

EUCLIDEAN EMBEDDING OF THE POISSON WEIGHTED INFINITE TREE AND APPLICATION TO MOBILITY MODELS

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Abstract

Continuous time branching models are used to create random fractals in a Euclidean space, whose Hausdorff dimension is controlled by an input parameter. Finite realizations are applied in modelling the set of sites visited in models of human and animal mobility.

Keywords: branching process, random fractal, mobility model, Poisson weighted infinite tree, Ulam-Harris tree

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1. Introduction and motivation

1.1. Context: stochastic mobility models

Zoologists, social scientists, and online advertisers which use locations inferred from communication metadata to target content, are building algorithms which process data sets containing sporadic reports of animal or human mobility. Such data sets are expensive to create by scientific means, and their distribution is limited by privacy and security concerns. This leads to the need to create realistic synthetic data for algorithm testing.

After examining data sets from several sources, the authors observed that (1) when a large set of places visited was projected on the Euclidean plane, the empirical Hausdorff dimension of the set was always less than 2, and sometimes less than 1; (2) whereas traditional observation models assume reports at regular or exponentially

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spaced intervals, in real situations reports may be sporadic and bursty; (3) stochastic models based on memory-less random walks in space, as presented in [11] for example, are inadequate to represent a sequence of adaptive choices by an intelligent agent, such as a sequence of places or URLs visited. These observations are not new; see [5], [7], [12], and references cited therein for related work. Assertion (1) is supported by inspection (www.mrlc.gov/nlcd2011.php) of areas marked as *developed high intensity* in the U.S. National Land Cover Database [13], and by analysis (courtesy of Dylan Molho) of trip record data (obtained by the website fivethirtyeight.com via Freedom of Information Law request on July 20, 2015) of Uber taxi pick-ups in New York City.

1.2. A stochastic mobility simulator

The first author has built and published a stochastic mobility simulator called `FRACTALRABBIT` [9], with three tiers:

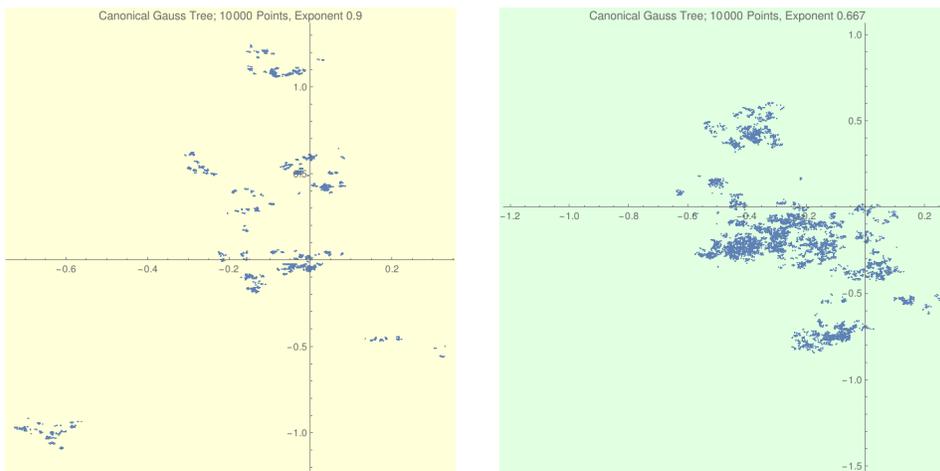
1. An *Agoraphobic Point Process* generates a set V of points in \mathbb{R}^d , whose limit is a random fractal, representing sites that could be visited.
2. A *Retro-preferential Process* generates a trajectory X through V , with strategic homing and self-reinforcing site fidelity as observed in human/animal behavior.
3. A *Sporadic Reporting Process* to model time points T at which the trajectory X is observed, with bursts of reports and heavy tailed inter-event times.

The present paper gives mathematical details about the first tier only. The second and third tiers will be described fully in later works.

1.3. High level view of the agoraphobic point process

Our aim is to analyze the metric and dimensional properties of a certain random sets of points in d -dimensional euclidean space. These random sets are models for geographical clustering in which new points are likely to be near to old points. The models are not time homogeneous. Rather, as time progresses, the typical distance between a new point and the set of existing points diminishes. The parameters of the model will ensure that the set of points is finite at all times and countably infinite at time infinity; the diameter remains bounded, whence there is a nontrivial limit set; we

FIGURE 1: *Discrete Time Spatial Branching with Gaussian Displacements in \mathbb{R}^2* : Displacements follow a Gaussian law (5). Trees were truncated at 10,000 vertices. Two different exponents in (3) were used, giving random fractals with Hausdorff dimensions $10/9$ and $3/2$, respectively.



are interested in how dense this set is, as measured by its Hausdorff dimension. We begin with an informal description, then proceed to a formal construction.

The process begins with a single point at the origin. A density f on step sizes is fixed. Each existing point x gives rise to new points according to a spacetime Poisson process, where the total intensity at time t (integrated over space) is proportional to $1/t$ and the displacement of the new point y from the parent x is isotropic with $|y - x|$ drawn from the density f rescaled by $t^{-1/d}$. We will see (Corollary 1 below) that the total number of points $\mathfrak{n}(t)$ grows like a random multiple of t^ρ where ρ can be any positive constant and may be computed from f . When the number of points reaches n , one has $t \asymp n^{1/\rho}$ while the displacements of new points from old ones is of order $t^{-1/d} \asymp n^{-1/(d \cdot \rho)}$. Therefore, taking $\rho := \alpha/d$, our model is capable of emulating a purely spatial model in which the $(n + 1)^{st}$ point is added by choosing uniformly among the existing points and adding an isotropic displacement with size distribution f scaled by any exponent $n^{-1/\alpha}$, as described in Section 5. In particular, the model can be fitted to spatial data consisting of a static collection of locations.

2. Definitions and notation

Because each new point is associated with a particular old point, it will be useful to represent the process as a tree, considering the new point to be the offspring of the old point. We begin with notation for trees, then give a formal construction of the process on an appropriate probability space.

2.1. Notation for trees

A rooted tree T with vertices in a space Ξ is a triple $(\mathbf{0}, V, E)$ with V a subset of Ξ (the vertex set of T), $\mathbf{0}$ an element of V (the root of T) and E a subset of $V \times V$ (the set of oriented edges of T). The set E must satisfy the in-degree condition: there is no edge $(v, \mathbf{0})$ in E and for all $v \neq \mathbf{0}$ in V there is a unique $u \in V$ with $(u, v) \in E$. This vertex u is called the parent of v and denoted $\text{par}(v)$. The resulting graph must be connected, or in other words, E must be well founded: for all $v \in V$ there exists a non-negative integer n such that iterating the parent map n times returns the root. We denote this number by $|v|$; thus $\text{par}^{|v|}(v) = \mathbf{0}$. This formalism is sufficiently general to allow the number of children of a vertex to be finite, countably infinite or uncountable, however all of our trees will have vertex degrees that are at most countable.

For v, w vertices of T , write $v \rightarrow w$ to denote the relation that v is the parent of w , and write $v \leq w$ to denote the relation that v is an ancestor of w (the transitive closure of $v \rightarrow w$). Let $v \wedge w$ denote the most recent common ancestor (the notation is consistent with the lattice meet in the ancestry partial order). If T is an infinite tree, let ∂T denote the set of infinite directed paths $\gamma = (\mathbf{0}, \gamma_1, \gamma_2, \dots)$ from the root of T . For $\gamma \neq \gamma'$, let $\gamma \wedge \gamma'$ denote γ_n where n is the maximum of values such that $\gamma_n = \gamma'_n$. Recall some facts from [14] about the topology of ∂T . The space ∂T is topologized by a basis of clopen sets of the form $\partial T(v) := \{\gamma : \gamma_n = v\}$. The Borel σ -field with respect to this topology is denoted \mathcal{B} . If T has finite degrees then this topology makes ∂T a compact Hausdorff space. In general, one can compactify by embedding in the space $\partial^+ T := \partial T \cup V(T)$ where a neighborhood basis of v are the sets $T(v; F) := \{\gamma : \gamma_n = v, \gamma_{n+1} \notin F\}$ for finite subsets F of the children of v . This makes $\partial^+ T$ compact Hausdorff; it is still compact if one removes the vertices of finite degree, as these are isolated points in $\partial^+ T$.

The tree representing processes of interest here might naturally be taken to have vertices in \mathbb{R}^d or in $\mathbb{R}^d \times \mathbb{R}^+$, representing locations or locations together with birth times. However, because the graph structure is not random (each vertex has countably many children), it will be more convenient to fix a particular tree and define the random process as a random function on the vertices of the canonical tree. Accordingly, we use the Ulam-Harris tree \mathcal{U} , whose vertex set is the set $\mathbf{V} := \bigcup_{n=1}^{\infty} \mathbb{N}^n$, with the empty sequence $\mathbf{0} = \emptyset$ as the root, with edges from any sequence $\mathbf{a} = (a_1, \dots, a_n)$ to any extension $\mathbf{a} \sqcup j = (a_1, \dots, a_n, j)$ (see [1]). Our construction will yield random maps $\tau : \mathbf{V} \rightarrow \mathbb{R}^+$ and $\chi : \mathbf{V} \rightarrow \mathbb{R}^d$ interpreted as birth times and locations. Ultimately we will be interested in the range of χ , or in its closure or limit points.

2.2. Probabilistic constructions

Fix a positive integer d which will be the spatial dimension. Fix positive real intensity parameters β and θ and a spatial decay profile $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$. One can without loss of generality take $\beta = 1$, as the process, up to a linear time change, depends only on β/θ . It is necessary that f be integrable but not that it have total mass 1. We assume f is bounded and has finite moments of all orders. This holds, for example, if f has exponential tails, $f(x) < Ae^{-Bx}$ for some $A, B > 0$.

Let \mathbf{m} denote d -dimensional Lebesgue measure. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is constructed a collection of IID Poisson point processes $\{N_v : v \in \mathbf{V}\}$ indexed by the canonical vertex space of the Ulam-Harris tree. The space of points for each point process is $\mathbb{R}^d \times \mathbb{R}^+$ and the common intensity is

$$F(y, t) := f \left[(t/\beta)^{1/d} |y| \right] \frac{d\mathbf{m}(y) dt}{\theta}. \quad (1)$$

Let

$$c_d := c_d(f) := \int_{\mathbb{R}^d} f(|x|) d\mathbf{m}(x) \quad (2)$$

denote the total mass of the measure $F(y, 1) d\mathbf{m}(y)$ in dimension d in the case where $\beta/\theta = 1$. By scaling, the total mass of $F(y, t) d\mathbf{m}(y)$ is c_d/t when $\beta/\theta = 1$, and $c_d\beta/(t\theta)$ in general. Define

$$\rho := c_d \frac{\beta}{\theta}. \quad (3)$$

The random maps τ and χ are constructed recursively. Begin with $\tau(\mathbf{0}) := 1$ and $\chi(\mathbf{0})$ equal to the origin. For the recursion, suppose that $\tau(v) = t$ and $\chi(v) = x$. The

values of τ and χ on the children of v are constructed from the Poisson process N_v on $\mathbb{R}^d \times (t, \infty)$. For any $t' > t$, the total mass of F on $\mathbb{R}^d \times (t, t')$ is finite, therefore $N_v(\mathbb{R}^d \times (t, t')) < \infty$ almost surely for each $t' > t$, hence the points of the Poisson process N_v restricted to $\mathbb{R}^d \times (t, \infty)$ can be enumerated in increasing order of the time coordinate: $(x_1, t_1), (x_2, t_2), \dots$. Now define

$$\begin{aligned}\tau(v \sqcup j) &= t_j \\ \chi(v \sqcup j) &= x + x_j,\end{aligned}$$

thereby extending the definitions of τ and χ to the children of v and completing the recursion.

The random map τ is equivalent to a well known branching random walk (BRW), called the Poisson Weighted Infinite Tree (PWIT), introduced in [2], with the name PWIT bestowed in [3] (see also [4]). The PWIT is a BRW in the sense that there are variables $\{S(v) : v \in V\}$ with the collection $\{S(v) - S(\text{parent}(v))\}$ independent as $\text{parent}(v)$ varies over V . In the PWIT, the values $S(v \sqcup 1), S(v \sqcup 2), \dots$ at the children of v are $S(v) + Y_1, S(v) + Y_2, \dots$ where $\{Y_n\}$ are the successive points of a rate 1 Poisson process. We state the equivalence as a proposition.

Proposition 1. (PWIT.) *For $v \in \mathbf{V}$ and $n \geq 1$, let*

$$Y_n(v) := \log \frac{\tau(v \sqcup n)}{\tau(v)}.$$

- (i) *The vector $\rho \mathbf{Y}(v) = (0, \rho Y_1(v), \rho Y_2(v), \dots)$ has IID mean 1 exponential increments, where $\rho = c_d \beta / \theta$.*
- (ii) *The vectors $\mathbf{Y}(v)$ are independent as v varies over \mathbf{V} . Consequently, \mathbf{Y} are the increments of a PWIT.*
- (iii) *Let \mathcal{Y} be the σ -field generated by the PWIT, that is, by all the vectors $\mathbf{Y}(v)$. Conditional on \mathcal{Y} , the increments $\Delta \chi(v) := \chi(v) - \chi(\text{par}(v))$ are independent with laws $F(x, \tau(v)) d\mathbf{m}(x)$ normalized to probability distributions.*

PROOF: Each $\mathbf{Y}(v)$ is constructed from the Poisson process N_v , making (ii) automatic. Projecting $\mathbb{R}^d \times \mathbb{R}^+$ to \mathbb{R}^+ , the intensity of $N_v(\mathbb{R}^d \times (t, \infty))$ is $\frac{\rho}{t} \mathbf{1}_{t \geq \tau(v)} dt$. Making a time change from t to $\log t$ results, after elementary change of variables, in a Poisson

process of intensity $\rho \mathbf{1}_{x \geq \log \tau(v)} dx$ and (i) follows immediately. For (iii), condition on a point of N_v projecting to $[u, u + h]$ and let $h \downarrow 0$. \square

Some examples of the spatial distributions we consider are as follows. In all the examples, without loss of generality, β is taken to be 1. The volume of the unit ball is denoted $V_d := \pi^{d/2} / \Gamma(d/2 + 1)$.

Example 1. (*Exponential.*) Let $f(x) = e^{-x}$. Then

$$F(y, t) := \exp \left[-t^{1/d} |y| \right] \frac{d\mathbf{m}(y) dt}{\theta} \quad (4)$$

and displacements of children from the parent will be exponential with mean $t^{-1/d}$. The constant c_d is given by

$$c_d = \int_0^\infty e^{-x} V_d \cdot dx^{d-1} dx = d! V_d.$$

Example 2. (*Gaussian.*) Let $f(x) = e^{-x^2/2}$. Then

$$F(y, t) := \exp \left[-\frac{|y|^2}{2t^{-2/d}} \right] \frac{d\mathbf{m}(y) dt}{\theta} \quad (5)$$

and displacements of children from the parent will be centered Gaussians with variance $t^{-2/d}$. One obtains $c_d = \int e^{-|x|^2/2} d\mathbf{m}(x) = (2\pi)^{d/2}$.

Example 3. (*Hard cutoff.*) Let $f(x) = \mathbf{1}_{x \leq 1}$. Then

$$F(y, t) := \mathbf{1}_{|y| \leq t^{-1/d}} \frac{d\mathbf{m}(y) dt}{\theta} \quad (6)$$

Here $c_d = V_d$, and displacements of children from the parent will be uniform within the ball of radius $t^{-1/d}$.

2.3. Markov process representation

To connect this construction to what was mentioned in the introduction, we give an alternate description as a continuous time Markov process, started at time $t = 1$. For $t \geq 1$, let \mathcal{T}_t be the subtree of \mathcal{U} induced on those vertices v satisfying $\tau(v) \leq t$. Without ambiguity, the restrictions of τ and χ to \mathcal{T}_t are also denoted τ and χ . Let \mathcal{F}_t denote the σ -field generated by the restrictions of all Poisson processes N_v to $\mathbb{R}^d \times [1, t]$. Let \mathcal{J}_0 be the set of finite trees with vertices in $\mathbb{R}^d \times \mathbb{R}^+$. Without loss of generality, we identify \mathcal{T}_t with the tree in \mathcal{J}_0 whose vertex set is the range of (χ, τ) restricted to

\mathcal{T}_t with edges such that (χ, τ) is a graph isomorphism. Let \emptyset_* denote the rooted tree whose unique vertex is the pair $(\mathbf{0}, 1)$. The following result is self-evident.

Proposition 2. *(i) The process $\{\mathcal{T}_t : t \geq 1\}$, together with the maps τ and χ , is Markov with respect to the filtration $\{\mathcal{F}_t\}$. It is a time-inhomogeneous pure jump process on \mathcal{J}_0 with initial state \emptyset_* whose jumps at time t add vertices (x, t) with kernel $F(x - v, t) d\mathbf{m}(x)$.*

(ii) Projecting $\mathbb{R}^d \times \mathbb{R}^+$ to the last coordinate yields a Markov process which is a BRW in the sense that any vertices alive at time t independently give birth to new vertices, marked with the timestamp t , at rate ρ/t .

(iii) The size \mathbf{n}_t of \mathcal{T}_t is a pure birth chain with birthrate $\rho\mathbf{n}_t/t$.

The following consequence makes precise the heuristic discussion in the introduction

Corollary 1. (growth exponent.) *With probability 1, the limit*

$$W := \lim_{t \rightarrow \infty} t^{-\rho} \mathbf{n}(t)$$

exists and is nonzero.

PROOF: Let $L_t := \log \mathbf{n}(t)$. The jump rate $\rho \mathbf{n}(t)/t$ for $\{\mathbf{n}(t)\}$ implies a jump rate $\rho \exp(L(t))/t$ for $\{L(t)\}$; the increment is $\log[1 + \exp(-L(t))]$. Then L_t is an increasing pure jump process with infinitesimal drift and variance give by

$$\begin{aligned} \mu_t &= \frac{\rho}{t} e^{L_t} \log(1 + e^{-L_t}) \\ \sigma_t^2 &\leq \frac{\rho}{t} e^{L_t} e^{-2L_t}. \end{aligned}$$

We first establish a crude bound: for some $\alpha > 0$,

$$W' := \sup_t t^\alpha e^{-L_t} < \infty. \quad (7)$$

Indeed, $\mu_t > \rho/(2t)$ for any $t \geq 1$, whence for any $\alpha < \rho/2$, for sufficiently large t we have $L_t > \alpha \log t$ and (7) follows. From (7), it follows that if $\tau_c := \inf\{t : t^\alpha e^{-L_t} \geq c\}$ then the events $\{\tau_c < \infty\}$ converge down to zero as $c \rightarrow \infty$. Taking Taylor series, it follows on $\{\tau_c = \infty\}$ that

$$\mu_t = \frac{\rho}{t} + O(t^{-1-\alpha}) \quad (8)$$

$$\sigma_t^2 \leq c\rho t^{-1-\alpha}. \quad (9)$$

Let $M_t := L_t - \int_1^t \mu_s ds$. Then $M_{t \wedge \tau_c}$ is a martingale in L^2 for every c . It follows from (9) and the fact that $\tau_c = \infty$ for $c > W'$ that the almost sure limit $M_\infty := \lim_{t \rightarrow \infty} M_t$ exists for all t . Then from (8) we get $\int_1^\infty (\mu_t - \rho/t) dt$ converges almost surely to a finite random variable Δ , whence

$$L_t - \rho \log t \rightarrow M_\infty + \Delta$$

proving the proposition with $W = \exp(M_\infty + \Delta)$. \square

2.4. Metric definitions

Let δ be a metric on $\partial^+ \mathcal{U}$ compatible with the topology generated by the clopen sets $\partial T(v)$. One such metric is the metric δ defined by $\delta(\gamma, \gamma') := \tau(\gamma \wedge \gamma')^{-1/d}$. Finiteness of the set $\{v : \tau(v) < t\}$ for every t , which occurs with probability one, ensures that this random metric is indeed a metric and generates the correct topology, which is compact. Any metric such as δ which depends only on $\gamma \wedge \gamma'$ is an *ultrametric*, meaning that among the three values $\delta(\gamma, \gamma')$, $\delta(\gamma, \gamma'')$ and $\delta(\gamma', \gamma'')$, the greatest two are always equal.

We recall the definition of the Hausdorff dimension of a closed subset S of a compact metric space.

Definition 2.1. (*Hausdorff dimension.*) The α -dimensional *Hausdorff content* $H_\alpha(S)$ is the infimum value of $\sum_{j=1}^\infty \text{diam}(B)^alpha$ over collections of sets $\{B(x_j, r_j) : j \geq 1\}$ covering S . The *Hausdorff dimension* $\dim(S)$ is the supremum of α for which $H_\alpha(S) = \infty$ and also the infimum of α for which $H_\alpha(S) = 0$.

Remarks 1. Finer information can be obtained by using gauge functions other than the α power. We will not need these here. When $\alpha = \dim(S)$, it can happen that $H_\alpha(S)$ is zero, or finite nonzero. The α -dimensional *Hausdorff measure* of S , defined by the increasing limit as $\varepsilon \rightarrow 0$ of $H_\alpha(S)$ when the infimum over covers is restricted to balls of diameter less than ε , may be zero, finite or infinite, according to how the content behaves; while Hausdorff measure is more complicated than Hausdorff content by one limit, it has the benefit when finite of being a measure, that is, additive over disjoint sets.

We recall from [14] an equivalent definition of Hausdorff dimension. First they show

(e.g. as a consequence of [14, Theorem 15.7]) that in the infimum over covers, it suffices to consider only covers by sets of the form $\partial T(v)$. Indeed for ultrametrics, as here, one can assume without loss of generality that every element of a finite cover is of the form $\partial T(v)$.

Next, consider finite directed flows on T defined by placing positive real numbers on the directed edges so that inflow equals outflow at every vertex, except at the root where total outflow is finite. Such flows are obviously in bijective correspondence with finite Borel measures on ∂T . Given α , consider the class of *admissible* flows, constrained never to exceed $\delta(v)^\alpha$ at the vertex v ; here $\delta(v)$ is short for $\delta(\gamma, \gamma')$ for any/every pair with $\gamma \wedge \gamma' = v$.

By the max-flow min-cut theorem for countable directed networks [14, Theorem 3.1]), the supremum of volume for admissible flows has magnitude equal to the infimum of the sum of constraints over cutsets, and there is a flow attaining this supremum. But the sum of constraints over cutsets is precisely $H_\alpha(\partial T)$. We have therefore shown:

Lemma 1. (Hausdorff dimension in terms of flows.) *Suppose δ is the metric for which $\delta(\gamma, \gamma')$ is $\tau(\gamma \wedge \gamma')^{-1/d}$. The Hausdorff dimension of ∂T is the critical α for whether it is possible to have a nonzero flow constrained to be at most $\delta(v)^\alpha$ at each vertex v .*
□

One further equivalent definition will be useful, in terms of capacity. Define the α -dimensional energy of a measure μ on ∂T by

$$\mathcal{E}_\alpha(\mu) := \int \int \delta(\gamma, \gamma')^{-\alpha} d\mu(\gamma) d\mu(\gamma'). \quad (10)$$

The following criterion for bounding dimension from below based on existence of measures of finite energy may be found in [14] or [6].

Lemma 2. (Capacity definition of Hausdorff dimension.) *Let A be a closed subset of ∂T . If there is a nonzero measure μ on a set A with $\mathcal{E}_\alpha(\mu) < \infty$ then $\dim(A) \geq \alpha$.*

PROOF: By definition, we take $\delta(\gamma, \gamma) = \infty$, hence any finite energy measure must be non-atomic. Thus $\mu \times \mu$ is supported on pairs (γ, γ') with $\gamma \neq \gamma'$. Therefore some small multiple of μ , interpreted as a flow, is admissible for the constraints $\{\delta(v)^\alpha\}$, and thus witnesses the criterion in Lemma 1. In fact, therefore, not only is α no more than

the critical dimension, but if α is the critical dimension, then the Hausdorff content (and hence Haudorf measure) in the critical dimension is positive. \square

3. Dimension of \mathcal{T}_∞ in metric δ

Theorem 3.1. *With respect to the metric δ , the dimension of $\partial\mathcal{U}$ is $d \cdot \rho$.*

PROOF: The set of constraints $\{\delta(v)^\alpha : v \in \mathbf{V}\}$ form a *Galton-Watson* network in the sense of [14, Section 5.9]; a slight generalization is required to allow infinitely many children. The weights $\{A_j\}$ in [14] are the constraints $(t_1^{-\alpha/d}, t_2^{-\alpha/d}, \dots)$. Theorem 5.35 of [14] states an intuitively obvious result, namely water flows if $\mathbb{E} \sum A_i > 1$ and not if $\mathbb{E} \sum A_i < 1$ (the case $\mathbb{E} \sum A_i = 1$ is not settled by that result). In particular, the critical dimension for water flow is the α for which $\mathbb{E} \sum A_i = 1$.

Recall that the birth times for children of the root is a Poisson point process with intensity $\rho dt/t$. We may therefore compute

$$\mathbb{E} \sum_i A_i = \int_1^\infty \rho t^{-\alpha/d} \frac{dt}{t} = \rho \int_1^\infty t^{-\alpha/d-1} = \rho \frac{d}{\alpha}. \quad (11)$$

We see that $\alpha = \rho d$ is critical for $\mathbb{E} \sum_i A_i \geq 1$, hence for water flow with capacities $t^{-\alpha/d}$. By Lemma 1, the dimension of $\partial\mathcal{U}$ with metric δ is thus shown to be $\rho \cdot d$. \square

Before continuing, we give a second proof of the lower dimension bound in Theorem 3.1. This will provide a framework we will use to prove Theorem 4.1 below. First, we construct a martingale for the Galton-Watson network from the proof of Theorem 3.1 analogous to the normalized size martingale for Galton-Watson trees. Define $Z(v) := \tau(v)^{-\rho}$. We have seen in (11) that $\mathbb{E} \sum_{|v|=1} Z(v) = 1$, where $|v|$ denotes the depth of v in the tree. A similar computation shows that

$$\mathbb{E} \sum_{|v|=1} \tau(v)^{-\lambda} = \rho \int_1^\infty t^{-\lambda-1} dt = \rho/\lambda, \quad (12)$$

and, by induction, that

$$\mathbb{E} \sum_{|v|=n} \tau(v)^{-\lambda} = \left(\frac{\rho}{\lambda}\right)^n.$$

Applying this with $\lambda = 2\rho$ shows that $\text{Var} \sum_{|v|=1} Z(v) = \mathbb{E} \sum_{|v|=1} \tau(v)^{2\rho} = 1/2$. Define $W_n := \sum_{|v|=n} Z(v)$. Letting $z \geq y$ denote the relation holding when y is an

ancestor of z , generalize this by defining

$$W_n(y) := \sum_{z \geq y: |z|=|y|+n} \frac{Z(z)}{Z(y)}.$$

For y varying over vertices at a fixed depth k , the variables $W_n(y)$ are IID with the same distribution as W_n . By convention, if $n = 0$, we take $W_n(y) = 1$. From the definitions and induction, it is easy to see that for any $1 \leq k \leq n$,

$$W_n(x) = \sum_{y \geq x, |y|=k+|x|} Z(y)W_{n-k}(y), \quad (13)$$

summing over descendants y of x . It follows from this that $\mathbb{E}(W_n | \mathcal{F}_{n+1}) = W_{n+1}$ where \mathcal{F}_n is the σ -field of information in the first n levels of the tree. Thus $\{W_n\}$ is a martingale. These variables are square integrable, as shown by the following computation.

$$\begin{aligned} \mathbb{E}W_n^2 &= \mathbb{E} \left(\sum_{|x|=1} Z(x)W_{n-1}(x) \right)^2 \\ &= \sum_{|x|=1} \mathbb{E}Z(x)^2 W_{n-1}(x)^2 + \sum_{|x|=|y|=1, x \neq y} \mathbb{E}Z(x)Z(y)W_{n-1}(x)W_{n-1}(y) \\ &= \mathbb{E}W_{n-1}^2 \mathbb{E} \sum_{|x|=1} Z(x)^2 + \sum_{|x|=|y|=1, x \neq y} \mathbb{E}Z(x)Z(y) \\ &= \mathbb{E} \left(\sum_{|x|=1} Z(x)^2 \right)^2 + \left(\mathbb{E} \sum_{|x|=1} Z(x)^2 \right) (\mathbb{E}W_{n-1}^2 - 1) \\ &= \frac{3}{2} + \frac{1}{2} (\mathbb{E}W_{n-1}^2 - 1). \end{aligned}$$

Inductively, $\mathbb{E}W_{n-1}^2 = 2 - 2^{-n}$, hence $\lim_{n \rightarrow \infty} \mathbb{E}W_n^2 = 2$. Therefore the martingale $\{W_n\}$ is square integrable, and so it converges almost surely and in L^2 to some random variable W with mean 1 and $\mathbb{E}W^2 = 2$. Similarly, $W(v) := \lim_{n \rightarrow \infty} W_n(v)$ defines a limiting random variable associated with each vertex.

We summarize the foregoing computation in a proposition.

Proposition 3. (Limit uniform measure.) *For each $v \in \mathbf{V}$ there is a martingale $\{W_n(v)\}$ converging almost surely and in L^2 to a limit $W(v)$ with $\mathbb{E}W(v) = 1$ and $\mathbb{E}W(v)^2 = 2$. Define*

$$\mu(V) := Z(v)W(v). \quad (14)$$

It follows from (13) that μ is a measure on $\partial\mathcal{U}$. We call this the $d \cdot \rho$ -**dimensional limit uniform measure**. \square

SECOND PROOF OF LOWER DIMENSION BOUND: By Lemma 2 it suffices to show for every $a < d \cdot \rho$ that $\mathcal{E}_a(\mu) < \infty$. Bound \mathcal{E}_a from above by

$$\begin{aligned} L &:= \sum_v \mu(\partial T(v))^2 \tau(v)^{a/d} \\ &\geq \sum_v (\mu \times \mu) \{(\gamma, \gamma') : \gamma \wedge \gamma' = v\} \tau(v)^{a/d} \\ &= \mathcal{E}_a(\mu). \end{aligned}$$

Taking expectations and breaking down the sum according to depth,

$$\begin{aligned} \mathbb{E}L &= \sum_n \mathbb{E} \sum_{|v|=n} \mu(\partial T(v))^2 \tau(v)^{a/d} \\ &= \sum_n \mathbb{E} \sum_{|v|=n} \tau(v)^{-2\rho} W(v)^2 \tau(v)^{a/d} \\ &= 2 \sum_n \mathbb{E} \sum_{|v|=n} \tau(v)^{-\rho-\varepsilon} \end{aligned} \tag{15}$$

where $\varepsilon := (d \cdot \rho - a)/d > 0$ by assumption. In (12) we saw that the inner sum is $(\rho/(\rho+\varepsilon))^n$. Hence the outer sum is a geometric series summing to $1 + \rho/\varepsilon < \infty$. This completes the proof that $\mathcal{E}_a(\mu) < \infty$ for $a < d \cdot \rho$ and hence that $\dim(\partial\mathcal{T}_\infty) = d \cdot \rho$. \square

4. Dimension of the euclidean set

We turn now to the question of the dimension of the euclidean set, that is, the closure of the range of χ . It is least complicated still to work on $\partial\mathcal{U}$, identifying $\gamma \in \partial\mathcal{U}$ with the point $\chi(\gamma) := \lim_{n \rightarrow \infty} \chi(\gamma_n)$. We do this by lifting the euclidean metric to the metric δ' on $\partial\mathcal{U}$ defined by

$$\delta'(\gamma, \gamma') := |\chi(\gamma) - \chi(\gamma')|.$$

This is not an ultrametric, therefore using Lemma 1 can give only an upper dimension bound. This is not a problem because we are going to use Lemma 2 for the lower bound.

Our result is that mapping $\partial\mathcal{U}$ into euclidean space does not reduce the dimension unless it must, due to the dimension of $\partial\mathcal{U}$ being greater than d .

Theorem 4.1. (euclidean dimension.) *The Hausdorff dimension of $\partial\mathcal{U}$ with respect to δ' is $d \cdot \min\{1, \rho\}$.*

PROOF OF LOWER DIMENSION BOUND: Fix $a < \min\{d, d \cdot \rho\}$. We need to show that the Hausdorff dimension of $\partial\mathcal{T}_\infty$ in the metric δ' is at least a . Let μ denote the $d \cdot \rho$ -dimensional limit uniform measure. We show that the expected a -dimensional energy with respect to δ' , $\mathcal{E}'_a(\mu)$ is finite. Finite expectation implies almost sure finiteness, which by Lemma 2 implies $\dim(\partial T)$ in the metric δ' is almost surely at least a . We do this in two steps. Given $\gamma, \gamma' \in \partial T$, if $\gamma_j = \gamma'_j$ for $j \leq n$ but $\gamma_{n+1} \neq \gamma'_{n+1}$, denote

$$\delta''(\gamma, \gamma') := \min\{\tau(\gamma_{n+1}), \tau(\gamma'_{n+1})\}^{-1/d}.$$

By comparison, $\delta(\gamma, \gamma') = \tau(\gamma_n)^{-1/d}$, therefore $\delta'' < \delta$, because the $-1/d$ power is of the first time occurring in only one of the two branches, rather than of the last common time.

Lemma 3. *The measure μ has finite a -dimensional energy in metric δ'' :*

$$\mathbb{E}\mathcal{E}''_a(\mu) < \infty.$$

Assuming this for the moment, the lower dimension bound is proved as follows. Let \mathcal{G} be the σ -field defined generated by the time variables $\{\tau(v) : v \in \mathbf{V}\}$. Given γ and γ' , let $v = \gamma \wedge \gamma'$, let $n = |v|$, and let i and j be the distinct positive integers for which $\gamma_{n+1} = v \sqcup i$ and $\gamma'_{n+1} = v \sqcup j$. Write $\gamma \prec \gamma'$ if $i < j$. For any measure, and in particular for μ ,

$$\mathcal{E}''_a(\mu) = 2 \sum_v \sum_{i < j} \mu(\partial T(v \sqcup i)) \mu(\partial T(v \sqcup j)) \tau(v \sqcup i)^{a/d}. \quad (16)$$

By comparison,

$$\mathcal{E}'_a(\mu) = 2 \sum_v \sum_{i < j} \int |\chi(\gamma) - \chi(\gamma')|^{-a} d\mu|_{T(v \sqcup i)}(\gamma) d\mu|_{T(v \sqcup j)}(\gamma'). \quad (17)$$

We will show that the expectation given \mathcal{G} of (17) may be bounded term by term by the summands in (16). In fact we claim there is a constant K for which

$$\mathbb{E}(|\chi(\gamma) - \chi(\gamma')|^{-a} | \mathcal{G}) \leq K \tau(v \sqcup i)^{a/d}. \quad (18)$$

To see this, let $\mathcal{G}' \supseteq \mathcal{G}$ be the σ -field generated by all the variables $\tau(v)$ and all the variables $\Delta\chi(v)$ except for $\Delta\chi(v \sqcup i)$. It suffices to show (18) with \mathcal{G}' in place of \mathcal{G} . Let $x := \chi(\gamma) - \chi(\gamma') - \Delta\chi(v \sqcup i)$, and write

$$\mathbb{E}(|\chi(\gamma) - \chi(\gamma')|^{-a} \mid \mathcal{G}') = \mathbb{E}|x + \tau(v \sqcup i)^{-1/d}\theta|^{-a}$$

where θ is a random variable with density $c_d^{-1}f(|x|)d\mathbf{m}(x)$; see Proposition 1, part (iii). Evidently $\tau(v \sqcup i)$ is measurable with respect to $\mathcal{G} \subseteq \mathcal{G}'$. The quantity x is the sum of all displacements of children from parents along γ , minus the same sum from γ' , except that one summand from γ gets deleted, namely the displacement of $v \sqcup i$ from v . Therefore, x does not require this displacement to be evaluated and is measurable with respect to \mathcal{G}' also. Abbreviate $\tau(v \sqcup i)$ to τ , and noting that θ is the only random quantity on the left side, we have a tail bound

$$\begin{aligned} \mathbb{P}(|x + \tau^{-1/d}\theta|^{-a} > \lambda) &= \mathbb{P}(|x + \tau^{-1/d}\theta| < \lambda^{-1/a}) \\ &= \mathbb{P}(|\tau^{1/d}x + \theta| < \tau^{1/d}\lambda^{-1/a}) \\ &= \mathbb{P}(|\tau^{1/d}x + \theta|^d < \tau\lambda^{-d/a}) \\ &\leq 1 \wedge C\tau\lambda^{-d/a} \end{aligned}$$

because the density f is bounded. where the constant C depends only on f . By assumption $-d/a < -1$. Therefore, letting $\lambda_0 := (C\tau)^{a/d}$, we obtain (18) via

$$\begin{aligned} \mathbb{E}|x + \tau^{-1/d}\theta|^{-a} &\leq \int_0^\infty 1 \wedge C\tau\lambda^{-d/a} d\lambda \\ &= \lambda_0 + \int_{\lambda_0}^\infty C\tau\lambda^{-d/a} d\lambda \\ &= C'\lambda_0 = K\tau^{a/d}. \end{aligned}$$

Integrating the left side of (18) against μ restricted to $T(v \sqcup i)$ in the variable γ and μ restricted to $T(v \sqcup j)$ in the variable γ' then shows that

$$\mathbb{E}\left(\int |\chi(\gamma) - \chi(\gamma')|^{-a} d\mu|_{T(v \sqcup i)}(\gamma) d\mu|_{T(v \sqcup j)}(\gamma') \mid \mathcal{G}\right) \leq K\mu(v \sqcup i)\mu(v \sqcup j)\tau(v \sqcup i)^{-a},$$

bounding (17) above, term by term, by K times (16). Thus,

$$\mathcal{E}'_a(\mu) = \mathbb{E}[\mathbb{E}(\mathcal{E}'_a(\mu) \mid \mathcal{G})] \leq K\mathcal{E}''_a(\mu) < \infty$$

by Lemma 3, finishing the lower dimension bound. It remains to prove the lemma.

PROOF OF LEMMA 3: For μ or any measure, we may break up $\mathcal{E}_a''(\mu)$ as

$$\frac{1}{2}\mathcal{E}_a''(\mu) = \sum_n \sum_{|v|=n} \sum_{i<j} \mu(v \sqcup i)\mu(v \sqcup j)\tau(v \sqcup i)^{a/d}.$$

Let ε denote the positive quantity $\rho - a/d$ and let $\theta_i = \theta_i(v)$ denote $\tau(v \sqcup i)/\tau(v)$.

Plugging in $\mu(v) = \tau(v)^{-\rho}W(v)$ then gives

$$\begin{aligned} \mathcal{E}_a''(\mu) &= \sum_n \sum_{|v|=n} \tau(v)^{-2\rho}\tau(v)^{a/d} \sum_i \left[W_i\theta_i(v)^{-\rho+a/d} \sum_{j>i} W_j\theta_j(v)^{-\rho} \right] \\ &= \sum_n \sum_{|v|=n} \tau(v)^{-\rho-\varepsilon} \sum_i \left[W_i\theta_i^{-\varepsilon} \sum_{j>i} W_j\theta_j(v)^{-\rho} \right]. \end{aligned}$$

For each v , the quantities $\tau(v), \theta_i(v), \theta_j(v), W(v \sqcup i) =: W_i$ and $W(v \sqcup j)$ are independent. Also, $\theta_i(v)$ is always less than 1 and $\mathbb{E}W_i = 1$ for all i . Therefore, reversing the inner summations,

$$\begin{aligned} \mathbb{E}\mathcal{E}_a''(\mu) &< \sum_n \sum_{|v|=n} \mathbb{E}\tau(v)^{-\rho-\varepsilon} \mathbb{E} \sum_j \left[W(v \sqcup j)\theta_j(v)^{-\rho} \sum_{i<j} W_i \right] \\ &= \sum_n \sum_{|v|=n} \mathbb{E}\tau(v)^{-\rho-\varepsilon} \sum_j (j-1)\mathbb{E}\theta_j(v)^{-\rho}. \end{aligned}$$

From $\mathbb{E}\theta_j^{-\rho} = (\mathbb{E}\theta_1^{-\rho})^j$ and $\theta_1 < 1$, it follows that $\sum_j (j-1)\mathbb{E}\theta_j^{-\rho}$ is equal to the constant $M := (\mathbb{E}\theta_1^{-\rho}/(1 - \mathbb{E}\theta_1^{-\rho}))^2$. Therefore,

$$\mathbb{E}\mathcal{E}_a''(\mu) < M\mathbb{E} \sum_n \sum_{|v|=n} \tau(v)^{-\rho-\varepsilon}$$

which was seen to be finite in (15), finishing the lemma, hence the lower dimension bound. \square

PROOF OF UPPER DIMENSION BOUND. We may assume that $\rho < 1$ because otherwise the upper dimension bound of d is trivial. Fix $a > \rho \cdot d$. We need to show that the a -Hausdorff content of \mathcal{T}_∞ is zero. We use the covers $\{\partial\mathcal{T}(v) : |v| = n\}$. We will show that the expectations go to zero

$$\mathbb{E} \sum_{|x|=n} \text{diam}(\partial\mathcal{T}(v))^a < \infty \rightarrow 0$$

which implies with probability 1 the lim inf of the sums goes to zero, hence $H_a(\partial\mathcal{T}_\infty) = 0$. Because $\text{diam}(\partial\mathcal{T}(v))/\tau(v)$ is independent of $\tau(v)$ and distributed as $\text{diam}(\partial\mathcal{T}_\infty)$ we may write

$$\begin{aligned} \mathbb{E} \sum_{|x|=n} \text{diam}(\partial\mathcal{T}(v))^a &= \mathbb{E} \sum_{|x|=n} \tau(v)^a \left(\frac{\text{diam}(\partial\mathcal{T}(v))}{\tau(v)} \right)^a \\ &= \left(\mathbb{E} \sum_{|x|=n} \tau(v)^a \right) \mathbb{E} \text{diam}(\partial\mathcal{T}_\infty)^a \\ &= \left(\frac{d \cdot \rho}{a} \right)^n \mathbb{E} \text{diam}(\partial\mathcal{T}_\infty)^a. \end{aligned}$$

Because $d \cdot \rho/a < 1$, it suffices to show that $\text{diam}(\mathcal{U})$ has finite a -moment. We show in fact all moments of $\text{diam}(\mathcal{U})$ are finite. We use a cheap upper bound, namely

$$\text{diam}(\mathcal{U}) \leq 2 \sup_{\gamma \in \partial\mathcal{U}} \sum_{n=1}^{\infty} |\chi(\gamma_n) - \chi(\gamma_{n-1})| \leq 2 \sum_{n=1}^{\infty} \sup_{|v|=n} |\chi(v) - \chi(\text{par}(v))|. \quad (19)$$

Let M_n denote the supremum inside the sum. We need tail bounds on M_n . Recall from Proposition 1 that the values $S(v) := \log \tau(v)$ form a PWIT scaled by $1/\rho$. By construction, conditional on \mathcal{Y} (that is all the values $\tau(v)$) the displacements $|\chi(v) - \chi(\text{par}(v))|$ are independent, the one at each vertex v chosen from f scaled by $\exp(-S(v)/d)$. Thus the collection $\{\chi(v) - \chi(\text{par}(v))\}$ is just $\{Y(v) := \theta(v) \exp(-S(v)/d)\}$ where $\{\theta(v)\}$ are IID picks from the density described by f .

Lemma 3.2 of [1] says that

$$G_n(x) := \mathbb{E} \# \{v : |v| = n \text{ and } PWIT(v) \leq x\} = \frac{x^n}{n!}. \quad (20)$$

The expected sum of the m -powers at level n may be written as

$$E_n^m := \mathbb{E} \sum_{|v|=n} \{Y(v)^m : |v| = n\} = [\mathbb{E} \theta(v)^m] \mathbb{E} \sum_{|v|=n} e^{-(m/d)S(v)}$$

where $S(v)$ are $1/\rho$ times the values of the PWIT at v . Expressing the last expectation as an integral against the density $G_n(x)$ of values of the PWIT at level n and using (20)

then gives

$$\begin{aligned}
E_n^m &= [\mathbb{E}\theta(v)^m] \int_0^\infty e^{-\frac{a}{d\rho}x} dG_n(x) \\
&= [\mathbb{E}\theta(v)^m] \int_0^\infty e^{-\frac{a}{d\rho}x} \frac{x^{n-1} dx}{(n-1)!} \\
&= [\mathbb{E}\theta(v)^m] \left(\frac{a}{d\rho}\right)^n.
\end{aligned}$$

Let $c_{m,f}$ be the m th moment of a random variable with density f , normalized, which we have assumed to be finite for all m . Markov's inequality then yields, for any m ,

$$\mathbb{P}(M_n \geq \lambda) \leq \lambda^{-m} E_n^m = \lambda^{-m} c_{m,f} \left(\frac{m}{d \cdot \rho}\right)^{-n}. \quad (21)$$

Fixing any $m > d \cdot \rho$, we may choose $\varepsilon > 0$ such that $(1 + \varepsilon)^m d\rho/m < 1$. Let $y_n := (1 + \varepsilon)^{-n}$ and apply (21) to see that

$$\mathbb{P}(M_n \geq \lambda \varepsilon (1 + \varepsilon)^{-n}) \leq c_{m,f} \varepsilon^{-m} \lambda^{-m} \left(\frac{(1 + \varepsilon)^m}{m/(d \cdot \rho)}\right)^n. \quad (22)$$

Recall that $\sum_n M_n$ is an upper bound on $\text{diam}(\mathcal{U})$, which we are trying to show has all moments finite. This follows if $\mathbb{P}(\sum_n M_n > \lambda) = O(\lambda^{-c})$ for every $c > 0$ as $\lambda \rightarrow \infty$. Because $\sum_{n=1}^\infty \varepsilon(1 + \varepsilon)^{-n} = 1$, if $\sum_n M_n > \lambda$ then $M_n \geq \lambda \varepsilon (1 + \varepsilon)^{-n}$ for some $n \geq 1$. Thus for any m , using (22),

$$\begin{aligned}
\mathbb{P}\left(\sum_{n=1}^\infty M_n \geq \lambda\right) &\leq \sum_{n=1}^\infty \mathbb{P}(M_n \geq \varepsilon(1 + \varepsilon)^{-n} \lambda) \\
&\leq C \lambda^{-m}
\end{aligned}$$

where $C := c_{m,f} \varepsilon^{-m} \frac{(1 + \varepsilon)^m}{m/(d \cdot \rho) - (1 + \varepsilon)^m}$. This finishes the proof that $\text{diam}(\mathcal{U})$ has finite moments, hence the proof that the a -dimensional Hausdorff content of $\partial\mathcal{T}_\infty$ is zero for any $a > d\rho$, establishing the upper dimension bound. \square

As a corollary, we get the convergence of $\chi(\gamma_n)$ along every path γ .

Corollary 2. *With probability 1, for every $\gamma \in \partial\mathcal{T}_\infty$, the sequence $\{\gamma_n\}$ converges.*

PROOF: We have seen in (21) that $\mathbb{P}(M_n \geq \varepsilon(1 + \varepsilon)^{-n} \lambda)$ is summable for some $\varepsilon > 0$. For any γ , the triangle inequality gives $|\chi(\gamma_n) - \chi(\gamma_m)| \leq \sum_{j=m+1}^n M_j$. Together with Borel-Cantelli, this implies that $\{\chi(\gamma_n) : n \geq 1\}$ is a Cauchy sequence simultaneously for all γ . \square

5. Computational Implementation

5.1. The Agoraphobic Point Process in the Unit Ball

The preceding theory described a continuous time branching process embedded in \mathbb{R}^d . This section describes two finite discrete time constructions used in [9].

Fix a dimension $d \geq 2$, a desired fractal dimension $\alpha < d$, and an *innovation parameter* $\theta \geq 0$ which affects the number of clumps (but does not affect fractal dimension). Let B_d denote the closed unit ball in \mathbb{R}^d . Build a nested increasing sequence of finite random rooted trees $(\xi_n, E_n)_{n \geq 0}$ according to the following rules. Take $\xi_0 := \{\mathbf{0}\}$, i.e. the single point at the origin, which serves as the root, and $E_0 := \emptyset$.

Suppose $n \geq 1$, $\xi_{n-1} \subset B_d$ is a set consisting of n elements, and E_{n-1} is a collection of directed edges so (ξ_{n-1}, E_{n-1}) is a tree rooted at $\mathbf{0}$. To generate ξ_n , perform two random experiments, independent of each other and of ξ_{n-1} :

1. Sample X uniformly at random in B_d .
2. Perform a Bernoulli($\theta/(\theta + n - 1)$) trial (here $0/0 = 1$).

If the trial is a **success**, define $\xi_n := \xi_{n-1} \cup \{X\}$, declare the parent of X in the tree to be $\mathbf{0}$, and define E_n to be E_{n-1} together with this extra directed edge. In effect we are seeding the process with a new point which need not be close to a previous one; if $\theta = 0$, this happens only once.

If the trial is a **failure** (which cannot occur when $n = 1$) compute

$$\Delta := \min_{x \in \xi_{n-1} \setminus \{\mathbf{0}\}} |X - x|. \tag{23}$$

If ξ_{n-1} were a uniform random sample of n points in B_d , this minimum distance Δ would scale like $n^{-1/d}$. To force points to clump together we would like the distance of a new point to the closest previous point to scale like $n^{-1/\alpha}$, where by assumption $1/\alpha > 1/d$. In the **smooth minimum distance** formulation, we accept X with probability

$$e^{-\Delta n^{1/\alpha}}. \tag{24}$$

If X is accepted, define $\xi_n := \xi_{n-1} \cup \{X\}$, and define E_n to be E_{n-1} together with the directed edge $x \rightarrow X$, where $x \in \xi_{n-1} \setminus \{\mathbf{0}\}$ is the minimizing point in (23) (almost

surely unique).

If X is not accepted, keep sampling X again in the same way until acceptance occurs. The conditional law of ξ_n given ξ_{n-1} is independent of $\xi_0, \xi_1, \dots, \xi_{n-2}$, and so $\{\xi_n\}_{n \geq 0}$ is a Markov process.

Except when n is small, the formula (24) has the effect of making rejection likely when the proposed $(n+1)$ -st point is further than $n^{-1/\alpha}$ from any existing point in $\xi_{n-1} \setminus \{\mathbf{0}\}$. Clumps form, as shown in Figure 2. Inserting $\xi_0 := \{\mathbf{0}\}$ serves to create a centering effect. When $\theta > 0$, the number of distinct seed points, like the number of occupied tables in the Chinese restaurant process, grows according to $\theta \log \frac{n+\theta}{\theta}$ after n points, as explained in Pitman [15].

5.2. Hard Threshold Formulation of Agoraphobic Point Process

Simulation is slow using (24), because of high rejection rates. Here is an inequivalent alternative formulation with similar properties, whose rejection rate is much lower. Replace the smooth acceptance rate (24), which in (23) is a continuous function of the sampled point X , by the **hard threshold**

$$1_{\{\Delta \leq n^{-1/\alpha}\}}.$$

To achieve this efficiently, generate a new point which must lie within distance $n^{-1/\alpha}$ of an existing point, as follows:

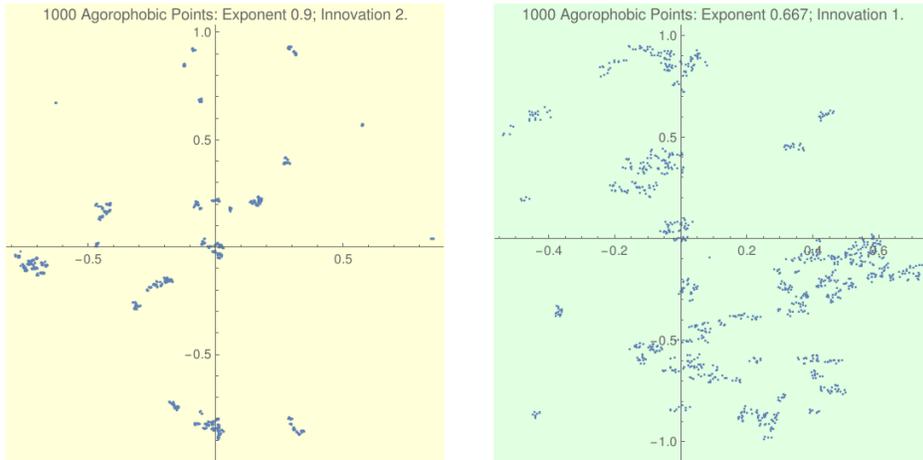
1. Select z uniformly from the $n-1$ points in $\xi_{n-1} \setminus \{\mathbf{0}\}$.
2. Sample Y uniformly from the ball of radius $n^{-1/\alpha}$, centered at z .
3. Accept Y with probability $1/k$, where

$$k := \sum_{x \in \xi_{n-1} \setminus \{\mathbf{0}\}} 1_{\{|Y-x| \leq 1/n\}}.$$

4. If Y is accepted, set $\xi_n := \xi_{n-1} \cup \{Y\}$, and declare z to be the parent of Y in the random tree. If Y is not accepted, keep sampling Y again in the same way until acceptance occurs.

Step 3. prevents oversampling in areas which are already dense. Not all the points need lie inside the unit disk. See Figure 2.

FIGURE 2: **Agoraphobic Point Process:** *Two instance of the Hard Threshold Version, 1000 points, with different exponents $1/\alpha$. Hausdorff dimensions of the limiting processes would be $10/9$ and $3/2$, respectively.*



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