the real part of the flat $\mathcal{V}_S$. Notice that the function $h_{\hat{\mathbf{r}}}$ is continous and strictly convex on each orthant O of $\mathbb{R}^d$. If $\mathcal{V}_{S,\mathbb{R}} \cap O$ is bounded, then there is a unique critical point on $\mathcal{V}_{S,\mathbb{R}}$ in the orthant O. Cnvexity means that it is a local minimum. If $\mathcal{V}_{S,\mathbb{R}} \cap O$ is unbounded, then we can move a variable z_i to arbitrarily large modulus and thus $h_{\hat{\mathbf{r}}}$ can be $-\infty$ on $\mathcal{V}_{S,\mathbb{R}} \cap O$. There is no critical point for $h_{\hat{\mathbf{r}}}$ on $\mathcal{V}_{S,\mathbb{R}} \cap O$.

Contributing points

Not all critical points will contribute to the final asymptotics. Intuitively speaking, the original Cauchy torus $T := \{\mathbf{z} \in \mathbb{C}^d : |z_i| = \epsilon, i = 1, \cdots, d\}$ can be set at an arbitrarily high height, higher than any critical points. When we push down T to lower heights, if a critical point is not contributing, then we can push the torus down below this critical point without needing to form an intersection class on the singular variety near this critical point. We give criteria on deciding a contributing point here, but defer explaining the reason of such criteria until we go into the topological arguments.

We first give definitions of **normal cones** and **lognormal cones**.

Definition 5.39 (normal cones)**.** *For any flat $\mathcal{V}_S$, the (positive) normal cone $N(S)$ is the cone*

$$N(S) = \left\{ \sum_{i \in S} a_j \mathbf{b}^{(i)} : a_j > 0 \right\},$$

the positive span of vectors $\mathbf{b}^{(i)}, i \in S$.

Definition 5.40 (lognormal cones)**.** *For any flat $\mathcal{V}_S$ and any $\sigma \in \mathcal{V}_S \cap \mathbb{R}^d$, the (positive) lognormal cone $\tilde{N}_\sigma(S)$ is the cone*

$$\tilde{N}_\sigma(S) = \left\{ \sum_{i \in S} a_j \tilde{\mathbf{b}}^{(i)}_\sigma : a_j > 0 \right\},$$

where $\tilde{\mathbf{b}}^{(i)}_\sigma = (b^{(i)}_1 \sigma_1, \cdots, b^{(i)}_d \sigma_d)$.

Remark. For critical points σ, we write $N(\sigma)$ to denote the normal cone of the flat corresponding to the stratum $\mathcal{S}(\sigma)$. Similarly, $\tilde{N}(\sigma)$ is the lognormal cone for that flat at point σ.

Definition 5.41 (contributing points)**.** *A critical point σ is contributing if $-\nabla h_{\hat{\mathbf{r}}}(\sigma) \in N(\sigma)$, or equivalently, $\hat{\mathbf{r}} \in \tilde{N}(\sigma)$. The collection of contributing points are denoted by* `contrib`.

5.3.2. Imaginary fibers and linking tori

We introduce two important ingredients in the analysis of generating functions with poles on hyperplane arrangements.

Going all the way back to the Cauchy integral where we have

$$a_{\mathbf{r}} = [\mathbf{z}^{\mathbf{r}}]F(\mathbf{z}) = \left(\frac{1}{2\pi i}\right)^d \int_T \frac{G(\mathbf{z})}{\prod_j L_j(\mathbf{z})^{p_j}} \frac{d\mathbf{z}}{\mathbf{z}^{\mathbf{r}+\mathbf{1}}}.$$

We can see this integral as an integral of a d-form ω over a d-chain C. In our case, the d-form $\omega = \dfrac{G(\mathbf{z})}{\prod_j L_j(\mathbf{z})^{p_j}} \dfrac{d\mathbf{z}}{\mathbf{z}^{\mathbf{r}+\mathbf{1}}}$ and the d-chain $C = T$. The form ω is holomorphic if and only if $\dfrac{G(\mathbf{z})}{\prod_j L_j(\mathbf{z})^{p_j} \mathbf{z}^{\mathbf{r}+\mathbf{1}}}$ is holomorphic. The poles are on the singular set $\mathcal{V}(z_1 \cdots z_d H)$. We define $\mathcal{M}$ to be $\mathbb{C}_*^d - \mathcal{V}(H)$. Therefore, $\mathcal{M}$ is the domain of holomorphy for ω since it avoids both $\{\mathbf{z} \in \mathbb{C}^d : z_i = 0 \text{ for some } i\ \}$ and $\{\mathbf{z} \in \mathbb{C}^d : H(\mathbf{z}) = 0\}$. We can deform the original Cauchy torus T for free in $\mathcal{M}$ without changing the integral value.

It is difficult to visualize $\mathcal{M}$ except when $d = 1$. We claim that $\mathcal{M} \cap \mathbb{R}^d$ captures enough information for us to deform the original Cauchy torus. If a real vector $\mathbf{x} \neq 0$, then $\mathbf{z} := \mathbf{x} + i\mathbf{y}$ is non-zero. Similarly, if $\mathbf{x} \in \mathbb{R}^d$ and $L_i(\mathbf{x}) \neq 0$, then $L_i(\mathbf{x} + i\mathbf{y}) \neq 0$. Indeed, $L_i(\mathbf{x} + i\mathbf{y}) = 1 - \mathbf{b}^{(i)} \cdot (\mathbf{x} + i\mathbf{y}) = 1 - \mathbf{b}^{(i)} \cdot \mathbf{x} - i\mathbf{b}^{(i)} \cdot \mathbf{y}$. Then $\mathrm{Re}(L_i(\mathbf{x} + i\mathbf{y})) = 1 - \mathbf{b}^{(i)} \cdot \mathbf{x} = L_i(\mathbf{x})$. This observation implies that if a point $\mathbf{x} \in \mathcal{M} \cap \mathbb{R}^d$, then $\mathbf{x} + i\mathbf{y} \in \mathcal{M}$ for all $\mathbf{y} \in \mathbb{R}^d$. We are then motivated to define the **imaginary fiber** at $\mathbf{x}$.

Definition 5.42 (imaginary fiber)**.** *Let $\mathbf{x} \in \mathbb{R}_*^d$ be any point not in $\mathcal{V}_H \cap \mathbb{R}^d$. The* **imaginary fiber** *$\mathcal{C}_{\mathbf{x}}$ is the chain*

$$\mathcal{C}_{\mathbf{x}} = \mathbf{x} + i\mathbb{R}^d = \{\mathbf{x} + i\mathbf{y} : \mathbf{y} \in \mathbb{R}^d\},$$

oriented by the standard orientation on $\mathbb{R}^d$. The point $\mathbf{x}$ is called the **basepoint** *of the fiber $\mathcal{C}_{\mathbf{x}}$.*

Remark. $\mathcal{C}_{\mathbf{x}}$ is in $\mathcal{M}$ since $\mathbf{x} \in \mathcal{M} \cap \mathbb{R}^d$.

What does $\mathcal{M} \cap \mathbb{R}^d$ look like?

Let's denote $\mathcal{M}_{\mathbb{R}} = \mathcal{M} \cap \mathbb{R}^d$. By definition, $\mathcal{M}_{\mathbb{R}} = \mathbb{R}^d_* - \mathcal{V}(H) \cap \mathbb{R}^d$. Since $H = \prod_j L_j$, $\mathcal{M}_{\mathbb{R}}$ is indeed a collection of convex polyhedra formed by coordinate axes and the real parts of hyperplanes.

Example 5.43. *Let $d = 1$ and $H = 1 - z$. Then $\mathcal{M}_{\mathbb{R}} = \mathbb{R}_* - \mathcal{V}(H) \cap \mathbb{R}^d = \mathbb{R} - \{0\} - \mathcal{V}(1 - x) = \mathbb{R} - \{0, 1\}$. Then $\mathcal{M}_{\mathbb{R}}$ is a collection of three intervals, $(\infty, 0), (0, 1)$, and $(1, \infty)$.*

For each point $\mathbf{x} \in \mathcal{V}(H) \cap \mathbb{R}^d$, not all $\mathbf{x}+i\mathbf{y}$ is in $\mathcal{V}(H)$. Instead, since $L_i(\mathbf{x}+i\mathbf{y}) = 1 - \mathbf{b}^{(i)} \cdot \mathbf{x} - i\mathbf{b}^{(i)} \cdot \mathbf{y}$, we need $\mathbf{y} \in \mathbb{R}^d$ to be in the line $\ell_i(\mathbf{y}) = \mathbf{b}^{(i)} \cdot \mathbf{y}$. When $d = 1$, $\mathcal{V}(L_i) \cap \mathbb{R}$ is just a 0-dimensional manifold, a set of isolated points. For each point $x \in \mathcal{V}(L_i) \cap \mathbb{R}$, there is only one choice for $x + iy \in \mathcal{V}(L_i)$, that is, $y = 0$. So singular varieties $\mathcal{V}(L_i)$ is just a collection of isolated points. When $d = 2$, $\mathcal{V}(L_i) \cap \mathbb{R}$ is a line, or a 1-dimensional manifold. For each point $\mathbf{x} \in \mathcal{V}(L_i) \cap \mathbb{R}$, there is another 1-dimensional manifold for $\mathbf{y}$ such that $\mathbf{x} + i\mathbf{y} \in \mathcal{V}(L_i)$. Therefore, $\mathcal{V}(L_i)$ is a 2-dimensional manifold and $\mathcal{V}(H)$ is the union of these 2-dimensional manifold, which is again 2-dimensional, with some points at the intersection of $\mathcal{V}(L_i)$ and $\mathcal{V}(L_j)$. For each point $\mathbf{x}$ in some coordinate axis, we assume without loss of generality that $x_1 = 0$. As long as $y_1 \neq 0$, $\mathbf{x} + i\mathbf{y} \in \mathcal{M}$.

Therefore, though two connected components in $\mathcal{M} \cap \mathbb{R}^d$ is disconnected either by a coordinate axis or the real part of a hyperplane, they are connected in $\mathcal{M}$.

Decompose T into imaginary fibers

An interesting observation is that we can decompose a small torus centered at the origin into a sum of alternating imaginary fibers. Let's start from the simplest case when $d = 1$ and $T = \{x + iy \in \mathbb{C} : x = \epsilon \cos(\theta), y = \delta \sin(\theta)\}$ where $\epsilon = \delta$ is sufficiently small so that $T \in \mathcal{M}$. We let $\delta \to \infty$. Then this circle is stretched along the imaginary axis and becomes an elongated ellipse. Intuitively, it looks like two imaginary fibers with basepoints ϵ and $-\epsilon$ such that they are connected infinitely far away from the complex plane. Since the connection parts are infinitely far away, the height $h_{\hat{\mathbf{r}}}$ in these positions are infinitely small so that they contribute to the final asymptotics exponentially small. To make it rigorous, we need to use the relative homology to quantify how much error we are introducing when we decome T into imaginary fibers. We will see how relative homology helps later when we introduce more topological definitions. Currently, let's assume that our intuition is right.

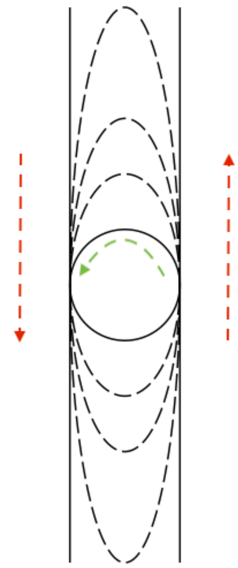


Figure 5.8: The deformation of T in one dimension to the sum of two imaginary fibers with orientations shown by the arrow

For general d, we can write

$$T = \{\mathbf{x} + i\mathbf{y} \in \mathbb{C}^d : \mathbf{x} = \epsilon(\cos(\theta_1), \cdots, \cos(\theta_d)) \in \mathbb{R}^d, \mathbf{y} = \delta(\sin(\theta_1), \cdots, \sin(\theta_d)) \in \mathbb{R}^d\}.$$

Letting $\delta \to \infty$, we get T is almost equal to

$$\sum_{B \in \mathrm{Adj}(\mathbf{0})} \mathrm{sgn}(B) \mathcal{C}_{\pm\epsilon}$$

where $\mathrm{Adj}(\mathbf{0})$ is the set of connected components in $\mathcal{M}_{\mathbb{R}}$ adjacent to the origin $\mathbf{0}$, and $\mathrm{sgn}(B) = \mathrm{sgn}(\prod_{j=1}^{d} x_j)$ for any $\mathbf{x} \in B$ is the **absolute sign** of B. The imaginary fiber $\mathcal{C}_{\pm\epsilon}$ is the fiber with basepoints $\mathbf{x} = (\pm\epsilon, \cdots, \pm\epsilon)$ such that $\mathbf{x} \in B$. For example, when $d = 2$, the above quantity is $\mathcal{C}_{\epsilon,\epsilon} + \mathcal{C}_{-\epsilon,-\epsilon} - \mathcal{C}_{\epsilon,-\epsilon} - \mathcal{C}_{-\epsilon,\epsilon}$.

Notice that we did not say T is equal to $\sum_{B \in \mathrm{Adj}(\mathbf{0})} \mathrm{sgn}(B)\mathcal{C}_{\pm\epsilon}$. Instead, they are equal only in a relative homology group that we defined later.

Linking torus at a critical point

Previously, we see how a torus around the origin can be decomposed into a sum of alternating imaginary fibers. Can we do the same if we have a torus around a critical point? More preciesly, if $\mathcal{A}$ is a simple arrangement and σ is a critical point on a stratum of codimension s. Let $\mathcal{S}(\sigma)$ be defined by s hyperplanes $L_1, \cdots, L_s$. Can we decompose the torus $\mathcal{T}_\sigma = \{\mathbf{z} \in \mathbb{C}_*^d : |L_1(\mathbf{z})| = \cdots = |L_s(\mathbf{z})| = \epsilon\}$? The answer is yes, up to introducing an exponentially small error to our final asymptotics. We can write the decomposition as a **linking torus** of σ. The definition is more involved because we need to take care of the orientation of each imaginary fiber in the linking torus.

Definition 5.44 (linking torus)**.** *[BMP24b, Definition 3.10] Let $\mathcal{A}$ be a simple arrangement, $\hat{\mathbf{r}}$ fixed, and $\boldsymbol{\sigma} \in \Omega$. Suppose s is the codimension of the stratum containing $\boldsymbol{\sigma}$ and let $k_1, \cdots, k_s$ be the indices such that the stratum $\mathcal{S}(\boldsymbol{\sigma})$ is defined by $L_{k_1} = \cdots = L_{k_s} = 0$. For each of the 2^k components B of $\mathcal{M}_{\mathbb{R}}$ with $\boldsymbol{\sigma} \in \partial B$, define the* **sign** *of B with respect to $\boldsymbol{\sigma}$ by*

$$\mathrm{sgn}_{\boldsymbol{\sigma}}(B) := \mathrm{sgn}\left(\prod_{i=1}^{s} L_{k_i}(\mathbf{x})\right)$$

where $\mathbf{x}$ is any point of B. The **linking torus** *of $\boldsymbol{\sigma}$ is then*

$$\tau_{\boldsymbol{\sigma}} := \sum_{B : \boldsymbol{\sigma} \in \partial B} \mathrm{sgn}(B)\mathrm{sgn}_{\boldsymbol{\sigma}}(B)\mathcal{C}_B$$

where $\mathcal{C}_B = \mathcal{C}_{\mathbf{x}}$ for any arbitrary $\mathbf{x} \in B$.

At first sight, this definition may not make sense because the choice of the basepoint for $\mathcal{C}_B$ is arbitrary. After we introduce the relative homology, we will see that it does not matter which basepoint we choose if we just want equality in the relative homology group.

5.3.3. Relative homology groups

We introduce the relative homology group in which we consider all equalities above hold. Fix the direction $\hat{\mathbf{r}}$ and let the height function $h : \mathbb{C}_*^d \to \mathbb{R}$ be defined as $h(\mathbf{z}) = h_{\hat{\mathbf{r}}}(\mathbf{z}) = -\hat{\mathbf{r}} \cdot \mathrm{Relog}(\mathbf{z})$. We denote $\mathcal{M}_{\leq a}$ to be the subset $\{\mathbf{z} \in \mathcal{M} : h(\mathbf{z}) \leq a\}$. We let $h_{\min}$ to be the lowest height among all critical points, $h_{\min} = \min\{h(\boldsymbol{\sigma}) : \boldsymbol{\sigma} \in \Omega\}$.

If $a < h_{\min}$ and C is a cycle supported in $\mathcal{M}_{\leq a}$, then the cycle C can be continously deformed in $\mathcal{M}$ so that it is supported in $\mathcal{M}_{\leq b}$ for any $b < a$. The reason is by a non-proper version of stratified More theory which says that the topology of $\mathcal{M}_{\leq a}$ changes only at critical points. Therefore, the relative homology groups $\mathrm{H}_d(\mathcal{M}, \mathcal{M}_{\leq a})$ for $a < h_{\min}$ are isomorphic. We denote all these equivalent topological pairs by $(\mathcal{M}, -\infty)$ and these isomorphic homology groups by $\mathrm{H}_d(\mathcal{M}, -\infty)$. If C and C' are two cycles supported in $(\mathcal{M}, -\infty)$, we say $C \doteq C'$ if $C - C' \doteq 0$ in $\mathrm{H}_d(\mathcal{M}, -\infty)$.

In the integral

$$\left(\frac{1}{2\pi i}\right)^d \int_C \frac{G(\mathbf{z})}{\prod_j L_j(\mathbf{z})^{p_j}} \frac{d\mathbf{z}}{\mathbf{z}^{n\hat{\mathbf{r}}+\mathbf{1}}},$$

we view the integrand d-form ω_n as holomorphic form and we view the cycle C as a representative of a homology class in $\mathrm{H}_d(\mathcal{M}, -\infty)$. Corollary 2.4 tells us that if we replace C with another representative C' in the same class, then the error to the integral is exponentially small when $n \to \infty$. In other words, as $n \to \infty$,

$$\int_C \omega_n = \int_{C'} \omega_n + O(e^{na})$$

for any $a < h_{\min}$. If $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z})$ grows at most polynomially, [BMP24b, Proposition 3.7] tells us that the error term is actually zero for any n greater than some fixed number.

Rigorous decomposition of T

For any two imaginary fibers $\mathcal{C}_{\mathbf{x}}$ and $\mathcal{C}_{\mathbf{x}'}$ where $\mathbf{x}, \mathbf{x}'$ are in the same connected component B of $\mathcal{M}_{\mathbb{R}}$, we have a homotopy taking $\mathcal{C}_{\mathbf{x}}$ to $\mathcal{C}_{\mathbf{x}'}$ by

$$t\mathbf{x} + (1-t)\mathbf{x}' + i\mathbb{R}^d, t \in [0,1].$$

Since $\mathcal{C}_{\mathbf{x}}$ and $\mathcal{C}_{\mathbf{x}'}$ are both (relative) cycles in $(\mathcal{M}, -\infty)$, the homotopy implies that $\mathcal{C}_{\mathbf{x}} \doteq \mathcal{C}_{\mathbf{x}'}$ in $\mathrm{H}_d(\mathcal{M}, -\infty)$ because $\mathcal{C}_{\mathbf{x}} - \mathcal{C}_{\mathbf{x}'}$ is null-homotopic. All $\mathcal{C}_{\mathbf{x}}$ for $\mathbf{x} \in B$ are the same in $\mathrm{H}_d(\mathcal{M}, -\infty)$. Therefore, for each connected component B in $\mathcal{M}_{\mathbb{R}}$, we identify $\mathcal{C}_B$ to be the equivalence class of $\mathcal{C}_{\mathbf{x}}$ in $\mathrm{H}_d(\mathcal{M}, -\infty)$ for $\mathbf{x} \in B$.

For unbounded component B in $\mathcal{M}_{\mathbb{R}}$, any imaginary fiber $\mathcal{C}_{\mathbf{x}}$ where $\mathbf{x} \in B$ can be moved to places where some coordinate x_i is arbitrarily large so that $h(\mathbf{x}) < h_{\min}$ and thus all points on the fiber have heights less than $h_{\min}$. Then $\mathcal{C}_{\mathbf{x}} \doteq 0$. On the other hand, imaginary fibers with basepoints on components where h is bounded from below form a basis of $\mathrm{H}_d(\mathcal{M}, -\infty)$.

Now we can say rigorously in great confidence that the original small Cauchy torus is equal to the sum of alternating imaginary fibers in $\mathrm{H}_d(\mathcal{M}, -\infty)$. In particular,

$$T := \{\mathbf{z} \in \mathbb{C}^d : |\mathbf{z}| = \boldsymbol{\epsilon}\} \doteq \sum_{B \in \mathrm{Adj}(\mathbf{0})} \mathrm{sgn}(B)\mathcal{C}_{\pm\epsilon} \doteq \sum_{B \in \mathrm{Adj}(\mathbf{0})} \mathrm{sgn}(B)\mathcal{C}_B$$

The second equality comes from the equivalence between two imaginary fibers with basepoints in the same component B of $\mathcal{M}_{\mathbb{R}}$. When we deform T by stretching it along its imaginary axis, its intersection with $\mathcal{M}_{\geq h_{\min}}$ converges to the alternating sum of imaginary fibers $\mathcal{C}_{\pm\epsilon}$. For the complement of the intersection, they are below height $h_{\min}$ and thus are homologous to zero in $\mathrm{H}_d(\mathcal{M}, -\infty)$.

Similarly, the generator of the local homology group $G(\boldsymbol{\sigma})$, namely the torus $\mathcal{T}_{\boldsymbol{\sigma}} = \{\mathbf{z} \in \mathbb{C}_*^d : |L_1(\mathbf{z})| = \cdots = |L_s(\mathbf{z})| = \epsilon\}$ is equal to the linking torus at $\boldsymbol{\sigma}$ in $\mathrm{H}_d(\mathcal{M}, -\infty)$ up to a multiplicative constant ± 1 due to orientations. Here $L_1, \cdots, L_s$ are hyperplanes that interesect at $\boldsymbol{\sigma}$.

5.3.4. Slide and replace

From now on, we assume that the hyperplane arrangement $\mathcal{A}$ is simple, unless otherwise noted. We have all ingredients and we are going to piece everything together. We have seen that to get the asympotic, only the homology class in relative holomogy group matters. Therefore, we only care the class of $T := \{\mathbf{z} \in \mathbb{C}^d : |\mathbf{z}| = \boldsymbol{\epsilon}\}$ in $\mathrm{H}_d(\mathcal{M}, -\infty)$. The ultimate goal is to show that

$$T \doteq \sum_{\boldsymbol{\sigma} \in \text{contrib}} \tau_{\boldsymbol{\sigma}}.$$

That is, T is homologous (in $\mathrm{H}_d(\mathcal{M}, -\infty)$) to a sum of linking tori at contributing points, a subset of critical points.

The first observation is that the height function $h_{\hat{\mathbf{r}}}$ is strictly convex. Therefore, on each bounded component $B \in \mathcal{B}$, since B is convex, $h_{\hat{\mathbf{r}}}$ has a unique minimizer in $\overline{B}$ and this minimizer is a critical point. Conversely, every critical point is a minimizer for some component $B \in \mathcal{B}$.

Proposition 5.45. *[BMP24b, Proposition 4.1] For $B \in \mathcal{B}$, let $\mathbf{x}_B$ be the unique $\mathbf{x} \in \overline{B}$ on which $h_{\hat{\mathbf{r}}}$ is minimized. Then $B \to \mathbf{x}_B$ is a bijection between $\mathcal{B}$ and Ω.*

Therefore, we write $B(\boldsymbol{\sigma})$ as the component where h achieves minimum in $\overline{B}$ at $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}(B)$ as the minimizer of h in $\overline{B}$. We denote $\mathcal{C}_{\boldsymbol{\sigma}} := \mathcal{C}_{B(\boldsymbol{\sigma})}$.

[OT94, Theorem 5.2] implies that $\{\mathcal{C}_B : B \in \mathcal{B}\}$ is a basis for $\mathrm{H}_d(\mathcal{M}, -\infty)$ with coefficients in $\mathbb{Z}$. Therefore, $\{\mathcal{C}_{\boldsymbol{\sigma}} : \boldsymbol{\sigma} \in \Omega\}$ is a basis because each $\mathcal{C}_{\boldsymbol{\sigma}}$ corresponds to each $\mathcal{C}_B$ for $B = B(\boldsymbol{\sigma})$. Therefore, we can write T as an integer combination of $\mathcal{C}_{\boldsymbol{\sigma}}$.

Proposition 5.46. *[BMP24b, Proposition 4.3] The $\mathbb{Z}$-module generated by $\{\mathcal{C}_{\boldsymbol{\sigma}} : \boldsymbol{\sigma} \in \Omega\}$ is the same as that generated by $\{\tau_{\boldsymbol{\sigma}} : \boldsymbol{\sigma} \in \Omega\}$, when we view elements in both modules as equivalence classes in $\mathrm{H}_d(\mathcal{M}, -\infty)$.*

To determine the coefficients in the decomposition of T in $\mathrm{H}_d(\mathcal{M}, -\infty)$ in terms of the homology generators $\{\tau_{\boldsymbol{\sigma}}\}$, there is a method called 'slide and replace'. First of all, $T \doteq \tau_{\mathbf{0}}$ where $\tau_{\mathbf{0}}$ is consisting of exactly 2^d imaginary fibers $\mathcal{C}_B$ with alternating orientations. Each B in the summand is the component of $\mathcal{M}_{\mathbb{R}}$ whose closure contains the origin. For each B in the summand, there is a critical point $\boldsymbol{\sigma}(B)$ on the boundary of B such that h achieves its minimum in $\overline{B}$ there. We replace $\mathcal{C}_B$ by $\tau_{\boldsymbol{\sigma}} - \mathrm{sgn}(\boldsymbol{\sigma}) \sum_{B' \in \mathrm{Adj}(\boldsymbol{\sigma}) \setminus B(\boldsymbol{\sigma})} \mathrm{sgn}_{\boldsymbol{\sigma}}(B') \mathcal{C}_{B'}$. Then for each B' in the summand, we continue to do so until we get unbounded B' where $\mathcal{C}_{B'} \doteq 0$.

Slide: The 2^d imaginary fibers in the decomposition of T can be thought as having basepoints very close to the origin in $\mathbb{R}^d$. Then we slide the basepoint of the imaginary fiber within B until it is near $\boldsymbol{\sigma}(B)$, the minimizer of h in $\overline{B}$. This minimizer is unique and so there is no ambiguity. This point is also critical so that $\tau_{\boldsymbol{\sigma}(B)}$ is well-defined.

Replace: We replace $\mathcal{C}_B$ by $\tau_{\boldsymbol{\sigma}(B)}$ and simultaneously introduce more imaginary fibers in components adjacent to $\boldsymbol{\sigma}(B)$. These fibers with basepoints in B' can be slided to lower heights and replaced by linking tori of critical points of lower heights. If B' is unbounded, then the fiber $\mathcal{C}_{B'}$ can be slided to height $-\infty$ and thus disappear in $\mathrm{H}_d(\mathcal{M}, -\infty)$.

Example 5.47 (Two lines in two dimensions)**.** *Let $H(x,y) = (3 - x - 2y)(3 - 2x - y)$ and let $\hat{\mathbf{r}} = (1/2, 1/2)$. There are three critical points of $h_{\hat{\mathbf{r}}}$, namely $\boldsymbol{\sigma}_1 = (3/2, 3/4)$, $\boldsymbol{\sigma}_2 = (3/4, 3/2)$, and $\boldsymbol{\sigma}_3 = (1,1)$ shown as the three solid dots in Figure 5.9. We also label the four components of $\mathcal{M}_{\mathbb{R}}$ in the first quadrant by A, B, C, and D.*

The torus T in $\mathcal{D}$ is homologous in $\mathrm{H}_d(\mathcal{M}, -\infty)$ to a sum of four imaginary fibers with basepoints close to the origin in the four quadrants respectively. The orientations of these four imaginary fibers are alternative, as noted by the plus and minus sign in the left graph of Figure 5.9. Since components in the second, third, and the fourth quadrants are unbounded, the three imaginary fibers with base points there are homologus to zero in $\mathrm{H}_d(\mathcal{M}, -\infty)$. We then slide the imaginary fiber with basepoint in component A toward the critical point $\boldsymbol{\sigma}_3$ as shown by the dotted arrow. We use '$+A$' in Figure 5.9 to denote the imaginary fiber $\mathcal{C}_A$ with positive orientation.

Once we slide $\mathcal{C}_A$ near $\boldsymbol{\sigma}_3$, we can replace it by the linking torus $\tau_{\boldsymbol{\sigma}_3} := \mathcal{C}_A + \mathcal{C}_D - \mathcal{C}_C - \mathcal{C}_B$ as shown *by the dotted green circle in the middle graph. We cannot do this for free. To keep the equivalence in* $\mathrm{H}_d(\mathcal{M}, -\infty)$, *we need to introduce three extra imaginary fibers* $\mathcal{C}_B, \mathcal{C}_C$, *and* $-\mathcal{C}_D$ *to compensate the replacement. Now we can slide the two imaginary fibers* $\mathcal{C}_B$ *and* $\mathcal{C}_C$ *toward critical points* $\boldsymbol{\sigma}_1$ *and* $\boldsymbol{\sigma}_2$ *respectively along the dotted arrows in the middle graph.*

After sliding, we replace them by the linking tori $\tau_{\boldsymbol{\sigma}_1}$ *and* $\tau_{\boldsymbol{\sigma}_2}$. *The replacement introduces two extra copies of imaginary fibers* $\mathcal{C}_D$. *Now we have* $T \doteq \tau_{\boldsymbol{\sigma}_1} + \tau_{\boldsymbol{\sigma}_2} + \tau_{\boldsymbol{\sigma}_3} + \mathcal{C}_D$ *in* $\mathrm{H}_d(\mathcal{M}, -\infty)$. *Since* D *is unbounded,* $\mathcal{C}_D$ *can be pushed to infinitely lower height, along the dotted arrow in the last graph. That is,* $\mathcal{C}_D \doteq 0$. *Therefore* $T \doteq \tau_{\boldsymbol{\sigma}_1} + \tau_{\boldsymbol{\sigma}_2} + \tau_{\boldsymbol{\sigma}_3}$. □

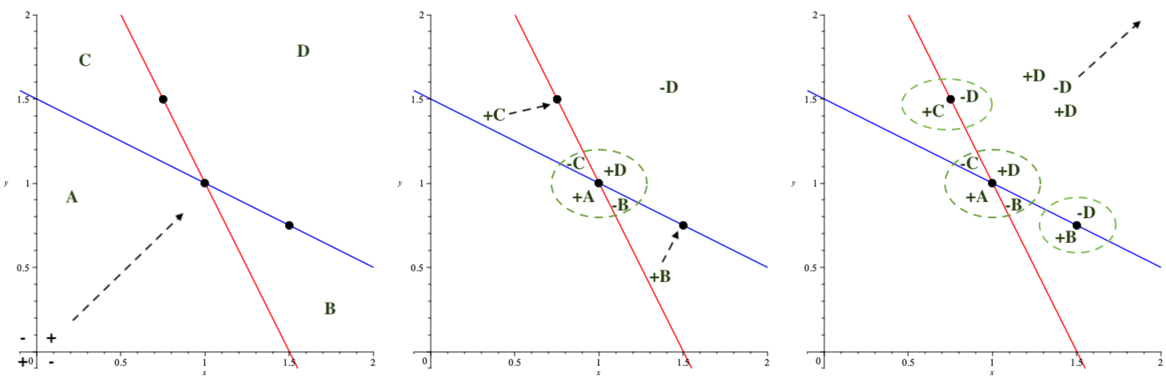


Figure 5.9: Procedure of 'slide and replace' in Example 5.47

It is easy to do when $d = 2$ or $d = 3$ because we can draw the pictures of $\mathcal{M}_{\mathbb{R}}$. It is though very hard to do or to visualize when $d > 3$. Fortunately, [BMP24b, Theorem 3.13] says that only linking tori at contributing points appear in the decomposition of T and these coefficients are all one.

Theorem 5.48. *[BMP24b, Theorem 3.13] If the hyperplane arrangement* $\mathcal{A}$ *is simple and* $\hat{\mathbf{r}}$ *is generic, then the domain of integration* $T = \{\mathbf{z} : |z_i| = \epsilon, i = 1, \cdots, d\}$ *(a product of sufficiently small positively oriented circles) satisfies*

$$T \doteq \sum_{\boldsymbol{\sigma} \in \texttt{contrib}} \tau_{\boldsymbol{\sigma}}$$

5.3.5. Integral

Theorem 5.48 tells us that it suffices to evaluate integral of the d-form $\omega = \dfrac{F(\mathbf{z})}{\mathbf{z}^{\mathbf{r}}}\dfrac{d\mathbf{z}}{\mathbf{z}}$ over linking tori at contributing points.

Suppose that $\boldsymbol{\sigma}$ is a contributing point on a stratum $\mathcal{S}$ of codimension s. Because we assume the hyperplane arrangement is simple and the direction $\hat{\mathbf{r}}$ is generic, we know that $\mathcal{S}$ is the intersection of s transverse hyperplanes. In particular, there exists coordinates $z_{\pi_1}, \cdots, z_{\pi_{d-s}}$ that locally parametrize $\mathcal{S}$.

Consider the change-of-coordinate map

$$\Phi(\mathbf{z}) = (L_{k_1}(\mathbf{z}), \cdots, L_{k_s}(\mathbf{z}), z_{\pi_1} - \sigma_{\pi_1}, \cdots, z_{\pi_{d-s}} - \sigma_{\pi_{d-s}}).$$

In particular, this is a linear invertible map and so it gives a global coordinate change.

Stratum of codimension $s = d$

When the contributing point $\boldsymbol{\sigma}$ is a complete intersection point, the linking torus at $\boldsymbol{\sigma}$ is homologus to $\Phi^{-1}(T_\epsilon)$ where $T_\epsilon = \{\mathbf{z} \in \mathbb{C}^d : |z_i| = \epsilon\}$ in $\mathrm{H}_d(\mathcal{M}, -\infty)$ up to a multiplicative constant ± 1 because they may be oriented oppositely. Let's assume that they are oriented in the same direction. In particular, we have for any $a < 0$,

$$\left(\frac{1}{2\pi i}\right)^d \int_{\tau_{\boldsymbol{\sigma}}} \omega = \left(\frac{1}{2\pi i}\right)^d \int_{\Phi^{-1}(T_\epsilon)} \omega + O(e^{an})$$

When $F(\mathbf{z})$ in ω is a rational function, the error term $O(e^{an})$ becomes $o(1)$. Then we can compute the iterated residue of ω on the 0-dimensional stratum $\mathcal{S} = \{\boldsymbol{\sigma}\}$ and $\left(\dfrac{1}{2\pi i}\right)^d \displaystyle\int_{\Phi^{-1}(T_\epsilon)} \omega = \mathrm{Res}(\omega; \boldsymbol{\sigma})$. In particular, this residue is just a constant. We can write down this residue explicitly by [PWM24, Theorem 10.12],

$$\begin{aligned}\left(\frac{1}{2\pi i}\right)^d \int_{\Phi^{-1}(T_\epsilon)} \omega :&= \left(\frac{1}{2\pi i}\right)^d \int_{\Phi^{-1}(T_\epsilon)} \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z}) d\mathbf{z} \\ &= \frac{\boldsymbol{\sigma}^{-\mathbf{r}}}{(\mathbf{p}-\mathbf{1})!} \frac{(-1)^{|\mathbf{p}-\mathbf{1}|} G(\boldsymbol{\sigma})}{\prod_{j \neq k_1, \cdots, k_d} L_j(\boldsymbol{\sigma}) \det \Gamma_\Phi(\boldsymbol{\sigma})} (\mathbf{r}\Gamma_\Phi^{-1}(\boldsymbol{\sigma}))^{\mathbf{p}-\mathbf{1}} \\ &\times \left(1 + O\left(\frac{1}{|\mathbf{r}|}\right)\right)\end{aligned} \tag{5.9}$$

where Γ_Φ is the **augmented lognormal matrix** corresponding to Φ as follows.

$$\Gamma_\Phi(\mathbf{z}) = \begin{bmatrix} \nabla_{\log} L_{k_1}(\mathbf{z}) \\ \vdots \\ \nabla_{\log} L_{k_s}(\mathbf{z}) \\ z_{\pi_1} e_{\pi_1} \\ \vdots \\ z_{\pi_{d-s}} e_{\pi_{d-s}} \end{bmatrix} \tag{5.10}$$

where e_i is the i-th elementary basis in $\mathbb{R}^d$. In particular, when $s = d$, the **augmented lognormal matrix** is

$$\Gamma_\Phi(\mathbf{z}) = \begin{bmatrix} z_1 b_1^{(k_1)} & \cdots & z_d b_d^{(k_1)} \\ \vdots & \ddots & \vdots \\ z_1 b_1^{(k_d)} & \cdots & z_d b_d^{(k_d)} \end{bmatrix} \tag{5.11}$$

The big O-term on the RHS of equation (5.9) is a polynomial in $\dfrac{1}{\mathbf{r}}$ of degree $|\mathbf{p}| - d$. In the general case when we don't know how $\Phi^{-1}(T_\epsilon)$ and $\tau_{\boldsymbol{\sigma}}$ are oriented, we put absolute sign on $\det \Gamma_\Phi(\boldsymbol{\sigma})$ in the denominator of equation (5.9) because $\det \Gamma_\Phi(\boldsymbol{\sigma})$ will be negative when they are oriented oppositely. [PWM24, Proof of Theorem 10.25(i)].

Stratum of codimension $s < d$

When a contributing point $\boldsymbol{\sigma}$ is on a stratum $\mathcal{S}$ of codimension $< d$, the linking torus at $\boldsymbol{\sigma}$ is homologus to $\Phi^{-1}(T_\epsilon \times i\mathbb{R}^{d-s})$ in $\mathrm{H}_d(\mathcal{M}, -\infty)$ where $T_\epsilon = \{\mathbf{z} \in \mathbb{C}^s : |z_i| = \epsilon\}$ [Mel21, Chapter 8.2.4]. Then for any $a < 0$, we have

$$\left(\frac{1}{2\pi i}\right)^d \int_{\tau_{\boldsymbol{\sigma}}} \omega = \left(\frac{1}{2\pi i}\right)^d \int_{\Phi^{-1}(T_\epsilon \times i\mathbb{R}^{d-s})} \omega + O(e^{an}).$$

Again $O(e^{an})$ becomes $o(1)$ when $F(\mathbf{z})$ is a rational function. By [PWM24, Corollary C.18(iii)], the above integral is equal to

$$\int_{\Phi^{-1}(\mathbf{0}\times i\mathbb{R}^{d-s})} \mathrm{Res}(\omega; \mathcal{S}).$$

Therefore, after taking the iterated residue, we need to do a saddle point integral on $\mathrm{Res}(\omega; \mathcal{S})$. The details can be found in [PWM24, Chapter 5, Chapter C.2] or [Mel21, Chapter 8.2.5]. We briefly give the result as below and we do not need this result in this paper.

Proposition 5.49. *[BMP24b, Proposition 4.14] Let $S = \{k_1, \cdots, k_s\}$ and suppose $\boldsymbol{\sigma}$ lies on the stratum $\mathcal{S}_S$ Let Γ_Φ be the augmented lognormal matrix in equation (5.10) and M be the $d \times d$ matrix whose first s rows are the coefficients $\{\mathbf{b}^{(j)} : j \in S\}$ and whose last $d-s$ rows are any $d-s$ standard basis vectors that complete all rows to a positively oriented basis for $\mathbb{R}^d$. Define the $(d-s)\times(d-s)$ matrix $\mathcal{H}$ to be the Hessian of*

$$\phi(\mathbf{y}) = \mathbf{r} \cdot \log(\boldsymbol{\sigma} - iM^{-1}\begin{bmatrix}\mathbf{0}\\ \mathbf{y}\end{bmatrix})$$

evaluated at $\mathbf{y} = (y_1, \cdots, y_{d-s}) = \mathbf{0}$, where the logarithm is taken coordinate-wise. Then there is an explicitly computable asymptotics series in $|\mathbf{r}| = |r_1| + \cdots + |r_d|$ beginning

$$\left(\frac{1}{2\pi i}\right)^d \int_{\tau_{\boldsymbol{\sigma}}} \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, d\mathbf{z} = \left[\frac{\boldsymbol{\sigma}^{-\mathbf{r}} G(\boldsymbol{\sigma})}{\prod_{j\notin S} L_j(\boldsymbol{\sigma})^{p_j} \prod_{j\in S}(p_{k_j}-1)! \sqrt{\det \mathcal{H}} |\det \Gamma_\Phi|}\right] \times (2\pi|\mathbf{r}|)^{-(d-s)/2} (\mathbf{r}\Gamma_\Phi)^{\mathbf{p}-\mathbf{1}} \times \left(1 + O\left(\frac{1}{|\mathbf{r}|}\right)\right)$$

All asymptotics terms in this series are uniform as $\mathbf{r}$ *varies without corsssing nongeneric directions and* $\boldsymbol{\sigma}$ *varies over a compact subset of the stratum* $\mathcal{S}_S$.

Coefficient asymptotics

After replacing T with a sum of linking tori at contributing points and evaluate the integral of $F(\mathbf{z})\mathbf{z}^{-\mathbf{r}-\mathbf{1}}d\mathbf{z}$ over each linking torus, we can get the following result, applied to simple arrangement and generic direction.

Theorem 5.50. *[BMP24b, Theorem 4.16] Suppose that* $H(\mathbf{z})$ *in* $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z})$ *gives a simple arrangement and* $\mathbf{r}$ *is a generic direction. Then as* $\mathbf{r} \to \infty$ *with* $\hat{\mathbf{r}}$ *staying in a compact subsect of* $\mathbb{R}^d_{>0}$ *consisting of only generic directions, there exist asymptotic series* $I_{\boldsymbol{\sigma}}(\mathbf{r})$ *such that*

$$[\mathbf{z}^{\mathbf{r}}]F(\mathbf{z}) = \sum_{\boldsymbol{\sigma} \in \mathtt{contrib}(\hat{\mathbf{r}})} I_{\boldsymbol{\sigma}}(\mathbf{r}).$$

If $\boldsymbol{\sigma}$ *is on the stratum* $\mathcal{S}_S$ *with* $S = \{k_1, \cdots, k_s\}$*, then for any positive integer* K*, there exist effective constants* $C_j^{\boldsymbol{\sigma}}$ *such that*

$$I_{\boldsymbol{\sigma}}(\mathbf{r}) = \boldsymbol{\sigma}^{-\mathbf{r}}|\mathbf{r}|^{p_{k_1}+\cdots+p_{k_s}-(s+d)/2}\left(\sum_{j=0}^{K} C_j^{\boldsymbol{\sigma}}|\mathbf{r}|^{-j} + O(|\mathbf{r}|^{-K-1})\right).$$

When $G(\boldsymbol{\sigma}) \neq 0$*, the leading asymptotic term of* $I_{\boldsymbol{\sigma}}$ *is given by Proposition 5.49. The error term varies uniformly as* $\hat{\mathbf{r}}$ *varies without crossing nongeneric directions and* $\boldsymbol{\sigma}$ *varies over a compact subset of the stratum* $\mathcal{S}_S$. □

In particular, when $\boldsymbol{\sigma}$ is on a stratum of codimension d formed by hyperplanes with indices in $S = \{k_1, \cdots, k_d\}$ and G is a polynomial,

$$I_{\boldsymbol{\sigma}}(\mathbf{r}) = \boldsymbol{\sigma}^{-\mathbf{r}}|\mathbf{r}|^{p_{k_1}+\cdots+p_{k_s}-d}\left(\sum_{j=0}^{p_{k_1}+\cdots+p_{k_s}-d} C_j^{\boldsymbol{\sigma}}|\mathbf{r}|^{-j}\right)$$

In other words, there is no error further error terms for $K > p_{k_1}+\cdots+p_{k_s}-d$. All $C_j^{\boldsymbol{\sigma}}$ in this case can be computed explicitly by [PWM24, Corollary C.23]. For example, $C_0^{\boldsymbol{\sigma}} = \dfrac{G(\boldsymbol{\sigma})}{\prod_{j\notin S} L_j(\boldsymbol{\sigma})\det(\Gamma_\Phi(\boldsymbol{\sigma}))}$.

If G is an analytic function instead, then there is an error term $O(e^{a|\mathbf{r}|})$ for arbitrarily small $a < 0$.

CHAPTER 6

PSEUDO MULTIPLE POINTS

In our application to ACSV, we mainly focus on a rational generating function $F = P/Q$ whose singular variety $\mathcal{V}$ is defined globally by the polynomial Q in the denominator. Therefore, readers who do not need most generality can replace every $\mathcal{V}$ by $\mathcal{V}_Q$ in the following text and neglect the small neighborhood $\mathcal{D}$. In the most general setting, $\mathcal{V}$ is not globally, but only locally defined by Q near each point $\mathbf{p} \in \mathcal{V}$.

6.1. Introduction

Let $\mathcal{V}$ be an analytic hypersurface (see Definition 2.7), and $\mathbf{p}$ be a point on $\mathcal{V}$. Let Q be a function such that Q is analytic in a small neighborhood $\mathcal{D}$ of $\mathbf{p}$ in $\mathbb{C}^d$ and $\mathcal{V} \cap \mathcal{D} = \mathcal{V}_Q \cap \mathcal{D}$. To determine if a point $\mathbf{p} \in \mathcal{V}$ is a multiple point, one needs to factorize Q into irreducibles in the local ring $\mathcal{O}_{\mathbf{p}}$. Unfortunately, factorization of a polynomial in $\mathcal{O}_{\mathbf{p}}$ is much harder than factorization in $\mathbb{C}[\mathbf{z}]$. An irreducible polynomial in $\mathbb{C}[\mathbf{z}]$ with zero gradient at $\mathbf{p}$ can be either reducible or irreducible in $\mathcal{O}_{\mathbf{p}}$. For example, $Q(x, y, z) = x^2 + y^2 + z^2$ is irreducible in $\mathbb{C}[x, y, z]$ and also irreducible in ${}_3\mathcal{O}$. On the other hand, $Q(x, y) = y^2 - x^2 + x^3$ is irreducible in $\mathbb{C}[x, y]$ but indeed reducible in ${}_2\mathcal{O}$.

Definition 6.1 (leading homogeneous term)**.** *The leading homogeneous term of Q at $\mathbf{p}$ is the sum of the lowest degree terms of the power series expansion $Q(\mathbf{p}+\mathbf{z})$ at $\mathbf{z} = 0$. We denote it by $\mathsf{hom}(Q, \mathbf{p})$.*

Proposition 6.2. *If $\mathbf{p}$ is a multiple point on an analytic hypersurface $\mathcal{V}$, then there exists an analytic function Q in a small neighborhood $\mathcal{D}$ of $\mathbf{p}$ in $\mathbb{C}^d$ such that $\mathcal{V}_Q \cap \mathcal{D} = \mathcal{V} \cap \mathcal{D}$ and $\mathsf{hom}(Q, \mathbf{p})$ is a product of linear terms $\ell_1, \cdots, \ell_n$.*

PROOF: The small neighborhood $\mathcal{D}$ and the analytic function Q are given by the definition of analytic hypersurface. If $\mathbf{p}$ is a multiple point, then $Q(\mathbf{z}) = u(\mathbf{z})Q_1(\mathbf{z})^{m_1} \cdots Q_s(\mathbf{z})^{m_s}$ where $Q_i(\mathbf{p}) = 0$, $u(\mathbf{p}) \neq 0$, and u, Q_i are in $\mathcal{O}_{\mathbf{p}}$. Then $\mathsf{hom}(Q, \mathbf{p}) = u(\mathbf{p}) \prod_{i=1}^{s} (\nabla Q_i(\mathbf{p}) \cdot \mathbf{z})^{m_i}$, which is a product of linear factors. $\square$

Remark. It is possible that $\mathsf{hom}(Q_i, \mathbf{p}) = \mathsf{hom}(Q_j, \mathbf{p})$ for some i and j. For example, when $Q = (x+y-2)(xy-1)$, $\mathbf{p} = (1,1)$ is a multiple point with tangential intersection. $\mathsf{hom}(x+y-2, \mathbf{1}) = x+y = \mathsf{hom}(xy-1, \mathbf{1})$ and thus $\mathsf{hom}(Q, \mathbf{1}) = (x+y)^2$. Therefore, we may have less distinct linear factors in $\mathsf{hom}(Q, \mathbf{p})$ than distinct analytic germs in the factorization of Q in $\mathcal{O}_{\mathbf{p}}$.

In this section, we relax the definition of multiple points to the above necessary condition. We call these points pseudo multiple points.

Definition 6.3 (pseudo multiple points)**.** *Let $\mathbf{p} \in \mathcal{V}$ where $\mathcal{V}$ is an analytic hypersurface and locally defined by an analytic function Q in a small neighborhood of $\mathbf{p}$ in $\mathbb{C}^d$. If $\mathsf{hom}(Q, \mathbf{p})$ factors in $\mathbb{C}[\mathbf{z}]$ into a product of linear polynomials $\ell_1, \cdots, \ell_n$, then we call $\mathbf{p}$ a pseudo multiple point on $\mathcal{V}$. By Proposition 6.2, it is immediate that a multiple point is a pseudo multiple point.*

Remark. We use ℓ to denote a linear polynomial without the constant term and L to denote a linear polynomial (possibly) with the constant term. In other words, ℓ_i defines a hyperplane going through the origin and L_i defines an affine hyperplane.

In this section, we study pseudo multiple points. It is much easier to determine whether or not a point is pseudo multiple because the factorization happens in the polynomial ring $\mathbb{C}[\mathbf{z}]$, instead of the ring of analytic germs at $\mathbf{p}$. Indeed, if Q can be written as $u\hat{Q}$ where $u \in \mathbb{C}_*$ and $\hat{Q} \in \mathbb{A}[\mathbf{z}]$, the polynomial ring over algebraic numbers, then there are effective algorithms to factor $\hat{Q}$ over $\mathbb{A}$, whose factorization is the same as that over $\mathbb{C}$. Therefore, whether a point is pseudo multiple or not is checkable at least in some cases, and fortunately many rational generating functions arising from combinatorial examples will have their denominator Q in $\mathbb{A}[\mathbf{z}]$. The term 'pseudo' comes from the fact that this condition is a necessary condition for a point to be a multiple point. A pseudo multiple point is a point that may or may not be a multiple point but at least passes the test that the leading homogeneous part factors into linear factors.

Unique factorization of $\mathsf{hom}(Q, \mathbf{p})$

Since $\mathbb{C}[\mathbf{z}]$ is a UFD, we can obtain a unique (up to units) factorization of $\mathsf{hom}(Q, \mathbf{p})$ in $\mathbb{C}[\mathbf{z}]$. In particular, if $\mathbf{p}$ is a pseudo multiple point, then

$$\mathsf{hom}(Q, \mathbf{p}) = u\ell_1^{m_1} \cdots \ell_s^{m_s} \tag{6.1}$$

where ℓ_i are non-associated linear polynomials in $\mathbb{C}[\mathbf{z}]$ with $\ell_i(\mathbf{0}) = 0$ and u is a non-zero complex number. That is, $\mathcal{V}_{\ell_i}$ are distinct hyperplanes through the origin in $\mathbb{C}^d$ and u is a non-zero complex number. If all m_i are zero, we say that $\mathsf{hom}(Q, \mathbf{p})$ factors into **simple linear factors**. We always require that ℓ_i's are distinct linear polynomials and so any factorization of $\mathsf{hom}(Q, \mathbf{p})$ is unique up to some complex number.

Definition 6.4 (distinct linear polynomials). *For two linear polynomials ℓ_1 and ℓ_2 in $\mathbb{C}[\mathbf{z}]$, we say they are distinct if there is no constant $u \in \mathbb{C}$ such that $u\ell_1 = \ell_2$.*

Definition 6.5 (order of vanishing). *Let R be a d-variate holomorphic function defined on $\mathbb{C}^d$. The order (of vanishing) at $\mathbf{p}$ of the function R is the smallest possible n such that all partial derivatives with order less than n vanish at $\mathbf{p}$ and at least one partial derivative with order n is non-zero at $\mathbf{p}$.*

Remark. The order (at the origin) of a monomial $z_1^{s_1} \cdots z_d^{s_d}$ with $s_i \geq 0$ is then $s_1 + \cdots + s_d$. The order of a homogeneous polynomial f at the origin is then the order of any monomial in f.

Pseudo multiple points arise from higher-order perturbation on hyperplane arrangements. Let $\mathbf{p}$ be a pseudo multiple point on $\mathcal{V}$. Since $\mathcal{V}$ is an analytic hypersurface, $\mathcal{V}$ is locally defined by an analytic function Q in a neighborhood $\mathcal{D}$ of $\mathbf{p}$ in $\mathbb{C}^d$. Let $\mathsf{hom}(Q, \mathbf{p}) = \ell_1^{m_1} \cdots \ell_s^{m_s}$ and $H(\mathbf{z}) = \ell_1(\mathbf{z} - \mathbf{p})^{m_1} \cdots \ell_s(\mathbf{z} - \mathbf{p})^{m_s}$. The variety $\mathcal{V}_H$ is an hyperplane arrangement. The order of H at $\mathbf{p}$ is $m_1 + \cdots + m_s$. Let $R = Q - H$ and thus the order of R at $\mathbf{p}$ is greater than $m_1 + \cdots + m_s$. We call R the higher-order perturbation term. Inside $\mathcal{D}$, the variety $\mathcal{V}_{H+tR}$ deforms from $\mathcal{V}_H$ to $\mathcal{V}_Q$ as t goes from 0 to 1.

Let's start with a hyperplane arrangement and then perturb it with higher order terms. Let $H(\mathbf{z}) =$

$L_1(\mathbf{z})^{m_1} \cdots L_s(\mathbf{z})^{m_s}$ where L_i are distinct linear polynomials so that $\mathcal{V}_{L_i} \neq \mathcal{V}_{L_j}$. Suppose that $L_i(\mathbf{p}) = 0$ for all i. Then $\mathsf{hom}(H, \mathbf{p}) = \prod_{i=1}^{s} \mathsf{hom}(L_i, \mathbf{p})^{m_i}$ where $\mathsf{hom}(L_i, \mathbf{p}) := L_i(\mathbf{p} - \mathbf{z}) = \ell_i(\mathbf{z})$. We perturb H by adding $R(\mathbf{z})$ where $\mathsf{hom}(R, \mathbf{p})$ is of order $m_1 + \cdots + m_s + 1$. We let $Q(\mathbf{z}) = H(\mathbf{z}) + R(\mathbf{z})$ and by definition, $\mathbf{p}$ is a pseudo multiple point for $\mathcal{V}_Q$. The reason we want $R(\mathbf{z})$ to have order at $\mathbf{p}$ greater than $m_1 + \cdots + m_s$ is to ensure that $\mathsf{hom}(Q, \mathbf{p}) = \mathsf{hom}(H, \mathbf{p}) = \prod_i \ell_i(\mathbf{z})^{m_i}$. Alternatively, if we have a pseudo multiple point $\mathbf{p}$ on $\mathcal{V}$ and $\mathcal{V} \cap \mathcal{D} = \mathcal{V}_Q \cap \mathcal{D}$, then we have a factorization of $\mathsf{hom}(Q, \mathbf{p})$ into $\ell_1(\mathbf{z})^{m_1} \cdots \ell_s(\mathbf{z})^{m_s}$ where ℓ_i are distinct. Then define $L_i(\mathbf{z}) = \ell_i(\mathbf{z} - \mathbf{p})$, $H(\mathbf{z}) = \prod_i L_i(\mathbf{z})$ and $R(\mathbf{z}) = Q(\mathbf{z}) - H(\mathbf{z})$. We see that $\mathsf{hom}(R, \mathbf{p})$ is order $m_1 + \cdots + m_n + 1$. We switch back and forth between these two viewpoints.

A pseudo multiple point $\mathbf{p}$ is nothing but a point on a hyperplane arrangement $\mathcal{V}_H$ with a higher-order perturbation term R added so that $\mathcal{V}_H$ becomes $\mathcal{V}$ in a small neighborhood of $\mathbf{p}$. The following two examples construct pseudo multiple points from this observation and they also motivate the study in Chapter 6.2 and Chapter 6.3.

Example 6.6 (perturbation of two lines in $\mathbb{C}^2$)**.** *Let* $H(x, y) = (y - x)(x + y - 2)$ *and let* $R(x, y) = (x - 1)^3$. *The point* $\mathbf{p} = \mathbf{1} = (1, 1)$ *is a multiple point on* $\mathcal{V}_H$. *It is easy to see that* $\mathsf{hom}(H, \mathbf{1}) = (y + x)(y - x)$ *and* $\mathsf{hom}(R, \mathbf{1}) = x^3$. *Therefore,* $\mathsf{hom}(Q, \mathbf{1}) = \mathsf{hom}(H + R, \mathbf{1}) = (y + x)(y - x)$. *By Definition 6.3,* $\mathbf{p} = \mathbf{1}$ *is a pseudo multiple point on* $\mathcal{V}_Q$.

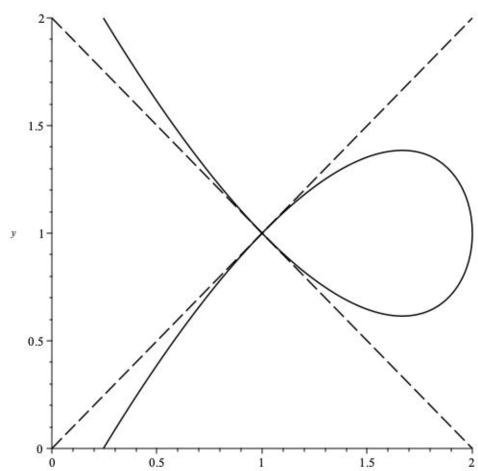


Figure 6.1: The real varieties defined by $H(x, y)$ and $H(x, y) + R(x, y)$.

It is more interesting to see that Q actually factors in the ring $\mathcal{O}_{(1,1)}$. Indeed,

$$Q(x,y) = \left[(y-1) + (x-1)\sqrt{2-x}\right]\left[(y-1) - (x-1)\sqrt{2-x}\right]$$

and thus $\mathbf{p} = (1,1)$ is even a multiple point.

Main Result in Chapter 6.2

One may wonder if this is an coincidence. In Theorem 6.13, we show that for $\mathbf{p}$ in an analytic hypersurface in $\mathbb{C}^2$ where $\mathcal{V}$ is locally defined by an analytic function Q at $\mathbf{p}$, as long as $\mathsf{hom}(Q,\mathbf{p})$ factors into simple linear factors (i.e. $m_i = 1$ for all i in equation (6.1)), $\mathbf{p}$ is a multiple point for $\mathcal{V}$. In other words, no matter what higher order perturbation term R we add, the multiple point $\mathbf{p}$ on $\mathcal{V}_H$ never ceases to be a multiple point on $\mathcal{V}_{H+R}$.

Example 6.7 (pseudo multiple point that is not a multiple point)**.** *Let $H(x,y,z) = (x-1)(y-1)(z-1)$ and $R(x,y,z) = (x-1)^4 + (y-1)^4 + (z-1)^4$. The point $\mathbf{p} = \mathbf{1} = (1,1,1)$ is a multiple point on $\mathcal{V}_H$. We observe that $\mathsf{hom}(H,\mathbf{1}) = \mathsf{hom}(H+R,\mathbf{1}) = xyz$. Therefore, $\mathbf{1}$ is a pseudo multiple point on $\mathcal{V}_Q$ where $Q = H + R$.*

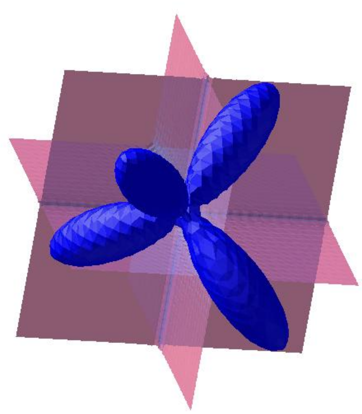


Figure 6.2: The real varieties defined by $H(x,y,z)$ and $H(x,y,z) + R(x,y,z)$.

Centering at the point $\mathbf{p} = \mathbf{1}$, we look at $q(\mathbf{z}) = Q(\mathbf{z}+\mathbf{1}) = xyz + x^4 + y^4 + z^4$. The variety

$\mathcal{V}_q$ is just a shift of $\mathcal{V}_Q$ by $\mathbf{1}$. Therefore, these two varieties should have the same geometry. In particular, what happens on $\mathcal{V}_Q$ around $\mathbf{1}$ is exactly what happens on $\mathcal{V}_q$ around $\mathbf{0}$. After computing the Groebner basis of the ideal generated by q, q_x, q_y, and q_z, there is only one singularity on $\mathcal{V}_q$, which is the origin. By Definition 5.2 of multiple points, a multiple point cannot be isolated once $d > 2$. Therefore, $\mathbf{0}$ cannot be a multiple point for $\mathcal{V}_q$ and thus $\mathbf{p} = \mathbf{1}$ is not a multiple point for $\mathcal{V}_Q$.

Main Result in Chapter 6.3

In Chapter 6.3, we study the coefficient asymptotics of a power series converging to a d-variate rational generating function $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$ when we have a minimal pseudo multiple point $\mathbf{p}$ such that $\mathsf{hom}(Q, \mathbf{p})$ factors into $\ell_1^{m_1} \cdots \ell_d^{m_d}$ where these d linear factors $\ell_1, \cdots, \ell_d$ have linearly independent gradients. Under suitable assumptions, we showed that the coefficient asymptotics for P/Q will be close to the coefficient asymptotics for P/H where H is equal to $H(\mathbf{z}) = \mathsf{hom}(Q, \mathbf{p})(\mathbf{z} - \mathbf{p})$. The difference between coefficient asymptotics of P/Q and P/H depends on the multiplicities $m_1, \cdots, m_d$.

6.2. Pseudo Multiple Points in $\mathbb{C}^2$

The main result of this section, Theorem 6.13, is that for an analytic hypersurface $\mathcal{V} \subset \mathbb{C}^2$ and a pseudo multiple point $\mathbf{p} \in \mathcal{V}$, when $\mathsf{hom}(Q, \mathbf{p})$ factors into simple linear factors, then this pseudo multiple point is an actual multiple point. Here, Q is an analytic function on a small neighborhood $\mathcal{D}$ of $\mathbf{p}$ in $\mathbb{C}^2$ such that $\mathcal{V} \cap \mathcal{D} = \mathcal{V}_Q \cap \mathcal{D}$. The proof is via an improved Weierstrass Preparation Theorem, Lemma 6.11.

Let $\mathsf{hom}(Q, \mathbf{p}) = \ell_1^{m_1} \cdots \ell_s^{m_s}$ with distinct ℓ_i's. We first see why we require simple factors (i.e. $m_i = 1$ for all i).

Example 6.8.

1. *(when some $m_i > 1$, a pseudo multiple point may not be a multiple point)*
   Let $Q(x, y) := x^2 + y^3$ and $\mathbf{p} = \mathbf{0}$. The leading homogeneous term of Q at $\mathbf{p}$ is x^2, a linear factor with multiplicity 2. The point $\mathbf{p} = \mathbf{0}$ is not a multiple point on $\mathcal{V}_Q$.

2. *(a multiple point may have some $m_i > 1$)*

   The point $\mathbf{1} = (1,1)$ on the variety defined by $Q = (x+y-2)(xy-1)$ is a multiple point but $\mathsf{hom}(Q, \mathbf{1}) = (x+y)^2$.

When we talk about multiple points, what we care is the local geometry of the variety $\mathcal{V}$ at $\mathbf{p}$. In other words, only the zero-locus of Q near $\mathbf{p}$ matters. Weierstrass Preparation Theorem is a tool to decompose Q into two parts in $\mathcal{O}_{\mathbf{p}}$, one part having essentially the same zero-locus of Q locally at $\mathbf{p}$ and another part having no effect on the local geometry. In one dimension, a locally analytic function $Q(z)$ at p can be written as $Q(z) = (z-p)^n u(z)$ where n is the order of vanishing of Q at p and $u(z)$ is locally analytic at p with $u(p) \neq 0$. Therefore, the local geometry of $\mathcal{V}$ at p is the same as that of the variety defined by $(z-p)^n$, essentially capturing the order n of the zero at p.

Definition 6.9 (Weierstrass polynomial)**.** *Let $U \subset \mathbb{C}^{d-1}$ be open, and let $\hat{\mathbf{z}} \in \mathbb{C}^{d-1}$ be the coordinates in $\mathbb{C}^{d-1}$. Let P be a monic polynomial of degree $m \geq 0$ with coefficients of holomorphic functions on U, that is,*

$$P(\hat{\mathbf{z}}, z_d) = z_d^m + \sum_{j=0}^{m-1} c_j(\hat{\mathbf{z}}) z_d^j$$

where c_j are holomorphic functions defined on U such that $c_j(0) = 0$ for all j. Such a polynomial P is called a Weierstrass polynomial of degree m.

It is fairly easy to do it in one dimension because ${}_1\mathcal{O}_p$ is a PID as we have shown in Lemma 5.6. Any ideal in ${}_1\mathcal{O}_p$ is generated by some monimials x^n. However, for $d \geq 2$, ${}_d\mathcal{O}_p$ is not a PID anymore. Weierstrass preparation theorem gives a more involved result: $u(z)$ in the one-dimensional case is replaced by a unit $u(\mathbf{z})$ in ${}_d\mathcal{O}_p$ and $(z-p)^n$ in the one-dimensional case is replaced by a polynomial in ${}_{d-1}\mathcal{O}_p[z_d]$. This polynomial is called a **Weierstrass polynomial**. It is the part that captures the local geometry of $\mathcal{V}_Q$ at $\mathbf{p}$. From now on, we assume without loss of generality that $\mathbf{p} = \mathbf{0}$ and thus we consider the ring ${}_d\mathcal{O}$, the ring of analytic germs at the origin.

Theorem 6.10 (Weierstrass Preparation Theorem)**.** *Suppose Q is a holomorphic function on U where $U \subset \mathbb{C}^{d-1} \times \mathbb{C}$, $\mathbf{0} \in U$, and $Q(\mathbf{0}) = 0$. Suppose $z_d \mapsto Q(\mathbf{0}, z_d)$ is not identically zero near the origin and its order of vanishing at the origin is $m \geq 1$.*

Then there exists an open polydisk $V = V' \times D \subset \mathbb{C}^{d-1} \times \mathbb{C}$ with $0 \in V \subset U$, a unique $u(\mathbf{z})$ holomorphic on V, $u(\mathbf{z}) \neq 0$ for all $\mathbf{z} \in V$, and a unique Weierstrass polynomial P of degree m with coefficients holomorphic in V' such that

$$Q(\hat{\mathbf{z}}, z_n) = u(\hat{\mathbf{z}}, z_n)P(\hat{\mathbf{z}}, z_n)$$

and such that all m zeros (counting multiplicities) of $z_n \mapsto P(\hat{\mathbf{z}}, z_n)$ lie in D for all $\hat{\mathbf{z}} \in V'$.

PROOF: Proof can be found in [Leb23, Chapter 6.2], [KK83, Chapter 2.2] or [BK86, Chapter 8.2]. Our statements of this theorem and Lemma 6.11 adopt the format in [Leb23, Theorem 6.2.3]. □

The **Weierstrass Preparation Theorem** tells us that for $Q \in \mathcal{O}$, (after a linear change of coordinates if needed) we can uniquely decompose Q into a product of a unit u and a Weierstrass polynomial P. A unit u in $\mathcal{O}$ has the property that $u(\mathbf{0}) \neq 0$. By the local analyticity of u, it is non-zero near $\mathbf{0}$. Therefore, the zero-locus near $\mathbf{0}$ of Q is the same as that of P.

The next result is a stronger special case of Weierstrass Preparation Theorem when $d = 2$. If $\mathsf{hom}(Q, \mathbf{0})$ factorizes into m distinct linear factors $\ell_1, \cdots, \ell_m$, Theorem 6.10 only says that $Q(x, y) = u(x, y)P(x, y)$ where $P(x, y)$ is a Weierstrass polynomial in y of degree m. Fix $\hat{\mathbf{z}}$ in V', there are exactly m roots (counting multiplicities) of $P(\hat{\mathbf{z}}, \cdot)$ since the coefficient of z_d^m is always one. Therefore, the Weierstrass polynomial encodes $\mathcal{V}_Q$ near $\mathbf{0}$ if we vary $\hat{\mathbf{z}}$ in V'. However, we cannot distinguish these m roots. We don't know when these m roots will coalesce when we vary $\hat{\mathbf{z}}$.

There is no guarantee that one can further factor P into a product of m analytic factors. Indeed, a Weierstrass polynomial is not necessarily reducible in $\mathcal{O}$. For example, $Q(x, y) = y^2 - x$ is itself a Weierstrass polynomial in y but the two roots of $Q(x, \cdot)$, namely $\pm\sqrt{x}$, are not analytic at $x = 0$. Fortunately, under the following assumption, we can factor P as a product of m locally analytic functions, each of which has the same algebraic tangent cone as the linear factor $\ell_j = y - \alpha_j x$.

We do not find any previously existing proof on this stronger version of Weierstrass Preparation Theorem. Therefore, we write the proof in detail. The very beginning and the very end of the proof are the same as in [Leb23, Theorem 6.2.3]. The originiality of the proof is on the decomposition of the Weierstrass polynomial P in Theorem 6.10 into m analytic factors.

Lemma 6.11 (stronger result in Weierstrass Preparation Theorem when $d = 2$)**.** *Let $Q(x, y)$ be a bivariate function, analytic at the origin $(0, 0)$, i.e. $Q(x, y) \in C\{x, y\}$. Suppose that the leading homogeneous term in the power series expansion of $Q(x, y)$ at $(0, 0)$ is a product of m distinct linear factors $y - \alpha_j x$. That is, $H(x, y) = \mathsf{hom}(Q, (0, 0)) = \prod_{j=1}^{m}(y - \alpha_j x)$ where $\alpha_i \neq \alpha_j$ for any $1 \leq i < j \leq m$.*

Then there exists a bi-disk $D = D_x \times D_y \in \mathbb{C} \times \mathbb{C}$ with $(0, 0) \in D$, a unique $u(x, y)$ holomorphic on D, and m unique (up to permutations) $\phi_j(x)$ holomorphic on D_x such that

$$Q(x, y) = u(x, y) \prod_{j=1}^{m} Q_j(x, y) \tag{6.2}$$

where

1. *$u(x, y) \neq 0$ for $(x, y) \in D$*

2. *$Q_j(x, y) = y - \alpha_j x - x^2 \phi_j(x)$*

In particular, for each non-zero $x_0 \in D_x$, there are exactly m distinct roots of $y \mapsto \prod_{j=1}^{m} Q_j(x_0, y)$, each of which corresponds to a root of $y \mapsto Q_j(x_0, y)$.

Remark. Since we assume that the leading homogeneous term of the power series expansion of Q at $(0, 0)$ is exactly $\prod_{j=1}^{m}(y - \alpha_j x)$, the constant term in $u(x, y) \in \mathbb{C}\{x, y\}$ must be 1. That is, $u(0, 0) = 1$.

PROOF: Since $Q(x, y)$ is analytic at the origin, there exists a bi-disk $U = U_x \times U_y \in \mathbb{C}^2$ such that $(0, 0) \in U$ and $Q(x, y)$ can be represented by a convergent power series on U. That is, there exists another analytic functions $g(x, y)$ with at least $(m+1)$-th order of vanishing at the origin such that $Q(x, y) = g(x, y) + \prod_{j=1}^{m}(y - \alpha_j x)$ for $(x, y) \in U$. One can think of $g(x, y)$ as the remaining terms in the convergent power series except the leading homogeneous term $\prod_{j=1}^{m}(y - \alpha_j x)$.

We will then shrink the bi-disk U such that for each x, there are exactly m zeros (counting multiplicity) of $y \mapsto Q(x, y)$. Let $x = 0$, then $Q(0, y) = g(0, y) + y^m$. Therefore, the univariate holomorphic function $y \mapsto Q(0, y)$ is not identically zero near $y = 0$. There then exists a small disk $D_y \subset U_y \Subset \mathbb{C}$ such that $\{0\} \times D_y \Subset U$ and $Q(0, y) \neq 0$ for $y \in \overline{D_y} \backslash \{0\}$. By continuity of $Q(x, y)$, there exists a small disk $D_x \subset U_x$ and thus a small bi-disk $D = D_x \times D_y$ such that $\overline{D} \Subset U$, and $Q(x, y)$ does not vanish on $D_x \times \partial D_y$. The new bi-disk D is smaller than U and we claim that this is the shrunk bi-disk we want.

For each $x \in D_x$, let's look at the number of zeros of $y \mapsto Q(x, y)$ in D_y. In particular, when $x = 0$, we have exactly m zeros at $y = 0$ because $Q(0, y) = g(0, y) + y^m = y^m(1 + d_1 y + d_2 y^2 + \cdots)$. For general $x \in D_x$, let's use the argument principle on the univariate holomorphic function $y \mapsto Q(x, y)$ to see that the number of zeros insides D_y is

$$\frac{1}{2\pi i} \int_{\partial D_y} \frac{\frac{\partial}{\partial y} Q(x, \xi)}{Q(x, \xi)} \, d\xi.$$

Since $Q(x, \xi) \neq 0$ when $(x, \xi) \in D_x \times \partial D_y$, the integral above is a continuous integer-valued function for $x \in D_x$. Since D_x is connected, the above integral is equal to the number of zeros for $y \mapsto Q(0, y)$, which is m.

We have now shown that for each $x \in D_x$, there are exactly m zeros (counting multiplicity) in D_y for $y \mapsto Q(x, y)$. We then argue that these m zeros can be made distinct (i.e. simple zeros) except when $x = 0$. That is, if we shrink D_x and D_y even more, these m zeros will finally split into distinct simple zeros and only coalesce when $x = 0$. The one-line reason is that the algebraic cone is equal to the limiting secant cone. The detailed proof is as follows.

Let $Q_\epsilon(x, y) = \epsilon^{-m} Q(\epsilon x, \epsilon y)$. The zero set of Q_ϵ is a ϵ-zoom-in of the zero set of Q at the origin $(0, 0)$. Let S_1 be the unit complex sphere $\{(x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 = 1\}$. Let V_ϵ be the intersection of the zero set of Q_ϵ with S_1. Let V_0 be the intersection of the zero set of $H(x, y) := \mathsf{hom}(Q, (0, 0)) = \prod_{j=1}^m (y - \alpha_j x)$ with S_1.

Let's first see what V_0 looks like. Since H is a product of linear terms $y - \alpha_j x$, then $(x, y) \in V_0$ must satisfy $y = \alpha_j x$. Moreover, since $V_0 \subset S_1$, we need $|x|^2 + |y|^2 = 1$. Therefore, $|x| = \sqrt{\dfrac{1}{1+|\alpha_j|^2}}$ and $y = \alpha_j x$. Let $M_j = \left\{(x,y) \in \mathbb{C}^2 : |x| = \sqrt{\dfrac{1}{1+|\alpha_j|^2}}, y = \alpha_j x\right\}$. Each M_j is a 1-dimensional compact manifold on S_1 and V_0 is a disjoint union of all M_j's since α_j are distinct.

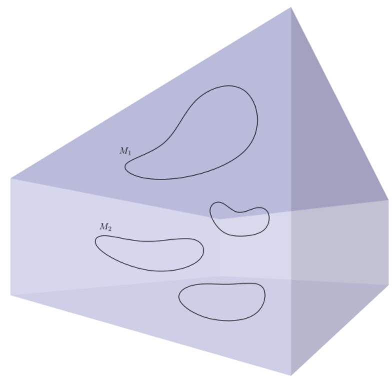


Figure 6.3: V_0 is a disjoint union of M_j's. The big box is the 3-sphere S_1 and each closed curve is M_j.

Then let's see what V_ϵ looks like. Theorem 6.48 on [PWM24] says that $V_\epsilon \to V_0$ in Hausdorff metric as $\epsilon \to 0$. Let δ be less than half of the minimum Hausdorff distance of any two M_i and M_j. There exists some ϵ_0 such that when $\epsilon < \epsilon_0$, the Hausdorff metric between V_0 and V_ϵ is less than δ.

Shrinking D_x and D_y so that for each $x \in D_x$ and $y \in D_y$, we always have $\left|\dfrac{x}{\epsilon_0}\right|^2 + \left|\dfrac{y}{\epsilon_0}\right|^2 \le 1$. This indeed can be done if we first choose a small enough D_y with radius less than half of ϵ_0, and then choose a small D_x accordingly. For any non-zero $x_0 \in D_x$ and any root y_0 in D_y of $y \mapsto Q(x_0, y)$, there is always some $\epsilon(x_0, y_0) < \epsilon_0$ depending on x_0 and y_0 such that $\left|\dfrac{x_0}{\epsilon}\right|^2 + \left|\dfrac{y_0}{\epsilon}\right|^2 = 1$. Since $Q(x_0, y_0) = 0$, then $Q_\epsilon\left(\dfrac{x_0}{\epsilon}, \dfrac{y_0}{\epsilon}\right) = 0$. That is, $\left(\dfrac{x_0}{\epsilon}, \dfrac{y_0}{\epsilon}\right) \in V_\epsilon$ and thus $\left(\dfrac{x_0}{\epsilon}, \dfrac{y_0}{\epsilon}\right)$ is in a δ-neighborhood of some M_i. Notice that when x varies continuously on D_x, some roots in D_y of $y \mapsto Q(x, y)$ may not vary continuously. However, the set $\left\{\left(\dfrac{x}{\epsilon_i}, \dfrac{y_i(x)}{\epsilon_i}\right) : y_i(x) \text{ is a root in } D_y \text{ of } y \mapsto f(x,y)\right\}_{i=1}^m$ is continuous in terms of Hausdorff metric, as x varies continuously in $D_x \backslash \{0\}$. [7] Suppose on the contrary that for some non-zero $x \in D_x$, none of elements in $\left\{\left(\dfrac{x}{\epsilon_i}, \dfrac{y_i(x)}{\epsilon_i}\right)\right\}_{i=1}^m$ is in a δ-neighborhood of

[7] Here we use ϵ_i because the scale depends on both x and the specific root in D_y of $y \mapsto Q(x, y)$. The reason for continuity is embedded in the proof of Weierstrass Preparation Theorem [Leb23, Theorem 6.2.3]: the expression $\prod_{i=1}^m (y - y_i(x))$ is indeed a Weierstrass polynomial $y^m + c_{m-1}(x)y^{m-1} + \cdots + c_0(x)$ where each $c_i(x)$ is holomorphic in D_x.

some M_j. By the following three facts that (1) the above set is continuous, (2) each element in the above set is in a δ-neighborhood of some M_i, and (3) the distance between any two M_i and M_j are larger than 2δ, we have that for any non-zero $x \in D_x$, none of elements in $\left\{\left(\frac{x}{\epsilon_i}, \frac{y_i(x)}{\epsilon_i}\right)\right\}_{i=1}^{m}$ is in a δ-neighborhood of some M_j. Now let ϵ be less than the radius of D_x and D_y so that the ϵ-ball in $\mathbb{C}^2$ is contained in $D_x \times D_y$. The Hausdorff metric between V_ϵ and V_0 is greater than δ because there is some $M_j \subset V_0$ whose δ-neighborhood has no elements from V_ϵ. We arrive at a contradiction. Therefore, for any non-zero $x \in D_x$, the m elements of $\left\{\left(\frac{x}{\epsilon_i}, \frac{y_i(x)}{\epsilon_i}\right)\right\}_{i=1}^{m}$ are respectively in the δ-neighborhood of each M_i. Since each M_i are separated by a distance of greater than 2δ, then $y_1(x), \cdots, y_m(x)$ are distinct.

Now let's show that these m roots of $y \mapsto Q(x, y)$, namely $y_1(x), \cdots, y_m(x)$, are indeed analytic on D_x. By implicit function theorem, since all of them are distinct, they are locally analytic on the punctured disk $D_x \backslash \{0\}$. They may fail to be analytic on the punctured disk because the punctured disk is not simply connected; when we do an analytic continuation along a loop $\gamma : [0, 1] \to \mathbb{C}$ around the origin, $y_i \circ \gamma(0)$ may not be the same as $y_i \circ \gamma(1)$ [8]. Let x_0 be the starting point of γ. Let $y_1(x_0), \cdots, y_m(x_0)$ be the m distinct roots of $y \mapsto Q(x_0, y)$. By previous arguments, the m elements of $\left\{\left(\frac{x_0}{\epsilon_i}, \frac{y_i(x_0)}{\epsilon_i}\right)\right\}_{i=1}^{m}$ are respectively in the δ-neighborhood of each M_i. By the local analyticity of $y_i(x)$, it is continuous until possibly at the endpoint of γ. That is, $y_i \circ \gamma$ is continuous on $[0, 1)$. Therefore, the point $\left\{\frac{\gamma(t)}{\epsilon_i}, \frac{y_i \circ \gamma(t)}{\epsilon_i}\right\}$ is continuously moving on S_1 as $t \in [0, 1)$. Suppose that when we do analytic continuation of y_i at the endpoint, $y_i \circ \gamma(1)$ becomes $y_j \circ \gamma(0)$. Since the ending position $y_i \circ \gamma(1) = y_j(x_0)$ and the starting position $y_i \circ \gamma(0) = y_i(x_0)$ belongs to two disjoint δ-neighborhoods of M_i and M_j respectively, at some moment $t \in (0, 1)$, the point $\left(\frac{\gamma(t)}{\epsilon_i}, \frac{y_i \circ \gamma(t)}{\epsilon_i}\right)$ will fall in the gap between the two δ-neighborhoods of M_i and M_j on S_1. This point on V_{ϵ_i} then has a distance larger than δ with V_0. This is a contradiction. Therefore, only trivial monodromy exists and thus y_i are analytic on the punctured disk $D_x \backslash \{0\}$.

If $y_i(x)$ has poles at $x = 0$, then $y_i(x)$ goes out of D_y, and in particular to infinity as $x \to 0$. This violates the fact that all $y_i(x)$ stays inside D_y. If $y_i(x)$ has an essential singularity at $x = 0$,

[8] Here by $y_i \circ \gamma$, we mean the analytic continuation of y_i along the path γ.

then $y_i(x)$ will take every values but one in $\mathbb{C}$ by Picard theorem. This is again a contradiction. Therefore, $x = 0$ is a removable singularity for $y_1, \dots, y_m$. Therefore, $y_1(x), \dots, y_m(x)$ are analytic on the disk D_x. In other words, $y - y_1(x), \dots, y - y_m(x)$ are analytic on the bi-disk $D = D_x \times D_y$. Therefore $\mathsf{hom}(Q, \mathbf{0}) = (y - y_1'(0)x) \cdots (y - y_m'(0)x)$ and thus we can choose $y_j(x) = \alpha_j x + \phi_j(x)x^2$ for some analytic function $\phi_j(x)$ at $x = 0$. That is, $Q_j(x, y) = y - y_j(x) = y - \alpha_j x - x^2\phi_j(x)$ in equation (6.2).

Finally, we show the existence and uniqueness of $u(x, y)$. Fix $x_0 \in D_x$, then the function $y \mapsto \dfrac{Q(x_0, y)}{\prod_j Q_j(x_0, y)}$ has removable singularities and it has no zeros inside $\overline{D_y}$ because these Q_j in the denominator eliminate all zeros of $y \mapsto Q(x_0, y)$ inside $\overline{D_y}$. We define $u(x, y)$ to be

$$u(x, y) = \frac{1}{2\pi i} \int_{\partial D_y} \frac{Q(x, \xi)}{\prod_j Q_j(x, \xi)(\xi - y)}\, d\xi$$

and by Cauchy's integral formula, $u(x, y) = \dfrac{Q(x, y)}{\prod_j Q_j(x, y)}$. For each fixed x, $u(x, y)$ is continuous and holomorphic in y. Differentiating with respect to x inside the integral, $u(x, y)$ is also holomorphic in x. Therefore, $u(x, y)$ is holomorphic on the bidisk $D_x \times D_y$ and never vanishes. $\square$

Remark 6.12. We can see that this proof does not work when $d \geq 3$. The main reason is that $S_1 = \{|\mathbf{z}_1| + \cdots + |\mathbf{z}_d| = 1\}$ is a $(2d-1)$-sphere. The manifold $M_j = \ell_i \cap S_1$ is of dimension $(2d-3)$. We need M_j to be disjoint from each other to argue no coalescing of analytic factors. Therefore, we need $2(2d - 3) < (2d - 1)$, which requires $d < 2.5$.

It is not far from Lemma 6.11 to the following result.

Theorem 6.13. *Supose that $\mathbf{p} \in \mathcal{V}$ where $\mathcal{V}$ is an analytic hypersurface in $\mathbb{C}^2$ and there is an analytic function Q in a small neighborhood $\mathcal{D}$ of $\mathbf{p}$ in $\mathbb{C}^d$ such that $\mathcal{V} \cap \mathcal{D} = \mathcal{V}_Q \cap \mathcal{D}$. If $\mathsf{hom}(Q, \mathbf{p}) = \ell_1 \cdots \ell_m$ for distinct linear factors ℓ_i, then $\mathbf{p}$ is a multiple point on $\mathcal{V}$. Moreover, $\mathcal{V} \cap U = (\mathcal{V}_1 \cap U) \cup \cdots \cup (\mathcal{V}_m \cap U)$ for every sufficiently small neighborhood U of $\mathbf{p} \in \mathbb{C}^2$ and $T_{\mathbf{p}}(\mathcal{V}_i) = \mathcal{V}_{\ell_i}$.*

PROOF: Let $\mathsf{hom}(Q, \mathbf{p}) = \ell_1 \cdots \ell_m$ where $\ell_i(x, y) = a_i x + b_i y$. If some $b_i = 0$, we can rotate coordinate axes so that ℓ_i is no longer perpendicular to x-axis. We can also scale the coordinate axes so that $\ell_i(x,y) = y - \alpha_i x$. Since rotation and scaling are both linear and invertible, there is an invertible linear transformation T_1 such that $\ell_i \circ T_1(x,y) = y - \alpha_j x$. We can also apply another invertible linear transformation T_2 on the coordinates to move the point $\mathbf{p} = (p_1, p_2)$ to the origin. Explicitly, $T_2(x,y) = (x - p_1, y - p_2)$. Let $T = T_1 \circ T_2$. Then $Q \circ T \in {}_2\mathcal{O}$ and $\mathsf{hom}(Q \circ T, \mathbf{0}) = \mathsf{hom}(Q \circ T_1, \mathbf{p}) = \mathsf{hom}(Q, \mathbf{p}) \circ T_1 = (\ell_1 \cdots \ell_m) \circ T_1 = (y - \alpha_1 x) \cdots (y - \alpha_m x)$.

Now apply Lemma 6.11 on $Q \circ T$. We can factorize $Q \circ T(x,y) = u(x,y) \prod_{i=1}^m Q_i(x,y)$ in the ring $\mathcal{O}$, where $u(0,0) = 1$ and $\mathsf{hom}(Q_i, \mathbf{0}) = y - \alpha_i x$. Then $Q = Q \circ T \circ T^{-1} = (u \circ T^{-1}) \prod_{i=1}^m (Q_i \circ T^{-1})$, where $u \circ T^{-1}(\mathbf{p}) \neq 0$ and $Q_i \circ T^{-1}$ is locally analytic at $\mathbf{p}$. The above factorization is in the ring $\mathcal{O}_{\mathbf{p}}$ and thus is good in every sufficiently small neighborhood U of $\mathbf{p}$ in $\mathbb{C}^2$. In particular, choose U to be smaller than $\mathcal{D}$, then $\mathcal{V} \cap U = \mathcal{V}_Q \cap U = (\mathcal{V}_1 \cap U) \cup \cdots \cup (\mathcal{V}_m \cap U)$. Moreover, $\mathsf{hom}(Q_i \circ T^{-1}, \mathbf{p}) = \mathsf{hom}(Q_i \circ T_1^{-1}, \mathbf{0}) = (y - \alpha_i x) \circ T_1^{-1} = \ell_i$. Therefore, the tangent plane of the variety $\mathcal{V}_i$ defined by $Q_i \circ T^{-1}$ at $\mathbf{p}$ is the hyperplane defined by ℓ_i. □

6.3. Pseudo Multiple Point With Exactly d Transverse Factors

When dimension $d > 2$, there are no sufficient conditions on the factorization of $\mathsf{hom}(Q, \mathbf{p})$ that imply that $\mathbf{p}$ is a multiple point, and in fact this can be very hard to tell. In this section, we prove a result that allows us to compute coefficient asymptotics in the case that $\mathsf{hom}(Q, \mathbf{p})$ factors into exactly d distinct linear factors, in the polynomial ring; factorization of Q in the ring of analytic germs is not needed. In this section, multiplicities of these distinct linear factors are allowed to be more than one.

6.3.1. Assumptions and results

The assumptions of this section are:

Assumption 6.1 (power series expansion)**.** *We consider a d-variate rational generating function $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$ with coprime polynomials $P, Q \in \mathbb{C}[\mathbf{z}]$ and $F(\mathbf{z})$ has a convergent* **power series** *$F(\mathbf{z}) = \sum_{\mathbf{r} \in \mathbb{N}^d} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ with domain of convergence $\mathcal{D}$ containing the origin.*

Assumption 6.2 (pseudo multiple point)**.** *Let* $\mathbf{p} \in \mathcal{V}_Q$ *such that*

$$\mathsf{hom}(Q, \mathbf{p}) = \prod_{i=1}^{d} \ell_i^{m_i}(\mathbf{z}),$$

as a product of d *distinct linear polynomials* $\ell_1, \cdots \ell_d \in \mathbb{C}[\mathbf{z}]$ *with multiplicities* $m_1, \cdots, m_d$*. Assume that* $\ell_i(\mathbf{p}) \neq 0$*,* $\ell_i(\mathbf{z})/\ell_i(\mathbf{p})$ *is a real polynomial and the gradients of* $\ell_i(\mathbf{z})/\ell_i(\mathbf{p})$ *are linearly independent in* $\mathbb{R}^d$*.*

Assumption 6.3 (minimality)**.** *Let* $L_i(\mathbf{z}) = \ell_i(\mathbf{z} - \mathbf{p})$*,* $H(\mathbf{z}) = L_1(\mathbf{z})^{m_1} \cdots L_d(\mathbf{z})^{m_d}$*, and* $R(\mathbf{z}) := Q(\mathbf{z}) - H(\mathbf{z})$*. The point* $\mathbf{p}$ *is minimal for* $H + tR$ *as* $t \in [0, 1]$*. In other words,* $H + tR$ *is nonvanishing on any torus* $cT(\mathbf{p})$ *when* $0 \leq t \leq 1$ *and* $0 \leq c < 1$*.*

Remark 6.14. Assumption 6.2 is equivalent to saying that $Q(\mathbf{z})$ has the form $u\hat{Q}(\mathbf{z})$ where u is a non-zero complex number and $\hat{Q}(\mathbf{z}) \in \mathbb{R}[\mathbf{z}]$ such that

$$\mathsf{hom}(\hat{Q}, \mathbf{p}) = \prod_{i=1}^{d} \hat{\ell}_i^{\,m_i}(\mathbf{z}) \qquad \text{in } \mathbb{R}[\mathbf{z}]$$

and $\hat{L}_i(\mathbf{z}) = \hat{\ell}_i(\mathbf{z} - \mathbf{p}) = 1 - \mathbf{b}^{(i)} \cdot \mathbf{z}$ for $\mathbf{b}^{(i)} \in \mathbb{R}^d$. Indeed, if Assumption 6.2 is satisfied, then $u = \prod_{i=1}^{d}(-\ell_i(\mathbf{p}))^{m_i}$, $\hat{\ell}_i(\mathbf{z}) = -\ell_i(\mathbf{z})/\ell_i(\mathbf{p})$, and $\mathbf{b}^{(i)} = \nabla \ell_i(\mathbf{z})/\ell_i(\mathbf{p})$. Moreover, we can see from this equivalent assumption that $\mathbf{p}$ is implicitly assumed to be a real point because it is the common zero of d real linear polynomials $\hat{\ell}_i$ with linearly independent gradients.

The second assumption may look strange the first time. We use results on hyperplane arrangements in Chapter 5.3 and indeed [BMP24b] in the proof of Theorem 6.16. Results on hyperplane arrangements currently works for hyperplanes defined by real polynomials only. We adopt the canonical representation $1 - \mathbf{b}^{(i)} \cdot \mathbf{z}$ for each hyperplane L_i so as to be consistent with the representation in Chapter 5.3.

You may want to make sure that there is actually any non-trivial example that fits all these assumptions. By non-trivial example, we mean that $\mathbf{p}$ is indeed a pseudo multiple point but not a multiple point. Example 6.7 satisfies Assumption 6.1 and 6.2 but not Assumption 6.3. We construct an example below that will satisfy all our assumptions.

Example 6.15. *[Bar23] Let* $\ell_1(x, y, z) := 2x + y + z$*,* $\ell_2(x, y, z) := x + 2y + z$*, and* $\ell_3(x, y, z) := x + y + 2z$*. Let* $q(x, y, z) := \ell_1\ell_2\ell_3 + c(x + y + z)^6$ *and* $Q(x, y, z) := q(x - 1, y - 1, z - 1)$*. For* c *sufficiently small,* $F = 1/Q$ *satisfies all assumptions and* $(1, 1, 1)$ *is the minimal pseudo multiple point.*

The only non-smooth point on $\mathcal{V}_Q$*, i.e.* $Q = Q_x = Q_y = Q_z = 0$*, is* $(1, 1, 1)$*. Since it is on a stratum of dimension* 0*, it is automatically a critical point. Our construction shows that it is a pseudo multiple point because* $\mathsf{hom}(Q, \mathbf{1}) = \ell_1\ell_2\ell_3$*. Assumption 6.2 is also automatically satisifed by our construction. Since a multiple point must not be an isolated singularity when* $d > 2$*, the point* $(1, 1, 1)$ *must not be a multiple point. Therefore, this example is not a trivial one.*

The next mission is to choose appropriate c *so that* $(1, 1, 1)$ *is minimal. By Proposition 2.37, it suffices to show that there is no point* (x, y, z) *with* $|x| = |y| = |z| = r, 0 < r < 1$ *such that* $Q(x, y, z) = 0$*. Since* $Q(x, y, z) = q(x - 1, y - 1, z - 1)$*, we need to show that* $q(x, y, z) \neq 0$ *on the torus* $\{(x, y, z) : |x - 1| = |y - 1| = |z - 1| = r\}$*.*

Let S *be a circle centered at* 1 *with radius* r *in* $\mathbb{C}$*. For* $x, y, z \in S$*, their real parts are positive. Then we have* $|\ell_i| \geq Re(\ell_i) \geq Re(x)$*. For any* $u + iw \in S$*, we have* $|v| < \sqrt{2u}$ *because the circle* S *is inside the parabola* $u^2 = v^2/2$*. Therefore,* $|x|^2 = Re(x)^2 + Im(x)^2 < Re(x)^2 + 2Re(x) = (Re(x) + 2)Re(x)$*. On the other hand,* $|\ell_i| < 8$ *on* S *and thus* $Re(x) < 8$*. Therefore, we have* $|x|^2 < 10Re(x) < 10|\ell_i|$ *and* $|x| < \sqrt{10}|\ell_1\ell_2\ell_3|^{1/6}$*. Similarly,* $|y|^2, |z|^2 < \sqrt{10}|\ell_1\ell_2\ell_3|^{1/6}$*. There then exists a constant* C *such that* $|x + y + z|^6 < C|\ell_1\ell_2\ell_3|$*. If we choose* c *smaller than* $1/C$*, then the term* $\ell_1\ell_2\ell_3$ *dominates the term* $c(x + y + z)^6$*. Therefore,* $q(x, y, z) \neq 0$ *on* S^3*. Therefore, Assumption 6.3 is also satisfied.* □

The goal of this section is to show that asymptotics for P/H gives a good approximation for P/Q. In other words, the leading homogeneous term of Q at the minimal pseudo multiple point is enough for us to say something for the coefficient asymptotics. We give a weaker version of what we want to show. It shows that asymptotics for P/H is a good approximation for $P/(H + tR)$ for sufficiently small t. When $t = 1$, it is the case that we finally want to prove. Unfortunately, the next proposition only says that a small perturbation on the hyperplane arrangement H will not give too much influence on asymptotics, but it does not say how small the perturbation is.

We need Assumption 6.1, 6.2, and 6.3 in the following result. To make the statement self-contained, we explicitly write down all assumptions and take the normalization as noted in Remark 6.14 by dividing both denominator and numerator of F by $\prod_{i=1}^{d}(-\ell_i(\mathbf{p}))^{m_i}$ in Assumption 6.2.

Theorem 6.16. *Let $F(\mathbf{z}) = P(\mathbf{z})/Q(\mathbf{z})$ be a d-variate rational generating function with coprime polynomials $P, Q \in \mathbb{C}[\mathbf{z}]$. Assume that there is a convergent power series $F(\mathbf{z}) = \sum_{\mathbf{r}\in\mathbb{N}^d} a_{\mathbf{r}}\mathbf{z}^{\mathbf{r}}$ with domain of convergence $\mathcal{D}$. Assume that $\mathbf{p}$ is a point on $\mathcal{V}_Q$ such that*

$$\mathsf{hom}(Q,\mathbf{p}) = \prod_{i=1}^{d} \ell_i^{m_i}(\mathbf{z}),$$

for a product of d polynomials $\ell_1, \cdots \ell_d \in \mathbb{R}[\mathbf{z}]$ with multiplicities $m_1, \cdots, m_d$ such that $L_i(\mathbf{z}) := \ell_i(\mathbf{z}-\mathbf{p}) = 1 - \mathbf{b}^{(i)}\cdot\mathbf{z}$ and $\{\mathbf{b}^{(i)}\}$ are linearly independent in $\mathbb{R}^d$. Let $H(\mathbf{z}) = L_1(\mathbf{z})^{m_1}\cdots L_d(\mathbf{z})^{m_d}$ and $R(\mathbf{z}) := Q(\mathbf{z}) - H(\mathbf{z})$. Assume that the point $\mathbf{p}$ is minimal for $H + tR$ as $t \in [0,1]$.

Define

$$F_{H+tR}(\mathbf{z}) = \frac{P(\mathbf{z})}{H(\mathbf{z}) + tR(\mathbf{z})} \text{ and } F_H(\mathbf{z}) = \frac{P(\mathbf{z})}{H(\mathbf{z})}.$$

There exists a sufficiently small δ such that the following holds for any $t < \delta$ as $\mathbf{r}\to\infty$ with $\mathbf{r}$ inside the lognormal cone at $\mathbf{p}$, i.e.

$$\left\{\sum_{i=1}^{d} a_i\tilde{\mathbf{b}}_{\mathbf{p}}^{(i)} : a_i > 0\right\} \text{ where } \tilde{\mathbf{b}}_{\mathbf{p}}^{(i)} = (b_1^{(i)}p_1, \cdots, b_d^{(i)}p_d)$$

defined in Definition 5.40.

(i) If $\mathbf{m} = \mathbf{1}$, then $[\mathbf{z}^{\mathbf{r}}]F_{H+tR}(\mathbf{z}) = [\mathbf{z}^{\mathbf{r}}]F_H(\mathbf{z}) + O(e^{a|\mathbf{r}|})$ for some $a < -\hat{\mathbf{r}}\cdot\log|\mathbf{p}|$.

(ii) If any $m_i > 1$, then $[\mathbf{z}^{\mathbf{r}}]F_{H+tR}(\mathbf{z}) = [\mathbf{z}^{\mathbf{r}}]F_H(\mathbf{z}) + O(\mathbf{p}^{-\mathbf{r}}|\mathbf{r}|^{|\mathbf{m}|-d-1})$.

Remark. In some cases, the big O term in (ii) can be even smaller.

6.3.2. Proof of Theorem 6.16

We give an outline of the proof first and then introduce necessary lemmas before finally piecing everything together. We define the height function as $h(\mathbf{z}) := -\hat{\mathbf{r}} \cdot \mathrm{Relog}(\mathbf{z}) + \hat{\mathbf{r}} \cdot \mathrm{Relog}(\mathbf{p})$ so that $h(\mathbf{p}) = 0$. Since $\mathbf{p}$ is on a stratum of dimension zero, it is a stratified critical point. The point $\mathbf{p}$ is a minimal contributing point in asymptotic analysis of F_H, when the singular variety is a hyperplane arrangement H. Hence the initial Cauchy torus T is homologous to a torus $\mathcal{T}_{\mathbf{p}}$ around $\mathbf{p}$ in the relative homology $\mathrm{H}_d(\mathcal{M}^H, \mathcal{M}^H_{\leq -\epsilon})$ where $\mathcal{M}^H := \mathbb{C}^d_* - \mathcal{V}_H$. Here $\mathcal{T}_{\mathbf{p}}$ is $\{\mathbf{z} \in \mathbb{C}^d_* : |L_i| = c, \forall i\}$ for some sufficiently small c.

1. The first thing we need to show is that this torus $\mathcal{T}_{\mathbf{p}}$ is indeed a homology generator in the relative homology $\mathrm{H}_d(\mathcal{M}^{H+tR}, \mathcal{M}^{H+tR}_{\leq -\epsilon})$ where $\mathcal{M}^{H+tR} := \mathbb{C}^d_* - \mathcal{V}_{H+tR}$.

2. Then we show that T is indeed homologous to $\mathcal{T}_{\mathbf{p}}$ in $\mathrm{H}_d(\mathcal{M}^{H+tR}, \mathcal{M}^{H+tR}_{\leq -\epsilon})$. This is easy if we just want to show for sufficiently small t. It is hard to show for large t, for example $t = 1$.

3. The last step is an analysis step. We compare the difference of integrals $\mathbf{z}^{-\mathbf{r}-\mathbf{1}} F_{H+tR}(\mathbf{z})\, d\mathbf{z}$ and $\mathbf{z}^{-\mathbf{r}-\mathbf{1}} F_H(\mathbf{z})\, d\mathbf{z}$ over $\mathcal{T}_{\mathbf{p}}$. When $\mathbf{m} = \mathbf{1}$, the difference is exponentially smaller. When some $m_i > 1$, the difference is polynomially smaller. This analysis step heavily relies on hyperplane arrangements in Chapter 5.3.

We introduce the first ingredient we need in the first step. By a linear transformation, each L_i becomes z_i and so the next lemma is exactly what we need in the first step.

Lemma 6.17. *Let $H(\mathbf{z}) = z_1^{m_1} \cdots z_d^{m_d}$ be a function on $\mathbb{C}^d$. Let $R(\mathbf{z})$ be a polynomial in $\mathbf{z}$ with order of vanishing at least $m_1 + \cdots + m_d + 1$ at the origin. Then there exists sufficiently small $\epsilon > 0$ such that the torus T_ϵ avoids $\mathcal{V}_{H+tR}$ for all $t \in [0, 1]$.*

Proof: Any torus $T_\epsilon = \{\mathbf{z} \in \mathbb{C}^d : |z_i| = \epsilon\}$ will avoid $\mathcal{V}_H$. Therefore, we only need to show the existence of a T_ϵ avoiding $\mathcal{V}_{H+tR}$. Consider the quantity

$$\frac{(H + tR)(\epsilon \mathbf{z})}{\epsilon^{m_1 + \cdots + m_d}}$$

where $\epsilon\mathbf{z} \in T_\epsilon$ and thus $\mathbf{z} \in T_1$. The above quantity can be rewritten into

$$H(\mathbf{z}) + \epsilon t R'(\mathbf{z})$$

where $R'(\mathbf{z})$ is a polynomial with order $m_1 + \cdots + m_d + 1$. Since $\mathbf{z} \in T_1$, $H(\mathbf{z})$ is always modulus one. Since $tR'(\mathbf{z})$ is a polynomial in $\mathbf{z}$ and t, it has bounded modulus as $\mathbf{z}$ varies in T_1 and t varies in $[0,1]$. Let $\epsilon \to 0$. The above quantity has a limiting modulus one and thus never zero. Therefore, there exists an ϵ small enough so that $H + tR(\mathbf{z}) \neq 0$ for $\mathbf{z} \in T_\epsilon$ uniformly for all $t \in [0,1]$. $\square$

The next lemma is to measure the difference between $\int_{\mathcal{T}_\mathbf{p}} F_{H+tR}(\mathbf{z})\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\, d\mathbf{z}$ and $\int_{\mathcal{T}_\mathbf{p}} F_H(\mathbf{z})\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\, d\mathbf{z}$.

Lemma 6.18. *Let $L_1, \cdots, L_d$ be linear polynomials in $\mathbb{R}[\mathbf{z}]$ with $L_i(\mathbf{z}) = 1 - \mathbf{b}^{(i)} \cdot \mathbf{z}$ where $\{\mathbf{b}^{(i)}\}$ are linearly independent. Assume $\mathcal{V}_{L_1}, \cdots, \mathcal{V}_{L_d}$ intersect at a point $\mathbf{p} \in \mathbb{C}_*^d$.*

Let $H(\mathbf{z}) = L_1(\mathbf{z})^{m_1} \cdots L_d(\mathbf{z})^{m_d}$ for some multi-index $\mathbf{m} \in \mathbb{N}_{>0}^d$. Let $R(\mathbf{z})$ be a convergent power series centered at $\mathbf{p}$ with order of vanishing at least $|\mathbf{m}|+1$ at $\mathbf{p}$. Let $\mathcal{T}_\mathbf{p}$ be a d-chain $\{\mathbf{z} \in \mathbb{C}^d : |L_i| = \epsilon, i = 1, \cdots, d\}$ such that $\mathcal{T}_\mathbf{p}$ is in $\mathcal{M}^H := \mathbb{C}_*^d - \mathcal{V}_H$. Define $F_H = P/H$ and $F_{H+R} = P/(H+R)$. The following statements hold for the quantity

$$\int_{\mathcal{T}_\mathbf{p}} \left(F_H(\mathbf{z}) - F_{H+R}(\mathbf{z})\right) \mathbf{z}^{-\mathbf{r}-\mathbf{1}}\, d\mathbf{z}$$

as $\mathbf{r} \to \infty$ with $\mathbf{r}$ inside a compact subset of the cone

$$\left\{ \sum_{i=1}^d a_i \tilde{\mathbf{b}}_\mathbf{p}^{(i)} : a_i > 0 \right\} \text{ where } \tilde{\mathbf{b}}_\mathbf{p}^{(i)} = (b_1^{(i)} p_1, \cdots, b_d^{(i)} p_d).$$

1. *If $\mathbf{m} = \mathbf{1}$, then the above quantity decays super exponentially. In other words, the quantity is $O(e^{-a|\mathbf{r}|})$ for any $a > 0$.*

2. *If any $m_i > 1$, the above quantity is $O(\mathbf{p}^{-\mathbf{r}}|\mathbf{r}|^{|\mathbf{m}|-d-1})$.*

PROOF: Define the linear map

$$\Phi(\mathbf{z}) = (L_1(\mathbf{z}), \cdots, L_d(\mathbf{z}))$$

Then $\mathcal{T}_{\mathbf{p}} = \Psi^{-1}(T_\epsilon)$ for some torus $T_\epsilon := \{\mathbf{z} \in \mathbb{C}^d : |z_i| = \epsilon, \forall i\}$. We look at the quantity

$$\int_{\mathcal{T}_{\mathbf{p}}} \left[\frac{P}{H} - \frac{P}{H+R}\right] \mathbf{z}^{-\mathbf{r}-\mathbf{1}} d\mathbf{z} \tag{6.3}$$

Since

$$\frac{1}{H+R} = \frac{1}{H}\frac{1}{1+R/H} = \frac{1}{H}\left(1 - \frac{R}{H} + \frac{R}{H}^2 - \cdots\right),$$

the quantity (6.3) becomes

$$\int_{\mathcal{T}_{\mathbf{p}}} \left[\frac{P}{H}\left(\frac{R}{H} - \left(\frac{R}{H}\right)^2 + \ldots\right)\right] \mathbf{z}^{-\mathbf{r}-\mathbf{1}}\ d\mathbf{z}.$$

Let's look at the k-th term $\int_{\mathcal{T}_{\mathbf{p}}} \frac{P}{H}\left(\frac{R}{H}\right)^k \mathbf{z}^{-\mathbf{r}-\mathbf{1}}\ d\mathbf{z}$. Applying the map Φ, we perform a global linear change-of-coordinate and the above integral becomes $\int_{T_\epsilon} \frac{\tilde{P}}{\tilde{H}}\left(\frac{\tilde{R}}{\tilde{H}}\right)^k \Phi(\mathbf{z})^{-\mathbf{r}-\mathbf{1}} J_\Phi(\mathbf{z})\ d\mathbf{z}$ where J_Φ is the Jacobian, $\tilde{P} \circ \Phi = P$, $\tilde{H} \circ \Phi = H$, and $\tilde{R} \circ \Phi = R$. More explicitly, $\tilde{H}$ is just $\mathbf{z}^{\mathbf{m}}$ and $\tilde{R}(\mathbf{z})$ is $\sum_{\mathbf{n}:|\mathbf{n}| \geq |\mathbf{m}|+1} a_{\mathbf{n}} z_1^{n_1} \cdots z_d^{n_d}$ where $a_{\mathbf{n}}$ are some constants. Rewriting the above integral, we got

$$\int_{T_\epsilon} \frac{\tilde{P}(\mathbf{z}) \sum_{|\mathbf{n}| \geq k(|\mathbf{m}|+1)} c_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}}{\mathbf{z}^{(k+1)\mathbf{m}}} \Phi(\mathbf{z})^{-\mathbf{r}-\mathbf{1}} J_\Phi(\mathbf{z})\ d\mathbf{z} \tag{6.4}$$

Now if $\mathbf{m} = \mathbf{1}$, every term in the numerator $\sum_{|\mathbf{n}| \geq k(|\mathbf{m}|+1)} c_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}$ is of degree at least $k(d+1)$ because $|\mathbf{m}| = d$. On the other hand, the denominator is $z_1^{k+1} \cdots z_d^{k+1}$. By the pigeonhole principle, for any $\mathbf{n}$ such that $|\mathbf{n}| \geq k(d+1)$, $\mathbf{z}^{\mathbf{n}}\mathbf{z}^{-(k+1)\mathbf{1}}$ is equal to $\mathbf{z}^{\mathbf{s}}$ such that some $s_i \geq 0$.

Therefore, we can rewrite the expression (6.4) as

$$\sum_{\mathbf{s}:\text{ some } s_i \geq 0} \int_{T_\epsilon} \tilde{P}(\mathbf{z}) d_{\mathbf{s}}\ \mathbf{z}^{\mathbf{s}}\ \Phi(\mathbf{z})^{-\mathbf{r}-\mathbf{1}} J_\Phi(\mathbf{z})\ d\mathbf{z} \tag{6.5}$$

where $d_{\mathbf{s}} = c_{\mathbf{s}+(k+1)\mathbf{1}}$. After applying Φ^{-1}, each term in the sum (6.5) becomes

$$\int_{\mathcal{T}_{\mathbf{p}}} d_{\mathbf{s}} P(\mathbf{z}) L_1^{s_1} \cdots L_d^{s_d} \mathbf{z}^{-\mathbf{r}-\mathbf{1}}\ d\mathbf{z}.$$

The torus $\mathcal{T}_{\mathbf{p}}$ is homologous in $\mathrm{H}_d(\mathcal{M}^H, -\infty)$ to the sum of alternating imaginary fibers (Definition 5.42, 5.44) with base points in components formed by $L_1, \cdots, L_d$ in $\mathbb{R}^d$. Since there is some i such that $s_i \geq 0$, without loss of generality, assume $s_d \geq 0$. Now the singular variety is defined only by $L_1, \cdots, L_{d-1}$ and therefore alternating imaginary fibers will cancel out with each other. That is, $\mathcal{T}_{\mathbf{p}}$ is null-homologous in $\mathrm{H}_d(\mathcal{M}^{H'}, -\infty)$ where $H' = L_1 \cdots L_{d-1}$. Therefore, the integral is growing less than any exponential order. Summing these super exponentially decaying terms together, the quantity (6.4) grows less than any exponential order and so does the quantity (6.3).

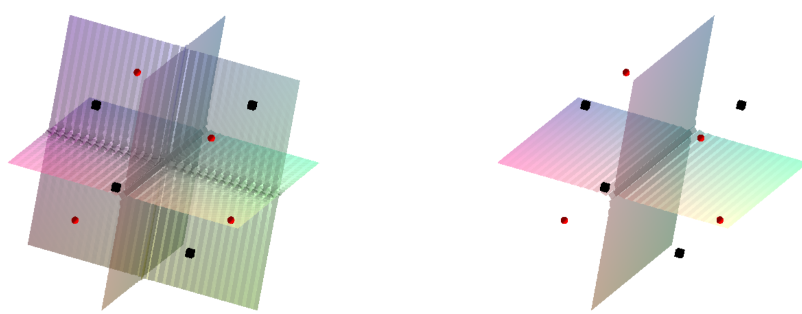


Figure 6.4: (left) When $d = 3$, we have eight imaginary fibers with basis points (red spheres and black boxes indicating different orientations) in the components formed by L_1, L_2, and L_3. (right) When one of the hyperplane L_3 disappear from the singular variety, imaginary fibers with different orientations now have their basis points in the same component and thus the sum of all eight imaginary fibers are null-homologous in $\mathrm{H}_d(\mathcal{M}, -\infty)$.

Now if some $m_i > 1$, then there is some $\mathbf{n}$ in the expression (6.4) such that $\mathbf{z}^{\mathbf{n}}\mathbf{z}^{-(k+1)\mathbf{m}} = \mathbf{z}^{\mathbf{s}}$ where

$s_i < 0$ for $i = 1, \cdots, d$. For example, if $m_d > 1$, then we can choose $\mathbf{n} = (km_1, \cdots, km_{d-1}, k(m_d+1))$ such that $\mathbf{z}^{\mathbf{n}}\mathbf{z}^{-(k+1)\mathbf{m}} = z_1^{-m_1} \cdots z_{d-1}^{-m_{d-1}} z_d^{k-m_d}$ where all exponents are less than zero if $k < m_d$. Let $\mathcal{I}$ be the set of all indices $\mathbf{n}$ in the expression (6.4) such that $\mathbf{z}^{\mathbf{n}}\mathbf{z}^{-(k+1)\mathbf{m}} = \mathbf{z}^{\mathbf{s}}$ where $s_i < 0$ for all $i \in \{1, \cdots, d\}$ and thus the integral (6.4) becomes

$$\sum_{\mathbf{n}\in\mathcal{I}} \int_{T_\epsilon} \frac{\tilde{P}(\mathbf{z}) c_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}}{\mathbf{z}^{(k+1)\mathbf{m}}} \Phi(\mathbf{z})^{-\mathbf{r}-\mathbf{1}} J_\Phi(\mathbf{z}) \, d\mathbf{z} + \sum_{\mathbf{n}\notin\mathcal{I}} \int_{T_\epsilon} \frac{\tilde{P}(\mathbf{z}) c_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}}{\mathbf{z}^{(k+1)\mathbf{m}}} \Phi(\mathbf{z})^{-\mathbf{r}-\mathbf{1}} J_\Phi(\mathbf{z}) \, d\mathbf{z} \tag{6.6}$$

The second summation is $O(e^{-a|\mathbf{r}|})$ for any $a > 0$ by the same reasoning when $\mathbf{m} = \mathbf{1}$. Each term in the first summation is indeed equal to, after applying Φ^{-1},

$$\int_{\mathcal{T}_{\mathbf{p}}} d_{\mathbf{s}} P(\mathbf{z}) L_1^{s_1} \cdots L_d^{s_d} \mathbf{z}^{-\mathbf{r}-\mathbf{1}} \, d\mathbf{z}$$

where $\mathbf{s} = \mathbf{n} - (k+1)\mathbf{m}$ and thus $d_{\mathbf{s}} = c_{\mathbf{s}+(k+1)\mathbf{m}}$. Since $\mathbf{n} \in \mathcal{I}$, by our definition of $\mathcal{I}$, each $s_i < 0$. Therefore, the singular variety is still the hyperplane arrangement formed by $L_1, \cdots, L_d$. By equation (5.9), the integral has asymptotics $O(\mathbf{p}^{-\mathbf{r}}|\mathbf{r}|^{-|\mathbf{s}|-d})$. Since $|\mathbf{s}| = |\mathbf{n}| - (k+1)|\mathbf{m}| \geq k(|\mathbf{m}| + 1) - (k+1)|\mathbf{m}| = k - |\mathbf{m}|$, the first summation in equation (6.6) has asymptotics $O(\mathbf{p}^{-\mathbf{r}}|\mathbf{r}|^{|\mathbf{m}|-d-k})$. In particular, the quantity (6.3) is dominated by $O(\mathbf{p}^{-\mathbf{r}}|\mathbf{r}|^{|\mathbf{m}|-d-1})$ with $k = 1$. $\square$

Remark 6.19. The asymptotic order we give in the case when some $m_i > 1$ can be smaller in some cases.

Proof of Theorem 6.16

Since $\mathbf{p}$ is on a stratum of dimension zero, it is a stratified critical point on the singular variety $\mathcal{V}_{H+tR}$ for all $t \in [0, 1]$. Our job is to show that $\mathbf{p}$ is a contributing point to the asymptotics by choosing the correct generator in the local homology group of $\mathcal{M}^{H+tR}$ at $\mathbf{p}$ and show that the original Cauchy torus T is homologous to it.

Let $\Phi(\mathbf{z}) = (L_1(\mathbf{z}), \cdots, L_d(\mathbf{z}))$. Let $\tilde{H} = H \circ \Phi^{-1} = z_1^{m_1} \cdots z_d^{m_d}$. Similarly, $\tilde{R} = R \circ \Phi^{-1}$ is a polynomial of $\mathbf{z}$ with order of vanishing at least $|\mathbf{m}| + 1$ at the origin. The torus $T_c := \{\mathbf{z} \in \mathbb{C}^d : |z_i| = c, \forall i\}$ does not intersect the variety $\mathcal{V}_{\tilde{H}}$ for any c. By Lemma 6.17, there exists sufficiently small $c > 0$ such that T_c does not intersect $\mathcal{V}_{\tilde{H}+t\tilde{R}}$ for any $t \in [0,1]$. Therefore, $\mathcal{T}_{\mathbf{p}} := \Phi^{-1}(T_c)$ does not intersect $\mathcal{V}_{H+tR}$ for any $t \in [0,1]$.

The next step is to show that T is homologous to $\mathcal{T}_{\mathbf{p}}$ in the relative homology $\mathrm{H}_d(\mathcal{M}^{H+tR}, \mathcal{M}^{H+tR}_{\le -\epsilon})$ where $\mathcal{M}^{H+tR} := \mathbb{C}^d_* - \mathcal{V}_{H+tR}$ and $-\epsilon$ is chosen between zero and the next largest critical value. When $t = 0$, $\mathcal{V}_H$ is a hyperplane arrangement by $L_1, \cdots, L_d$ which intersect at a common point $\mathbf{p}$. We can decompose T into a sum of 2^d alternating imaginary fibers with basis points at each component of $\mathcal{M}_{\mathbb{R}}$ adjacent to the origin $\mathbf{0}$. Here $\mathcal{M} := \mathbb{C}^d_* - \mathcal{V}_H$ and $\mathcal{M}_{\mathbb{R}} := \mathcal{M} \cap \mathbb{R}^d$. Since $\mathbf{r}$ is inside the lognormal cones at $\mathbf{p}$ and $\mathbf{p}$ is minimal, the point $\mathbf{p}$ is the minimizer of $h_{\hat{\mathbf{r}}}$ in the component of $\mathcal{M}_{\mathbb{R}}$ containing both $\mathbf{p}$ and $\mathbf{0}$. Other components of $\mathcal{M}_{\mathbb{R}}$ adjacent to $\mathbf{0}$ are bounded by the minimality of $\mathbf{p}$. Therefore, by Chapter 5.3.3, we know that T is homologous to $\mathcal{T}_{\mathbf{p}}$ in $\mathrm{H}_d(\mathcal{M}^H, \mathcal{M}^H_{\le -\epsilon})$.

That being said, there exists a cobordism $\mathcal{H}$ in $\mathcal{M}^H$ with boundary $T - \mathcal{T}_{\mathbf{p}} + \alpha$ where α is a d-cycle on which the height is less than $-\epsilon$. There exists a small ball B_r with radius r centered at $\mathbf{p}$ such that $\mathcal{T}_{\mathbf{p}}$ is inside B_r. The cobordism $\mathcal{H}$ can be kept inside the open polydisk $\{\mathbf{z} \in \mathbb{C}^d_* : |z_i| < 1\}$ unioned with B_r and $\mathcal{M}^H_{\le -\epsilon}$. By minimality of $H + tR$, the part of $\mathcal{H}$ inside the open polydisk $\{\mathbf{z} \in \mathbb{C}^d_* : |z_i| < 1\}$ will not intersect with $\mathcal{V}_{H+tR}$ for any $t \in [0,1]$. There exists sufficiently small δ such that the part of $\mathcal{H}$ inside the ball B_r will not intersect with $\mathcal{V}_{H+tR}$ for $t < \delta$. For the part of $\mathcal{H}$ that has height less than or equal to $-\epsilon$, they are mod out in relative homology. Therefore, there exists a sufficiently small δ such that if $t < \delta$, then T is homologous to $\mathcal{T}_{\mathbf{p}}$ in $\mathrm{H}_d(\mathcal{M}^{H+tR}, \mathcal{M}^{H+tR}_{\le -\epsilon})$. In other words, for $t < \delta$, by Corollary 2.4

$$[\mathbf{z}^{\mathbf{r}}]F_{H+tR}(\mathbf{z}) = \int_T F_{H+tR}(\mathbf{z})\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\, d\mathbf{z} = \int_{\mathcal{T}_{\mathbf{p}}} F_{H+tR}(\mathbf{z})\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\, d\mathbf{z} + O(e^{b|\mathbf{r}|})$$

for some $b < -\hat{\mathbf{r}} \cdot \operatorname{Relog} \mathbf{p}$.

We can replace $\int_T F_{H+tR}(\mathbf{z})\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\, d\mathbf{z}$ on the right hand side by $\int_{\mathcal{T}_{\mathbf{p}}} F_H(\mathbf{z})\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\, d\mathbf{z}$ at the expense of the error,

$$\int_{\mathcal{T}_{\mathbf{p}}} \left(F_H(\mathbf{z}) - F_{H+tR}(\mathbf{z})\right)\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\, d\mathbf{z}.$$

Lemma 6.18 gives the asymptotic order of this error term regarding to whether $\mathbf{m} = \mathbf{1}$ or not. In particular, when $\mathbf{m} = \mathbf{1}$,

$$[\mathbf{z}^{\mathbf{r}}]F_{H+tR}(\mathbf{z}) = \int_T F_{H+tR}(\mathbf{z})\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\ d\mathbf{z} = \int_{\mathcal{T}_{\mathbf{p}}} F_H(\mathbf{z})\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\ d\mathbf{z} + O(e^{b|\mathbf{r}|}) + O(e^{-c|\mathbf{r}|}). \tag{6.7}$$

for any $c > 0$. When $\mathbf{m} \neq \mathbf{1}$,

$$[\mathbf{z}^{\mathbf{r}}]F_{H+tR}(\mathbf{z}) = \int_T F_{H+tR}(\mathbf{z})\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\ d\mathbf{z} = \int_{\mathcal{T}_{\mathbf{p}}} F_H(\mathbf{z})\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\ d\mathbf{z} + O(e^{b|\mathbf{r}|}) + O(\mathbf{p}^{-\mathbf{r}}|\mathbf{r}|^{|\mathbf{m}|-d-1}). \tag{6.8}$$

Since $\mathbf{p}$ is minimal on $\mathcal{V}_H$, $\mathbf{p}$ is the highest contributing point in the hyperplane arrangement $\mathcal{V}_H$ by Proposition 5.45. Therefore, $[\mathbf{z}^{\mathbf{r}}]F_H(\mathbf{z}) = \int_{\mathcal{T}_{\mathbf{p}}} F_H(\mathbf{z})\mathbf{z}^{-\mathbf{r}-\mathbf{1}}\ d\mathbf{z} + O(e^{d|\mathbf{r}|})$ for some $d < -\hat{\mathbf{r}} \cdot \operatorname{Re}\log \mathbf{z}$. Now equation (6.7) becomes

$$[\mathbf{z}^{\mathbf{r}}]F_{H+tR}(\mathbf{z}) = [\mathbf{z}^{\mathbf{r}}]F_H(\mathbf{z}) + O(e^{b|\mathbf{r}|}) + O(e^{d|\mathbf{r}|}) + O(e^{-c|\mathbf{r}|})$$

and equation (6.8) becomes

$$[\mathbf{z}^{\mathbf{r}}]F_{H+tR}(\mathbf{z}) = [\mathbf{z}^{\mathbf{r}}]F_H(\mathbf{z}) + O(e^{b|\mathbf{r}|}) + O(e^{d|\mathbf{r}|}) + O(\mathbf{p}^{-\mathbf{r}}|\mathbf{r}|^{|\mathbf{m}|-d-1}).$$

Now let $a = \max(b, d)$ and let c be arbitrarily large. We thus prove the theorem. $\square$.

The unfortunate thing about Theorem 6.16 is that it only works for $t \leq \delta$ and we don't even know how small δ should be. In other words, even if we have a pseudo multiple point $\mathbf{p}$ satisfying Assumption 6.1, 6.2, and 6.3, we can't make sure if the perturbation term R is small enough to fit into Theorem 6.16. Assumption 6.3 is also more than we need in the proof of Theorem 6.16. Indeed, we only need the minimality to hold for $\mathcal{V}_{H+tR}, t \leq \delta$. But again, since we don't know how large δ can be, we just keep the stronger assumption 6.3 in place.

6.3.3. Future directions

To make $\delta = 1$ in Theorem 6.16, we need to impose a more strict version of minimality of $\mathbf{p}$ in Assumption 6.3. That is, in addition to the current running assumptions, we need to require

$T(\mathbf{p}) \cap \mathcal{V}_{H+tR}$ contains only one critical point, that is, $\mathbf{p}$. If the following conjecture is true, then we can make $\delta = 1$. The proof is exactly the same as Theorem 6.16 by replacing the homologous argument.

Conjecture 6.20. *Under Assumption 6.1, 6.2, and 6.3 (satisfying the new version of minimality), the initial torus T is homologous to $\mathcal{T}_{\mathbf{p}} := \Phi^{-1}(T_c)$ in $\mathrm{H}^d(\mathcal{M}^{H+tR}, \mathcal{M}^{H+tR}_{\leq -\epsilon})$ as $t \in [0,1]$*

Beyond the conjecture, possible further directions include studying cases when

- $T(\mathbf{p}) \cap \mathcal{V}_{H+tR}$ contains finitely many critical points and all these points satisfy Assumption 6.1, 6.2, and 6.3.
- $\{\mathbf{b}^{(i)}\}$ in Theorem 6.16 are not linearly independent
- $\mathsf{hom}(Q, \mathbf{p})$ factorizes in less than or more than d distinct factors.

The methods in [ABG70] or more explicitly [BP11] may be helpful in these cases. In particular, one can consider the original cauchy torus T as $\exp(\mathbf{x} + i\mathbf{y})$ where $\mathbf{x}$ is in the component B of $\mathsf{amoeba}(Q)^c$ containign the origin, and $\mathbf{y} \in \mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d$. Let $q := Q \circ \exp$ be the log version of Q. The Cauchy integral

$$\int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} \frac{1}{Q(\mathbf{z})} d\mathbf{z}$$

becomes

$$\int_{\mathbf{x}+i\mathbb{T}^d} \exp(-\mathbf{r} \cdot \mathbf{z}) \frac{1}{q(\mathbf{z})} d\mathbf{z} \tag{6.9}$$

If $\mathbf{p} = \exp(\mathbf{x}_{\min} + i\mathbf{w})$, then by minimality, we can move the integral d chain in (6.9) to $\mathbf{x}_{\min} - \mathbf{u} + i\mathbb{T}^d$ without changing the integral value. Here $\mathbf{u}$ is chosen by first picking a vector with starting point at $\mathbf{x}_{\min}$ pointing toward inside B and then translating the starting point to the origin in $\mathbb{R}^d$. One can try to construct a vector field η on $\mathbb{T}^d$ like the one in [BP11, Lemma 5.3] such that there is a homotopy from $\mathbf{x}_{\min} - \mathbf{u} + i\mathbb{T}^d$ to a another chain following the vector field η. Explicitly, the homotopy is given by $\Phi_t(\mathbf{y}) := \mathbf{x}_{\min} + iy + [(1-t)\mathbf{u} + t\eta(\mathbf{y})]$. This homotopy will avoid $\mathcal{V}_Q$ and gives us an explicit deformation of T. Then we stop the homotopy early in a neighborhood of each critical points [BP11, Definition 5.7]. Analysis can be done locally near each critical points on the chain $\mathbf{x}_{\min} + i\mathbb{T}^d$.

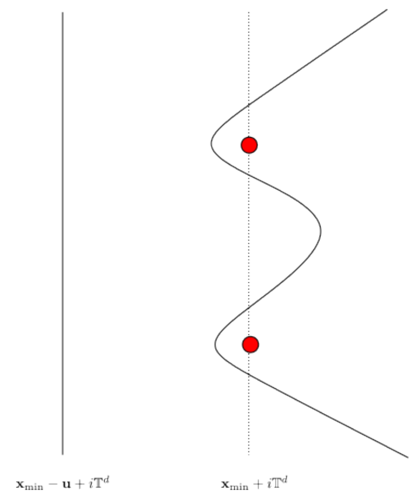


Figure 6.5: The deformation of the chain $\mathbf{x}_{\min}-\mathbf{u}+i\mathbb{T}^d$ given by Φ_t that stops early in a neighborhood of critical points (red).

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