Counting partitions inside a rectangle

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Abstract

We consider the number of partitions of n whose Young diagrams fit inside an $m \times \ell$ rectangle; equivalently, we study the coefficients of the q-binomial coefficient $\binom{m+\ell}{m}_q$. We obtain sharp asymptotics throughout the regime $\ell = \Theta(m)$ and $n = \Theta(m^2)$. Previously, sharp asymptotics were derived by Takács [Tak86] only in the regime where $|n - \ell m/2| = O(\sqrt{\ell m(\ell + m)})$ using a local central limit theorem. Our approach is to solve a related large deviation problem: we describe the tilted measure that produces configurations whose bounding rectangle has the given aspect ratio and is filled to the given proportion. Our results are sufficiently sharp to yield the first asymptotic estimates on the consecutive differences of these numbers when n is increased by one and m, ℓ remain the same, hence significantly refining Sylvester's unimodality theorem.

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1 Introduction

A partition λ of n is a sequence of weakly decreasing nonnegative integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ whose sum $|\lambda| = \lambda_1 + \lambda_2 + \cdots$ is equal to n. The study of integer partitions is a classic subject with applications ranging from number theory to representation theory and combinatorics. Integer partitions with various restrictions on properties, such as part sizes or number of parts, occupies the field of partition theory [And76]. The generating functions for integer partitions play a role in number theory and the theory of modular forms. In representation theory, integer partitions index the conjugacy classes and irreducible representations of the symmetric group S_n; they are the signatures of the irreducible polynomial representation of GL_n. Partitions also give a basis for the ring of symmetric functions. More recently, partitions have appeared in the study of interacting particle systems and other statistical mechanics models.

The number of partitions of n, typically denoted by $p(n)$ but here unconventionally[1] by N_n, was implicitly determined by Euler via the generating function

$$\sum_{n=0}^{\infty} N_n q^n = \prod_{i=1}^{\infty} \frac{1}{1-q^i} .$$

There is no exact explicit formula for the numbers N_n. The asymptotic formula

$$N_n := \#\{\lambda \vdash n\} \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) , \tag{1}$$

obtained by Hardy and Ramanujan [HR18], is considered to be the beginning of the use of complex variable methods for asymptotic enumeration of partitions (the so-called circle method).

Our goal is to obtain asymptotic formulas similar to (1) for the number of partitions λ of n whose Young diagram fits inside an $m \times \ell$ rectangle, denoted

$$N_n(\ell, m) := \#\{\lambda \vdash n : \lambda_1 \leq \ell, \quad \text{length}(\lambda) \leq m\} .$$

These numbers are also the coefficients in the expansion of the q-binomial coefficient

$$\binom{\ell+m}{m}_q = \frac{\prod_{i=1}^{\ell+m}(1-q^i)}{\prod_{i=1}^{\ell}(1-q^i)\prod_{i=1}^{m}(1-q^i)} = \sum_{n=0}^{\ell m} N_n(\ell, m) q^n .$$

The q-binomial coefficients are themselves central to enumerative and algebraic combinatorics. They are the generating functions for lattice paths restricted to rectangles and taking only north and east steps under the area statistic, given by the parameter n. They are also the number of ℓ-dimensional subspaces of $\mathbb{F}_q^{\ell+m}$ and appear in many other generating functions as the q-analogue generalization of the ubiquitous binomial coefficients. Notably, the numbers $N_n(\ell, m)$ form a symmetric unimodal sequence

$$1 = N_0(\ell, m) \leq N_1(\ell, m) \leq \cdots \leq N_{\lfloor m\ell/2 \rfloor}(\ell, m) \geq \cdots \geq N_{m\ell}(\ell, m) = 1,$$

a fact conjectured by Cayley in 1856 and proven by Sylvester in 1878 via the representation theory of sl_2 [Syl78]. One hundred forty years later, no previous asymptotic methods have been able to prove this unimodality.

Our main result is an asymptotic formula for $N_n(\ell, m)$ in the regime $\ell/m \to A$ and $n/m^2 \to B$ for any fixed $A > B > 0$. This is the regime in which a limit shape of the partitions exists: $\ell/m \to A$ implies the

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[1] We use the notation N_n to distinguish scenarios of probability with those of enumeration, both of which occur in the present manuscript.

aspect ratio has a limit and $n/m^2 \to B \in (0, A)$ implies the portion of the $m \times \ell$ rectangle that is filled approaches a value that is neither zero nor 1. The existence of a limit shape was not previously verified for this regime but follows from our methods (see Section 3 below). By "asymptotic formula" we mean a formula giving $N_n(\ell, m)$ up to a factor of $1 + o(1)$; such asymptotic equivalence is denoted with the symbol $\sim$. By the obvious symmetry $N_n(\ell, m) = N_{m\ell-n}(\ell, m)$ it suffices to consider only the case $A \geq 2B > 0$.

To state our results, given $A \geq 2B > 0$ we define three quantities c, d and Δ. The quantities c and d are the unique positive real solutions (see Lemma 7) to the simultaneous equations

$$A = \int_0^1 \frac{1}{1 - e^{-c-dt}}\,dt - 1 = \frac{1}{d} \log\left(\frac{e^{c+d} - 1}{e^c - 1}\right), \tag{2}$$

$$B = \int_0^1 \frac{t}{1 - e^{-c-dt}}\,dt - \frac{1}{2} = \frac{d \log(1 - e^{-c-d}) + \operatorname{dilog}(1 - e^{-c}) - \operatorname{dilog}(1 - e^{-c-d})}{d^2}, \tag{3}$$

where we recall the dilogarithm function

$$\operatorname{dilog}(x) = \int_1^x \frac{\log t}{1 - t}\,dt = \sum_{k=1}^{\infty} \frac{(1 - x)^k}{k^2}$$

for $|x - 1| < 1$. The quantity Δ, which will be seen to be strictly positive, is defined by

$$\Delta = \frac{2Be^c(e^d - 1) + 2A(e^c - 1) - 1}{d^2(e^{d+c} - 1)(e^c - 1)} - \frac{A^2}{d^2}. \tag{4}$$

Theorem 1. *Given m, ℓ and n, let $A := \ell/m$ and $B := n/m^2$ and define c, d and Δ as above. Let K be any compact subset of $\{(x, y) : x \geq 2y > 0\}$. As $m \to \infty$ with ℓ and n varying so that (A, B) remains in K,*

$$N_n(\ell, m) \sim \frac{e^{m\left[cA + 2dB - \log(1 - e^{-c-d})\right]}}{2\pi m^2 \sqrt{\Delta\,(1 - e^{-c})\,(1 - e^{-c-d})}}, \tag{5}$$

where c and d vary in a Lipschitz manner with $(A, B) \in K$.

Remark. In the special case $B = A/2$, the parameters take on the elementary values

$$d = 0, \qquad c = \log\left(\frac{A + 1}{A}\right), \qquad \text{and} \qquad \Delta = \frac{A^2(A + 1)^2}{12}.$$

In this case we understand the exponent and leading constant to be their limits as $d \to 0$, giving

$$N_{Am^2/2}(Am, m) \sim \frac{\sqrt{3}}{A\pi m^2} \left[\frac{(A + 1)^{A+1}}{A^A}\right]^m.$$

To explain Theorem 1 we consider the special case $B = A/2$, when the Young diagram fills half the area of the rectangle. Takács [Tak86] observed that for typical partitions of this type, the gaps between part sizes behave like independent geometric random variables with mean A. Counting the partitions is therefore equivalent (as further explained below) to computing the probability that these $m + 1$ geometric random variables will sum to precisely ℓ and that the area of the ensuing Young diagram will be precisely n. A local central limit theorem immediately yields a sharp asymptotic estimate. With further work, Takács obtained bounds on the relative error that are of order $(m + \ell)^{-3}$. These error bounds are meaningful for n differing from $m\ell/2$ by up to a few multiples of $\log(m + \ell)$ standard deviations. If $\ell = \theta(m)$, this means that $|B - A/2|m^2 = \Theta(m^{3/2} \log m)$. When $|B - A/2| \gg m^{-1/2} \log m$, the error is much bigger than the main term of the Gaussian estimate provided by the LCLT.

We circumvent this limitation on the use of the LCLT using a technique from the theory of large deviations. Specifically, we employ a so-called tilted measure for which maximum likelihood occurs at any desired pair (A, B). The tilted measure replaces the IID geometric random variables by independent but not identically distributed geometric random variables, where the parameter $1-p_i$ for the ith variable varies in a log-linear manner. This requires extending a two-variable lattice LCLT to handle non-identically distributed summands. While this result, Lemma 4, is completely predictable, we could not find it in the literature, hence we include a proof in the Appendix.

Previous results on $N_n(m)$ and $N_n(\ell, m)$

There are numerous previous results on $N_n(m)$, the number of partitions of n with part sizes bounded by m. Erdös and Lehner [EL41] showed $N_n(m) \sim \frac{n^{m-1}}{m!(m-1)!}$ for $m = o(n^{1/3})$ in 1941, which was generalized by Szekeres, and others, ultimately leading to asymptotics of $N_n(m)$ for all m in 1953 [Sze53]. Szekeres simplified his arguments a number of times, ultimately giving asymptotics using only a saddle-point analysis, without needing results on modular functions. His argument has been referred to as the Szekeres circle method.

Mann and Whitney [MW47] showed, through the study of an equivalent statistical problem, that the size of a uniform random partition in the $\ell \times m$ rectangle satisfies a normal distribution. As previously noted, Takács [Tak86] used a local central limit theorem to show that

$$N_n(\ell, m) \sim \binom{\ell+m}{\ell} \phi\left(\frac{n-\ell m/2}{\sigma_{\ell,m}}\right) \frac{1}{\sigma_{\ell,m}} \tag{6}$$

when

$$|n-\ell m/2| < K\sigma_{\ell,m} = O\left(\sqrt{\ell m(\ell+m)}\right),$$

where $\sigma_{\ell,m} = \sqrt{lm(\ell+m+1)/12}$, $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$, and K is a positive constant; see also [AA91]. Figure 1 shows the predicted exponential growth of Takács compared to the actual exponential growth of partitions outside the valid asymptotic region.

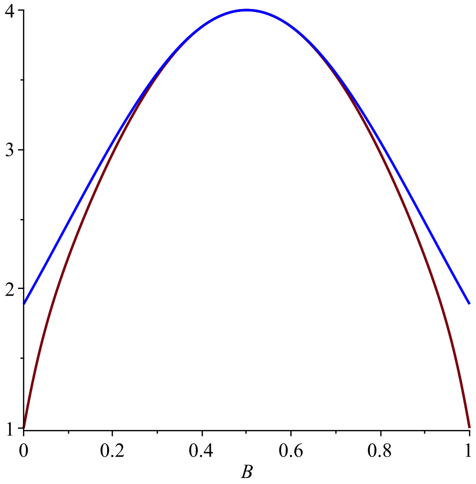


Figure 1: Exponential growth of $N_{Bm^2}(m, m)$ predicted by Takács' formula (blue, above) compared to the actual exponential growth given by Theorem 1 (red, below).

The results of Erdös and Lehner [EL41] imply that the expected maximal part (and thus also number of parts) in a partition of size n is typically $f_n \log(f_n)$, where $f_n := \frac{\sqrt{6n}}{\pi}$. Near this expected maximum, $N_n(m)$

behaves like a doubly-exponential distribution in m [Sze90]. When

$$\frac{\sqrt{6n}\,\log n}{4\pi} < \ell, m < \frac{\sqrt{6}\,n^{3/4}}{\pi \log n}$$

Szekeres [Sze90, Theorem 1] used saddle-point techniques to express $N_n(\ell, m)$ in terms of N_n, $\lambda := \frac{\pi\ell}{\sqrt{6n}}$ and $\mu := \frac{\pi m}{\sqrt{6n}}$. If, in fact,

$$\frac{\sqrt{6n}}{\pi}\left(\frac{1}{4}+\varepsilon\right)\log n < \ell, m < \frac{\sqrt{6n}\,\log n}{\pi}$$

for some $\varepsilon > 0$, then the distributions defined by ℓ and m are independent and equal, and Szekeres' formula simplifies to

$$N_n(\ell, m) \sim N_n\, \exp\left[-(\lambda+\mu) - \sqrt{\frac{6n}{\pi}}\left(e^{-\lambda}+e^{-\mu}\right)\right].$$

The Szekeres circle method has more recently been applied to the study of partitions by Richmond [Ric18].

Unimodality

Sylvester's proof of unimodality of $N_n(\ell, m)$ in n [Syl78], and most subsequent proofs [Sta84, Sta85, Pro82], are algebraic, viewing $N_n(\ell, m)$ as dimensions of certain vector spaces, or their differences as multiplicities of representations. While there are also purely combinatorial proofs of unimodality, notably O'Hara's [O'H90] and the more abstract one in [PR86], they do not give the desired symmetric chain decomposition of the subposet of the partition lattice. These methods do not give ways of estimating the asymptotic size of the coefficients or their difference. It is now known that $N_n(\ell, m)$ is strictly unimodal [PP13], and the following lower bound on the consecutive difference was obtained in [PP17, Theorem 1.2] using a connection between integer partitions and Kronecker coefficients:

$$N_n(\ell, m) - N_{n-1}(\ell, m) \geq 0.004\frac{2^{\sqrt{s}}}{s^{9/4}}, \tag{7}$$

where $n \leq \ell m/2$ and $s = \min\{2n, \ell^2, m^2\}$. In particular, when $\ell = m$ we have $s = 2n$.

Any sharp asymptotics of the difference appears to be out of reach of these algebraic methods, however as a consequence of Theorem 1 we obtain the following estimate.

Theorem 2. *Given m, ℓ and n, let $A := \ell/m$ and $B := n/m^2$ and define d as above. Suppose m, ℓ and n go to infinity so that (A, B) remains in a compact subset of $\{(x, y) : x \geq 2y > 0\}$ and*

$$m^{-1}\,|n - lm/2| \to \infty.$$

Then

$$N_{n+1}(\ell, m) - N_n(\ell, m) \sim \frac{d}{m} N_n(\ell, m).$$

Remark. The condition $m^{-1}\,|n - lm/2| \to \infty$ is equivalent to $m\,|A - B/2| \to \infty$ and also to $d \notin O(m^{-1})$. It is automatically satisfied whenever K is a compact subset of $\{(x, y) : x > 2y > 0\}$.

2 A discretized analogue to Theorem 1

Large deviations heuristic

Although we do not need to invoke any results from the theory of large deviations, it might be helpful to know the LD origins of the probability model by which the proof of Theorem 1 is reduced to a local central limit theorem. The proof of Takács' result may be summarized as follows. Let $\{\lambda_j : 1 \leq j \leq m\}$ denote the parts, in decreasing size, of a partition of n into at most m parts of size at most ℓ, padded with zeros at the end if necessary. Defining $\lambda_0 := \ell$ and $\lambda_{m+1} := 0$, the numbers $x_j := \lambda_j - \lambda_{j+1}$ satisfy the following two identities (see Figure 2),

$$\sum_{j=0}^{m} x_j = \ell \, ; \qquad \sum_{j=0}^{m} j x_j = n \, . \tag{8}$$

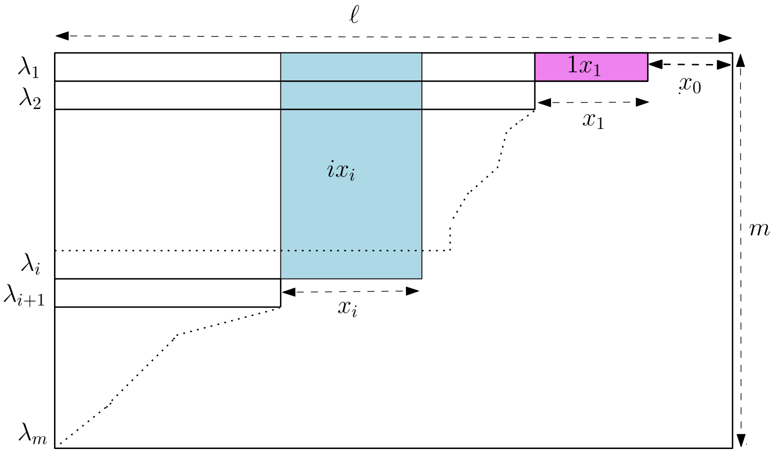


Figure 2: The total area n of a partition is composed of rectangles of area jx_j

By the *reduced geometric distribution with parameter* p we mean one less than a geometric with mean $1/p$; that is, X has this distribution if $\mathbb{P}(X = k) = p \cdot q^k$ where $q := 1 - p$. Let $\{X_j : 0 \leq j \leq m\}$ be a collection of independent reduced geometric random variables with parameter $p = 1/2$. This distribution has the crucial property that for any set of values $x_0, \ldots, x_m$, the probability $\mathbb{P}(X_j = x_j : j = 0, \ldots, m)$ depends only on the sum $\ell := \sum_{j=1}^m x_j$ and is equal to $p^{m+1} q^\ell$. Let $P(\ell)$ denote the sum of $\mathbb{P}(\mathbf{X} = \mathbf{x})$ over all $(m+1)$-vectors $\mathbf{x}$ with coordinate sum ℓ. If $N(\ell)$ denotes the number of such vectors summing to ℓ, we see immediately that $1 \geq P(\ell) = p^{m+1} q^\ell N(\ell)$, hence $N(\ell) \leq [p^{m+1} q^\ell]^{-1}$. In fact, this inequality is good because $P(\ell)$ is not all that small: it is of order $m^{-1/2}$. Now let $N(\ell, n)$ count those vectors satisfying both identities in (8) and $P(\ell, n)$ be the probability that the geometric variables lie in this set. Takács gave a sharp asymptotic estimate of $P(\ell, n) \sim cm^{-2}$ when $n = m^2/2 + O(m^{3/2})$, thereby showing that $N(\ell, n) \sim cm^{-2} [p^{m+1} q^\ell]^{-1}$.

Central limit theorems do not provide a sharp estimate when $|n - m^2/2| \gg m^{3/2}$. However, because the constraints on the vectors counted by $N(\ell, n)$ are linear, the theory of large deviations [DZ98] implies that $P(\ell, n)$ is well estimated by "tilting" the independent laws of the $\{X_j\}$ so that one is in the central limit regime of the tilted laws. Having solved for the correct tilt, we may dispense with the LD theory and prove the estimates directly. Tilting preserves the reduced geometric family, altering only the parameters; it turns out that the correct tilt makes $q_j := 1 - p_j$ log-linear.

Statement of discretized result

With c_m, d_m to be specified later, let $q_j := e^{-c_m - jd_m/m}$, let $p_j := 1 - q_j$ and let

$$L_m := \sum_{j=0}^{m} \log p_j .$$

Let $\mathbb{P}_m$ be a probability law making the random variables $\{X_j : 0 \le j \le m\}$ independent reduced geometrics with respective parameters p_j. Define random variables S_m and T_m by

$$S_m := \sum_{i=0}^{m} X_i \,; \qquad T_m := \sum_{i=1}^{m} iX_i \,, \tag{9}$$

corresponding to the unique partition λ satisfying $X_j = \lambda_j - \lambda_{j+1}$. We first prove a result similar to Theorem 1, except that the parameters c and d that solve integral Equations (2) and (3) are replaced by c_m and d_m satisfying the discrete summation Equations (10) and (11) below. These equations say that $\mathbb{E}S_m = \ell$ and $\mathbb{E}T_m = m$. Writing this out, using $\mathbb{E}X_j = 1/p_j - 1 = 1/\left(1 - e^{-c_m - d_m j/m}\right) - 1$, gives

$$\ell = \sum_{j=0}^{m} \frac{1}{1 - e^{-c_m - d_m j/m}} - (m+1) \tag{10}$$

$$n = m \sum_{j=0}^{m} \frac{j/m}{1 - e^{-c_m - d_m j/m}} - \frac{m(m+1)}{2} \,. \tag{11}$$

Let M_m denote the covariance matrix for (S_m, T_m). The entries may be computed from the basic identity $\mathrm{Var}\,(X_j) = q_j/p_j^2$, resulting in

$$\mathrm{Var}\,(S_m) = \sum_{j=0}^{m} \frac{e^{-c_m - d_m j/m}}{\left(1 - e^{-c_m - d_m j/m}\right)^2} \tag{12}$$

$$\mathrm{Cov}\,(S_m, T_m) = \sum_{j=0}^{m} j \frac{e^{-c_m - d_m j/m}}{\left(1 - e^{-c_m - d_m j/m}\right)^2} \tag{13}$$

$$\mathrm{Var}\,(T_m) = \sum_{j=0}^{m} j^2 \frac{e^{-c_m - d_m j/m}}{\left(1 - e^{-c_m - d_m j/m}\right)^2} \,. \tag{14}$$

Theorem 3 (discretized analogue)**.** *Let c_m and d_m satisfy (10) – (11). Define α_m, β_m and γ_n to be the normalized entries of the covariance matrix*

$$\alpha_m := m^{-1}\mathrm{Var}\,(S_m) \,; \qquad \beta_m := m^{-2}\mathrm{Cov}\,(S_m, T_m) \,; \qquad \gamma_m := m^{-3}\mathrm{Var}\,(T_m) \,,$$

which are $O(1)$ as $m \to \infty$. Again, let $A := \ell/m$ and $B := n/m^2$ and $\Delta_m := \alpha_m\gamma_m - \beta_m^2$. Then

$$N_n(\ell, m) \sim \frac{1}{2\pi m^2 \sqrt{\Delta_m}} \exp\left\{ m \left(-\frac{L_m}{m} + c_m A + d_m B \right) \right\} \,. \tag{15}$$

Proof of discretized result

The atomic probabilities $\mathbb{P}_m(\mathbf{X} = \mathbf{x})$ depend only on S_m and T_m as

$$\log \mathbb{P}_m(\mathbf{X} = \mathbf{x}) = \sum_{j=0}^{m} (\log p_j + x_j \log q_j)$$

$$= L_m - \sum_{j=0}^{m} \left(c_m + j \frac{d_m}{m} \right) x_j$$

$$= L_m - c_m \left(\sum_{j=0}^{m} x_j \right) - \frac{d_m}{m} \left(\sum_{j=0}^{m} j x_j \right) .$$

In particular, for any $\mathbf{x}$ satisfying (8),

$$\log \mathbb{P}(\mathbf{X} = \mathbf{x}) = L_m - c_m \ell - \frac{d_m}{m} n \,. \tag{16}$$

These three things are equivalent: (i) the vector $\mathbf{X}$ satisfies the identities (8); (ii) the pair (S_m, T_m) is equal to (ℓ, n); (iii) the partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ defined by $\lambda_j - \lambda_{j+1} = X_j$ for $2 \leq j \leq m-1$, together with $\lambda_1 = \ell - X_0$ and $\lambda_m = X_m$, is a partition of n fitting inside a $m \times \ell$ rectangle. Thus,

$$\begin{aligned} N_n(\ell, m) &= \mathbb{P}_m \left[(S_m, T_m) = (\ell, n) \right] \exp \left(-L_m + c_m \ell + \frac{d_m}{m} n \right) \\ &= \mathbb{P}_m \left[(S_m, T_m) = (\ell, n) \right] \exp \left[m \left(-\frac{L_m}{m} + c_m A + d_m B \right) \right] . \end{aligned} \tag{17}$$

Comparing (15) to (17), the proof is completed by an application of the following LCLT, for which a proof is given in the Appendix. This result is stated for an arbitrary sequence of parameters $p_0, \ldots, p_m$ bounded away from 0 and 1, though we need it only for $p_j = 1 - e^{-c_m - d_m j/m}$. For a 2×2 matrix M, denote by $M(s,t) := [s\,,\ t]\, M\, [s\,,\ t]^T$ the corresponding quadratic form.

Lemma 4 (LCLT)**.** *Fix $0 < \delta < 1$ and let $p_0, \ldots, p_m$ be any real numbers in the interval $[\delta, 1-\delta]$. Let $\{X_j\}$ be independent reduced geometrics with respective parameters $\{p_j\}$, $S_m := \sum_{j=0}^m X_j$, and $T_m := \sum_{j=0}^m j X_j$. Let M_m be the covariance matrix for (S_m, T_m), written*

$$M_m = \begin{pmatrix} \alpha_m m & \beta_m m^2 \\ \beta_m m^2 & \gamma_m m^3 \end{pmatrix} ,$$

Q_m denote the inverse matrix to M_m, and $\Delta_m = m^{-4} \det M_m = \alpha_m \gamma_m - \beta_m^2$. Let μ_m and ν_m denote the respective means $\mathbb{E} S_m$ and $\mathbb{E} T_m$. Denote $p_m(a,b) := \mathbb{P}((S_m, T_m) = (a,b))$. Then

$$\sup_{a,b \in \mathbb{Z}} m^2 \left| p_m(a,b) - \frac{1}{2\pi (\det M_m)^{1/2}} e^{-\frac{1}{2} Q_m (a - \mu_m, b - \nu_m)} \right| \to 0 \tag{18}$$

as $m \to \infty$, uniformly in the parameters $\{p_j\}$ in the allowed range. In particular, if the sequence (a_m, b_m) satisfies $Q_m(a_m - \mu_m, b_m - \nu_m) \to 0$ then

$$\mathbb{P}(S_m = a_m, T_m = b_m) = \frac{1}{2\pi \sqrt{\Delta_m} m^2} \left(1 + O\left(m^{-3/2} \right) \right) .$$

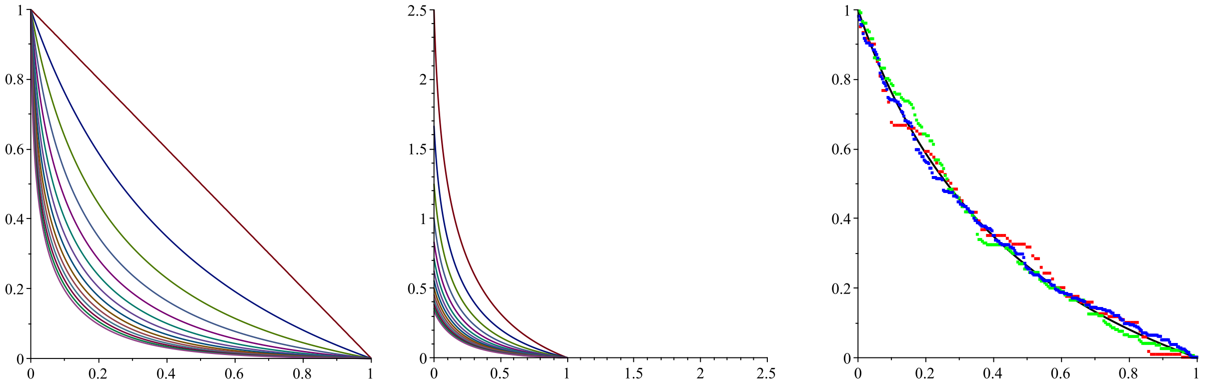
The following consequence will be used to prove Theorem 2.

Corollary 5 (LCLT consecutive differences)**.** *Let $\mathcal{N}(a,b) := \frac{1}{2\pi (\det M)^{1/2}} e^{-\frac{1}{2} Q(a-\mu, b-\nu)}$ be the normal approximation in Equation (18). Using the notation of Lemma 4,*

$$\sup_{a,b \in \mathbb{Z}} \left| p(a, b+1) - p(a,b) - \big(\mathcal{N}(a, b+1) - \mathcal{N}(a,b) \big) \right| = O(m^{-4}).$$

3 Limit shape

Using the independent random variables X_i we can derive the limit shape for the partitions of n inside a rectangle; i.e., the curve which approximates most Young diagrams of $\lambda \vdash n$. The existence of the limit curve follows from the easy fact that the maximum discrepancy $\max_{j\le m}\left|\sum_{i=0}^{j}(X_i - 1/p_i)\right|$, conditional on $\sum_{i=0}^{m} X_i = \ell$, is $o(m)$ in probability. We now discuss how to compute the limit shape function $x \mapsto m^{-1}\mathbb{E}\lambda_{\lfloor mx\rfloor}$.



$(A, B) = (1, 1/k)$
$k = 2, \dots, 15$

$(A, B) = (5/k, 1/k)$
$k = 2, \dots, 15$

Limit curve of $(A, B) = (1, 1/3)$ and random partitions of size 120, 201 and 300.

Figure 3: Limit shapes of scaled partitions as $m \to \infty$.

Let λ be a partition defined by $X_0, X_1, \dots$ in our setup. Then $\lambda_i = \ell - (X_0 + X_1 + \cdots + X_{i-1})$ and hence $\mathbb{E}[\lambda_i] = \ell - \sum_{j=0}^{i-1}(1/p_j - 1)$. Using the integral approximation, setting $i := xm$ we have that

$$y := \mathbb{E}\left[\frac{\lambda_i}{m}\right] = A + x - \int_0^x \frac{1}{1 - e^{-c-dt}}\,dt = A + x - \frac{1}{d}\ln\left(\frac{e^{xd+c} - 1}{e^c - 1}\right),$$

and the limit curve of partitions scaled to the $1 \times A$ rectangle is given by points (x, y) satisfying this equation. It can be rewritten, after expressing c in terms of d, as

$$\begin{aligned} e^{(A+1)d} - 1 &= (e^d - 1)e^{d(A-y)} + (e^{Ad} - 1)e^{d(1-x)} \\ \iff \qquad 1 &= (1 - e^{-c})e^{d(A-y)} + e^{-c}e^{-dx}\,, \end{aligned} \tag{19}$$

when $A > 2B$, and

$$y = A(1 - x)$$

when $A = 2B$ and $d = 0$; see Figure 3 for some examples of limit curves. Note that the transformation $(x, y) \mapsto (1 - x, A - y)$ gives the shape of the complementary partition inside the $1 \times A$ rectangle.

4 Existence and Uniqueness of c, d

We now show that for any $A \ge 2B > 0$ there exists unique positive constants c and d satisfying Equations (2) and (3). If $A = B/2$ then $d = 0$ and c can be determined uniquely, so we may assume $A > 2B > 0$. The following lemma will be used to show uniqueness.

Lemma 6. *Let ψ denote the map taking the pair (c, d) to (A, B) defined by the two integrals in Equations (2) and (3), and let K be a compact subset of $\{(x, y) : x > 2y > 0\}$. The Jacobian matrix $J := D[\psi]$ is negative definite for all $(c, d) \in (0, \infty)^2$, and all entries of ψ and J (respectively ψ^{-1} and J^{-1}) are Lipshitz continuous on $\psi^{-1}[K]$ (respectively K).*

Proof. Differentiating under the integral sign shows that the partial derivatives comprising the entries of $D[\psi]$ are given by

$$\begin{aligned} J_{A,c} &= \int_0^1 \frac{-e^{-(c+dt)}}{(1-e^{-(c+dt)})^2}\,dt \\ J_{A,d} &= \int_0^1 \frac{-t\,e^{-(c+dt)}}{(1-e^{-(c+dt)})^2}\,dt \\ J_{B,c} &= \int_0^1 \frac{-t\,e^{-(c+dt)}}{(1-e^{-(c+dt)})^2}\,dt \\ J_{B,d} &= \int_0^1 \frac{-t^2\,e^{-(c+dt)}}{(1-e^{-(c+dt)})^2}\,dt\,; \end{aligned}$$

note that each term is negative. Let ρ denote the finite measure on $[0, 1]$ with density $e^{-(c+dt)}/(1-e^{-(c+dt)})^2$ and let $\mathbb{E}_\rho$ denote expectation with respect to ρ. Then

$$J_{A,c} = \mathbb{E}_\rho[-1], \qquad J_{A,d} = J_{B,c} = \mathbb{E}_\rho[-t], \qquad J_{B,d} = \mathbb{E}_\rho[-t^2],$$

and

$$\det J = \mathbb{E}_\rho[1] \cdot \mathbb{E}_\rho[t^2] - (\mathbb{E}_\rho[t])^2 = \mathbb{E}_\rho[1]^2 \cdot \mathrm{Var}_\sigma[t],$$

where $\mathrm{Var}_\sigma[t]$ denotes the variance of t with respect to the normalized measure $\sigma = \rho/\mathbb{E}_\rho[1]$. In particular, $\det J$ is positive, and bounded above and below when c and d are bounded away from 0, implying the stated results on Lipshitz continuity. As J is real and symmetric, it has real eigenvalues. Since the trace of J is negative while its determinant is positive, the eigenvalues of J have negative sum and positive product, meaning both are strictly negative and J is negative definite for any $c, d > 0$. □

Lemma 7. *For any $A > 0$ and $B \in (0, A/2)$ there exist unique $c, d > 0$ satisfying Equations (2) and (3). Moreover, for a fixed A, when B decreases from $A/2$ to 0, d increases strictly from 0 to ∞ and c decreases strictly from $\log\left(\frac{A+1}{A}\right)$ to 1.*

Proof. Solving Equation (2) for c (assuming $d \geq 0$) gives

$$c = \log\left(\frac{e^{(A+1)d} - 1}{e^{(A+1)d} - e^d}\right).$$

Substituting this into Equation (3) gives an explicit expression for B in terms of A and d, and shows that for fixed $A > 0$ as d goes from 0 to infinity B goes from $A/2$ to 0. By continuity, this implies the existence of the desired c and d. It also shows that, for a fixed A, c is a decreasing function of d with the given maximal and minimal values as d goes from 0 to ∞.

To prove uniqueness, we note that for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ Stokes' theorem implies

$$\psi(\mathbf{y}) - \psi(\mathbf{x}) = \int_0^1 D[\psi]\,(t\mathbf{x} + (1-t)\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})\,dt$$

so that

$$(\mathbf{x} - \mathbf{y})^T \cdot (\psi(\mathbf{y}) - \psi(\mathbf{x})) = \int_0^1 \left[(\mathbf{x} - \mathbf{y})^T \cdot D[\psi]\,(t\mathbf{x} + (1-t)\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})\right] dt.$$

When $\mathbf{x} \neq \mathbf{y}$, negative-definiteness of $D[\psi]$ implies that the last integrand is strictly negative on $[0,1]$, and $\psi(\mathbf{y}) \neq \psi(\mathbf{x})$. Thus, distinct values of c and d give distinct values of A and B.

To see the monotonicity, let A be fixed and let $F_B(d) = B$ be the equation obtained after substituting $c = c(A,d)$ above in equation (3), i.e. $F_B(d) = \psi_2(c(A,d),d)$. Then d is a decreasing function of B and vice versa since

$$\frac{\partial F_B(d)}{\partial d} = \frac{J_{B,d}J_{A,c} - J_{A,d}J_{B,c}}{J_{A,c}} = \frac{\det D[\psi]}{J_{A,c}} < 0\,.$$

□

5 Proof of Theorem 1 from the discretized result

Here we show how c_m and d_m from the discretized result are related to c, d defined independently of m. The proof below also shows that c_m and d_m exist and are unique.

The Euler-MacLaurin summation formula [dB58, Section 3.6] gives an expansion

$$\begin{aligned}\frac{L_m}{m} &= \int_0^1 \log(1 - e^{-c_m - d_m t})\,dt + \frac{\log(1-e^{-c_m}) + \log(1-e^{-c_m-d_m})}{2m} + O(m^{-2}) \\ &= \frac{\operatorname{dilog}(1-e^{-c_m-d_m}) - \operatorname{dilog}(1-e^{-c_m})}{d_m} + \frac{\log(1-e^{-c_m}) + \log(1-e^{-c_m-d_m})}{2m} + O(m^{-2}) \qquad (20)\end{aligned}$$

of the sum L_m in terms of c_m and d_m. Assume that there is an asymptotic expansion

$$c_m = c + um^{-1} + O(m^{-2}) \qquad (21)$$
$$d_m = d + vm^{-1} + O(m^{-2}) \qquad (22)$$

as $m \to \infty$, where u and v are constants depending only on A and B. Under such an assumption, substitution of Equations (21) and (22) into Equation (20) implies

$$\begin{aligned}\frac{L_m}{m} &= \frac{\operatorname{dilog}(1-e^{-c-d}) - \operatorname{dilog}(1-e^{-c})}{d} + \frac{uA + vB}{m} + O(m^{-2}) \\ &= \log(1-e^{-c-d}) - dB + \frac{uA+vB}{m} + O(m^{-2}). \qquad (23)\end{aligned}$$

Substituting Equations (21)–(23) into Equation (15) of Theorem 3 and taking the limit as $m \to \infty$ then gives Theorem 1, as

$$\Delta_m \to \left(\int_0^1 \frac{e^{-c-dt}}{(1-e^{-c-dt})^2}\,dt\right)\left(\int_0^1 \frac{t^2 e^{-c-dt}}{(1-e^{-c-dt})^2}\,dt\right) - \left(\int_0^1 \frac{te^{-c-dt}}{(1-e^{-c-dt})^2}\,dt\right)^2 = \Delta.$$

It remains to show the expansions in Equations (21) and (22). For $x, y > 0$, define

$$\overline{S}_m(x,y) := \frac{1}{m}\sum_{j=0}^{m} \frac{1}{1-e^{-(x+yj/m)}} - 1\,,$$

$$\overline{T}_m(x,y) := \frac{1}{m}\sum_{j=0}^{m} \frac{j/m}{1-e^{-(x+yj/m)}} - \frac{1}{2}\,.$$

Another application of the Euler-MacLaurin summation formula implies

$$\overline{S}_m(c,d) = A + A_1(c,d)m^{-1} + O(m^{-2})\,, \tag{24}$$

$$\overline{T}_m(c,d) = B + B_1(c,d)m^{-1} + O(m^{-2})\,, \tag{25}$$

with

$$A_1 = \frac{1}{2}\left(\frac{1}{1-e^{-c}} + \frac{1}{1-e^{-c-d}}\right) \qquad \text{and} \qquad B_1 = \frac{1}{2(1-e^{-c-d})}.$$

Let $\mathcal{J}$ denote the Jacobian $D[\psi]$ of the map ψ with respect to c and d, and let

$$(c'_m, d'_m) = (c,d) - m^{-1}\mathcal{J}^{-1}\cdot(A_1 - 1, B_1 - 1/2)^T.$$

A Taylor expansion around the point (c,d) gives

$$\begin{aligned}\begin{pmatrix}\overline{S}_m(c'_m,d'_m)\\ \overline{T}_m(c'_m,d'_m)\end{pmatrix} &= \begin{pmatrix}\overline{S}_m(c,d)\\ \overline{T}_m(c,d)\end{pmatrix} - \left(\mathcal{J} + O\left(m^{-1}\right)\right)\cdot\left(m^{-1}\mathcal{J}^{-1}\begin{pmatrix}A_1\\ B_1\end{pmatrix}\right) + O(m^{-2})\\ &= \begin{pmatrix}A - 1/m\\ B - 1/2m\end{pmatrix} + O\left(m^{-2}\right)\\ &= \begin{pmatrix}\overline{S}_m(c_m,d_m)\\ \overline{T}_m(c_m,d_m)\end{pmatrix} + O\left(m^{-2}\right),\end{aligned}$$

where Equations (24) and (25) were used to approximate the Jacobian of $\psi_m : (x,y) \mapsto \left(\overline{S}_m(x,y), \overline{T}_m(x,y)\right)$ with respect to x and y.

The map ψ_m is Lipschitz for a similar reason as its continuous analogue. Namely, consider the partial derivatives

$$\begin{aligned}J_{S,x} &= \frac{1}{m}\sum_{j=0}^{m} -\frac{e^{-x-yj/m}}{(1-e^{-x-yj/m})^2}\\ J_{S,y} &= \frac{1}{m^2}\sum_{j=0}^{m} -\frac{j\ e^{-x-yj/m}}{(1-e^{-x-yj/m})^2}\\ J_{T,x} &= \frac{1}{m^2}\sum_{j=0}^{m} -\frac{j\ e^{-x-yj/m}}{(1-e^{-x-yj/m})^2}\\ J_{S,y} &= \frac{1}{m^3}\sum_{j=0}^{m} -\frac{j^2\ e^{-x-yj/m}}{(1-e^{-x-yj/m})^2}.\end{aligned}$$

Let ρ_m be a discrete finite measure on $R_m := \{0, 1/m, 2/m, \ldots, 1\}$ with density $e^{-x-yt}/(1-e^{-x-yt})$ for $t \in R_m$ and 0 otherwise, and let $\mathbb{E}_{\rho_m}$ be the expectation with respect to ρ_m. Then

$$J_{S,x} = \mathbb{E}_{\rho_m}[-1]\,, \qquad J_{T,x} = J_{S,y} = \mathbb{E}_{\rho_m}[-t]\,, \qquad J_{T,y} = \mathbb{E}_{\rho_m}[-t^2]$$

and

$$\det D[\psi_m] = \mathbb{E}_{\rho_m}[1]\mathbb{E}_{\rho_m}[t^2] - \mathbb{E}_{\rho_m}[t]^2 = \mathbb{E}_{\rho_m}[1]^2\mathrm{Var}_{\,\sigma_m}[t]\,,$$

where σ_m is the probability function $\rho_m/\mathbb{E}_{\rho_m}[1]$. For any fixed m and (x,y) in a compact neighborhood of (A,B), both the variance and the expectation are finite and bounded away from 0, as is the Jacobian determinant. Moreover, the trace $\mathrm{Tr}D[\psi] = -\mathbb{E}_{\rho_m}[1+t^2]$ is bounded away from 0 and infinity, so the Jacobian is negative definite with locally bounded eigenvalues, and hence ψ_m is locally Lipschitz. Since the norm of the Jacobian is bounded away from 0 and infinity, we have that the inverse map ψ_m^{-1} is also locally Lipschitz in a neighborhood of $\psi^{-1}(A,B)$. Moreover, similarly to proof of existence and uniqueness of c and

d in Section 4, we have that there indeed are c_m and d_m as unique solutions of Equations (10) and (11) since the Jacobian is negative semi-definite.

The trapezoid formula implies $|J_{S,c} - J_{A,c}| = O(m^{-1})$, and similar bounds for the other differences of partial derivatives in the continuous and discrete settings. Hence, the bounds for the norms and eigenvalues of $D[\psi_m]$ are within $O(m^{-1})$ of the ones for $D[\psi]$, and ψ_m (and its inverse) is Lipschitz with a constant independent of m. Thus,

$$O(m^{-2}) = \|\psi_m(c'_m, d'_m) - \psi_m(c_m, d_m)\| \geq C^{-1}\|(c'_m - c_m, d'_m - d_m)\|$$

for some constant C, so that the expansions (21) and (22) hold. □

6 Proof of Theorem 2

We will prove Theorem 2 from Equation (17) and Corollary 5. Let $p_m(\ell, n) = \mathbb{P}_m\left[(S_m, T_m) = (\ell, n)\right]$ and let

$$L_m(x,y) := \sum_{j=0}^{m} \log(1 - e^{-x-yj/m}) \ , \tag{26}$$

$$A_m(x,y) := \sum_{j=0}^{m} \frac{1}{1 - e^{-x-yj/m}} - (m+1) \ , \tag{27}$$

$$B_m(x,y) := \sum_{j=0}^{m} \frac{j/m}{1 - e^{-x-yj/m}} - \frac{m+1}{2} \ . \tag{28}$$

Then c_m and d_m are the solutions to

$$A_m(c_m, d_m) = \ell = Am \, , \qquad B_m(c_m, d_m) = n/m = Bm \, .$$

Let c'_m, d'_m be the solutions to $A_m(c'_m, d'_m) = \ell$ and $B_m(c'_m, d'_m) = (n+1)/m$, and let $\Delta x = c'_m - c_m = O(m^{-2})$ and $\Delta y = d'_m - d_m = O(m^{-2})$ by the Lipschitz properties proven in Section 4. Observe that

$$\frac{\partial L_m(x,y)}{\partial x} = A_m(x,y) \quad \text{and} \quad \frac{\partial L_m(x,y)}{\partial y} = B_m(x,y). \tag{29}$$

Using the Taylor expansion for $L_m(c'_m, d'_m)$ around (c_m, d_m) and the L_m partial derivatives from Equation (29),

$$-L_m(c'_m, d'_m) = -L_m(c_m + \Delta x, d_m + \Delta y) = -L_m(c_m, d_m) - \Delta x \, A_m(c_m, d_m) - \Delta y \, B_m(c_m, d_y) + O(m^{-3}),$$

so that

$$-L_m(c'_m, d'_m) + (c_m + \Delta x)\ell + (d_m + \Delta y)(n+1)\, m^{-1} = -L_m(c_m, d_m) + c_m\ell + d_m(n+1)\, m^{-1} + O(m^{-3}).$$

To lighten notation, we now write $L_m := L_m(c_m, d_m)$ and $L'_m := L_m(c'_m, d'_m)$. Then

$$N_{n+1}(\ell, m) - N_n(\ell, m) = p_m(\ell, n+1) \exp\left[-L'_m + c'_m\ell + \frac{d'_m}{m}(n+1)\right] - p_m(\ell, n) \exp\left[-L_m + c_m\ell + \frac{d_m}{m}n\right]$$

$$= p_m(\ell, n) \exp\left[-L_m + c_m\ell + \frac{d_m}{m}n\right]\left[e^{d_m/m} - 1\right] \tag{30}$$

$$+ \left[p_m(\ell, n+1) - p_m(\ell, n)\right] \exp\left[-L_m + c_m\ell + \frac{d_m}{m}(n+1)\right] \tag{31}$$

$$+\; p_m(\ell, n+1)\left(e^{-L'_m + c'_m \ell + d'_m (n+1)/m} - e^{-L_m + c_m \ell + d_m (n+1)/m}\right). \tag{32}$$

We now bound each of these summands.

- Since $d_m = d + O(m^{-1})$, Equation (17) implies that the quantity on line (30) equals

  $$N_n(\ell, m)\left(\frac{d}{m} + O(m^{-2})\right)$$

  as long as $d \notin O(m^{-1})$. This holds when $|A - B/2| \notin O(m^{-1})$ as $d = 0$ when $A = B/2$ and the map taking (A, B) to (c, d) is Lipschitz.
- By Corollary 5,

  $$\begin{aligned}
  [p_m(\ell, n+1) - p_m(\ell, n)] &\le |\mathcal{N}(\ell, n+1) - \mathcal{N}(\ell, n)| + O(m^{-4}) \\
  &= O\left(m^{-2} \cdot \left|1 - e^{\frac{1}{2} Q_m(0,1)}\right|\right) + O(m^{-4}) \\
  &= O(m^{-4}),
  \end{aligned}$$

  where Q_m is the inverse of the covariance matrix of (S_m, T_m). Thus, the quantity on line (31) is $O(m^{-4} \cdot m^2 N_n(\ell, m)) = O(m^{-2} N_n(\ell, m))$.
- Let

  $$\psi_m := \exp\left[-L'_m + c'_m \ell + d'_m (n+1)\, m^{-1} - \left(-L_m + c_m \ell + d_m (n+1)\, m^{-1}\right)\right] - 1 = O(m^{-3}).$$

  As $p_m(\ell, n+1) = p_m(\ell, n) + O(m^{-4})$, it follows that the quantity on line (32) is

  $$\begin{aligned}
  p_m(\ell, n+1)\, e^{-L_m + c_m \ell + d_m (n+1)/m}\, \psi_m &= N_n(\ell, m)\, \psi_m\, e^{d_m/m} + O(m^{-4}\, e^{d_m/m}\, e^{-L_m + c_m \ell + d_m n/m}\, \psi_m) \\
  &= O(m^{-3} N_n(\ell, m)).
  \end{aligned}$$

Putting everything together,

$$N_{n+1}(\ell, m) - N_n(\ell, m) = N_n(\ell, m)\left(\frac{d}{m} + O(m^{-2})\right),$$

as desired. □

Appendix: Proof of the Local Central Limit Theorem

Throughout this section, $1/2 \ge \delta > 0$ is fixed and $\{p_j : 0 \le j \le m\}$ are arbitrary numbers in $[\delta, 1-\delta]$. The variables $\{X_j\}$ and (S_m, T_m) are as in Lemma 4; we drop the index m on the remaining quantities $\alpha_m, \beta_m, \gamma_m, \Delta_m, \mu_m, \nu_m, p_m(a, b)$ and the matrices M_m and Q_m. Recall the quadratic form notation $M(s, t) := [s\ ,\ t]\, M\, [s\ ,\ t]^T$.

Lemma 8. *The constants α, β, γ and Δ are bounded below and above by positive constants depending only on δ.*

Proof: Upper and lower bounds on α, β and γ are elementary: $\alpha \in \left[\frac{\delta}{(1-\delta)^2}, \frac{(1-\delta)}{\delta^2}\right]$, $\beta \in \left[\frac{\delta}{2(1-\delta)^2}, \frac{(1-\delta)}{2\delta^2}\right]$ and $\gamma \in \left[\frac{\delta}{3(1-\delta)^2}, \frac{(1-\delta)}{3\delta^2}\right]$. The upper bound on Δ follows from these.

For the lower bound on Δ, let $\tilde{M} = \begin{pmatrix} \alpha_n & \beta_n \\ \beta_n & \gamma_n \end{pmatrix}$ denote M without the factors of m. We show Δ is bounded from below by the positive constant $(4-\sqrt{13})\delta/6$. A lower bound for the determinant Δ of $\tilde{M}$ is $|\lambda|^2$ where λ is the least modulus eigenvalue of $\tilde{M}$; note that $|\lambda|^2 = \inf_\theta \tilde{M}(\cos\theta, \sin\theta)$. We compute

$$\begin{aligned} \tilde{M}(\cos\theta,\sin\theta) &= m^{-1}\mathbb{E}\left(\cos\theta S + m^{-1}\sin\theta T\right)^2 \\ &\geq \delta m^{-1}\sum_{k=0}^{m}\left(\cos\theta + \frac{k}{m}\sin\theta\right)^2 \\ &> \delta\cdot\left(\cos^2\theta + \cos\theta\sin\theta + \frac{1}{3}\sin^2\theta\right). \end{aligned}$$

This is at least $\dfrac{4-\sqrt{13}}{6}\delta$ for all θ, proving the lemma. ☐

Lemma 9. *Let X_p denote a reduced geometric with parameter p. For every $\delta \in (0,1/2)$ there is a K such that simultaneously for all $p \in [\delta, 1-\delta]$,*

$$\left|\log\mathbb{E}\exp(i\lambda X_p) - \left(i\frac{q}{p}\lambda - \frac{q^2}{2p^2}\lambda^2\right)\right| \leq K\lambda^3.$$

Proof: For fixed p this is Taylor's remainder theorem together with the fact that the characteristic function $\phi_p(\lambda)$ of X_p is thrice differentiable. The constant $K(p)$ one obtains this way is continuous in p on the interval $(0,1)$, therefore bounded on any compact sub-interval. ☐

Proof of the LCLT: The proof of Lemma 4 comes from expressing the probability as an integral of the characteristic function, via the inversion formula, and then estimating the integrand in various regions.

Let $\phi(s,t) := \mathbb{E}e^{i(sS+tT)}$ denote the characteristic function of (S,T). Centering the variables at their means, denote $\widehat{S} := S - \mu$, $\widehat{T} := T - \nu$, and $\widehat{\phi}(s,t) := \mathbb{E}e^{i(s\widehat{S}+t\widehat{T})}$ so that $\phi(s,t) = \widehat{\phi}(s,t)e^{is\mu+it\nu}$. Then

$$\begin{aligned} p(a,b) &= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{-isa-itb}\phi(s,t)\,ds\,dt \\ &= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{-is(a-\mu)-it(b-\mu)}\widehat{\phi}(s,t)\,ds\,dt. \end{aligned} \tag{33}$$

Following the proof of the univariate LCLT for IID variables found in [Dur10], we observe that

$$\frac{1}{2\pi(\det M)^{1/2}}e^{-\frac{1}{2}Q(a-\mu,b-\nu)} = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-is(a-u)-it(b-v)}\exp\left(-\frac{1}{2}M(s,t)\right)ds\,dt. \tag{34}$$

Hence, comparing this to (33) and observing that $e^{-is(a-\mu)-it(b-\nu)}$ has unit modulus, the absolute difference between $p(a,b)$ and the left-hand side of (34) is bounded above by

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\mathbf{1}_{(s,t)\in[-\pi,\pi]^2}\widehat{\phi}(s,t) - e^{-(1/2)M(s,t)}\right|ds\,dt. \tag{35}$$

Fix positive constants L and ε to be specified later and decompose the region $\mathcal{R} := [-\pi,\pi]^2$ as the disjoint union $\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3$, where

$$\begin{aligned} \mathcal{R}_1 &= [-Lm^{-1/2}, Lm^{-1/2}] \times [-Lm^{-3/2}, Lm^{-3/2}] \\ \mathcal{R}_2 &= [-\varepsilon,\varepsilon] \times [-\varepsilon m^{-1}, \varepsilon m^{-1}] \ \setminus\ \mathcal{R}_1 \end{aligned}$$

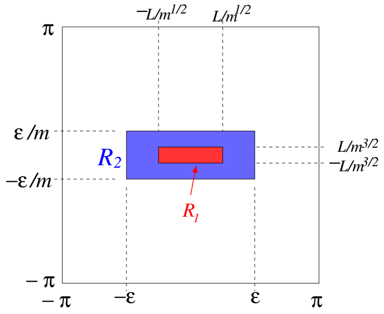


Figure 4: The regions $\mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \mathcal{R}$ in the proof of the LCLT.

$$\mathcal{R}_3 = \mathcal{R} \setminus (\mathcal{R}_1 \cup \mathcal{R}_2)\,;$$

see Figure 4 for details.

As $\int_{\mathcal{R}_2^c} e^{-(1/2)M(s,t)}\,ds\,dt$ decays exponentially with m, it suffices to obtain the following estimates

$$\int_{\mathcal{R}_1} \left|\widehat{\phi}(s,t) - e^{-(1/2)M(s,t)}\right| \,ds\,dt = O\left(m^{-5/2}\right) \tag{36}$$

$$\int_{\mathcal{R}_2} \left|\widehat{\phi}(s,t) - e^{-(1/2)M(s,t)}\right| \,ds\,dt = O(m^{-5/2}) \tag{37}$$

$$\int_{\mathcal{R}_3} \left|\widehat{\phi}(s,t)\right| \,ds\,dt = o\left(m^{-3}\right). \tag{38}$$

By independence of $\{X_j\}$,

$$\log \widehat{\phi}(s,t) = \sum_{j=0}^{m} \log \mathbb{E} e^{i(s+jt)(X_j - \mu_j)}\,.$$

Using Lemma 9 with $p = p_j$ gives

$$\left|\log \mathbb{E} e^{i(s+jt)(X_j - q_j/p_j)} + \frac{q_j}{2p_j^2}(s+jt)^2\right| \le K|s+jt|^3\,.$$

The sum of $(q_j/p_j^2)(s+jt)^2$ is $M(s,t)$, therefore summing the previous equation over j gives

$$\left|\log \widehat{\phi}(s,t) + \frac{1}{2}M(s,t)\right| \le K \sum_{j=0}^{m} |s+jt|^3\,. \tag{39}$$

On $\mathcal{R}_1$ we have the upper bound $|s+jt| \le |s| + m|t| \le 2Lm^{-1/2}$. Thus,

$$\sum_{j=0}^{m} |s+jt|^3 \le (m+1)(8L^3)m^{-3/2} = O\left(m^{-1/2}\right)\,.$$

Plugging this into (39) and exponentiating shows that the left hand side of (36) is at most $|\mathcal{R}_1| \cdot O(m^{-1/2}) = O(m^{-5/2})$.

To bound the integral on $\mathcal{R}_2$, we define the sub-regions

$$S_k := \left\{(x,y) : k \le \max\left(m^{1/2}|x|, m^{3/2}|y|\right) \le k+1\right\}.$$

As the area of S_k is $(8k+4)m^{-2}$,

$$\begin{aligned}\int_{\mathcal{R}_2} \left|\widehat{\phi}(s,t) - e^{-(1/2)M(s,t)}\right| \, ds\, dt &\le \sum_{k=L}^{\lceil \epsilon\sqrt{m}\,\rceil} \int_{S_k} \left|\widehat{\phi}(s,t) - e^{-M(s,t)/2}\right| dsdt \\ &\le m^{-2} \sum_{k=L}^{\lceil \epsilon\sqrt{m}\,\rceil} (8k+4) \max_{(s,t)\in S_k} \left|\widehat{\phi}(s,t) - e^{-M(s,t)/2}\right| . \qquad (40)\end{aligned}$$

We break this last sum into two parts, and bound each part. For $(s,t) \in \mathcal{R}_2$, we have $|s+jt| \le |s|+m|t| \le 2\varepsilon$ so that

$$\sum_{j=0}^{m} |s+jt|^3 \le 2\varepsilon \sum_{j=0}^{m} (|s|+j|t|)^2 \le (2\varepsilon\Delta^{-1})M(|s|,|t|).$$

Comparing this to (39) shows we may choose ε small enough to guarantee that

$$\left|\log\widehat{\phi}(s,t) + \frac{1}{2}M(s,t)\right| \le \frac{1}{4}M(s,t)\,,$$

so $|\widehat{\phi}(s,t)| \le e^{-(1/4)M(s,t)}$. Lemma 8 shows there is a positive constant c such that the minimum value of $M(s,t)$ on $\mathcal{R}_2$ is at least ck^2. Thus, for $(s,t) \in S_k$,

$$\left|\widehat{\phi}(s,t) - e^{-M(s,t)/2}\right| \le \left|e^{-M(s,t)/4}\right| + \left|e^{-M(s,t)/2}\right| \le 2e^{-ck^2}.$$

If $r_m := \left\lceil \sqrt{(\log m)/c} \right\rceil$ then

$$\begin{aligned}\sum_{k=r_m}^{\infty} (8k+4)(k+1) \max_{(s,t)\in S_k} \left|\widehat{\phi}(s,t) - e^{-M(s,t)/2}\right| &\le 2\sum_{k=r_m}^{\infty} (8k+4)(k+1)e^{-ck^2} \\ &= O(m^{-1}\,\mathrm{polylog}(m)) \\ &= O(m^{-1/2}). \qquad (41)\end{aligned}$$

Furthermore, for $(s,t) \in S_k$ there exist constants C and C' such that

$$\left|\log\widehat{\phi}(s,t) + M(s,t)/2\right| \le C\sum_{j=0}^{m} |s+jt|^3 \le C\left(2(k+1)m^{-1/2}\right)^3 (m+1) = C'k^3m^{-1/2}.$$

This implies the existence of a constant $K > 0$ such that for $0 \le k \le r_m$ and $(s,t) \in S_k$,

$$\begin{aligned}\left|\widehat{\phi}(s,t) - e^{-M(s,t)/2}\right| &= \left|e^{-M(s,t)/2}\right| \left|1 - e^{\log\widehat{\phi}(s,t)+M(s,t)/2}\right| \\ &\le Ke^{-ck^2}k^3m^{-1/2}.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{k=L}^{r_m} (8k+4)(k+1) \max_{(s,t)\in S_k} \left|\widehat{\phi}(s,t) - e^{-M(s,t)/2}\right| &\le Km^{-1/2}\sum_{k=L}^{r_m} (8k+4)(k+1)k^3e^{-ck^2} \\ &= O(m^{-1/2}). \qquad (42)\end{aligned}$$

Combining (40)–(42) gives (37).

Finally, for (38), we claim there is a positive constant c for which $|\widehat{\phi}(s,t)| \le e^{-cm}$ on $\mathcal{R}_3$. To see this, observe (see [Dur10, p. 144]) that for each p there is an $\eta > 0$ such that $|\phi_p(\lambda)| < 1-\eta$ on $[-\pi,\pi]\setminus[-\varepsilon/2,\varepsilon/2]$.

Again, by continuity, we may choose one such η valid for all $p \in [\delta, 1-\delta]$. It suffices to show that when either $|s|$ or $m|t|$ is at least ε, then at least $m/3$ of the summands $\log \mathbb{E} e^{i(s+jt)(X_j-\mu_j)}$ have real part at most $-\eta$. Suppose $s \geq \varepsilon$ (the argument is the same for $s \leq -\varepsilon$). Interpreting $s+jt$ modulo 2π always to lie in $[-\pi, \pi]$, the number of $j \in [0, m]$ for which $s + jt \in [-\varepsilon/2, \varepsilon/2]$ is at most twice the number for which $s + jt \in [\varepsilon/2, \varepsilon]$, hence at most twice the number for which $s + jt \notin [-\varepsilon/2, \varepsilon/2]$; thus at least $m/3$ of the $m+1$ values of $s+jt$ lie outside $[-\varepsilon/2, \varepsilon/2]$ and these have real part of $\log \mathbb{E} e^{i(s+jt)(X_j-\mu_j)} \leq -\eta$ by choice of η. Lastly, if instead one assumes $\pi \geq t \geq \varepsilon/m$, then at most half of the values of $s+jt$ modulo 2π can fall inside any interval of length $\varepsilon/2$. Choosing η such that the real part of $\log \mathbb{E} e^{i(s+jt)(X_j-\mu_j)}$ is at most $-\eta$ outside of $[-\varepsilon/4, \varepsilon/4]$ finishes the proof of (38) and the LCLT. □

Proof of Corollary 5. In order to estimate the error terms in the approximation of $p(a,b)$ we will consider the partial differences and repeat the approximation arguments above. Changing b to $b+1$ in Equations (33) and (34) implies

$$\left| p(a, b+1) - p(a,b) - \big(\mathcal{N}(a, b+1) - \mathcal{N}(a,b)\big) \right| = \int_{[-\pi,\pi]^2} \left|1 - e^{-it}\right| \left|\widehat{\phi}(s,t) - e^{-1/2M(s,t)}\right| ds dt. \tag{43}$$

For $(s,t) \in \mathcal{R}_3$, the proof of the LCLT shows that the integral in Equation (43) decays exponentially with m. As $\left|1 - e^{-it}\right| = \sqrt{2 - 2\cos(t)} \leq |t| = O(m^{-3/2})$ for $(s,t) \in \mathcal{R}_1$, the proof of the LCLT shows that the integral in Equation (43) grows as $O(m^{-3/2} \cdot m^{-5/2}) = O(m^{-4})$. Finally, since $\left|1 - e^{-it}\right| \leq |t| \leq (k+1)m^{-3/2}$ for $(s,t) \in S_k$ following the proof of the LCLT shows that

$$\begin{aligned} \int_{\mathcal{R}_2} \left|1 - e^{-it}\right| \left|\widehat{\phi}(s,t) - e^{-1/2M(s,t)}\right| ds dt &\leq m^{-7/2} \sum_{k=L}^{\lceil \epsilon\sqrt{m} \rceil} (8k+4)(k+1) \max_{(s,t)\in S_k} \left|\widehat{\phi}(s,t) - e^{-M(s,t)/2}\right| \\ &= O(m^{-4}). \end{aligned}$$

□

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