Math 132

Activities and Instructor's Guide

(Revised Summer 1999)

Chapter 1

Activities

This chapter contains the worksheets for Math 132, arranged by unit. Below is a table of contents for the chapter.

Arithmagic

As part of a field trip, you take your class to see the famous mind reader Maya the Mind. To prove that she can read your mind, Maya has you do the following. Choose two whole numbers less than ten, without telling anyone what they are. Add one to the first number, multiply the result by two, add three, and multiply the result by five[1]. Now add the second number to the result, multiply by five, subtract three, multiply by two, and subtract one. You tell Maya the end result, and she immediately reads the original numbers out of your mind.

Some members of the audience are not convinced by her first trick, so Maya chooses another volunteer, and has him do the following. First think of a three digit number whose first digit is at least two more than its last digit[2], reverse the order of the digits in the chosen number, and find the difference between the two numbers. Then add this difference to the number you get by reversing its digits. Once the volunteer has done these steps, Maya is able to read the result out of his mind.

- Is Maya really a mind reader? If not, then how do these tricks work?
- Why does Maya make the restrictions she does on the numbers people can choose?
- Suppose you perform Maya's first trick in base b, rather than in base ten. What answer will you get?

[1]Of course, the numbers two, three, and five have certain mystical powers, since they are they are the first three primes.

[2]Other three digit numbers, like 666 or 487, cause psychic interference.

Letters

In algebra we gradually make more and more use of letters (such as $a, b, c, \ldots, x, y, z$, etc.). Traditionally, the use of letters is considered one of the most difficult topics in the school mathematics curriculum. We hope that you never had difficulties understanding "what all those letters mean", but we still wish to give you some advice. If you ever want to explain the meaning of letters to your classmates, brothers and sisters, your students, your parents or your children (some day), just say that the letters are abbreviations for words. To see what we mean, read the following problem and spend five minutes working on it. Then read the three solutions below.

Problem. A small pitcher and a larger pitcher contain (together) 5 liters. Two small pitchers and three large pitchers together contain 13 liters. What are the volumes of the pitchers?

Arithmetic Solution. A small pitcher and a large pitcher together contain 5 liters. Therefore, two small pitchers and two large pitchers together contain 10 liters. However, we are also told that two small pitchers and three large pitchers contain 13 liters. So we get 13 liters instead of 10 by adding one large pitcher. Therefore the volume of a large pitcher is 3 liters. Now it is easy to find the volume of a small pitcher: together a large pitcher and a small pitcher contain 5 liters, so a small pitcher contains 5 - 3 = 2 liters. So the volume of a small pitcher is 2 liters and the volume of a large pitcher is 3 liters.

Solution with Abbreviations. Let's use "Vol.SP" to stand for "volume of a small pitcher" and "Vol.LP" to stand for "volume of a large pitcher". According to the statement of the problem,

$$\text{Vol.SP} + \text{Vol.LP} = 5$$

and therefore

$$2 \cdot \text{Vol.SP} + 2 \cdot \text{Vol.LP} = 10.$$

We also know that

$$2 \cdot \text{Vol.SP} + 3 \cdot \text{Vol.LP} = 13.$$

If we subtract the preceeding equality from the last one we find that Vol.SP = 3. Now the first equality implies that Vol.SP = 5 - 3 = 2.

Solution with Letters. Denote the volume of a small pitcher by x and the volume of a large pitcher by y. We get the following system of equations.

$$x + y = 5$$

$$2x + 3y = 13$$

Multiplying the first equation by 2 we get

$$2x + 2y = 10,$$

and subtracting the last equation from the second equation of our system gives us

$$y = 13 - 10 = 3.$$

Now the first equation tells us that

$$x = 5 - y = 2.$$

Therefore, y = volume of a large pitcher = 3 liters and x = volume of a small pitcher = 2 liters.

Self-Check Problems

1. Suppose a spaceship travels in "jumps" that are eleven light-years long. Write an expression for the distance the spaceship travels in j jumps.

2. Write an expression for the nth term in the sequence $1, 3, 5, 7, \ldots$, which is the sequence of positive odd numbers.

3. If $a + 5 = 8$ then what is a?

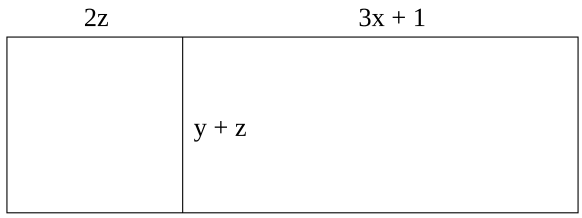
4. If $c + d = 10$ and $c < d$ then what is c?

5. Which is larger, $2n$ or $n + 2$?

6. Suppose p is the perimeter of a certain square. Write an expression for the area A of that square in terms of p.

7. Blue pencils cost 5 cents each and red pencils cost 6 cents each. Let b denote the number of blue pencils bought and let r denote the number of red pencils bought. Altogether, 90 cents were spent. Write an equation that describes this purchase.

8. What is the perimeter of this rectangle? What is its area?



9. Suppose $m + n = 7$ and $3m = 7n + 1$. What is m?

10. What is the difference between an equation and an expression?

Using Variables

When students begin to learn algebra, one of the things that is tough to understand is the use of letters in expressions and equations. One of the reasons for students' confusion is that letters are used in many different ways to represents a number of different things. Below is a list of some of the ways letters are used. Read through these and as you do the problems that follow, decide which way(s) letters are used in each problem. Some problems may use letters in more than one way.

1. *Letter as evaluated:* The letter is part of an equation, and in order for the equation to be true, the letter must be equal to a certain number. For instance, if $3x + 2 = 11$ then x must be 3.

2. *Letter as ignored:* The letter(s) are simply placeholders and the problem can be solved without ever determining what number(s) the letter(s) stand for. For example, if $x^2 + 3 = 2$ then $(x^2 + 3)^2 = 4$.

3. *Letter as object:* The letter is used as a name for something, such as the side length of a figure, and is then used in writing an expression for something like perimeter or area.

4. *Letter as specific unknown:* The letter stands for a single specific number that is unknown at this time. For example, if a person takes n steps, each of which is three feet long, then they have gone $3n$ feet.

5. *Letter as generalized number:* The letter represents a group of numbers, any of which would work in the expression or equation.

6. *Letter as variable:* The letter can take on an infinite number of values, often depending on the value of a second letter.

Self-Check Problems

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2. Write an expression for the nth term in the sequence $1, 3, 5, 7, \ldots$, which is the sequence of positive odd numbers.

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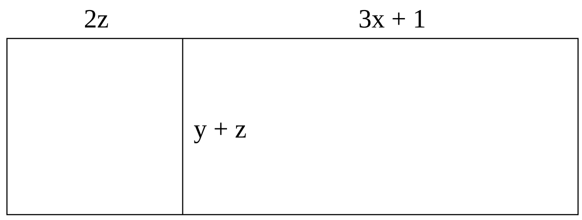

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It's All Relative

1. How hot is an oven whose temperature in degrees Fahrenheit is double the temperature in degrees Celsius? (Remember that 0 Celsius, the freezing point of water, is 32 Fahrenheit and 100 Celsius, the boiling point of water, is 212 Fahrenheit.)

2. You can drive from Madison to St. Louis in 6 hours. If you drive 13 MPH faster, you can make the drive in 5 hours. How fast do you drive when it takes you 6 hours to complete the trip?

3. After a cyclist has gone $\frac{2}{3}$ of her route, she gets a puncture. Finishing on foot (and carrying the bicycle!), she spends twice as long walking as she did riding. How many times faster does she ride than she walks?

4. Donna is now twice as old as Jalen was when Donna was 30 years older than Jalen is now. How old are they now?

Squares Near Fifty

While you are teaching times tables to your third grade class, you spend a day talking about squares, like 49 and 64. After class, one student tells you that she knows how to find the square of any number between 40 and 50 in her head. To do it, she explains, you start with 25 and you subtract how far your number is less than fifty. Then you append the square of how far your number is less than fifty. The number you get is the answer.

- Is the student's method correct? If so then justify it. If not then fix it.
- Describe and justify a method for squaring numbers between 50 and 60.
- Can you find a similar method for squaring numbers near 100?

Finger Products

While you are teaching times tables to your third grade class, one student tells you he only has to learn them for numbers less than five. For a product like 9×6, he puts $6 - 5 = 1$ finger up on his left hand and $9 - 5 = 4$ fingers up on his right hand. The first digit of the answer is the sum of the number of fingers down on each hand and the second digit is the product of the number of fingers up on each hand.

- Is this method correct? If not, then explain why not. If so, then justify it.

After class you describe your student's method to a colleague. She says that the first digit of the answer is actually the sum of the number of fingers up on each hand and the second digit is the product of the number of fingers down on each hand.

- Is her method correct? If not, then explain why not. If so, then justify it.

- Describe all of the products for which both methods give the same answer.

The Sorcerer's Brooms

Due to some misjudgements on the part of his apprentice, the Sorcerer in charge of the Castle Fantasia has a number of extra brooms. To get rid of some of them, he is giving them to his neighbors, Merlin and Glinda. Merlin already had 41 brooms, while Glinda had none. Merlin agreed to accept two brooms on the first day, four on the second, six on the third, etc. Glinda agreed to take one on the first day, three on the second, five on the third, etc.

- Glinda has found that on each of the first five days, she was able to arrange her collection of brooms into a square pattern, like this:

  $$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

  Will she always be able to do this?

- How many brooms will Merlin have after the fifth day? After the tenth day? After the nth day?

- After the tenth day, Merlin notices that every day so far his total number of brooms has been prime. He surmises that his total number of brooms will always be prime. When will he realize he is wrong?

- Unbeknownst to the Sorcerer, his apprentice is still making brooms. On the first day he makes three hundred and sixty seven, on the second day three hundred and seventy, on the third day three hundred and seventy three, etc. Before his apprentice started making more, the Sorcerer had 1466 brooms. Will he ever get rid of all of his brooms? Explain.

Geometric Progressions

In the sequence of numbers

$$3, 6, 12, 24, \ldots$$

each term is two times the preceeding term. In the sequence

$$6, 4, \frac{8}{3}, \frac{16}{9}, \frac{32}{27}, \ldots$$

each term is two thirds of the preceeding term.

Definition. A *geometric sequence* (or *geometric progression*) is a sequence in which each term is the product of a fixed number and the preceeding term. This fixed number is called the *common ratio* of the sequence.

One setting in which geometric progressions appear is in the sequence of frequencies of the keys of a piano. The book *Algebra* by I.M. Gelfand and A. Shen contains an interesting, easy to read discussion of this connection.

Practice Problems, Part I

1. Find the third term of the geometric progression that begins $4, 5$.
2. Find the second term of the geometric progression whose first term is 2 and whose third term is 4.
3. Find the two thousand three hundred and fifth term of a geometric progression whose first term is a and whose last term is r.
4. Is there a geometric progression whose second term is 2 and whose third term is -3? If so, what is its common ratio?
5. Is there a geometric progression whose second term is 2 and whose fourth term is -4? If so, what is its common ratio?
6. In a geometric progression whose common ratio is r, every other term is deleted. Is the result a geometric progression? If so, what is its common ratio? What if only every third term is deleted?
7. If a geometric progression whose common ratio is r is written in reverse order, is the result a geometric progression? If so, what is its common ratio?

Suppose we wish to compute the following sum[3]:

$$2 + 6 + 18 + 54 + 162 + \cdots + 20{,}920{,}706{,}406$$

in which each term is three times the preceeding term. (In other words, the terms in this sum form a geometric progression whose first term is 2 and whose common ratio is 3.) Of course, we could

[3]Each term in this sum might represent the government's deficit for a certain year. If so, then the sum represents the total national debt.

do it the tedious way, which is to figure out what all of the terms between 162 and 20,920,706,406 are, and then add them up. However, there is an easier way.

First, let's call the sum we are trying to find S, so that we have

$$S = 2 + 6 + 18 + \cdots + 20,920,706,406.$$

Because the terms in the sum form a geometric progression, if we multiply both sides of this equation by the common ratio we should move every term to the right one space. That is, we get

$$3S = 6 + 18 + \cdots + 20,920,706,406 + 3 \times 20,920,706,406.$$

Now we subtract S from both sides of this last equation. On the left we obtain $3S - S = 2S$. On the right we think of S as the sum $2 + 6 + 18 + \cdots + 20,920,706,406$, so that we can cancel some terms. We obtain

$$2S = 3 \times 20,920,706,406 - 2.$$

Dividing both sides by two, we find that the sum is $\frac{3}{2} \times 20,920,706,406$.

Practice Problems, Part II: The Sequel

8. Find the sum
$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2048}$$
in which each term is one half of the preceeding term.

9. The first term of a certain geometric progression is a and the common ratio is r. Find a formula for the sum of the first 20 terms of the progression.

Trollbooths

There is a tax on ale for all travelers on the Queen's highway. This tax is collected by trolls at various stopping points on the highway; these stopping points are called trollbooths. The tax at the first trollbooth is one third of whatever ale you are carrying. At each subsequent trollbooth, the tax is one third of the tax at the previous trollbooth. How much will an ale merchant travelling on the Queen's highway pay in taxes if she passes two trollbooths? What if she passes five? n? an infinite number?

The Queen has decided to change the tax rate on her highway so that if someone passes through an infinite number of trollbooths they will have exactly three fifths of their ale left. What should the tax rate be, instead of one third?

The Queen's advisor has recommended the following taxation system. At the first trollbooth the tax is two thirds of whatever ale you have. At the second trollbooth you are given back two thirds of the amount taken at the first trollbooth. At the third trollbooth the tax is two thirds of what you received at the last trollbooth. If this pattern continues then how much ale will you have after an infinite number of trollbooths?

Phone Plans

Ameritech's basic cellular phone plan costs \$19 for the first 15 (or fewer) minutes each month, and 55 cents for each additional minute. Cellular One has a plan that charges \$30 for the first 30 (or fewer) minutes each month and 39 cents for each additional minute.

- How many minutes before Cellular One becomes a better deal?
- Suppose that the rates are

  Ameritech: \$A for B minutes, then C cents per minute

  Cellular One: \$D for E minutes, then F cents per minute,

  and assume that Ameritech starts out cheaper. That is, assume that $A < D$. Under what conditions is Cellular One a better deal? When is Ameritech better?

Concavity

A graph is called *concave up* if it bends upward and *concave down* if it bends downward. A straight line is neither concave up nor concave down. The graph of $y = x^2$, which looks like the smile $\smile$, is concave up, while the graph of $y = -x^2$, which looks like the frown $\frown$, is concave down. Here are three self-check problems, after which are more interesting problems for group discussion.

- Sketch an example of each of the following possibilities.
  1. A graph which is increasing and concave up.
  2. A graph which is increasing and concave down.
  3. A graph which is decreasing and concave up.
  4. A graph which is decreasing and concave down.
- If a graph is increasing and concave up, what can you say about the *rate* at which it is increasing? In other words, is it increasing at a faster and faster rate, or a slower and slower rate?
- Sketch an example of a graph that is concave up in one part and concave down in another part.

1. Here are some sequences of points in the plane, $\mathbf{R}^2$, listed in increasing order by x-coordinate. For each one, say whether the trajectory is concave up or concave down. If it's not obvious, as in (c), make sure you carefully justify your answer.

   (a) $(2, 2),\ (5, 5),\ (10, 6),\ (17, 3)$.

   (b) $(-4, -2)\ (-1, 0),\ (8, 3)$.

   (c) $(-4, 1),\ (0, -2),\ (3, -4)$.

2. Sketch the graph of a function which is decreasing everywhere, but concave down for negative values of x and concave up for positive values of x.

3. Suppose a colony of slime mold doubles each hour. If you plot a graph of populations vs. time, would you expect this to be concave up or down? Why?

4. New employees at the 1-800-TEACHER hotline are monitored for efficiency. On the first day of the job, they tell you how many calls they've handled by 10:00, 12:00, 2:00, 4:00. Do you expect the graph of total calls vs. time to be concave up or concave down? Do you have to make any additional assumptions to answer this question? If so, what are they?

5. The table below gives the number of abortions given in the US for certain years.

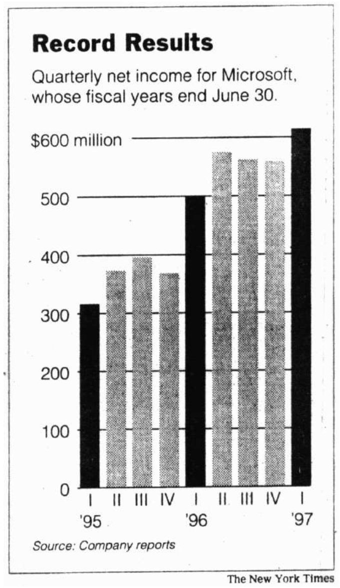
Year	1972	1976	1980	1985
Number of abortions reported	586,760	988,267	1,297,606	1,328,570

   Are these data concave up or concave down? Explain what this means practically in terms of yearly rates of increase.

6. Can you find a condition to tell whether the sequence of points $(1, l),\ (2, m),\ (3, n)$ is concave up or down? Explain. Can you do the same thing for the sequence $(1, a), (2, b), (10, c)$?

Microsoft's Profits

Consider the graphic "Record Results" from *The New York Times* of October 1996, which gives the quarterly net income for Microsoft.



1. Why is the most recent data listed for "the first quarter of 1997"?

2. Where is the graph of net income versus time concave up? concave down?

3. Consider the graph of the function TI = "total income earned by Microsoft since the first quarter of 1995." Sketch a possible graph for TI versus time.

4. Where would the graph of TI be concave up? Concave down?

5. Describe what condition on net income makes total income concave down. In other words, how can you tell *without* first drawing the graph of TI where it will be concave down?

6. Suppose that at a competing company TI increased by 50% from the first quarter of 1995 to the second quarter, then by 40% from the second to the third quarter, then by 38% from the third to the fourth quarter, then by 36% from the fourth to the first quarter of 1996, and continued in the same way through the first quarter of 1997. Would the graph of TI be concave up or concave down for that period?

Introduction to Probability

Suppose we roll a six sided die; then the set of *outcomes* is $\{1, 2, 3, 4, 5, 6\}$. The set of possible outcomes is called the *sample space for a probability experiment.* If the die is fair, then the probability of each outcome is $\frac{1}{6}$ or $16\frac{2}{3}\%$.

Now suppose we roll two fair six sided dice and we consider the sum of the two results. Then the sample space is $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, but these are not equally likely. Figure out the probabilities of each of these outcomes and fill in the following distribution table. (Don't forget that the probabilities must add up to 1 or 100%.) The filled out table is called a *probability distribution.*

Outcome	Probability	Outcome	Probability
2		8	
3		9	
4		10	
5		11	
6		12	
7			

The *expected value* for a probability experiment (with numerical outcomes) is calculated as follows. Take each outcome and multiply it by the probability of that outcome. The sum of all of these results is the expected value, or mean, of the probability experiment. Practice finding expected values by doing the following problems.

1. Find the expected values for the experiment of rolling one die and for the experiment of rolling two dice and then adding the results, as described above.

2. Suppose now that we have an unfair six sided die. For this die, $Pr(1) = \frac{1}{2}$ (in words, the probability of a 1 appearing when you roll this die is $\frac{1}{2}$) and the probability of each of the other numbers is $\frac{1}{10}$. Find the expected value for the experiment of rolling this die one time. For the experiment of rolling this die twice and adding the results, write down the probability distribution and compute the expected value.

As you may already know, if a probability experiment has outcomes A,B,C,D, and E and you want to know the probability of getting either A or B, then you simply add the individual probabilities. This is true provided there is no "overlap" in the outcomes A and B. That is, if A occurs then B doesn't occur and vice versa. If I want to know that probability I get both A and B then I multiply the individual probabilities. This is true provided there is no "dependence" between A and B. That is, the probability of A occuring is not influenced by whether or not B occured and vice versa.

As an example, suppose I have the following Dice Product Bingo Array.

12	15	9
1	36	16
10	6	4

What is the probability that on three successive rolls of the dice, the numbers result in a Bingo across the first row? To answer this question, let's first assume that we get 12 on the first roll, 15 on the second, and 9 on the third. The probability of getting twelve is $\frac{4}{36}$, the probability of 15 is $\frac{2}{36}$, and the probability of 9 is $\frac{1}{36}$. (Why are these correct?) Since these events are independent, we multiply the probabilities to get approximately .00017. All told, there are six different ways we can get a Bingo across the first row (what are they?) and we have considered only one. They each have a probability of .00017, so the probability of a getting a Bingo across the first row on the first three rolls is approximately $6 \times .00017 = .00103$.

In the same way, we could calculate the probability of every other possible Bingo on the above array. Then we would add the results to find the probability of getting a Bingo on exactly three rolls of the dice.

Practice computing with probabilities by doing the following problems.

3. Suppose you roll two fair six sided dice.

   (a) What is the probability that their sum is 8?

   (b) What is the probability that their product is 12?

   (c) What is the probability that both of these outcomes occur? Why is this probability not the product of the answers to (a) and (b)?

   (d) What is the probability that (at least) one of these outcomes occurs? Why is this probability not the sum of the answers to (a) and (b)?

4. Suppose we roll two fair six sided dice. What is the expected value of the product of the results?

Pay to Play

Suppose you were going to play each of the following games repeatedly. Furthermore, suppose that you didn't want to lose any money (in the long run), but you didn't want to win any either; you just wanted to break even. How much should you pay to play each game?

1. This game is played by rolling two fair six sided dice repeatedly, until at least one of them is a six. If the other die is a six then you win \$36; otherwise you win the amount shown on the other die.

2. In this game we roll two fair four sided dice. (What shape might these dice have?) You win a number of dollars equal to the product of the resulting numbers.

3. This game is the same as the last one, except that we are rolling two fair n sided dice.

4. This game is played by flipping a coin repeatedly. If the coin comes up heads on the first flip, then you win nothing. If it comes up tails on the first flip and heads on the second, then you win one dollar. After this, with each successive flip that the coin comes up tails, your winnings double. For example, if the coin is tails for the first three flips and then comes up heads, you win four dollars. However, the coin is not a fair one; the probability of tails is one in three.

5. This game is the same as the last one, except with a fair coin.

Introduction to Conditional Probability

Conditional probability situations are worded like this: given that situation A has occured, what is the probability of situation B? For example, suppose you roll two fair twelve sided dice. (What shape might these have?) Given that the sum is 22, what is the probability that a 10 shows? To answer this, first observe that there are three equally likely ways that the sum could be 22. Namely, the first die could be 10 and the second 12, both dice could be 11, or the first die could be 12 and the second 10. In two out of these three cases a ten shows, so the answer is two thirds.

Here are two practice problems for you to do.

1. Suppose you roll two (fair) six sided dice. Given that the sum is 8, what is the probability that a 4 shows?

2. Suppose you roll two (fair) six sided dice. Given that the sum is even, what is the probability of two 1's showing?

Although it is useful in certain gambling situations, conditional probability is even more important in testing situations. For example, suppose that there is a disease and a test for the disease. If a person has the disease then the test is positive 95% of the time. On the other hand, it has a false positive rate of 3.5%. That is, 3.5% of the people without the disease will still test positive. What most people are interested in is the answer to this question:

Given that I test positive, what are the chances I have the disease?

We can answer this question in the situation described above by considering the following chart, which shows the results of the test for a typical group of 1000 people.

	Have the Disease	Don't Have the Disease
Test Positive	190	28
Test Negative	10	772

To practice with conditional probability, first verify that this chart is consistent with the information given above, and then answer the following questions.

3. According to the chart, what percent of the people selected have the disease?

4. Given that someone in this group tests negative, what is the probability that they do not have the disease?

5. Given that someone in this group tests positive, what is the probability that they have the disease?

Test Positive!

A routine blood test during your thirtieth birthday physical comes up positive for HIV. You are shocked, since the incidence rate in your relatively low-risk group is only 1 in 100,000. The doctor claims the test is 99.9% accurate.

1. How worried should you be? In other words, given that you have tested positive once, what is the probability that you are actually infected with HIV?

2. You go back for a second test, and it comes up positive again. Now how worried should you be[4]?

[4] For the purposes of this question, you should assume that the two tests are independent. That is, you should assume that the fact that you tested positive the first time will not affect the probability of an error on the second test.

Let's Make a Deal!

On the popular game show *Let's Make a Deal!*, host Monty Hall plays the following game with one contestant every day. First, he shows the contestant three doors. Behind one of them is a huge cash prize, but there is nothing behind the other two. Monty then asks the contestant to pick one of the doors. After the contestant has chosen a door, Monty opens one of the other doors. Invariably, the door Monty opens has nothing behind it. Then he gives the contestant a choice: keep whatever is behind the door she or he originally chose, or keep what is behind the other unopened door. What should the contestant do to maximize his or her winnings? (Feel free to simulate this game before answering the question.) Be prepared to justify your answer.

Chickens & Rabbits, Cows & Wranglers

It is important to be able to explain something in several different ways. See how many genuinely different ways you can find to solve the following problems (e.g., numeric, algebraic, graphical, ...).

In the barnyard there are some chickens and some rabbits. There are 50 heads and 120 legs on these chickens and rabbits. How many chickens and how many rabbits are there in the barnyard?	At a roundup the greenhorn counted the total number of heads and legs she saw there, instead of counting the number of cows and the number of wranglers. Her totals were 38 heads and 98 legs. How many cows were there? How many wranglers?

Algebra Puzzlers

1. Here is an algebraic expression. It can actually be reduced to a simpler one. Evaluate the expression by trying various values of x and y. Then guess what the simpler expression is. After doing that, try to explain why the simpler expression is always equal to the one here.

$$\frac{x^2 + xy - y - 1}{x + y + 1}$$

2. The function f defined by $f(x) = \dfrac{1}{1-x}$ is defined for all rational numbers except $x = 1$. (Why not 1?) Here is a table of some of its values. Can you fill in the blanks?

Input	Output
12	$-1/11$
2/5	5/3
-3	1/4
2/3	
$-2/7$	
	1/2

3. Take a number x and compute $f(f(f(x)))$ using f from problem 2 (make sure you know what is being asked here). Try it with another number. Can you explain the result?

Invent Your Own Homework

Invent a problem whose main point is the use of algebraic variables. Make it at least as hard as the cows and wranglers problem but not much harder than the problem about traveling to St. Louis.

Now explain how to solve your problem. How do you go about assigning variables to the quantities in the problem? What do you think is hard about it?

Within your group trade problems and answer the same questions about the problem that you receive. Compare your answers to those of the problem author.

The Field Trip

To enter White Sands National Monument, just south of Alamagordo, New Mexico, and drive down Sand Dune Dr. to see the magnificent 80 foot white gypsum sand dunes, it costs \$5 per vehicle and an additional \$3 per person. Suppose you are leading a school field trip, taking x students to see the dunes.

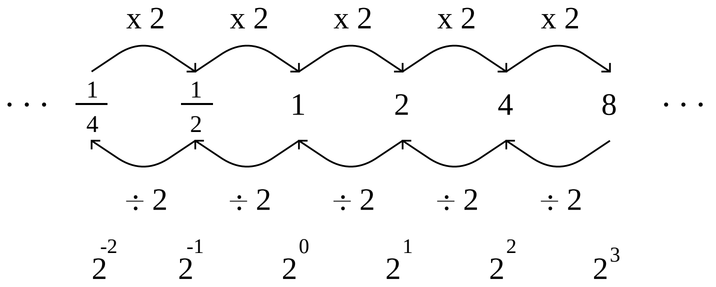
1. What is $C(x)$, the cost *per student* of driving a bus out to see the sand dunes? Remember that each student has to chip in to pay for the driver's \$3, your \$3, and one more chaperone's \$3. Give a formula for $C(x)$, explaining each part of it.

2. State the domain of $C(x)$, remembering what x represents.

3. How many students should come so that the cost per student is \$3.50? What is a formula for $N(y)$, the number of students who should go if they want to pay exactly \$y each?

Exponents

Remember that we use exponents to indicate repeated multiplication. For example, 2^5 is the number we get by multiply five twos together. That is, $2^5 = 2 * 2 * 2 * 2 * 2 = 32$. Recall that we extend the notion of exponents to include negative exponents:



Use the picture above to answer the following questions.

1. (a) What is another name for 2^{-4}? Can you give a decimal representation?
   (b) What about 3^{-4}? 10^{-5}?

2. Suppose we multiply two numbers written with exponents, such as 2^a and 2^b. What is a simple way of writing the answer, again using exponents? Check your answer for various values of a and b. Does the same method work for multiplying 2^a and 3^b? What about 5^a times 5^b?

3. Without using your calculator, explain what number $9^{\frac{1}{2}}$ stands for. (Hint: you might want to use your answer to the last question.) What about $9^{\frac{1}{3}}$? $9^{\frac{3}{5}}$?

Scientific Notation

Many scientific measurements are written in *scientific notation*, which looks like $m \times 10^n$, where m is between 1 and 10. So instead of saying there are 86,400 seconds in a day, we would say there are 8.64×10^4 seconds in a day.

1. Try converting the following numbers into, or out of, scientific notation.

   (a) 0.2

   (b) 314159265

   (c) 1

   (d) 1.18×10^{-3}

   (e) 6.023×10^{23}

   (f) 4.5×10^{-11}

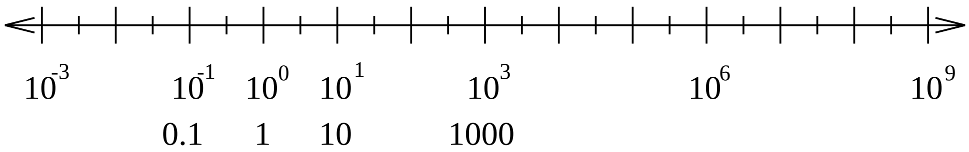
   (g) Convert 0.018×10^3 into scientific notation.

2. Suppose you wish to find the product of two numbers written in scientific notation, such as 3.2×10^{34} and 7.3×10^{41}. How would you do it? Give an algorithm for multiplying $a \times 10^b$ and $c \times 10^d$ and writing the result in scientific notation. Justify your algorithm using the laws of arithmetic and what you know about exponents. Can you do the same for division?

3. Suppose you have two numbers $a \times 10^b$ and $c \times 10^d$ written in scientific notation. Now suppose their product is written in scientific notation as $x \times 10^y$. What are the possiblities for y, in terms of a, b, c, d? Give an example for each possiblity. What if you divide your two numbers?

Logarithmic Scales

Many everyday things like intensity of sound (deciBels) and earthquakes (the Richter scale... well, in some places it's an everyday thing) are measured on what is called a *logarithmic scale*. Logarithms are exponents. On a logarithmic scale, we count by exponents instead of by the whole number, as below:



1. On the number line, mark the locations of the first four numbers from *Scientific Notation*.
2. What number is exactly halfway between 1 and 10 on this number line? (Can it be written as a power of 10?) What about the number exactly one third of the way from 1 to 10? (That is, closer to 1 than 10.)
3. Can you use your answer to question two to find the number halfway between 10 and 100 on the scale?
4. Where is zero on this scale?
5. Bels (B) are actually the exponents of 10 in the measurement of a sound's intensity. A deciBel (dB) is one tenth of a Bel. Thus a 40 dB sound is 10 times as intense as a 30 dB sound (it is 10 dB = 1 B higher, so the exponent is 1 higher). How much more intense is a 50 dB sound than a 30 dB sound? How much more intense is a 31 dB sound than a 30 dB sound?

Estimation

Estimation is one of the most common mathematical activities in our daily lives, yet it often gets ignored in school mathematics. Try to come up with reasonable estimates for the following quantities.

1. How much your hair grows in an hour
2. How much a single sheet of paper weighs
3. The total amount of tuition paid by students at this university
4. The total amount (volume) of human blood in the world
5. The number of dentists in the United States
6. The thickness of one sheet of paper
7. One person's share of the national (U.S.) debt
8. The number of books in a library (it may help to choose a particular library)
9. The number of words you speak in a day
10. The number of gas stations in the U.S.

Chapter 2

Activity Notes

This chapter includes a discussion of (some of) the activities presented in the course pack for Math 132, including the main point of the activity, the solution, and possible "leading questions" to give to groups that are stuck at various points. The discussion is segmented by unit, with a brief overview of the unit (and suggested activities) for each.

Expected time for an activity is given in parentheses following its name. Be aware, however, that if an activity is marked as untested, then this estimate is very rough. Activities marked with an asterisk are not crucial to the exploration of a core topic and can therefore be regarded as optional if time runs short (as it will). Activities marked with two asterisks are doubly so. Any external materials needed are listed in square brackets at the beginning of each description; much of the time, these will be materials you as the instructor provide for use during class, but in other cases you should be sure to give the students advance warning.

One more note: it is quite possible that at some point during the semester an activity will fail to get the point across to your students, despite your best efforts. If or when this happens, be willing to drop the matter for the time being and move on to something that will break the tension. Then go home and write your own worksheet to hand out at the next class meeting which brings into focus the stumbling block the class collectively encountered. If you're short on ideas, consult with a colleague or the course coordinator.

UNIT ONE: Algebra (4 weeks)

The main goal of this section is to get students to use algebra as a tool to solve problems, understand algorithms, and justify arithmetic calculations. Many of these students have had a difficult time with algebra in the past, and so will approach this unit with some fear and little confidence. It might be helpful for these students to focus on letters as abbreviations, so that the algebra is a shorthand for saying things that we could say in English, if we had the time and inclination. Some students will have very poor algebra skills, though these may not be the same students who lack confidence. Careful thought on their part will sometimes help here as well. However, if one is using algebra in some larger (and probably more complicated) context, it is counterproductive to have to stop to think about what operations are legal. As a result, sometimes practice is what's required.

Arithmagic (2 hours) (UNTESTED) — These two problems are standard arithmetic "tricks", which can be understood by introducing variables. If students don't know where to start, encourage them first to try various examples, looking for a relationship between inputs and outputs, or a pattern in the outputs. If students become convinced that Maya can in fact read minds, then casually mention that you too can read minds, and give them a demonstration. Either they will believe you are telepathic (not necessarily a bad thing, all in all), or more likely they will conclude that there is a trick. It is likely at least one group will think of introducing variables, and this will allow them to see through the first trick. This is a good place to emphasize that variables are abbreviations, and that we need to be sure we know what they are abbreviations for. By the same token, we must "unabbreviate" at the end, in order to make sense of our solution in terms of the original problem.

The second trick is not as straightforward as the first, and draws on Math 130 ideas about place value in base ten. If students are struggling at the start of this trick (e.g., they know they will always get 1089, but have no way to start justifying it) then you might want to suggest that they use $100a+10b+c$ to represent the starting number. Given this, it is likely some groups will conclude from the algebra that you always get zero, which they will (we hope) recognize as incorrect. If they get stuck here, you might ask them what the expression in front of them is supposed to represent, and what (if anything) they are assuming about it at each step. You could also have them work an example side by side with the algebra, to see where their numerical computations do not agree with their algebraic computations. This worksheet will most likely take at least a day and a half (two days if you decide to consider other bases), and this might be a good point to send them home to think about the problem, should it work out that way.

A good follow-up question for those groups who have finished the second trick in base ten is whether it is really necessary that the first digit be *two* larger than the second digit. In other words, what is wrong with 413? Can we say anything for numbers like this? Does the algebra tell us anything?

Solutions: For the first trick, let x denote the first number and let y denote the second number. Then the result of Maya's operations is $100x + 10y + 243$, so Maya subtracts $24*$ from your result and reads the first two digits to you. (Here * can be 0,1,2, or 3.)

For the second trick, let $100a + 10b + c$ be your chosen number. The difference between the number and its reversal is $100(a-c)+(c-a)$. Since $a > c+1$, we see that $a-c > 0$, but $c-a < 0$. So in order to reverse the digits, we must first find the digits in terms of a and c. To do this we regroup, as in *Addition and Subtraction Problems.* We obtain

$$\begin{aligned}(a-c)100+(c-a) &= (a-c-1)100+10\cdot 10+(c-a)\\ &= (a-c-1)100+9\cdot 10+(10+c-a).\end{aligned}$$

This last line really is a representation of our partial result in base ten, so we can reverse the digits to obtain $(10+c-a)100+9\cdot 10+(a-c-1)$. Adding this to our last partial result gives 1089, as desired. If this is done in base b, the result will read $1\ 0\ (b-2)\ (b-1)$. For example, in base eight the result will be 1067_{eight}.

Letters (outside of class, half an hour or less in class) (UNTESTED) — This worksheet is intended to be read outside of class; the problems at the end are intended to be refresher homework. It is primarily aimed at those students who have had algebra before and never really "got it"; we need to get as many of these students as we can on board at the outset. (You should feel free to skip this worksheet if you are fortunate enough to have a class in which *all* of the students have strong algebra skills.) As a result, some of your better students will be bored with it, seeing it as trivial, and others may be downright hostile. In this last case it may be worthwhile to appeal to their teacher instincts, explaining that not everyone is as practiced or confident with algebra as they are, and they should take this opportunity to assist their less adept classmates.

Answers: 1. $11j$. 2. $2n-1$. 3. $a=3$. 4. c can be any real number less than 5. 5. Neither. If $n=.1$ then $n+2>2n$. If $n=5$ then $2n>n+2$. 6. $A=(\frac{p}{4})^2$ 7. $.05b+.06r=.9$. 8. The perimeter is $2(2z+3x+1)+y+z$. The area is $(y+z)(2z+3x+1)$. 9. $m=5$. 10. An equation involves an equal sign, an expression does not.

Using Variables (outside of class, half an hour in class) (UNTESTED) — This worksheet is an alternative to *Letters*; you should use one or the other but not both. If you use this worksheet, then you can refer to it throughout the semester to help students link variables to the quantities they represent. That is, if a student is stuck on an algebra problem, it may be helpful to ask them how the various variables are used in the problem: are they ignored, or specific unknowns, etc.

Answers: The questions are the same as in *Letters*; therefore so are their answers. Below we describe the ways variables are used in each question. Although your students may not come up with all of these, an argument can be made for each. We don't claim this list is complete, however; it is possible that your students will argue persuasively for something not on the list for a particular problem.

1. as ignored, as object, as specific unknown, as generalized number, as variable. 2. as object, as specific unknown, as generalized number, as variable. 3. as evaluated. 4. as generalized number, as variable. 5. as generalized number, as variable. 6. as ignored, as object, as specific unknown, as generalized number, as variable. 7. as ignored, as object, as specific unknown, as generalized number. 8. as ignored, as object, as generalized number, as variable. 9. as evaluated, as generalized number, as variable.

It's All Relative (1 hour) — These problems are typical of those students would encounter in a standard algebra course, and so are likely to induce the largest number of flashbacks. The oven problem is reasonably straightforward, though there is no guarantee the students will find it so. The two rate-related problems take a little more time to think about.

The age problem at the end is unsolvable — there just isn't enough information given. However, it is still possible to test whether a solution is correct, so it is worthwhile to ask your students questions such as, "Is it possible Donna is 65 and Jalen is 25?" or "Is it possible that Jalen is 25 and Donna is some other age?" In discussion, solicit their opinions fo the problem. As teachers, they are going to be responsible for choosing problems (and other larger chunks of material) out of textbooks, etc., and should be able to think critically rather than be led like lambs.

Solutions: 1. The problem gives $F=2C$, and substitution into the canonical conversion formula $F=\frac{9}{5}C+32$ yields $\frac{1}{5}C=32$, whence $C=160$, $F=320$. 2. The equation one needs to recall here is $d=rt$, with d distance, r rate or speed, and t time. Thus, letting r represent the speed (in mph) on the six-hour trip, we write $d=r6$ and $d=(r+13)5$, whence $6r=5r+65$ and $r=65$ mph. 3. Here again use $d=rt$ and let d be the full route, t the time spent riding, r the riding speed, and w the walking speed. Then $\frac{2}{3}d=rt$ and $\frac{1}{3}d=w(2t)$. Solving for d/t in each equation yields $\frac{3}{2}r=6w$, so that $r=4w$. 4. Let d be Donna's current age and j Jalen's, and let t be the amount of time elapsed since the "then" to which the problem refers. We have that $d=2(j-t)$ and $d-t=30+j$. Two equations, three unknowns — forget it.

Squares Near Fifty (1 hour) (UNTESTED) — The point of this worksheet is to get students to think in terms of introducing a variable, computing, and then interpreting their results. Most likely, they will let x denote the number to be squared. If they get stuck here, then encourage them to write all of the other important quantities in terms of x (if possible). You will need to bring out the subtlety involving numbers greater than fifty versus numbers less than fifty; you may want to ask them point blank whether the student's method works for numbers greater than fifty, and if not, then whether it can be modified so that it does.

There is some redundancy amongst *Arithmagic*, *Squares Near Fifty*, and *Finger Products*. As a result, you may wish to drop *Squares Near Fifty* altogether, or assign it to be done completely outside of class. If you choose that latter, then you can use *Finger Products* as a platform to discuss *Squares Near Fifty* a few days before it is due.

Solutions: The student's method is correct for numbers *less than* or equal to fifty. One justification goes like this. Let x denote the number to be squared and assume $x \leq 50$, so that $50 - x$ is the distance from your number to fifty. Then the result of the student's algorithm is $2500 - 100(50 - x) + (50 - x)^2 = x^2$, as claimed. This computation makes it clear that if one says "fifty minus the number" wherever the student has said "distance from the number to fifty", then the method works for squaring any number.

The method for numbers near 100 is the same, except you start at $100 = 100^2/100$ and you add/subtract twice the distance to one hundred.

Finger Products (1 hour) (UNTESTED) — By now, many students will assign variables first and start thinking later; these students are likely to get stuck in the middle of the first problem. Encourage such students to try some examples to see whether the proposed algorithm is correct; after spending some time to understand the algorithm itself, they should quickly conclude that it is not. Upon reading the second method, they may decide it is the same as the first; encourage them to read carefully and try some examples. After doing this, they will probably conclude that the second method is in fact different, and that it is correct. This is your opportunity to encourage justification beyond reasoning from examples. Some groups may do all possible examples; encourage them to seek an algebraic proof as well.

To justify the second algorithm, students will quickly assign variables to the numbers to be multiplied, and then get stuck. Encourage them to be organized, and write every quantity in the problem in terms of their variables, starting with the number of fingers up and down on each hand, then the claimed product, etc. Once they have done this, the justification is an algebraic manipulation. The point they should take away from this (and, in fact, from *Squares Near Fifty* and *Arithmagic* as well) is that the real work is in organizing the given information; once this is done, the justification falls out.

The last question requires them to return to the first algorithm and perform the same analysis. They should see that this question amounts to asking for which products the first method happens to give the correct answer, which should be interesting to them since one such product is given as an example.

Once they have finished this worksheet, you might ask your students how far they could have gotten without techniques of algebra; we hope they will be moved to say not very far at all.

Solutions: The first algorithm only works for products ab such that $a + b = 15$. The second algorithm works for all products ab for which a and b are both between 5 and 9 inclusive. As a justification of the last assertion, we have the following table.

quantity	abbreviation
factors	a, b
fingers up on left	$a - 5$
fingers up on right	$b - 5$
fingers down on left	$5 - (a - 5) = 10 - a$
fingers down on right	$10 - b$
product	$10(a + b - 10) + (10 - a)(10 - b) = ab$

The Sorcerer's Brooms (1.5 hours) (UNTESTED) — This worksheet is aimed at getting students to use algebraic techniques to understand how to sum an arithmetic series. Students will find it more challenging than this suggests, however, because it uses algebraic thought processes in several ways and at several junctures. As a result, you should wait to use it until you are either confident at least some groups can finish it or willing to provide much more assistance than usual.

Some groups may be able to answer the first question geometrically, without actually computing $1 + 3 + 5 + \cdots + (2n - 1)$, by observing that each day another "layer" is added to the top and left (say) sides of the square. This explanation is worth bringing out in large group discussion, even if no group has found it, because it requires some understanding of how to translate the series into a picture and back again.

The second question will likely stop everyone. Some students will have looked for a pattern and not found one; this is a good time to remind them that just because they don't see a pattern doesn't mean there isn't one. Once students have reached this point, you may want to take a few minutes to remind them how to compute $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ by adding the left side to $n + (n-1) + \cdots + 3 + 2 + 1$ and regrouping terms appropriately. You should also remind them that in Math 131 they found this sum geometrically, in *Proofs Without Words*.

Once they can compute $1+2+3+\cdots+n$, they should be able to find that $2+4+\cdots+2n = n^2+n$, either by emulating your work or by multiplying its result by two. Encourage them to write down a polynomial for the actual number of brooms Merlin has on day n; this will make it possible (for you) to answer the next question at a glance. It may not be so easy for your students to answer, however, so you should have some leading questions in mind. For example, you might ask them "Can you find an odd number x that makes $32778x+1619$ composite? How about $32778x^2+1619$?" When they do address the question of the primality of Merlin's brooms, they should realize that to answer the question they need to check that he has a prime number of brooms on days 1 through 39. Don't make them check these, though.

The last question is by far the hardest on the worksheet. When your students get stuck, have them account for all of the brooms the Sorcerer has on the nth day, and use this accounting to write down this number as a polynomial in n. You will probably want to give them the quadratic formula when they get to the point of solving the resulting equation. This equation has a positive solution and a negative solution; if there is time at the end you may wish to discuss how one can make sense of the problem setting for negative day numbers, since this ties back to *Double Negatives*.

Solutions: Yes, Glinda will always be able to arrange her brooms in a square. After all, $1 + 3 + \cdots + (2n - 1) = n^2$. Merlin will have $n^2 + n + 41$ brooms after the nth day, 71 after the fifth day, and 151 after the tenth day. Day 40 is the first day he will not have a prime number of brooms, and so (we assume) this is when he will realize his conjecture is incorrect. To decide whether the Sorcerer will ever run out of brooms, we can find an expression for how many brooms he had on day n. We have the following data.

Person	Number of Brooms on Day n
Glinda	n^2
Merlin	$n^2 + n + 41$
Apprentice	$\frac{731}{2}2 + \frac{3}{2}n^2$

Merlin had 41 brooms to start with, and the Sorcerer had 1466, so on day n the Sorcerer has $1466 + \frac{729}{2}n - \frac{1}{2}n^2$ brooms. Setting this equal to zero and solving we find that $n = -4$ or $n = 733$.

One can also explain why the Sorcerer eventually runs out of brooms without determining exactly when, by arguing as follows. Each day, the number of extra brooms the Sorcerer acquires decreases by one. Therefore, there will come a day when the Sorcerer acquires no new brooms. Every day after that, the Sorcerer loses at least one broom, so eventually he must run out.

Geometric Progessions (one hour outside of class, half an hour in class for discussion) This worksheet is intended to be a warm-up for *Trollbooths.* Students may not be able to answer all of the practice problems before class, in which case you can have them work on them for a few minutes before starting *Trollbooths.* Although students need only understand the algorithm for summing a geometric series in order to succeed with *Trollbooths*, our goal is for them to become familiar enough with geometric series' to reconstruct the algorithm on their own, and to explain the connection with converting repeating decimals to fractions. You may wish to have your students read section 44 of Gelfand and Shen's *Algebra*, which is entitled *The Well Tempered Clavier* and which explains the connection between geometric sequences and how one tunes a piano.

Answers: 1. $\frac{25}{4}$ 2. $2\sqrt{2}$ or $-2\sqrt{2}$ 3. $a\frac{r^{2306}-1}{r-1}$ 4. Yes, $r = -\frac{3}{2}$. 5. No, since in this case we would have $r^2 = -2$. 6. Yes, with common ratio r^2. No. 7. Yes, if $r \neq 0$. The common ratio is $\frac{1}{r}$. 8. $2 - \frac{1}{2048}$ 9. $a\frac{r^{21}-1}{r-1}$

Trollbooths (1 hour) (UNTESTED) — This worksheet is aimed at getting students to use algebraic techniques to understand how to sum a (finite or infinite) geometric series. Students will find this worksheet challenging, though perhaps not as much as *The Sorcerer's Brooms*, since it doesn't bring together quite as many ideas. After your students have worked on the first question for a while you will want to tell them how to sum a geometric series, particularly if you have chosen not to use *Geometric Progressions.* One good way to motivate this is to remind them how one converts a repeating decimal into a fraction (as in *Common Fractions and Repeating Decimals*) and how this method generalizes to finding the sum of a geometric series. It is a good idea to point out here that how we learned to convert repeating decimals to fractions greatly affected how we learned to sum geometric series, and more generally that how one teaches a certain topic is often influenced by what one's students will be learning later on.

Once they have the idea for summing geometric series, they should be able to answer the rest of the questions by keeping their goals in mind and using the thought processes they have developed in doing the previous worksheets.

Solutions: To answer the questions in the first paragraph, we have the following table.

Number of Booths	Proportion Taken in Tax
2	$\frac{4}{9}$
5	$\frac{121}{243}$
n	$\frac{3}{2}[\frac{1}{3} - (\frac{1}{3})^{n+1}]$

To address the question in the second paragraph, let r denote the proportion of ale taken at each booth. Then the amount one has left after an infinite number of booths is $1 - \frac{r}{1-r}$. Setting this equal to $\frac{3}{5}$ and solving, we obtain $r = \frac{2}{7}$.

For the last question, the amount you will have after an infinite number of booths is $1 - \frac{2}{5} = \frac{3}{5}$.

UNIT TWO: Graphs (2 weeks)

The point of this unit is for students to become adept at creating, interpreting and using graphs. At the moment this unit consists of three worksheets; there is certainly room for more. For example, one could include some material on histograms and distributions, since this would tie in well with the unit on probability. Most of the worksheets in this unit assume the students have a certain facility with algebra and that they will use this facility with little or no prompting. As a result, this unit should come after the unit on algebra.

Phone Plans (1 hour) — This worksheet presents a problem which is algebraically painful, but whose qualitative solution yields easily to an analysis using graphs. Students will work well on this worksheet, but as you visit each group, encourage them to include all of the information they are given in their graph. It should be easy for them to find all three (qualitatively) different scenarios, but actually describing the conditions on A, B, C, D, E, and F that make each plan cheaper and when will probably tax their algebra and graphical skills to the limit. Nevertheless, you should push them to get a solution that does not require one to draw the graph first.

Solution: For both questions, we consider the graph with cost in dollars on the y-axis and time in minutes on the x-axis. The graph for Cellular One is a horizontal line until $(30, 30)$ and then is a line with slope .39. The graph for Ameritech is a horizontal line until $(15, 19)$ and then is a line with slope .55; this line passes through the point $(30, 27\frac{1}{4})$. Writing down equations for these lines and solving we find that Ameritech becomes cheaper at $47\frac{3}{16}$ minutes.

In the second question, it is clear that if $C > F$ then Cellular One will eventually become cheaper than Ameritech, and once it does it will stay cheaper forever. This can happen in one of two ways. If $EC > 100D - 100A + BC$ then it will happen when Cellular One is still at its constant rate. Otherwise, it will happen after Cellular One has reached its variable rate. On the other hand, if $C \leq F$ then eventually Ameritech will stay cheaper forever. However, if $EC > 100D - 100A + BC$ then there will be an interval in which Cellular One is cheaper; this interval will start at a time when Cellular One is still at its constant rate and end at a time when Cellular one has reached its variable rate. If this condition does not hold, then Ameritech will always be cheaper than Cellular One.

Concavity (1 hour) — Although ostensibly about concavity, this worksheet is in fact aimed at translating between properties of a curve and properties of the data that curve represents. It's a good idea to tell students this up front, since many will wonder (some very vocally) what all of this concavity stuff is good for anyway.

You may wish to emphasize the answer to the second warm-up question, since it can be taken as an axiom on which the answers to all of the group questions can be based. That is, you might want to tell them explicitly that a graph is concave up iff its rate of change is increasing and concave down iff its rate of change is decreasing. However, some students will want to make a distinction between when the graph itself is increasing and when it is decreasing, in part because they will have trouble with the idea that something like -4, -2, -1 is an increasing sequence. This unease will probably permeate the class, so it is unlikely many of your students will use the principle above to its full capacity, preferring instead to rely on simpler ad hoc methods.

Students will answer most of the questions by producing graphs of the relevant data and then deciding on concavity visually. However, 1c and 6 will force them to look for other methods. Some will solve 1c with various ad hoc methods, but 6 will bring the need for a systematic way to the forefront. At some point you will either need to give them the formula for the slope of a line segment, or have them help you produce it. Question four requires some additional assumptions, as the shape of the graph depends on how many incoming calls there are to handle. Nevertheless, you might encourage students to consider the various possiblities, including the (rare) case in which there are exactly as many calls as the operators are able to handle.

Solutions: 1. (a) concave down, (b) concave down, (c) concave up. 2. $\sqrt[3]{x}$ is one such function. 3. The rate of change is proportional to the current population, which is increasing. So the rate of change is increasing and therefore the graph must be concave up. 4. If you assume that there are exactly as many incoming calls as can be handled, then the graph will be concave up since each operator can handle more calls in later hours than in earlier hours. If you don't make this assumption then anything is possible. 5. This data is concave down, which means yearly rates of increase are decreasing. 6. The first condition is this: if $n - m > m - l$ then the graph is concave up. The second condition is if $\frac{c-b}{8} > b - a$ then the graph is concave up.

Microsoft's Profits (1 hour) — This worksheet is another case in which students are asked to transform and reformulate the information in a given graph. If time is short, you may wish to have students do this worksheet outside of class, and turn in write ups for either this worksheet or *Concavity*. Which worksheet to write up can be your choice or theirs.

Questions one and two are warm-ups, though you may have to point out to one or two groups the sentence above the graph about the end of a fiscal year. Questions three, four, and five are concerned with the graph which shows Microsoft's cumulative income; it may take several minutes for students to realize this. In the end, we want students to understand that the graph displayed shows the rate of change of Microsoft's income, so using the principle from *Concavity*, we can easily read off when the TI graph is concave up and when it is concave down. A few students will understand this eventually, though likely not all. The last question is unrelated to the actual definition of TI, which will confuse a few students at first. It is supposed to point out that we cannot decide on the concavity of a graph by looking at percentage increases and decreases, since these are generally percentages of the current total, which is changing with time.

Answers: 1. Because the first quarter of the fiscal year is July through September. 2. Concave down: roughly I-IV 95, IV 95 - II 96. Concave up: roughly III 95 - I 96, II 96 - I 97. 4. Concave up: I - III 95, IV 95 - II 96, IV 96 - I 97. Concave down: III - IV 95, II - IV 96. 5. TI is concave up iff net income is increasing and TI is concave down iff net income is decreasing. 6. Concave up.

UNIT THREE: Probability (2 weeks)

The purpose of this unit is to introduce students to the rudiments of probability, as well as to some of the subtle (and therefore counterintuitive) phenomema which occur in this area.

Introduction to Probability and Expected Value (outside of class) (UNTESTED) — This worksheet is fairly self-explanatory. Students should read it outside of class, though you may wish to spend a few minutes in class discussing the answers to the practice problems.

Answers: 0. For $n \leq 7$, we have $Pr(n) = \frac{n-1}{36}$. For $n \geq 7$, we have $Pr(n) = Pr(14-n)$. 1. 3.5 and 7 respectively. 2. Expected value for one roll is 2.5. For $n \geq 8$, we have $Pr(n) = \frac{13-n}{100}$. For $3 \leq n \leq 7$ we have $Pr(n) = \frac{n+7}{100}$, and we also have $Pr(2) = \frac{1}{4}$. The expected value of the sum is 5. 3.(a) $\frac{5}{36}$. (b) $\frac{1}{9}$. (c) $\frac{1}{18}$. Because these events are not independent. (d) $\frac{7}{36}$. Because these events are not disjoint. 4. $12\frac{1}{4}$.

Pay to Play (1 hour) (UNTESTED) [materials: Two six sided dice for each group and some play money.] — This worksheet is intended to give students practice computing expected value in various situations, as well as to introduce them to some of the vagaries of probability. In considering the first game, most students will quickly conclude that each value on "the other die" is equally likely. After some computation, they will then say that one should be willing to pay \$8.50. If this happens, divide the group into two and have them play the game at whatever price they think will let them break even. Better yet, offer to let them play the game with you for \$8 (what a deal!); after a few rounds you will have taken much of their money and you can leave them to figure out how. The second game is straightforward, but students will benefit from carefully organizing this computation when they consider the third game. Games three through five all involve summing either arithmetic or geometric series, so you should be sure students are familiar enough with these that at least some of them will be able to sum the series without help from you. If desired, game four can be simulated with dice by (for instance) letting 1 and 2 stand for tails and 3,4,5,6 all stand for heads. The results for game five may seem counterintuitive to your students, since the expected value of the game is infinite. If they argue that since the series doesn't converge, you shouldn't be willing to pay anything, offer to pay a dollar a game to play a few games with them.

Solutions: 1. There are six possiblities for the (unordered) results of the two rolls, assuming at least one of the rolls is a six. They are $(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)$. The first five of these can happen in two different ways and the last can happen in only one way. So we have eleven equally likely outcomes which produce winnings totalling \$66; therefore the expected value of this game is \$6. 2. $6\frac{1}{4}$. 3. The expected value of this game is $\frac{(1+2+\cdots+n)^2}{n^2} = \frac{(n+1)^2}{4}$. 4. The expected value of this game is $\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\cdots = 1$. 5. The expected value of this game is $0+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots = \infty$. In words, you should be willing to pay any amount to play this game.

Introduction to Conditional Probability (outside of class) (UNTESTED) — This worksheet is intended to be an easy introduction to conditional probability and crosstabulation tables, as well as a warmup for *Test Positive*. Students should read it outside of class.

Answers: 1. $\frac{1}{5}$. 2. $\frac{1}{18}$. 3. 20%. 4. $\frac{772}{782}$. 5. $\frac{190}{209}$.

Test Positive! (1 hour) — This worksheet is intended both to improve students' facility with crosstabulation tables and to expose them to the fact that even for a test with a tiny false positive rate, most of the positives will be false if the condition being tested for is rare in the population. See the CHANCE database at `http://www.dartmouth.edu/ chance` for information on this and other problems involving conditional probability.

Answers: 1. After one positive test, the probability that you are infected is approximately one in one hundred. 2. After two positive tests, the probability that you are infected is approximately ten in eleven, assuming the two tests are independent.

Let's Make a Deal! (1 hour) (UNTESTED) [materials: Some play money.] — This problem is generally called the Monty Hall Problem, after the host of the show, and is very well known. We have included it here to demonstrate one of the misconceptions that can affect almost everyone who thinks about probability. Be warned, however, that this problem can lead to a discussion that will not end on its own, which you would have to terminate. Some of your students may of heard of this problem before, but we hope that none know the answer. Even if someone does, it is very unlikely that they will be able to justify it to anyone's satisfaction. Upon first reading the problem most people will conclude (erroneously) that there is a fifty percent chance that the money is behind the door chosen, so that it doesn't matter if you switch or not. Encourage groups who believe this to play the game repeatedly, as fairly as possible, keeping track of the results as well as the strategy used. By combining the data from several groups, you should be able to get a large enough sample to see that it really does matter what choice one makes. If you are feeling particularly nefarious, you might ask them if they are willing to play the game (repeatedly) with them as the host, you as the contestant, and a pot of money provided half by them and half by you. By the end, most people should be convinced that the correct strategy is to switch doors, and in particular, that although the money could be behind either of two doors, the possiblities are not equally likely.

This problem was made famous by Marilyn Vos Savant in Parade magazine. For the correspondence in her column on this problem, see the book *Ask Marilyn*, published in 1993 by St. Martins Press.

Solution: The most convincing solution to this problem that I have heard (aside from enumerating all of the possible cases) is the following. You have a one in three chance of choosing the door with the money when you make your first choice. In other words, you have a two in three chance of choosing a door with nothing behind it. Therefore, two out of three times it is to your advantage to switch doors.

UNIT FOUR: Miscellaneous

This unit contains all of the worksheets that don't seem to belong in any other unit, but should remain in the course. They can be interspersed in the course if you so desire; for instance, *Algebra Puzzlers* could easily be used in the algebra unit.

Chickens & Rabbits, Cows & Wranglers (1 hour) — The point of these two essentially identical problems is not just to solve the simple algebra involved, but to find as many different paths to the solution as possible. Students will not mind, as it applies directly to their future jobs (be able to explain something in many different ways). There are indeed many ways to solve the problem including the following:

- straightforward algebra
- trial and error
- graphing the two lines (from the two equations) and finding their point of intersection
- pictorially: drawing the correct number of heads, putting two legs on each head, and then drawing the remaining number of legs, in pairs, on as many of the two-legged creatures as necessary

Solutions: For Chickens & Rabbits, the equations are $c+r=50$, $2c+4r=120$, and the solution is $c=40$, $r=10$. For Cows & Wranglers, the equations are $c+w=38$, $4c+2w=98$, which yield $c=11$, $w=27$. (Too many wranglers, if you ask me.)

Algebra Puzzlers* (1/2 hour) — These two problems can also be doled out piecemeal; the second deals with some rudimentary function concepts.

Solutions: 1. The expression simplifies to $x-1$ by factoring the top (by polynomial long division if desired), although simply substituting values as suggested in the problem will give the simplification. 2. $f(2/3)=3$, $f(-2/7)=7/9$, $f^{-1}(1/2)=-1$. Algebraic simplification will show that $f(f(f(x)))\equiv x$.

Invent Your Own Homework* (1/2 hour, possibly out of class) — This gives the students a chance to write their own problem, something they will need to do as teachers. The good thing about this problem is that some groups will be very creative (both contextually and mathematically); the bad thing is that some groups will essentially copy the chickens and rabbits problem, and no one is likely to provide meaningful commentary on another group's problem.

The Field Trip (1 hour) (Tested only in 112) — Although very few students will realize it, this worksheet is a very simple example of inverse functions. At the same time, it forces students to translate back and forth between the mathematics of the situation and the real problem they're trying to address. The answer, of course, is that cost is a linear function of the number of students, and thus so is $N(x)$, its inverse. The domain of $C(x)$ is a little trickier; be prepared to discuss why it's restricted to the integers, and how the capacity of the bus affects it, since the equation only models the situation if you take one bus.

Exponents (1 hour) (UNTESTED) — This activity deals with the basics of exponents. Students may display a wide range of knowledge of these topics. For those who seem to know it well, you should emphasize that they may have to explain it to their own students, and so should be able to use the picture to justify the fact that $2^a2^b=2^{a+b}$, for example. Encourage them to test their explanatory powers by helping those who do not understand exponents as well. You may be surprised to find that students are not familiar with exponents; they may know that $10^{-1}=0.1$ but think that $2^{-1}=0.2$ instead of $1/2$.

Answers: 1. (a) $1/16 = 0.0625$ (b) $1/81 = 0.\overline{012345679}$; $1/100,000 = 0.00001$ 2. $2^a \times 2^b = 2^{a+b}$, and the same thing works when 2 is replaced by 5. The same thing does not work, however, if one of the twos is replaced by a 3. 3. $9^{\frac{1}{2}}$ should be 3, since it has the property that its square is 9. Similarly, $9^{\frac{1}{3}}$ should be the cube root of three, and $9^{\frac{3}{5}}$ is the cube of the fifth root of 9.

Scientific Notation (1 hour) (UNTESTED) — As its title suggests, this worksheet is intended to help students become familiar with scientific notation. It relies heavily on ideas from *Exponents*, and serves in part to strengthen students' understanding of these ideas.

Answers: 1. a. 2×10^{-1} b. 3.14159265×10^{8} c. 1×10^{0} d. .00118 e. 602,300,000,000,000,000,000,000 f. .000000000045 g. 1.8×10^{1}

2. The product is $ac \times 10^{b+d}$. However, this may not be in scientific notation, since ac may be greater than or equal to 10. In this case, the answer is $\frac{ac}{10} \times 10^{b+d+1}$.

3. There are two possiblities: $y = b + d$ iff $ac < 10$ and $y = b + d + 1$ if $ac \geq 10$. If you divide your two numbers, then the possiblities for y are $b + d$ and $b + d - 1$.

Logarithmic Scales (1 hour) (UNTESTED) — As its title suggests, this worksheet is intended to help students become familiar with logarithmic scales. It relies heavily on ideas from *Exponents*, and serves in part to strengthen students' understanding of these ideas. Most of the questions are routine, but beware of question four and the last part of five; these will test your students' understanding of exponents.

Answers: 2. $\sqrt{10} \approx 3.16228$ or $10^{1/2}$ 3. It should be 10 times as great: $10^{1.5} = 10\sqrt{10} \approx 31.6228$. 4. Zero is not on this scale, since there is no number x such that $10^x = 0$. 5. A 50 dB sound is $100 = 10^{\frac{20}{10}}$ times as intense as a 30 dB sound. A 31 dB sound is $10^{\frac{1}{10}} \approx 1.258925$ as intense as a 30 dB sound.

Estimation (2 hours) — Estimation is something we all do at one time or another; leaving a tip at a restaurant is an example that may not occur to the students (as being estimation). This activity has several quantities to estimate, and you cannot hope to get through all of them in an hour, or even in two, so pick (or have the students pick) the most interesting ones, and leave the rest as review exercises. Make it clear that since this is estimation, you should not expect every group or person to come up with the same figure (in fact, you'll be lucky if they're all within an order of magnitude); the important thing is to be able to reason out each step of the estimation.

Two useful references are John Harte's *Consider a Spherical Cow: A Course in Environmental Problem Solving* (William Kaufman, Inc., Palo Alto, 1985) and Douglas Hofstadter's *Metamagical Themas* (Basic Books, Inc., New York, 1985). Harte's book goes into more detail in environmental problems than is appropriate for a course like this, but his first chapter has some interesting and accessible problems (how long until all the fossil fuel is consumed? until the entire world surface has the population density of New York?), and the appendices contain a multitude of useful Earth-related statistics that you may wish to have on hand for help improving estimates (some excerpts are reproduced below). Hofstadter makes the case (in the section "On Number Numbness") that really understanding large numbers helps you to understand things like the federal budget in terms of trade-offs. He also estimates things like the number of cigarettes smoked in America in a year, or how many letters in a good-sized bookstore.

As always, with these problems, you'll want students to come up with as much of the solution as possible on their own. However, there's no shame in allowing them access to some hard-to-guess statistics, and some which might be useful are below.

weight of 1 kg	2.2 pounds
world population	5×10^9, or 5 billion, people
U.S. population	2.5×10^8, or 250 million, people
average daily dietary intake, per person	2500 calories
world energy consumption	$10^{13} W$ (watts of power)
U.S. energy consumption	$2.5 \times 10^{12} W$
mass of Earth	5.98×10^{24} kg
mass of Earth's atmosphere	5.14×10^{18} kg
mass of Earth's oceans	1.4×10^{21} kg
average radius of Earth	6.37×10^6 m
surface area of land on Earth	$1.48 \times 10^{14} m^2$
average annual rainfall on land	$1.08 \times 10^{14} m^3/yr$
... on U.S. land	$5.6 \times 10^{12} m^3/yr$
average amount of blood in a human	5.5 kg (men), 4.1 kg (women)

Some hints for the estimations, if the students (and you) get stuck:

1. How much your hair grows in an hour: Estimate how often you get your hair cut, how much is cut off when you get it cut, and divide.
2. How much a single sheet of paper weighs: Estimate the weight of a large number of sheets, and divide.
3. The total amount of tuition paid by students at this university: Students should know both the amount of their tuition and the number of students at the university.
4. The total amount (volume) of human blood in the world: Estimate (perhaps using the table above) the volume of blood per human and multiply by the world population. A heuristic estimate for the first quantity may be less accurate, but if the students enjoy the problem, it may be more interesting.
5. The number of dentists in the United States: Estimate the average time for a dentist to see one patient, number of patients (or appointments) a dentist can have in a year, number of times a person sees a dentist per year, number of appointments needed (total U.S.) per year, and put them together.
6. The thickness of one sheet of paper: Again, estimate in large quantity and divide.
7. One person's share of the national (U.S.) debt: Statistics are available for the debt, if attempts at reasoning it out fail.
8. The number of books in a library: multiply number of floors times number of bookcases per floor times number of shelves per bookcase times number of books per shelf. Many libraries can tell you the number of books in their possession if you call them up, so you can have someone check the actual figure.
9. The number of words you speak in a day: Estimate number of conversations per day and length of each one. This one can go either by some kind of word count (transcribing an "average" conversation) or by time spent talking, multiplied by number of words said per minute.
10. The number of gas stations in the U.S.: See the dentist estimation above.