Math 130 Instructor's Guide

by Eric Egge, David Kung, Christopher Kribs & Robin Pemantle
Department of Mathematics, University of Wisconsin

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Chapter 1

What is Math 130?

Math 130 is the first in a three-course sequence of required mathematics courses for future elementary school teachers. Its primary goal is to educate mathematically. We cannot expect schoolchildren to learn anything beyond the rote execution of algorithms if the first ten years of their education is at the hands of teachers who can't themselves go beyond this. By providing elementary educators with a good mathematical foundation, we hope to reverse the downward spiral into which many perceive mathematics education to have fallen. A secondary goal is to introduce students to the cooperative learning environment. Keep in mind, though, that the course is a content course, not a pedagogy course.

The content covered in Math 130 has a *how* and a *what* component. Students will be taught how to think and speak mathematically, many for the first time in their lives. They will solve problems they have not been shown how to do, they will learn to put their ideas into precise language and to prove their assertions when possible. They will learn to break down problems into smaller problems, use trial and error, generalize when appropriate, and to harness their (atrophied) common sense. It is the *how* content that determines the format of the course, necessitating the problem-solving theme and the socratic style of discussion. The *what* content, i.e., the choice of topics, is designed both to help the students cope with mathematics that will arise in the classroom setting, and to give them a solid mathematical base. We believe a solid mathematical base at roughly the high school level is essential even for teachers whose students are four to eight years old. These teachers will be given a good deal of independence in designing lessons and choosing curriculum, and will also be shaping their pupils' attitudes toward the subject. Thus in addition to such obviously applicable topics as fractions, decimals, place value, sets and factoring, we also include some algebra, logic and combinatorics, and require that the students be able to justify their solutions with some rigor.

Some specifics about the student body here at UW-Madison are worth noting. College algebra (Math 112) is *not* a prerequisite for this course, which means that along with the problem solving will by necessity come some algebra review, and indeed a continual reinforcing of the use and meaning of algebraic notation. There will also be some resistance on the students' part to the way the class is run, though probably the students here are less resistant than one would find on average nationwide. The classroom format turns traditional norms upside down by making students take responsibility and authority for determining whether a problem has been solved correctly[1].

Teaching a course by means of cooperative learning, Socratic discourse, and the like involves decentralizing yourself (the instructor) as the focus for learning and authority. This is a skill which takes some time to develop, and it's not always easy. This is why it's best to start preparing to

[1]We do not, in the end, withhold information and techniques they need to know or judgment on what is correct, but we do insist that they develop their own judgment and we train them to do so.

teach a course like this well in advance. Don't worry, though — you're not on your own. If you're expecting to teach in this sequence, you should already have looked into observing others' classes, and the rest of the informal instructor preparation program. Part of this is the extensive set of notes included in Chapter 3 of this guide. Reading and discussing them with folks who have had experience in the sequence will help defray a lot of the anxiety you may have.

This guide is designed to answer any questions you may have, large or small, about this class (although of course conferring with other people who have taught it is an irreplaceable resource), and to provide all the help you need with the details until you feel comfortable enough to stand on your own. Even once you've come into your own, it's a good idea to refer to the guide throughout your first time teaching the course, as we have collected the potential pitfalls of several semesters' worth of teaching it, and it's always best to be prepared. Chapter 2 gives a detailed blow-by-blow of each of the activities in the course pack (including solutions), along with suggested times and a couple of sample syllabi. Chapter 3 discusses how to teach this kind of class. Chapter 4 goes into detail on the write-ups: which activities have traditionally had write-ups associated with them, how often to assign them, which ones are important to assign, how to make the assignments in class, and specifically what to look for in critiquing them. Chapter 5 discusses another type of write-up: the reflections, which get the students to be meta-cognitive (to think about what they've been doing, and what their students will be doing). Chapter 6 discusses exams and evaluation, including a large bank of sample exams from a variety of past instructors. Finally, Chapter 7 details the organization of the directories and LaTeX files used to develop this guide and the course pack, so that you can take things you like and adapt them for your own handouts, exams and worksheets.

Chapter 2

The Activities

2.1 Activity Notes

In this section, we discuss each of the activities presented in the course pack, including the main point of the activity, the solution, and possible "leading questions" to give to groups that are stuck at various points. The discussion is segmented by unit, with a brief overview of the unit (and suggested activities) for each. Sample syllabi are given following the activity notes; when composing your own syllabus, you should be sure to allow time for activities which end up taking longer than expected (there are always several of these) and allowing the less crucial activities either to stand as place-holders for this overflow, or to fill in time if an activity ends sooner than expected.

> **How to use these materials to construct your own coursepack:** There are two ways to do this. The easiest is to download the activities set as is, and add to it a course information page, adapted from the sample given in Section 3, and a syllabus. If you number the pages of the course information handout i, ii, ..., then the activities set will be correctly page-numbered. Then you privately go through the activities and plan a day-to-day syllabus, detailing which activities you plan to do and in what order. The coursepack will contain some activities you don't plan to use, but that is not a problem.
>
> The second way to construct a coursepack is to create your syllabus and then re-order the materials in the order you plan to use them, possibly regrouping units as well. Unless you are certain you don't want to use a given worksheet, it's safest to include it in the coursepack at the end of the appropriate unit. You may choose to add new worksheets of your own design, and will certainly also add your own course information sheet. The advantages of the second method are that you are forced to go through all the materials before the semester starts, that you will probably better internalize a sense of direction for the semester's curriculum, and that you will feel less bound to stick to someone else's ordering and topic selection. The disadvantages are that you have to renumber the pages and that you have to make a new table of contents. You'll find that the first sample syllabus at the end of the activity notes is compatible with the idea of using the activities set as is, while the last has regrouped things enough that you'd probably want to reflect that in the coursepack.

Expected time for an activity is given in parentheses following its name. Activities marked with an asterisk are not crucial to the exploration of a core topic and can therefore be regarded as optional if time runs short (as it will). Activities marked with two asterisks are doubly so. Any external materials needed are listed in square brackets at the beginning of each description; much of the time, these will be materials you as the instructor provide for use during class, but in other cases you should be sure to give the students advance warning.

One more note: it is quite possible that at some point during the semester, an activity will fail to get the point across to your students, despite your best efforts. If or when this happens, be willing to drop the matter for the time being and move on to something that will break the tension. Then go home and write your own worksheet to hand out at the next class meeting which brings into focus the stumbling block the class collectively encountered. If you're short on ideas, consult with a colleague or the course coordinator.

UNIT ONE: Problem Solving (2 weeks)

The problems in this unit are very different mathematically, but chosen to bring up, and into discussion, important aspects of problem-solving: recognizing and explaining a pattern, being systematic, modeling, recognizing when a problem has either no solution or more than one, and what that means. If you decide to vary the small groups initially, this unit is a good time to do it: by the end of the unit, students should have not only a list of some basic problem solving techniques but a good idea of what works and what doesn't in small group dynamics. (See Chapter 3 for more details on this, and on what to do the first day.) Regardless of what you decide to do with group formation, you should end this unit with an in-class large group discussion of group dynamics and what qualities a good group member has (see the relevant reflection in Chapter 5, which you may wish to assign ahead of time).

Poison (2 hours) [materials: counters or chips for use in play] — Math 130 teachers have recognized this activity pretty universally as a great first activity in basically every way. (1) The final and most general solution is somewhat sophisticated and will challenge the students' explanatory powers. On the other hand, it is easy to start solving the problem simply by playing the game, and almost every student will find that s/he gains understanding by doing this. (2) The solution will typically emerge piece by piece. Students will generate several theories, some correct and some not, and the ensuing discussions will be good prototypes for how to argue mathematics. (3) The ultimate authority lies in being able to win the game. The teacher cannot win against a student who thoroughly understands the game. It is a tremendous confidence builder for students to see that something they have figured out is correct no matter what the teacher says. When the point is reached where several students understand completely how to win, teachers often play it up by issuing a challenge: "OK, so-and-so thinks she can win against me, no matter what I do — I am going to try my hardest to beat her; the rest of you vote on who you think will win." This can be done at several stages: when they think they can win the 10 counter game with you going first, when they think they can win the n counter game if they get to say who goes first, and when they (erroneously) think they can win the 10 or n counter game no matter who goes first. Poison should be the basis for the first write-up, and you should insist on continuing the discussion of the solution until everyone claims to understand why the pattern works.

Solution: The winning strategy for the general n counter game is: if n is one more than a multiple of three, choose to go second and always do the opposite of what the first player does. If n is two more than a multiple of three, go first and take one counter, then always do the opposite of what the other player does. If n is a multiple of three, go first and take two counters, then always do the opposite of what the other player does. Here's why it works. In any of these cases, you first force the other person to move when the number is one more than a multiple of three. By doing the opposite of what the other person does, you will together have removed a total of three counters, thus arriving at one more than another multiple of three (the next smallest multiple).

Thus you can force the other player to move when the number of remaining counters is 4 and finally 1, meaning that the other person ends up with the last counter.

Remember not to give any of this away in class! Old habits die hard, especially when the students become frustrated, but the whole point of the activity is that if the students struggle long enough they *will* discover the answer! Here is a list of typical milestones they will come to and blocks they will encounter.

Many students will fixate on odd-even parity and try to argue that the strategy depends on parity, even in the face of evidence to the contrary. It is hard to discover the $3n+1$ pattern, since the number 3 does not appear in the game description and since the pattern $3n+1$ is harder to recognize than the patterns $3n$ or $2n+1$. Initially they all play 10 counter games, and the first thing they discover is that leaving the opponent with 4 is a winning move. Some will now say they have found the strategy. This opens the door to a discussion of what a winning strategy is, and why "leave 4" is not a complete strategy. Some students will think that a strategy can include requiring the opponent to make certain moves; this can also come up in the discussion of what a strategy is.

In general, the students will have a very hard time at first saying exactly what they mean, and being precise (you will get $3x+1$ vs. $3(x+1)$ confusions, for example). Don't get too frustrated by this; just encourage them, and try to help them come up by themselves with a clear formulation of their thoughts. You may want to differentiate explicitly between understanding how to win and being able to explain how to win. A student understands if they can beat a perfectly smart opponent (you) every time. A student can explain how to win if they can give you a set of instructions which you will win if you follow, even if you play dumb and try hard to misinterpret them. Depending on what they come up with, it can be fun to demonstrate this.

Prompts: If groups get stuck right away, tell them just to try playing the game a few times. If they say they don't know how, tell them not to worry about winning or losing, just try it. If they don't seem to be recording the play-by-play, ask if they see any overall pattern — this should lead to a request to see (or hear) specific sets of moves from previous games. If a group appears to have solved the problem much faster than the others, ask to hear what their solution is. If there's a flaw in it, ask a question that will bring this flaw out. If it only works for the ten-counter game, tell them to move on to an eleven-counter game, or even down to a nine-counter game. If they've really hit on the solution without having played the game before, ask them for detailed justifications to "convince" you their solution is right (don't tell them they're right!). If a group finishes early, having obtained a complete solution they can justify, tell them to move on to "Rat Poison" (q.v.).

Also, at each step, as you move around, when you ask where a given group is, also ask if everyone in the group believes (or understands, or agrees with) the response you receive. It's important that the group members help each other keep up, and it probably won't yet be obvious to you which students are the quiet ones.

Large group discussion: When it comes time to discussing the solution in large group, remember that this is the first time for everyone, and students will not be used to explaining themselves (or convincing each other, or even speaking up at all). You will have to keep prompting them with, "Why?" Also try to get as many as possible to come to the board to write even a partial strategy, or a table of recorded moves, etc. — the first activity sets the tone for the semester, and you want them to be used to writing on the board, and talking to each other rather than to you. Encourage the notion that ideas can be road-tested: if two students claim winning strategies, one going first and the other going second, play them against each other. Same if two students suggest different moves from a given position. This is also a good time to start working on their verbal skills. When a student says something essentially correct but poorly stated, write it on the board. If the student immediately wishes to restate it, good. If not, the class as a whole can comment on it and refine it.

One question you should ask at some point (not necessarily right away) is "What does it mean to have a winning strategy?" The answer should include things like: you are in control of the game, you can always force your opponent to lose provided you have choice of going first or second, etc.

The inevitable question "Why are we doing it this way?" may come up as soon as this first large-group discussion, so be prepared to offer a rational, non-confrontational explanation of why "we" (and you) believe in it. Use the problem they've just solved as an example, and try to get some people to admit it gave them a little self-confidence to see that they could eventually come up with the answer by themselves.

Rat Poison** — This is a difficult extension of the Poison game, and is best used only for groups which happen to finish Poison ahead of the rest of the class.

Books and Pages (1 hour) [materials: you may want to bring some scrap paper and scissors, for use in direct modeling] — This activity is about modeling a physical problem; some groups will be systematic and try to think through the problem, but many will end up making physical models of the books. There are two sets of problems: the folio problem and the quarto problem. The quarto problem should be given to those groups which finish the folio problem early; if this means everyone, so be it, but don't spend an extra class period just on the quarto problems. This is the first problem in which there are several good paths to the solution; encourage diversity of explanations, and try to get the class to understand each, and enjoy the diversity.

Solution: In folio printing, each sheet of paper contains four numbered pages, so *The Golden Gate* requires $156/4 = 39$ sheets. Across from page 104 is page 53; on the back are pages 54 and 103. Some ways to discover this are: direct modeling, either partial or complete (yes, some groups will make little 156-page books); making a number line and using symmetry (104 is 52 pages from the last page, so its opposite number should be 52 pages from the first page); noting that the sum of opposite page numbers is always 157.

In quarto printing, each sheet of paper produces eight pages, so a 128-page book requires $128/8 = 16$ sheets. The answer to which pages will be on the top sheet will vary depending on how the folds are made (let the students sort this out themselves, and agree that we can have more than one answer here); one answer is that the upper left-hand quarter will have page 1 upside-down, the upper right-hand quarter will have page 128 upside-down, and the lower two pages will have pages 64 and 65 right-side-up. The answer to the last question, regarding page 13, can be obtained by making parallel counts (two of them up, two of them down) from all four of the pages on the top sheet.

Time to Weigh the Hippos (2 hours) — This starts out looking like a hard but otherwise ordinary algebra problem (here's your first shot at identifying their abilities with algebra) but ends up confusing almost everybody, because the problem has two solutions. Getting the groups to discover this fact, and to understand what it means in the context of the problem, is tricky but very important.

Solution: The first part of solving the problem is identifying variables, and this takes more work than you would expect. For one thing, students tend to assign variables randomly to elephants without thinking about what they mean (in the end, we really want to assign variables to "the weight of the lightest hippo", "the weight of the second lightest hippo", etc.). And even when the subject of assigning such meaning to the variables came up, several students were not sure if this was allowable. Also, the students usually fail to distinguish between the hippo itself and the hippo's weight.

The second, even more problematic, aspect of the solution is writing down equations. It is easy enough to assign random pairs of letters to each of the five given numbers, and unfortunately one of the simplest ways of doing so results in one of the two solutions. In fact, you have to be careful assigning summed pairs of variables to pair-masses, because doing so makes implicit claims about specific hippos. If Hippo X's weight appears in the equation with 312 kg and also the one whose right-hand side is unknown, for instance, we have a problem, because this says that Hippo X is simultaneously one of the two lightest hippos and one of the two heaviest hippos, which is impossible. It may take a surprisingly long time for the students to realize the trouble inherent in this method.

If we choose A to represent the mass of the lightest hippo (in kg), B the second lightest hippo's mass, C the second heaviest hippo's mass and D the heaviest hippo's mass (try to let the students come up with variable names), then we know that $A + B = 312$, $A + C = 356$, $B + D = 466$, and $C + D$ is the unknown mass. Since we don't know which of $A + D$ and $B + C$ is heavier, we don't know which of the two is 378 kg, and which is 444 kg. Thus we should try each of the two possible assignments, and see the result of each assumption. As it happens, both lead to meaningful solutions: If $A + D = 378$, then (after some algebra) we find $A = 112$, $B = 200$, $C = 244$, $D = 266$, and $C + D = 510$. This is the solution groups are most likely to find. On the other hand, if we let $A + D = 444$, then we find $A = 145$, $B = 167$, $C = 211$, $D = 299$, and $C + D = 510$. In the context of the problem, this is bad, because we cannot know what the individual hippo masses are, although we do know that the last pair mass is 510 kg regardless.

Prompts: If they have trouble just getting started, ask them what they're interested in finding out, and tell them to identify it mathematically. (Even so, at this point you may have to push them to make them think of variables.) Tell them to try to quantify the information given to them in the problem. If a group gets as far as the equations, you should probably make sure that they understand what they're doing, although of course it's your call as to whether or not to make them go back and re-examine their assumptions in claiming that "these are the equations" (they may only get confused, shaky as their progress is already). If they get stuck doing the algebra, you can either try to remind them of a particular technique (like substitution) for solving systems of linear equations, or tell them to sit tight with that (if the large group discussion is coming up) and think instead about whether there's any other way to write the equations. Finally, you're likely to have a "good" group or two which struggles through, finds one solution, and contents itself with it. If there's sufficient time remaining before the large group discussion, ask them to go back and write down the assumptions they made early on.

Large-group discussion: At least one group should have a solution by the time you go into large group. With luck, each of the solutions will have been found by at least one group. If this is so, have the two solutions put up on the board, and then ask the class to critique them, or say why one is right and one is wrong. Once they see that the algebra is right in both cases, someone may decide that there are an infinite number of solutions. Even if this doesn't happen, eventually you have to cement the notion of how to assign variable pairs to the pair masses. Preferably, of course, the students do the cementing, and as seen above, the crux is the assignment of the intermediate pair masses to variable pairs (following an ordering assumption along the lines of $A < B < C < D$).

The question of whether random assignment of letter pairs to pair masses always generates a real solution will come up, and it will take a while to separate the difference between a real permutation of pair assignments and one which effectively does a global variable replacement:

Original assignment	Equivalent assignment	Fundamentally different
$A + B = 312$	$D + C = 312$	$A + B = 312$
$A + C = 356$	$D + B = 356$	$A + D = 356$
$A + D = 378$	$D + A = 378$	$A + C = 378$
$B + C = 444$	$C + B = 444$	$B + C = 444$
$B + D = 466$	$C + A = 466$	$B + D = 466$
$C + D =???$	$B + A =???$	$C + D =???$

Once everybody agrees that there are exactly two solutions (and why), be sure to spend time discussing what this means in the context of the problem. Is it good or bad to have two solutions? How could you avoid having more than one? In what other context might it be good? If you carefully marked down the sequence of pairs of hippos and their weights, can you eliminate one of the solutions?

A Frame-Up (1 hour) — This is a reasonably straightforward algebra "story problem", except that it has no solution. Typically students will set it up, reach the solution (which makes a picture -1 inches wide), and then decide they must have made a mistake. There are a couple of ways to misread the problem so that it has a positive answer (e.g. that the border is 1.5 inches on each side), and where many groups end up is some people saying "but that's not the right way to interpret the problem", while others say "but your way of interpreting the problem doesn't yield a good answer." Students are more willing to believe that they've made an algebra mistake than that there's no answer.

Four Cuts* (1 hour or less) [materials: something like a block of cheese and a wire might be helpful — this activity requires three-dimensional visualization, which is difficult for many people] — This is a new addition to the syllabus. The idea is that you can slice a three-dimensional object (like a block of cheese) once lengthwise, once widthwise and once depthwise, each time doubling the number of pieces. Of course, you cannot do this a fourth time in a three-dimensional space; it turns out, however, that you can (if sufficiently clever) make a fourth cut which increases the number of pieces from eight to fifteen. The point of the exercise is more to get the students to reason out why you can keep doubling up to a point, and then not any more, than to get them to find exactly fifteen. In discussing this, it might be helpful to first work on the two dimensional problem (i.e. 3 lines cutting a square).

Squares & Paths (out of class) — These problems are about induction: generalizing and explaining patterns. The squares problem by itself can be done quickly in class as far as solving the 5×5 problem, and even putting up a somewhat generalized formula, but students have a very difficult time making explicit generalizations (using n instead of specific numbers), and not only no idea as to why the pattern can be generalized, but no idea that there should even be a reason why it does. For this reason, it may be tempting to show the students something like *Patterns That Aren't* (Unit Seven). An admittedly somber note: in the past, this assignment has sometimes led to the reflection on getting stuck, or the one on the value of hard problems. Use these if necessary.

Solutions: The number of squares of any size appearing in a 5×5 square grid is $1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$, as there are 5^2 1×1 squares, 4^2 2×2 squares, 3^2 3×3 squares, 2^2 4×4 squares, and only one 5×5 square. This generalizes to $1^2 + 2^2 + ... + (n-1)^2 + n^2$ squares for an $n \times n$ grid. (Students may not be happy with an answer of this form, with n general and the ellipsis there, but you should try to convince them that this is a valid mathematical formula. I once gave my students, in response to this complaint, the formula $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$, and it only confused them further, so I don't recommend it.) The formula can be justified (briefly) as follows: in an $n \times n$ grid, there are

$$\begin{array}{rlc}
1^2 & \text{squares of size} & n \times n \\
2^2 & \text{squares of size} & (n-1) \times (n-1) \\
3^2 & \text{squares of size} & (n-2) \times (n-2) \\
\ldots & \ldots & \ldots \\
(n-2)^2 & \text{squares of size} & 3 \times 3 \\
(n-1)^2 & \text{squares of size} & 2 \times 2 \\
n^2 & \text{squares of size} & 1 \times 1
\end{array}$$

For the paths problem, the solution is to start by counting first the number of paths that start at Y and end at Z (there are 3), then the number which go from X to Z (there are 7), then the number from W to Z (there are 15), and then generalizing. (Actually, it is easier to start at Z and move outward.) In general, there are $2^n - 1$ paths of length n which end at Z; hence the total answer is $2^{26} - 1$ paths from A to Z. There are several ways to justify this formula; one is to note that $2^n - 1 = \sum_{i=0}^{n-1} 2^i = 1 + 2 + 4 + ... + 2^{n-1}$ and account for the different kinds of paths piece by piece.

Group Dynamics Discussion — By the time you finish Unit One, your students should have a pretty good idea of how the class in general, and group work in particular, go, and it's worth taking at least half a period (more if you can spare it and the discussion is lively) to discuss in a large group what the benefits of group learning are, what drawbacks there may be, and what qualities one looks for in a teammate. You may wish to assign the relevant reflection ahead of time. There will likely be one small group in the class which hasn't worked out so well, and for them it will be beneficial to hear others' experiences as well as to discuss what they're looking for without having to name names and offend groupmates. You can even address such a situation explicitly, since they will be very aware that their group hasn't worked well. This also sets a good good example of being able to talk openly about such problems. If you decide to let the students choose their own (permanent or semipermanent) groups, you may want to let them do so after this discussion. Some qualities that may come up in discussion are: good communications, diversity, respect for everyone's ideas, balanced contributions, positive attitude, patience, being prepared.

You may also wish to include in the discussion the compilation of a short list of problem solving techniques and questions-to-ask-oneself which the class has encountered in the preceding problems, as a sort of review.

UNIT TWO: Sets (1 week)

The purpose of this unit is to introduce students to basic concepts in elementary set theory. Students in Math 130 typically have never seen a Venn diagram or used unions, intersections and complements, and by the end of this unit they should be able to do so (note that there is no formal treatment of subsets; this is intentional because of time constraints). The unit begins with a "word problem" which will prove tricky without the use of sets; the point is to motivate the discussion that follows. The introductory material remains somewhat dry after several attempts to spruce it up; the definitions remain for the simple reason that there is no official text for the class, and students need a reference of some sort for such things.

Some things to make sure the students understand before you finish: 1. "The universe" in a Venn diagram may have members not in any of the named sets, but it does include all the members in the named sets. 2. $(A \cup B) \cap C \neq A \cup (B \cap C)$. 3. $\overline{A \cup B} \neq \overline{A} \cup \overline{B}$ (expect a few students to have a surprisingly difficult time understanding this difference). Students should end this chapter with enough fluency about sets to translate among Venn diagrams, standard set notation, and verbal descriptions of sets. Something to make sure the students understand before you begin: Units Two and Three contain a higher proportion of "dry" material than those before and after it; this is because there's some overhead introducing terminology which we'll need to use during the rest of the semester.

Banned Book Survey (1 hour) — This problem, as noted above, is most easily solved via Venn diagrams and sets. Its placement, however, serves more as later motivation for learning sets than as indication of how students should be expected to solve it. Were this problem placed at the end of this unit, all groups would use Venn diagrams. As it is, expect a variety of methods (which is good), and possibly a lot of struggling (which can be good). Even without set theory, properly speaking, most groups will end up defining variables representing [sizes of] different classes of people. (Note that the problem is stated in terms of percentages; in some classes, this confused students, and prevented them from using guess-and-check methods.)

Solution: The most straightforward solution is to draw a Venn diagram with three intersecting circles, each one representing one of the books. If we label each of the eight regions so created, we can solve the resulting linear equations without (one hopes) too much trouble. If we let the number of *Huckleberry Finn* readers be $A + B + E + F$, the number of Bobbsey Twins readers be $D + E + F + G$, and the number of *Lady Chatterly's Lover* readers be $B + C + F + G$, then the information given in the problem can be written $A + B + E + F = .53$, $D + E + F + G = .42$, $B + C + F + G = .36$, $E + F = .25$, $F + G = .14$, $B + F = .10$, and finally $H = .12$, so that (using $A + B + C + D + E + F + G + H = 1$) $A + B + C + D + E + F + G = .88$. The solution (in percentages) is $A = .24$, $B = .04$, $C = .18$, $D = .09$, $E = .19$, $F = .06$, $G = .08$. Thus 6% (F) of the people surveyed had read all three books. In general, with linear equations we need 8 equations to solve 8 unknowns, and that is what we have here. Thus the last piece of information was necessary.

Introduction to Sets (out of class) — Have students read the definitions and work the diagnostic problems on their own, before you get into the activities that follow. Have them compare answers, and, as time allows, either give answers for them to check themselves or explain the ones which seem to be most problematic. In the past, I've usually over-estimated my students' abilities to work with sets, leading to uncomfortable dynamics in class. Note: In questions 2 and 3, (e) and (h) are the same, which leads to a statement of De Moivre's Law(s) in question 4: $\overline{A \cup C} = \overline{A} \cap \overline{C}$. (g), of course, should be the empty set ($A \cap \overline{A}$).

Sets of People (1 hour) — These activities are reasonably straightforward questions that nevertheless produce some good discussion. This is where you will see how well people are catching on to the ideas.

Solutions: 1. Many solutions are possible. Some solutions may take the following general forms (with appropriate definitions): (a) $A \cap B \cap C$, (b) $A \cap \overline{B} \cap \overline{C} \cap \overline{D}$, (c) $A \cup B$, (d) $A \cap (B \cup C)$, (e) $A \cap \overline{B}$.

2. (a) *i.* people who either have a name starting with B or don't have a name starting with A: Barbara Allen, Carrie Ann Baerman, Elise Black, Bill Cooper, Chris Danes, Joy Smith.

ii. people who do not have both a name starting with A and a name starting with B: Jennifer Ahrendt, Elise Black, Allison Camp, Bill Cooper, Chris Danes, Joy Smith.

iii. people who have neither a name starting with A nor a name starting with B: Chris Danes, Joy Smith.

iv. everyone except those people who either don't have a name starting with A or don't have a name starting with B; put more simply, those people who have a name starting with A and a name starting with B: Barbara Allen, Carrie Ann Baerman.

(b) Multiple answers for the first one, including $(A \cap \overline{B}) \cup (\overline{A} \cap B)$; for the second, $A \cap \overline{B}$.

Describing Sets of Numbers (out of class) — The point here is to reason out which set descriptions go with which diagrams (except for the last one, in which the students must write their own problems). It is suggested that you either start them off doing the first one together in class, or give them a chance to compare answers and ask questions in class. (This could be made homework if desired.) The answers match up as follows: 1. (b,i), 2. (h,a), 3. (h,c), 4. (g,i), 5. (g,b), 6. (d,f), 7. (e,g), 8. (c,f)

Extra Problems: Sets** — These problems are designed to give students extra practice working with set concepts and applying them. The first three problems are taken from old exams; the last is simply more Venn diagram practice. These are included not as an important part of the curriculum, but to save the instructor time writing problems if (s)he or the students think they need more practice.

Solutions: 1. Use a two-circle Venn diagram. The question asks for the size of the intersection of the two sets; this is 4 since $|A \cup B| = 20 - 8 = 12$, $|A| = 9$ and $|B| = 7$.

2. Split the class once by gender and once by handedness. boys: 2 LH, 9 RH. girls: 5 LH, 9 RH.

3. Let C be the set of students whose last name begins with a consonant, and M the set with a multisyllabic name. $|C| = 20$, $|M| = 25$, so $|\overline{C}| = 35 - 20 = 15$, $|\overline{M}| = 35 - 25 = 10$, and $10 \leq |C \cap M| \leq 20$. The first question asks about $M \cap \overline{C}$, whose size must be between 5 and 15; the second asks about $M \cap C$, whose size must be between 10 and 20. The point of this question is more to make students realize what can be said (an estimate: less than a precise answer, but better than nothing).

4. (a) $A \cap (B \cup C)$, (b) $A \cap C$, (c) $((A \cap B) \cup (A \cap C) \cup (B \cap C)) \cap \overline{(A \cap B \cap C)}$, (d) $(A \cap B) \cup (A \cap C) \cup (B \cap C)$, (e) $(A \cup B \cup C) \cap \overline{(A \cap B) \cup (A \cap C) \cup (B \cap C)}$, (f) $(A \cap B \cap C) \cup \overline{(A \cap B \cap C)}$

UNIT THREE: Operations & Relations (2 weeks)

The main point of this unit is to make sure the students understand what a binary operation really is and does. All the activities dealing with relations (the second half of the unit) are starred because relations is not one of the core topics and can be glossed over or omitted depending on time. The first couple of pages are text-heavy introductory material; one of the important things to make sure everyone understands in this unit before moving on is the structure(s) of the four basic arithmetic operations, the algorithms for which will undergo some serious scrutiny in Unit Four. "Clock Arithmetic" also provides a good opportunity to compare and contrast the four basic arithmetic operations.

Introduction to Operations (out of class plus 1/2 hour) — This sheet gives some basic definitions and a couple of diagnostic exercises. Tell your students to read it at home, along with Properties of Operations, and try the questions. Then spend an hour in class discussing the answers (you can give them more time in small groups if needed); make sure the discussion goes beyond mere reading of the answers. The algebraic rule for the second table is $a \odot b = 3a - 2b$.

Properties of Operations (out of class plus 1/2 hour) — Here also, there appears to be rather a lot of text. It might be a good idea to tell your students that there will be a short review of what the terms mean at the beginning of the hour you'll spend on this, but *they* will be giving it — that is, when the time comes, make them explain the various concepts. Together there should be enough comprehension that you can help a discussion get going. This is one of the points in the course where you can't avoid giving at least a little bit of lecture, because the students have no framework on which to build their understanding. Rather, this *is* the framework. It's not crucial to spend time in class answering every question on the page: better to spend a little time discussing some of them properly than just to read through the answers. You can always put the answers on the board or on a handout if need be. Speaking of which:

Solutions: CLOSURE: The set given is not closed under $\odot$, but the integers are. COMMUTATIVE: The visual clue is that the table is symmetric (like a symmetric matrix). $\uplus$ is clearly commutative, but subtraction isn't. ASSOCIATIVE: The $\bowtie$ operation is associative since it is effectively a maximum function. Associativity has no visual clues in a table. One noncommutative associative operation is multiplication of quaternions (alternatively, cross product on three standard basis vectors), which can be presented in a sufficiently simple way that the students understand the example (even if they don't know what it's good for). A binary operation which is instead commutative but not associative is the distance function $|a - b|$. IDENTITY: $\bigcirc$ is the identity for $\bowtie$ since it has fewer sharp points than anything else (also visible in the table). $\uplus$ does not. The visual clue is that the row and column corresponding to a given element are identical to the "header" row and column. Of course an identity must be unique, and you should be able to show this to them (by contradiction), if you can't get them to show it to you. INVERSES: $\bowtie$ has no inverses. $\odot$ has all sorts of nice properties, including inverse(s), on the trivial set consisting solely of zero.

If this sheet (and/or the one preceding it) don't go over too well, you may want to give your students the sheet "Examples of Operations" included among the extra activities at the end of this section.

Analyzing Operations (1 hour) — Analysis of the four basic arithmetic operations is the key part of this worksheet; the other two are provided for comparison. The operation in the second question is really addition modulo 4, to which you may want to refer back after doing "Clock Arithmetic", as there are obvious similarities.

Clock Arithmetic (2 hours) — This is a good activity which also allows groups to be creative. Start out by allowing them to experiment with different definitions and follow the consequences of these differences. Before you wrap up, however, you should bring them to understand the more canonical, modulo 12 arithmetic (it is also possible to define clock subtraction if you have the time and inclination, but don't try to go any further), including an analysis of its properties. A reference back to Poison (working mod 3) will be helpful to some students, and completely baffling to others. Be sure to allow and encourage different representations of the definition(s): algebraic rules (two cases), a table, pictures, etc. Their ability to create a clear, precise, complete definition (regardless of how they decide to define the operation) is as important as anything else. Be sure they realize that the set on which one operates must be part of the definition. In the discussion, don't let groups get too bogged down trying to decide how to include minutes and seconds in their definition.

For especially precocious students: have them try to see if they can define multiplication and division in clock arithmetic. Of course, they should discover that they cannot define "clock division", though they will be able to do so on (e.g.) a five-hour clock.

Introduction to Relations* (1 hour, part out of class) — Here again there is some overhead involved, and the idea is that the students read the text and try the diagnostic questions at home, and then you spend some time discussing them in class. Try to maintain a balance between "mathematical" relations and "English" relations. Note, by the way, that it is rather difficult to come up with a relation which is symmetric and transitive but not reflexive; in order to get one, you must have an element of the set in question which is not related to any elements.

Family Relations* (1 hour) — This activity usually goes quickly and well; students are interested, but all the wrinkles usually get ironed out in small groups, and there may be little to say in large group beyond checking answers.

Solutions: Note that there are sometimes more complicated possibilities in this age of non-nuclear families, i.e., my wife's daughter need not be my daughter, etc. These solutions take the simplest path and leave discussions to you. FAMILY RELATIONS. 1. You are my sister-in-law (wife's sister). 2. You are my daughter. 3. You are my aunt (mother's sister). 4. You are my cousin. 5. You are my cousin also. 6. You are my niece or nephew (presumably the person in such a situation would know which!) 7. You are my son-in-law. 8. You are my mother-in-law. 9. You are my father. 10. You are my uncle. 11. You are my grandmother. 12. If there were such a term, you would be my brother-in-law-in-law. Our sisters are married. MORE FAMILY RELATIONS. 1. I made a phone call to your brother. 2. Your brother is my brother. 3. My brother telephoned your brother. 4. My brother telephoned both of us (you and me). 5. I made phone calls to two of your brothers. 6. I called someone who called your brother.

Equivalence Relations* (1/2 hour) — After giving a brief explanation of the idea, have the students read the text and try the questions. It shouldn't take too long. 1, 2 and 5 are; 3 and 4 aren't.

Alphabet Likenesses* (1/2 hour) — This one also goes quickly; the only catch will be the matter of whether or not line segments (and curves) can be shrunk to a point (e.g., whether or not P and R are equivalent). There is some value in letting the small groups argue this out for a little while, but in the end, the last question (which they won't have read yet) allows this, so the first question is meaningless unless you don't allow it.

UNIT FOUR: Arithmetic, Place Value & Bases (3.5 weeks)

Now we get to the core of the course. The point of this unit is to give the students a solid understanding of the place value system, the four common arithmetic algorithms based on it, and the ways kids think about arithmetic. This is done in part by working in other bases, a concept of which most of the students have heard before but which is unfamiliar (and different) enough to make the students look explicitly at the underlying structure and assumptions. It starts out rather free-form and exploratory and eventually focuses on an analysis of the aforementioned algorithms.

After the most recent revision of these course materials (1999), this unit now contains more worksheets than you'll need. Which activities you choose to leave out will largely depend on your philosophical approach. Be sure to think through the whole unit before you begin, so your students don't get a disjointed picture of these topics.

One instructor has noted that it might be beneficial to bring in outside reading, or provide references, for students on how other cultures have represented numbers. (One fictional reference is Richard Adams's *Watership Down*, wherein a culture of rabbits counts up to three, and then has the number *hrair* which represents anything over three, and basically means "a lot".) Certainly this is one unit where the students should see an excellent reason for learning material they may never have to teach, and you should make sure they see this. Other bases are used in other cultures, and in computers, and in any case provide an ideal environment in which to re-examine the old familiar arithmetic algorithms.

Blocks, Rods, Flats and Cubes (1 hour) — This is a new worksheet which is designed to give students a model in which to think about our place value system and other bases. Students will find the self check questions fairly straightforward; the second of these is meant to foreshadow *A Multiplication Problem*. Students will find the problems more challenging. Some will do the first by brute force, while others will reason (with varying levels of justification) that since it takes ten days to have no blocks, it should take one hundred days to have no blocks or rods. You might ask them whether this continues to be the case if the prize is different, say 5 blocks instead of 7. As for the second problem, they may at first decide that any number is possible, but a little probing should disabuse them of this notion. Be prepared to press them to explain that each assembly reduces the number of pieces by 9, so only numbers that differ from 2000 by a multiple of 9 are possible.

The ideas on this worksheet should become part of the lingua franca for any situation in which place value plays a role. In particular, whenever you are discussing the standard arithmetic algorithms, arithmetic tricks, alternative algorithms, and divisibility tests, you should ask them to express their ideas in terms of blocks, rods, flats, and cubes.

Introduction to Bases (1/2 hour) [materials: a large bucket of counters, at least 60 per group] — Distribute chips or counters (blocks have some obvious stacking advantages) to each group and have them work through the activity. This has been done before both by letting groups work entirely independently and by having a sort of large-group discussion. Use whatever feels comfortable. You may want to bring folks together halfway through to discuss adding using the chips. Don't worry about trying to get them to work without direct modeling... yet. You do, however, want to discuss an explicit scheme for converting back and forth between bases without direct modeling, if you haven't already. In particular, to get to base ten, simply list the base ten numeral corresponding to each place value (over the digit in question), multiply, and add. To get to the other base from base ten, divide the largest possible power of the base into the number, subtract, and iterate until you're down to the ones' place. (Other conversion algorithms are, of course, possible!)

Answers: 1. $45_{twelve} = 49_{eleven} = 53_{ten} = 58_{nine} = 65_{eight} = 104_{seven} = 125_{six} = 203_{five} = 311_{four} = 1222_{three} = 110101_{two}$. 2. In base eight, $45 + 17 = 64$; in base four, $211 + 33 = 310$; in base two, $100101 + 1111 = 110100$.

Driving in Septobasiland (1 hour) — This worksheet obliges students, little by little, to discover how to add and subtract in base seven. Many students will be tempted simply to convert repeatedly between base seven and base ten. Encourage them to work directly in "odometer readings".

Answers: 1. 0100 2. $3 \times 7^2 + 3 \times 7 + 5 = 166_{ten}$ miles 3. $7^4 = 2401_{ten}$ miles 4. $900_{ten} = 2 \times 343 + 4 \times 49 + 2 \times 7 + 4 = 2424_{seven}$ 5. in base seven, $1435 + 0324 = 2062$ 6. in base seven, $2153 - 1435 = 0415$

Addition and Subtraction Problems (UNTESTED) — The main point of this worksheet is to get students to begin using the language of place value and binary operations to talk about addition and subtraction with regrouping. By the end, students should recognize that in spite of the strange notation, the place value system is still in place in each example, and that this can be used to easily answer each question.

Plan for this worksheet to take about one hour of class time. Once the students have completed these two worksheets, they should spend some time outside of class thinking about how to explain and justify addition and subtraction with regrouping. To make sure this happens, you should assign *Addition and Subtraction with Regrouping* (which includes a write-up) or *A Base Four Lesson* immediately after doing this worksheet in class. In either case, students should at least read *Addition and Subtraction with Regrouping*, since this is the sort of thing they will be expected to do with multiplication, which is conceptually much more difficult.

Solution: Jeff would have carried one into the tens column and zero into the hundreds column in the first case. In the second, he would have carried two into the tens column and two into the hundreds column. He would have obtained 381 and 1113. If he got $a|b|c$ then he should have obtained $100a + 10b + c$. Alicin should have obtained 3054 and 121. If she obtained $a\ \underline{b}\ c$ then she should have obtained $100a - 10b + c$. The digit $\underline{8}$ belongs in the blank space.

Addition and Subtraction with Regrouping (UNTESTED) — This worksheet models two justifications for addition with regrouping, and has the students write a similar justification for subtraction with regrouping. It should be assigned when students have spent some time using the language of place value and properties of binary operations in the context of addition and subtraction. It could be done in pairs, or even in groups of three or four, to reduce the amount of grading it entails. It could be replaced by *A Base Four Lesson*, but students should read it in any case.

A Base Four Lesson (1 hour, part out of class) — This lesson is perhaps the one students will be happiest to do, as it relates in a very clear way to their future lives as teachers. Make it clear, however, that you will not necessarily be evaluating them on the format of their lesson plan, or other such technicalities, but rather how well (and creatively!) they convey the information in question. It is suggested that you give your small groups a little time at the end of class one day in which to plan strategy, then to have them work on ideas at home, and then compare ideas and assemble the lesson the next time class meets. Encourage them to use manipulatives (and even bring them to class; UW's School of Education has an Instructional Materials Center (IMC) where students can check out all sorts of things) and direct modeling, and to choose and specify the grade level of their lesson. This is an excellent candidate for a group write-up, though it can also be made simply as an ungraded group project (assuming your students are diligent enough to follow through in such conditions).

Ben and Kris's Arithmetic (UNTESTED) — This worksheet is intended to show students some of the nonstandard ways children will find to solve "easy" arithmetic problems, and to give students practice attaching meaning to arithmetic operations. After doing this worksheet in class (it should take an hour or less) have each student turn in their own word problem illustrating a subtraction problem of your choice. (This problem could involve fractions, or not, as you desire.) Place the problems in equivalence classes according to the meaning of subtraction used to write the problem, and give everyone copies of two or three interesting representatives from each equivalence class.

The Waupun License Plate Factory (UNTESTED) — This activity is designed to get students to synthesize ideas of place value, addition, and subtraction to solve a "natural" problem. It should be used while students are working on either *A Base Four Lesson* or *Addition and Subtraction with Regrouping* outside of class. Students will most likely find it to be a challenging worksheet; it is probably not worthwhile to spend more than a day on it, particularly if no one seems to be making progress.

In answering the first two questions, students will need to realize that only the thousands place and higher matter, since one thousand plates are printed per day. For example, the plates printed on the same day as Pat's are exactly those that begin with JUN. Effectively, this means that each day is "numbered" by a three letter sequence. They can then represent the given information on timelines: one for each factory. In this way it should become clear that subtraction is needed for the first question, and (after some thought) that addition is needed for the second question. The last problem can also be represented using timelines, but here it is more difficult to see that subtraction of the birthday plates is what's called for.

As far as justification, if the students view the two birthday plates in the third question as numbers to be subtracted in alternating base twenty six and base ten notation, then ask them to justify their subtraction step by step. How much do they get when they borrow from a number column? from a letter column? If students convert to base ten before subtracting, then ask them to explain this conversion. How much larger would the result be if E became F? If 6 became 7?

Solutions: Viewing MAY and JUN as base twenty six numbers, with A = 0, B = 1, C = 2, etc, we see that the desired plates for the first question will be those beginning with MAY - JUN = CGL. Similarly, the desired plates for the second question are those beginning with MAY + JUN = VVN. For the third question, the desired plate is S9E6P0T - J5U8L3Y = J3J8D6V.

A Multiplication Problem (UNTESTED) — This problem is taken from Liping Ma's thesis, which contains a fascinating account of the responses of various American and Chinese elementary school teachers to several questions, including this one. Our purpose in placing it here is to get students to discuss the standard multiplication algorithm using the language of place value, laws of arithmetic, and blocks, rods, flats, and cubes. By the end, students should recognize that there are many ways to justify the multiplication algorithm and they should be able to explain at least one of them well enough to transfer the explanation to another base. Be willing to spend a day and a half on this worksheet if necessary, and assign a write-up in which they justify the multiplication algorithm, as we have done in *Addition and Subtraction with Regrouping.* If your class is especially precocious then you might have them do this justification in another base. If even this is not challenging enough, you could have them consider how one might design a multiplication algorithm for numbers such as those in *The Waupun License Plate Factory.* However, it seems unlikely that any of your students will be up to this.

Lynna's Arithmetic (UNTESTED) — Like *Ben and Kris's Arithmetic*, this worksheet is intended to show students a nonstandard way a student does a division problem, and to give them more experience attaching meanings to arithmetic operations. It should take about an hour.

Students (and maybe even you) will have trouble with the last part of the last question; below are some example problems. After students have struggled with this question for a while, you should make sure that the problems below (or better yet problems of the students' devising that rely on similar ideas) become public knowledge. One way to do this is to have each group turn in the best problem they can at the end of the hour, and then hand out a "compilation" of these problems, filling any gaps with examples like the ones below.

> I have a rectangular garden which is $544\frac{2}{3}$ square feet. One side of my garden measures $16\frac{4}{7}$ feet. How long are the other sides?
>
> After throwing a pizza party for several hundred of your nearest and dearest friends, you have $544\frac{2}{3}$ liters of pop left over. If the pop fills $16\frac{4}{7}$ kegs, then how much does each keg hold?

After the same party, you decide to store the pop in vats, each of which holds $16\frac{4}{7}$ liters. How many vats of pop are left over?

Suppose that your car gets $16\frac{4}{7}$ miles per gallon. How many gallons will it take to drive $544\frac{2}{3}$ miles?

Arithmetic in Other Bases (1 hour, part out of class) — These problems are practice for performing the four basic arithmetic algorithms in other bases (both smaller and larger than ten). Don't spend time having the students do every single problem on the sheet in class. Rather, pick some (perhaps one of each type, but in any case including at least one division problem) and have the students work on these, with success perhaps coming only in the large group discussion. Sooner or later, the students will want to try their old familiar algorithms in these new bases, but will be unsure whether or not it's okay to do so. When they get to this point, ask them what it means when they do each step — what does it mean to "carry a 1"? "borrow a 1"? Force them to be explicit, and in the end somebody (preferably a student or students) should go through the solution in complete detail at the board: write place values over each place, carry or borrow as needed, cross out the original digit and write perhaps the word name for the resulting number, etc. Ask them questions like: Why can I multiply an n-digit number by a one-digit number by multiplying each individual digit of the n-digit number by the one-digit number? Why do we carry (as in addition)? Why do we shift rows and add when the bottom number has more than one digit? This is where the buck stops, and where the students must come to an understanding first of how the algorithms work in other bases (and why they transfer so naturally), and then (in retrospect) why they work the way they do in base ten. There should be some sort of write-up involved, in the end, perhaps of one of the four algorithms (why it works, how it works, etc.); some instructors have chosen just to make this part of the Base Four Lesson (q.v.).

By the way, if anyone argues that these are not just constructed algorithms, but somehow the only natural way to perform arithmetic calculations, you might want to show them the Russian Peasant algorithm for multiplication (see, for example, C. Reardin, Jr., "Understanding the Russian Peasant", *The Arithmetic Teacher* **20** (January 1973): 33-35).

Answers: 1. (a) 4400, (b) 1010000, (c) EB3; 2. (a) 1707, (b) 10110, (c) 189X; 3. (a) 102022, (b) 4543; 4. (a) 350, (b) 111.1 or 111 1/10

Optional Topic: Decimal Extension* (1 hour) — A natural extension of these ideas, and a reasonable question to consider, is how to extend other bases to non-integers. Of course, the place value system solves the problem immediately, in that to the right of what we normally call the decimal point, we have places representing negative powers of the base in question. In base six, for example, we have "heximal", so that 1/3 is represented as 0.2_{six} — a terminating decimal, which fact itself will spark a flurry of discussion. In general, you can extend the algorithms completely naturally, and have the students divide (e.g.) seventeen by three in base six. This topic will spark interest and discussion, so it's well worth it... unfortunately, you may not have the time, and this is expendable.

Double Negatives (1/2 hour) — A topic related to representations of number is the negative sign, and why two negative signs cancel each other out. The groups should try to come up with some sort of reason why a double negative is a positive, using basic facts you've already discussed. They will probably have difficulty separating the "basic" from the non-basic, since they have become accustomed to taking all of this stuff for granted. (If someone says "who cares why?", reply, "one of your future students probably will, and will ask you.")

One proof goes as follows (below). If you can't get your students to come up with anything, at least try to get them to understand this.

$$
\begin{array}{rrl}
-1 \text{ is defined as the additive inverse of 1, so} & (1 + \ -1) & =0. \\
\text{We can multiply both sides of the equation by } -1\text{:} & -1 \times (1 + \ -1) & = -1 \times 0 \\
\text{Now use the distributive law and zero property:} & (-1 \times 1) + (-1 \times -1) & =0 \\
\text{Since 1 is the multiplicative identity,} & -1 + (-1 \times -1) & =0
\end{array}
$$

Thus (-1×-1) is the additive inverse of -1. But 1 is also the additive inverse of -1, and in fact we know that inverses are unique. Thus these two inverses are the same: $(-1 \times -1) = 1$.

Another negative-related rule about which you might ask students who finish early is the one requiring an inequality to reverse direction when both sides of it are multiplied by a negative number.

UNIT FIVE: Logic (2 weeks)

This unit introduces students to some basic propositional logic, as well as quantifiers. Here, perhaps, the reason for developing an alternative to the often ambiguous and/or awkward nature of natural languages becomes evident. Students will have trouble parsing (or at least pinning down) some of the sentences in this unit; indeed, one activity is specifically designed to get them to acknowledge the need for quantifiers. You might want to mention ahead of time the fact that this is, in a way, the place where natural language breaks down a little, and that they shouldn't get too frustrated trying to figure all the sentences out, as it's natural that they will have to struggle here. On the other hand, they always love Propositional Logic Bingo.

Introduction to Logic (1 hour, part out of class) — This worksheet introduces students to the basics of propositional logic (excluding quantifiers) through some examples using everyday English sentences. The first page illustrates the need for mathematical (formal) logic and introduces the basic symbols; the second provides practice for parsing English sentences logically; the third introduces truth tables; and the fourth, the "Definition Page", serves as a reference. Depending on how much time you have, much of this can be assigned as out-of-class reading, though by the same token you should also always allow some in-class time for discussion. One possibility is to assign the first page as homework, and let the groups work through the second and third pages in class (though it may not be an effective use of class time to discuss each of the truth tables one after the other — have them put up on the board simultaneously and discuss only the problematic parts, or else cover just one or two in class). As you work through this unit, keep in mind that this is probably the most abstract thinking your students have been exposed to. The more you can tie this material in to things they have already learned (esp. properties of operations), the better they'll grasp it.

Glicks, Glucks, Glacks, and Glocks (1 hour) — [This activity was formerly officially called Deductive and Analytic Thinking, but nobody ever referred to it as such, so its name has been changed by popular acclamation.] Here there are four logically different kinds of statement, and students have typically had some difficulties (most, but not all, of which are ironed out on their own) distinguishing between them. In particular, only statement 3 is definitional, and students will have trouble recognizing first that the other statements are not definitional, and then that 3 is. Statement 1 says "if Glick and A, then B". 2 says "if A or B, then Gluck". 3 says "Glack iff A or B". 4 says "if Glock, then A and B". When going over the answers in large group, some instructors prefer to cover all the (a) questions, then all the (b) questions, etc., so that students observe more the contrast between the types of statement. Some of the questions will engender no discussion, but at least some should spark conversation, especially in agreeing that statement 3 really is an if-and-only-if. Leave question 5 for review.

Solutions: 1. (a) true (b) can't tell (c) can't tell (d) can't tell (e) can't tell (f) true 2. (a) can't tell (b) true (c) can't tell (d) false (e) can't tell (f) false 3. (a) true (b) false (c) true (d) false (e) true (f) exactly nine: false, at least nine: true 4. (a) false (b) true (c) can't tell (d) true (e) true (f) can't tell 5. (a) true (b) can't tell (c) can't tell (d) can't tell (e) can't tell

Propositional Logic Bingo (1 hour) [materials: bingo "calling" cards for instructor's use, counters for use in playing (student bingo cards are in course pack), coin/die for true/false] — This game always proves popular with students. Each group chooses a different bingo card (with seven, there should be more than enough for one per group — otherwise your class is too big!), and you (the instructor) call out true or false for different variables. The students have to keep track of which variables are true, etc., and put counters on a square as soon as it can be proven (note that all the center squares are tautologies; each card also has another tautology on its periphery). Each of the letters in the alphabet, except for l and o (which are confusable as 1 and 0), is used. Thus, as caller, you use a deck of special cards (with a different letter on each), or some other way, to choose which variable you are going to "call". Then flip a coin (or other random event) to determine whether the variable is true or false. Thus you might call "a is false", or "g is true". The first team to get five in a row wins.

This game may seem a little frivolous to some, but it gets the students to practice working with formal propositional logic, and, equally importantly, provides a light break in what can be heavy material.

What's True About the Whole Numbers? (1 hour or less) — This is the activity which will leave students ready (if not grateful) for quantifiers. Some good discussion usually comes of this (even if it's mostly the need for a better way to state things, especially with the last statement), and if you finish with time left over, you should introduce them to the two quantifiers, and get them to help rewrite the four statements using them. *Solutions:* 1. true 2. true 3. true 4. false (it says the set is closed under subtraction)

Using Quantifiers (1 hour) — If you have gotten as far in the previous worksheet as introducing them to the quantifiers and rewriting the four statements, tell your students to look this activity over (and answer at least one question) before coming to class. In large group discussion (either before or after small groups), be sure you point out that order makes a difference: that the sentence $\forall x\ \exists y\ y = x^2$, for example, is true, while the sentence $\exists y\ \forall x\ y = x^2$ is false, because in this case the same number y has to be simultaneously the square of every possible x. If time permits, feel free to stop the discussion after a question like 3 (g) or (h), reverse the order of quantifiers, and ask the students what the difference is. (e) and (f) force this discussion.

Solutions: 1. $\forall x\ \forall y\ \forall z\ x(yz) = (xy)z$, which is true for whole numbers. 2. $\forall x\ \exists y\ x + y = 0$, which is false on whole numbers. 3. (a) true, since all whole numbers are (strictly) positive (b) true ($c = 7$) (c) true (d) false (e) false (when $x = 1$) (f) true ($y = 1$ is the smallest element) (g) true (1 is the multiplicative identity) (h) false (see previous paragraph) (i) true (j) true ($a = 0$) (k) false: this sentence says that for each x, there is some y which is not a multiple of x, but if $x = 1$, y cannot help but be a multiple (l) true, since 3 and 5 are relatively prime: this sentence claims that every (whole) number can be written as a combination of positive multiples of 3 and -5. To see this without getting into a long discussion of relative primeness (which will come up in Unit Seven), just note that we can write $1 = 3 \times 7 - 5 \times 4$ ($r = 7$, $s = 4$), $2 = 3 \times 4 - 5 \times 2$ ($r = 4$, $s = 2$), $3 = 3 \times 6 - 5 \times 3$ ($r = 6$, $s = 3$). After that, just increase r by as much as necessary, since any higher number is some multiple of three more than 1, 2, or 3. [4. The first quote can be written $\neg(\exists x :$ value$(x) >$ value(eternal happiness)); value(ham sandwich)> 0. The second can be written $\forall x$ (x is a dog $\rightarrow x$ is an animal); $\exists y$: (y is an animal $\wedge y$ is in my yard).]

UNIT SIX: Fractions (1.5 weeks)

This unit has two main goals. The first is to get your students thinking about fractions, including how and *why* we use them. The second is to explore how grade school kids think about fractions, and how that should inform the teaching of fractions. The philosophy taken here is that students even at the first grade level know quite a bit about fractions, and that an effective teacher will use that knowledge and build from it instead of simply teaching algorithms which are disconnected from it.

Although the two goals are obviously related, the worksheets in this unit are quite clearly divided between the goals. Law and Order, the Condominium Problem, and $\frac{1}{3}$ and $\frac{2}{5}$ are tricky problems which designed to make 130 students think hard about what fractions are and how to manipulate them. The mathematics in Ratio and Proportion Problems and Fractions! is not very complex; you should try to focus the class on *why* things work out as they do and finding multiple explanations for them. This is especially true of "cross multiplying and dividing", which as far as I know, has only the algebraic justification and no context-based one.

Law and Order (1 hour) — This worksheet usually goes pretty well. The point is, in part, to address students' misconception that "fraction" = "number between 0 and 1" (at least, that's why it's in this unit), although it will be up to you to bring that out in large group discussion since the worksheet doesn't. [Example: Why is this worksheet in a unit on fractions? Where in this worksheet did we use the "fact" that the numbers involved are fractions? (Of course, they need not be.)] Perhaps the most interesting point raised in the actual problems is that multiplying need not increase the size of a number.

Solutions: 1. (a) $<$ (b) NMI (c) NMI (d) $>$ (e) $>$ (f) $>$ (g) NMI (h) $<$ 2. (a) C (or possibly B) (b) E (or possibly D) (c) D (arguably C or E) (d) about 17 (e) about 22

The Condominium Problem (1 hour) — This problem is deceptively complicated for students; the first, and perhaps biggest, obstacle, is that students instinctively try to solve for the number of men and women in the complex, which is impossible from the information given. When they inevitably fail, they give up. At this point you may wish to ask them what the problem asks for (a proportion), and point out that they may be able to solve for the proportion without solving for the variables they have invented. Tell them to write down an expression representing the proportion, and to identify (mathematically) all the information given in the problem. (Of course, if they happen to find the right path themselves, you needn't tell them so much.)

Solution: Suppose we let m be the number of men in the complex, and w the number of women. Then $\frac{2}{3}m$ is the number of married men, $\frac{3}{5}w$ is the number of married women, and $\frac{2}{3}m = \frac{3}{5}w$, so that $m = \frac{9}{10}w$. (Thus it is certainly not true that there are as many men as women in the complex.) The proportion of married residents to all residents is thus $(\frac{2}{3}m + \frac{3}{5}w)/(m + w)$. Now substitute $m = \frac{9}{10}w$ and simplify to get married/total $= 12/19$. Other possible solutions: arbitrarily setting the number of people; finding a common numerator (this can lead to a very productive discussion).

Ratio and Proportion Problems (1 hour) — These are relatively straight-forward fractions problems, many of which would be suitable in grade school classrooms. Students' first reactions will invariably be to set up a proportion and then cross multiply and divide. Few of them if any will be able to give a reasoned explanation for this. Force them to find solutions which would make sense to kids who had never seen fractions before.

For most standard problems, two sense-making explanations are available. For instance, in the second problem (15 cans of paint cover 18 chairs, how many chairs can be painted with 25 cans?) the solution is $\frac{18\times25}{15}$. You can make sense of this as $18 \times \frac{25}{15}$ where $\frac{25}{15}$ is the factor by which you're increasing the number of chairs, or as $\frac{18}{15} \times 25$ where $\frac{18}{15}$ is the number of chairs you can paint per can. As far as I know, no similar sense-making explanation exists for doing the multiplication in the numerator first (as in the cross-multiply and divide algorithm).

$\frac{1}{3}$ **and** $\frac{2}{5}$ (1 hour) (UNTESTED) — This worksheet gets at the heart of why many kids find fractions difficult. We use fractions to represent a host of disperate concepts, from part-to-whole relationships to ratios of different types of things (e.g. 2 girls to 3 boys) to rates (e.g. miles per hour). Different operations are required at different times, and this worksheet forces students to use many typical operations all within the same problem context. For two of the questions, you need more information; for number 4, you need to know the size of the class, and given the nature of the numbers, the only realistic possibilities are 15 and 30. This is a good point to refer back to the Hippo problem and the Frame Up.

Solutions: 1. $\frac{2}{15}$ 2. Need to know the ratio of girls to boys (or some other data). 3. $\frac{11}{15}$ 4. Need to know the size of the class. 5. $\frac{11}{30}$

Common Fractions & Repeating Decimals (1 hour) — These questions try to get students to discover the correspondence between common fractions and repeating decimals. Question 1 usually goes straightforwardly, with the proof coming via long division; you may want to try two examples, one a terminal decimal and the other repeating. In question 2, students will usually use their calculators, which show that 1/7 repeats but don't show that 5/17 repeats. Many groups which try long division will give up (or, equally likely, make a mistake) before the repetition begins. Feel free to leave them with this misconception for the moment, as long as there is at least one small group which "sees the light" and can present it in large group discussion. The answers to question 3 are (a) 432/1000 (b) 123/999 (c) 1/9 (d) 1/90 (e) $\frac{22}{10} + \frac{1}{10} \times \frac{43}{99} = 2221/990$ (f) can't be done. Students will most likely get the first few by guess-and-check with their calculator; at some point you should do one of the repeating decimals using the multiply-by-10^n-and-subtract method (e.g., for $0.\overline{123}$, call the number x, multiply by 1000 and subtract $1000x - x$, to align the repeating decimals and get $999x = 123$). Then, for example, with (d), extend this concept by taking the representation from (c) and multiplying by 1/10.

Question 4 brings the matter to a head. By now, students should agree that all repeating decimals can be written as common fractions, with a denominator consisting of as many nines as there are digits in the repeating string (at least, this description applies to the repeating part). The question is whether or not long division of common fractions always ends up with a repetition, and the key is the remainders. When you divide by a number n using long division, the remainder can be any of the numbers from 0 through $n-1$. If the remainder is zero, then the decimal representation terminates. For any other remainder, however, the step that follows will always send you to a unique remainder — for example, in dividing 1/7, every time the remainder is 3, you move down a position, the remainder becomes 30, and you subtract 28 to get a remainder of 2. Thus every remainder of 3 is always followed by a remainder of 2. Thus after $n-1$ steps (if not sooner), you *have* to run into a remainder that you have already seen before. At that point, you're caught in a cycle, and "doomed" to repeat that cycle ad infinitum. Thus every common fraction with denominator of n has a repeating decimal representation with a repeating digit string of length at most $n-1$.

By the way, if things are going well at this point, you may want to press the point that terminating decimals can be written as repeating decimals by repeating 0 at the end. It's a triviality, but some students may argue that not all common fractions can be written as repeating decimals "because some fractions are terminating decimals".

Going to Extremes (1 hour) — This activity may be best done as a large group discussion, although groups which finish the previous worksheet early can go on to this one. The answer to the first question, of course, is that 0.9999... = 1, and the answer to the other questions is no. The first question will almost certainly be most controversial (perhaps the most controversial of the entire course). As in the Frame-Up problem of Unit One, students will typically proceed with some fairly straightforward reasoning, reach a (correct) conclusion which they do not like (multiply $1/9 = 0.\overline{1}$ by 9), and conclude instead that they made an error, and spend the next twenty minutes checking their math for mistakes, or else abandon the argument altogether. Another way to try to convince students of the truth is to abuse the long division algorithm when dividing 1 into 1. Another idea: if 0.99999... and 1 are different numbers, then we ought to be able to subtract them and get something different than zero. But the difference will be smaller than any number you can write down: smaller than 0.01, than 0.001, than 0.0001, etc. Some students may say the difference is written 0.0000....01, with an infinite number of zeroes preceding the 1, but of course that's illegal, as you'll never get around to writing the 1. There are lots of ways you can go with this, and at some point the discussion of question 1 may naturally turn aside to consider one of the later questions (like number 3). Later on, when it comes time for review, a popular question for review sheets has been: Suppose you are teaching decimals to your fifth grade class and a student says, "$.\overline{9}$ is the next largest number below 1." How do you respond?

UNIT SEVEN: Number Theory & Number Systems (2 weeks)

This unit has some of the most enjoyable and interesting activities in the course, perhaps after Unit One. The principal topics to be covered here are factors and primeness (and relative primeness); a few activities illustrating the differences between number systems are sort of tacked on at the end. The stamps problem in particular is both interesting and complex, and likely to end up open-ended in class discussion.

One of the important concepts with which students should come away from this unit is the structure of numbers — that is, how many factors a number has. At some point you should have the students consider questions like "What kinds of numbers have exactly three factors? exactly four? exactly five? How many factors does p^q have, if p is a prime?" (the answers being, respectively, squares of primes; cubes of primes and products of two primes; fourth powers of primes; $q+1$). Eventually you should try to get them to realize (if the activities in the unit don't accomplish this) that one can find the number of factors of a given integer x by writing its prime factorization $x = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$, so that you can see x has $(n_1+1)(n_2+1)\cdots(n_m+1)$ factors, this product obtained by multiplying the number of choices you have for the quantity of p's appearing in the prime factorization of any factor of x. There is a worksheet among the extras at the end of this section of the guide, *Numbers in their Primes*, which may help with this discussion.

Factors & Prime Numbers (1 hour, part out of class) — The first half of this page provides definitions and some diagnostic questions; you should assign it as out-of-class reading, and have the students ready to discuss the definitions and compare answers in class (of course, as always, when introducing new material you should be more willing to give straight answers than otherwise). Answers: 1. (a) 4 (b) no (c) 2, 3, 5, 7, 11, 13, 17, 19, 23, 29 (d) 1, p, p^2, p^3, p^4 2. (a) 20 and 30; 2 and 5 (b) $a \times b = \text{GCF}(a,b) \times \text{LCM}(a,b)$ (c) 101 and 102 (101 is prime) (d) 2; 1; 40; 88; yes, {4, 8, 11} is relatively prime

The second half of this page contains several problems involving factors. Do not spend time in class trying to cover all of these; rather, give them to groups who finish early on any given day, or to the whole class if you finish an activity early and don't want to start something big that day. The first couple of questions deal with the structure of numbers: in particular, the number of factors a number has (see also note in introduction to unit, above), while the latter two questions deal with LCMs. *Note:* If you choose not to have the entire class do the second problem, at some point during this unit you should do something to get them to discover how many factors a given number has (as discussed in the introduction to this unit above), even if it means spending a day on the question.

Solutions to Related Problems: 1. All numbers smaller than 100 have 12 or fewer factors. However, the problem asks for the smallest n such that no number less than 100 has exactly n factors; the answer to this question is 11, since there are no tenth powers of a prime less than 100 (and only numbers of the form p^{10} have exactly 11 factors). The classification is as follows:

Number of factors	Numbers
1	1
2	primes: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97
3	squares of primes: 4, 9, 25, 49
4	cubes of primes and pq: 6, 8, 10, 14, 15, 21, 22, 26, 27, 33, 34, 35, 38, 39, 46, 51, 55, 57, 58, 62, 65, 69, 74, 77, 82, 85, 86, 87, 91, 93, 94, 95
5	fourth powers of primes: 16, 81
6	p^5 and p^2q: 12, 18, 20, 28, 32, 44, 45, 50, 52, 63, 68, 75, 76, 92, 98, 99
7	p^6: 64
8	24, 30, 40, 42, 54, 56, 66, 70, 78, 88
9	p^2q^2: 36, 100
10	p^4q: 48, 80
11	
12	60, 72, 84, 90, 96

2. To answer this question, we really need to understand the fact stated in the introduction to this unit (above) about how many factors a number has. Then we realize that we want to have as many different primes involved as possible, since multiplying a number by a "new" prime doubles the number of factors, while multiplying by an "old" (already used) prime p_i increases the corresponding $(n_i + 1)$ by less than a factor of 2. Thus we get $2 \cdot 3 \cdot 5 \cdot 7 = 210$. We can multiply by no more factors without passing 1000, so we multiply by as many of the "old" primes as possible. We can multiply by 3 once or by 2 twice. Doing the latter, we get $840 = 2^3 \cdot 3 \cdot 5 \cdot 7$, with 32 factors. Replacing any of the single-power primes with the next smallest prime, 11, increases the product beyond 1000, so the answer should be unique.

3. The question here is deliberately vague: does it ask "when will Jupiter and Saturn next be in the same position as they were in 1978?", or merely "in the same position as each other?" For the answer to the former, more straightforward question, the answer is 1978+LCM(12,30)= $1978 + 60 = 2038$, at which point Jupiter will have made 5 revolutions and Saturn 2. However, in

order for that to happen, Jupiter must have passed Saturn by twice before 2038 (the third time being when they realign in 2038), and if we assume that the planets' orbital speeds are close to constant (which need not be true, realistically speaking), then those passes should have happened at regular intervals. Hence Jupiter passed Saturn once after 20 years, in 1998, and again after 40 years, in 2018. It is possible that none of the small groups will come up with this interpretation of the problem; if you have time left in class and wish to discuss the problem further, you can ask them about this yourself.

4. (a) LCM(81,900) is $9 \cdot 9 \cdot 100 = 8100$ seconds. (b) During 3600 seconds, we get an increase of $3600/10 = 360$ births and $3600/81 = 44\ 4/9$ immigrants, and a decrease of $3600/16 = 225$ deaths and $3600/900 = 4$ emigrants, for a net increase of 175 4/9. You should ask the students what it means to have a fractional person; the correct interpretation, of course, is as an average, and you may get 0 or 1 depending on where the immigrant light was in its flashing period when the hour began.

The Stamps Problem (2 hours) — This is possibly the most complex and open-ended problem in the course. It revolves around relative primeness, although it will be some time before the students get any notion of this. The 4-cent and 9-cent problem eventually (beyond a certain point) allows one to make any amount of postage, while the 9-cent and 21-cent problem does not, for the simple reason that 9 and 21 are both multiples of 3, so only multiples of 3 can be made. It will take the students at least an hour to get even this far, however, so be patient.

The crux of the problem in the beginning appears to be the way one keeps track of the amounts "makable" and "unmakable"; more than any other problem in this set of materials, the ease of solution depends directly on the method of representation chosen. Making two lists, one of each type of amount, gives little insight into the underlying structure. The optimal representation appears to be a grid-style list of all numbers (starting with 0 or 1), with each row of length one of the two stamp denominations (e.g., 9). One can then circle each of the "makable" amounts, and see that once a number in a given column is circled, all the numbers [physically] below that number in the same column are also circled, as they are multiples of the row length added to a "makable" number. An intermediate strategy is to make several lists, organized by one of the denominations, e.g., "multiples of 9", "4 plus multiples of 9", "8 plus multiples of 9", etc. (This latter can also manifest as a grid, which seems to be more helpful for the students.) There are, of course, many possible strategies, but students may choose one which gives them little insight for the problem and eventually get stuck. In this case it is your decision whether or not to give them a push. One instructor suggested a grid representation but let the students decide the width; another gave no suggestion, with the result that discussion took an extra hour before students came to an agreement.

Try to keep the students from getting discouraged, and if they tell you they cannot make any generalizations (as asked in the problem) after solving the first two questions, tell them to (a) look for the differences between the two pairs of numbers, or (b) make up some more pairs of small numbers and try answering the same question with them. (Ideally these ideas would occur to them naturally, but their problem solving skills are not likely to be honed this sharp yet.) In any case, realize that you will not get as far as a complete solution with explanation in class, and take what you can get (don't spend more than a week on this, interesting though it is). You want to get to the point where students recognize that the important thing is the relative primeness of the denominations involved, and why.

Solution: In complete generality, the answer is that given stamps of denomination a and b, one can make amounts which are multiples of GCF(a,b), and the last such multiple which cannot be made is LCM(a,b)$-a-b$. Thus if a and b are relatively prime, one can make every amount greater than $ab - a - b$. (Note: Every part of the preceding solution was originally discovered by a Math 130 student!) Less generally, given denominations 4 and 9, one can make 4, 8, 9, 12, 13, 16, 17, 18, 20, 21, 22, and everything beyond (but not including) 23. Given denominations 9 and 21, one can make all multiples of 3 except 3, 6, 12, 15, 24 and 33. It should seem completely reasonable (though the students will probably not have thought of it in these terms) that since 9 and 21 are multiples of 3, you can only hope to make multiples of 3 using them.

As short a proof as I can find of why $ab - a - b$ is the last "unmakable" number given a and b relatively prime: Make a table of numbers in rows of b entries each:

$$\begin{array}{rrcrr}
1 & 2 & \ldots & b-1 & b \\
b+1 & b+2 & \ldots & 2b-1 & 2b \\
2b+1 & 2b+2 & \ldots & 3b-1 & 3b \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
(a-1)b+1 & (a-1)b+2 & \ldots & ab-1 & ab
\end{array}$$

Note that numbers in the same column differ by a multiple of b. Thus, the first (smallest, or highest-up in the table) multiple of a in each column will be the first "makable" amount in that column (by using, say, m a-stamps). Every number in the column below ma can be made by using m a-stamps and the same number of b-stamps as there are rows separating ma and the number in question (e.g., the number three rows below ma is $ma + 3b$ and can be made with m a-stamps and 3 b-stamps).

Thus the question of being able to make all numbers is equivalent to finding a multiple of a in each of the b columns (except for the last one, if you like, since they're all multiples of b in that column). So we need at least the first b multiples of a to hit all the columns (i.e., one multiple of a for each of the b columns). Suppose the first b multiples of a *aren't* all in different columns, i.e., suppose there's one column with two multiples of a in it. Since the numbers are both multiples of a, their difference is a multiple of a. Since the numbers are in the same column, their difference is a multiple of b. Since a and b are relatively prime, their difference must be a multiple of ab. Thus the greater number (lower down in the column) is at least b multiples of a after the smaller number, and hence is not in the first b multiples of a. So the first b multiples of a do it, and eventually we can make anything.

Note also that the bth multiple of a is, as previously mentioned, not especially important because it hits the bth column, which were all multiples of b anyway. So the last column to be hit is the one hit by the $(b-1)$th multiple of a, $(b-1)a$. Thus the last number that cannot be made is the one directly above $(b-1)a$: $(b-1)a - b$, or $ab - a - b$.

There are several other similar "tabular" ways to look at this problem which you will undoubtedly encounter from your students. In this activity more than most, it's vital to keep your mind open to solutions which never would have occured to you (but still have some validity).

There are several levels of understanding to this problem: Level 1. With 4 and 9 you can eventually make everything; with 9 and 21 you can't. Level 2. The difference is due to 4 and 9 being relatively prime; with 9 and 21 you can only make multiples of 3. Level 3. In general you can eventually make every number precisely when the two numbers are relatively prime. Level 4. In general you can make every number beyond $ab - a - b$ if a and b are relatively prime; otherwise you can eventually make multiples of GCF(a,b). Level 5. Proving $ab - a - b$ is the last number one cannot make; or knowing the generalized formula (above). Level 6. Proving the generalized formula.

You should try to get the class discussion to Level 3. Some individuals will only get to Level 2; others will get as far as Level 4. One or two may get to Level 5. I myself have not bothered getting to Level 6.

An interesting extension of the stamps problem is the Chicken McNuggets problem (Sample Exam 6a, problem 6).

The Lockers Problem (2 hours) — This is a good problem for finding and recognizing patterns as well. It can come before the stamps problem if desired. The bottom line is that squares are the only numbers with an odd number of factors; if students have recently solved Related Problem 1 at the beginning of the unit, they may have made this observation and remember it (not likely, but just a warning). Here as with the stamps problem, there will be a multitude of ways of keeping track of which lockers are open and which closed, some of which will prove more useful than others. Here, however, students are usually able to find a representation that serves sufficiently well.

Solution: The number of times the door of a given locker is moved is equal to the number of factors of the locker number. All numbers have pairs of factors (and hence an even number of factors) and thus end up closed, except for the ten squares 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, which have a square root which only counts once (it's paired with itself). Students are likely to start off modeling just the first twenty (e.g.) lockers and going through and making the twenty first passes, and then trying to see the pattern. Eventually they will see that only squares end up open, but by the end of the large group discussion they should be able to say why square-numbered lockers end up open and the rest don't.

Patterns That Aren't (1 hour, possibly less) — The point of this exercise is to illustrate that a pattern must be justified or explained before we can afford to believe that it really does go on forever. This exercise is placed at this point in the curriculum because the examples of cyclotomic polynomials and Mersenne primes both have to do with factoring. However, it is appropriate to use this activity earlier in the course if warranted — if students argue that recognizing a pattern should be enough (e.g., in the Squares and Paths problems in Unit One). Be prepared to discuss how to tell whether a number is prime or not — things like checking primes up to $\sqrt{n}$ and divisibility rules for 3.

If students seem interested enough in the problem, you might add that in 1971 an IBM researcher discovered that $2^{19,937}-1$ is prime, and was the largest known prime for some years, until two high school students discovered in 1978 (with the help of their computer) that $2^{23,209}-1$ is prime. (Five years later, another computer showed that $2^{132,049}-1$ is prime.)

Solution: The first nine numbers of the form 2^n-1, $n \geq 2$, have a correspondence between their primeness and the primeness of n, but the pattern breaks down at $n = 11$. Here is a table showing the first ten numbers and correspondences for the Mersenne prime problem:

n	n prime/composite	2^n-1	2^n-1 prime/composite
2	prime	$2^2-1=3$	prime
3	prime	$2^3-1=7$	prime
4	composite	$2^4-1=15$	composite (3×5)
5	prime	$2^5-1=31$	prime
6	composite	$2^6-1=63$	composite $(3^2\times 7)$
7	prime	$2^7-1=127$	prime
8	composite	$2^8-1=255$	composite $(3\times 5\times 7)$
9	composite	$2^9-1=515$	composite (5×103)
10	composite	$2^{10}-1=1023$	composite $(3\times 11\times 31)$
11	prime	$2^{11}-1=2047$	composite (23×89)

Counting Blocks (UNTESTED) — The point of this exercise is for students to (re)discover tests for divisibility by eleven, nine, and eight, and to recognize squares and cubes of small numbers. It should last about a day and can come immediately after *Factors & Prime Numbers* if desired. Some students may finish early; if so, you may wish to give them the following riddle.

> My social security number uses each of the digits 1,2,3,4,5,6,7,8,9 exactly once. Not only that, but the number formed by the first two digits of my social security number is divisible by two, the number formed by the first three digits is divisible by three, etc. What could my social security number be?

Solution: The number of red blocks has the form $a^3 + b^2$; the only such number beginning with 2 and ending with 0 is $260 = 14^2 + 4^3$. The number of blue blocks is a square; the only one in the seven twenties is $729 = 27^2$. The total number of blocks is divisible by 8, so the last digit must be 2. It is also divisible by 9, so the first digit must be 4. Subtracting, we find that the number of green blocks is 3763.

If the blocks come in packages of 22, then the total number must be divisible by 11 and 2. Therefore the last digit could be 0,2,4,6, or 8. Of these, 8 is ruled out since the resulting number cannot be divisible by 11; four cases remain. The corresponding first digits are 2,4,6, and 8. Of these, 2 is ruled out since the resulting number of green blocks is a three digit number. The remaining numbers of green blocks are 2763, 4765, and 6767.

Square Root of 2 (1 hour, possibly less) — This activity begins the part of this unit on number systems by proving that there is a number (namely, $\sqrt{2}$) which cannot be written as a rational number. Have the students answer the two diagnostic questions at the beginning, and spend 30 minutes reading and studying the proof given, for homework. Then, in class, have the students work through the proof line by line in small groups and write down answers to the questions for reflection at the end. You may choose not to have a large group discussion if the small groups do well enough. The students should not be made to memorize the proof, just to understand it, and the concepts involved. You may have to start off reminding the students of the difference between the rationals and the reals. The mail point of this is to familiarize students with what a rigorous mathematical "proof" is.

Life in Hell (1 hour or less) — This activity is about countability, and the bottom line is that the whole numbers, the integers and even the rationals are countably infinite while the reals are uncountably infinite. Students are likely to figure out strategies for whole numbers and integers, but you'll be lucky if a group figures out a strategy for the rationals (although you can give them a nudge by suggesting that they try to find an organized way to write fractions). Students will probably conclude, by virtue of failing to find a strategy that works, that the reals are uncountable, and those groups which fail to find a strategy for rationals will probably make the same claim there.

Solutions: The strategy for whole numbers is obvious: count up. For integers, one idea is to guess 0, 1, -1, 2, -2, 3, -3, ... For rationals, take the xy-plane and write the fraction x/y at each point (x,y) where x and y are integers (skipping, of course, the x-axis), and mark 0 at the origin. Then one can draw a continuous path starting at 0 and working its way around and outward (clockwise or counterclockwise) which eventually intersects all the named points (i.e., the Gaussian integers). Of course, there will be some duplication — for example, the points (1,2), (2,4), (3,6), ... all represent the number 1/2. Now, starting at 0, guess each of the numbers as you come to it along the path. (Naturally there are different ways to represent this strategy.)

If you decide to try to prove to your students that the real numbers are uncountable, you can use the diagonalization argument: Suppose the reals are countable. Then write a list, aligning the decimal points and writing the real as an infinite decimal (repeating if need be, padding zeroes if need be). Now go back and create a number by starting with the first decimal place of the first number and moving down and to the right diagonally, as shown in the table below. Then increase each of the digits by one (changing 9s to 0s). This new number does not appear anywhere in your infinite list of reals, since at each point the nth decimal place differs from that of the nth number in the list. Thus no list can possibly hope to contain all reals.

Ostensible list of reals:

0.**1**23465347074353...
1.0**0**0000000000000...
2.14**3**638943214352...
6.082**8**40406608256...
2.7182**8**1828459045...
3.14159**2**653589793...
9.750000**0**00000000...
4.4444444**4**4444444...
6.01860186**0**186018...
0.142857142**8**57142...
1.2312312312**3**1231...
0.93933933393**3**3339...
1.414213562373**0**95...
0.9999999699999**9**9...
8.72348794589548**7**...

Number made from diagonals:
0.103882040833097...

Number made by incrementing diagonals:
0.214993151944108...

Comparing Number Systems (1 hour) — This activity wraps up the unit and the section on number systems, and reviews the use of quantifiers. Depending on how quickly the small groups work, and whether they stumble, there may be little to say in large group beyond checking answers.

	stmt #	wholes	naturals	integers	rationals	reals	
	1	F	F	T	T	T	(additive inverses)
	2	F	F	F	T	T	(mult. inverses)
	3	F	T	T	T	T	(zero property)
	4	F	F	F	F	T	(square root of 2)
Solutions:	5	F	F	F	F	F	(square root of -2)
	6	T	T	T	T	T	(no upper bound)
	7	F	F	F	F	F	
	8	F	F	F	F	F	
	9	T	T	T	F	F	(squaring never decreases)
	10	F	F	F	F	F	

You can ask groups which finish early a question like the following: Find a statement using quantifiers ($\forall$ and $\exists$) that's true for $\mathbb{W}$, $\mathbb{N}$, and $\mathbb{Z}$, but not for $\mathbb{Q}$ or $\mathbb{R}$.

2.2 Sample Syllabi

Numbers in parentheses are unit numbers. Asterisks denote activities done either partially or completely out of class (see the main part of this section for details).

Syllabus 1

Week 1:
- (1) First day activities
- (1) Poison
- (1) Books and Pages

Week 2:
- (1) Time to Weigh the Hippos
- (1) A Frame-Up
- (1) Group Dynamics Discussion
- (1) Squares & Paths*

Week 3:
- (2) Banned Book Survey
- (2) Introduction to Sets*
- (2) Sets of People
- (2) Describing Sets of Numbers*

Week 4:
- (3) Introduction to Operations*
- (3) Properties of Operations*
- (3) Analyzing Operations
- (3) Clock Arithmetic

Week 5:
- (4) Blocks, Rods, Flats, and Cubes
- (4) Introduction to Bases
- (4) Driving in Septobasiland
- (4) Addition and Subtraction Problems

Week 6:
- (4) Addition and Subtraction with Regrouping*
- (4) Ben and Kris's Arithmetic
- (4) The Waupun License Plate Factory
- (4) A Multiplication Problem

Week 7:
- (4) Lynna's Arithmetic
- (4) Arithmetic in Other Bases*
- (4) Double Negatives
- review for midterm

Week 8:
- midterm
- (5) Intro. to Propositional Logic*
- (5) Glicks, Glucks, Glacks, and Glocks

Week 9:
- (5) Propositional Logic Bingo
- (5) What's True About the Whole Numbers?
- (5) Using Quantifiers

Week 10:
- (6) Law and Order
- (6) The Condominium Problem
- (6) Ratio and Proportion Problems

Week 11:
- (6) $\frac{1}{3}$ and $\frac{2}{5}$
- (6) Common Fractions & Repeating Decimals
- (6) Going to Extremes

Week 12:
- (7) Factors & Prime Numbers*
- (7) The Stamps Problem

Week 13:
- (7) The Lockers Problem
- (7) Patterns That Aren't
- (7) Square Root of 2*

Week 14:
- (7) Life in Hell
- (7) Comparing Number Systems

Week 15:
- review for final
- (and catch up)

Syllabus 2

Week 1:
- (1) First day activities
- (1) Poison
- (1) Books and Pages

Week 2:
- (1) Time to Weigh the Hippos
- (1) A Frame-Up
- (1) Squares & Paths*
- (7) Patterns That Aren't

Week 3:
- (1) Group Dynamics Discussion
- (2) Banned Book Survey
- (2) Introduction to Sets*
- (2) Sets of People*
- (2) Describing Sets of Numbers*

Week 4:
- (3) Introduction to Operations*
- (3) Properties of Operations*
- (3) Analyzing Operations
- (3) Clock Arithmetic

Week 5:
- (4) Blocks, Rods, Flats and Cubes
- (4) Introduction to Bases
- (4) Driving in Septobasiland
- (4) A Base Four Lesson*

Week 6:
- (4) Addition and Subtraction Problems
- (4) Ben and Kris's Arithmetic
- (4) The Waupun License Plate Factory
- (4) A Multiplication Problem

Week 7:
- (4) Lynna's Arithmetic
- (4) Arithmetic in Other Bases*
- review for midterm

Week 8:
- midterm
- (5) Introduction to Propositional Logic
- (5) Propositional Logic Bingo

Week 9:
- (5) Glicks, Glucks, Glacks, and Glocks
- (5) What's True About the Whole Numbers?
- (5) Using Quantifiers
- (6) Law and Order*

Week 10:
- (6) The Condominium Problem
- (6) Fractions!
- (6) Ratio and Proportion Problems

Week 11:
- (6) $\frac{1}{3}$ and $\frac{2}{5}$
- (6) Common Fractions & Repeating Decimals
- (6) Going to Extremes

Week 12:
- (7) Factors & Prime Numbers*
- (7) The Stamps Problem

Week 13:
- (7) The Lockers Problem
- (7) Counting Blocks
- (7) Square Root of 2*

Week 14:
- (7) Life in Hell
- (7) Comparing Number Systems

Week 15:
- review for final
- (and catch up)

Syllabus 3

UNIT ONE: Problem Solving (3 weeks)

- Poison (2 days, including time for course intro)
- Books & Pages (1 day)
- Time to Weigh the Hippos (2 - 3 days)
- A Frame-up (1 day)
- Four Cuts (1 day)
- Squares and Paths (assigned as out-of-class work)

UNIT TWO: Logic (2 1/2 weeks)

- Introduction to Propositional Logic (2 days)
- Propositional Logic Bingo (1 day)
- What's True About Whole Numbers? (2 days)
- Using Quantifiers (2 - 3 days)

UNIT THREE: Arithmetic, Place Value and Bases (2 1/2 weeks)

- Blocks, Rods, Flats, and Cubes (1 day)
- Introduction to Bases (1 day)
- Driving in Septobasiland (1 day)
- A Base Four Lesson (assigned out of class)
- Addition and Subtraction Problems (1 day)
- Ben and Kris's Arithmetic (1/2 day)
- A Multiplication Problem (1 1/2 days)
- Lynna's Arithmetic (1/2 day)
- Arithmetic in Other Bases (1 1/2 days)

UNIT FOUR: Sets, Operations and Relations (3 weeks)

- Introductions to Sets (1 day)
- Sets of People (assigned out of class; 1/2 day for comparing answers)
- Describing Sets of Numbers (1 - 2 days)
- Glicks, Glacks, Glocks and Glucks (1 - 2 days — only do first few)
- Introduction to Operations (1 day)
- Analyzing Operations (1 day)
- Clock Arithmetic (1 day)

UNIT FIVE: Fractions, Primes and Number Systems (3 weeks)

- Law and Order (1 day)
- The Condominium Problem (1 - 2 days)
- Ratio and Proportion Problems (1 day)
- $\frac{1}{3}$ and $\frac{2}{5}$ (1 day)
- Common Fractions and Repeating Decimals (1 day)
- Factors and Prime Numbers (2 days)
- The Locker Problem (1 day)
- Square Root of 2 (assign out of class plus 1 day in class)

2.3 Extra Activities

The following pages contain some extra activities that you may wish to use, as extra practice problems either as you cover material or in reviewing, or as an extra in-class activity. Notes on the activities are collected here below, so that you can use the activity sheets that follow for photocopying if desired.

Toothpick Squares: This activity is very similar to the Squares Problem in Unit One and could be given as practice, or if you wanted to give students yet a third alternative to Squares and Paths (for example, if each small group were to solve a different problem). The answers to the first two questions are 4 and 12, and the answer for general n is $2n^2 + 2n$. One way to see this is to see that any grid can be broken up by letting the top and left sides of each square be grouped together. Then we have n^2 objects of length 2, and to complete the picture we need the right and bottom sides of the whole grid, each of length n.

Guess the Base: These problems may be useful as extra practice getting students familiar with bases. *Answers:* (a) eight, (b) eleven, (c) five

Examples of Operations: Except for the last example, this sheet is not an activity, but rather a set of examples in case the introductory sheets (and class discussions) on operations don't make the matter clear.

Numbers in their Primes: This worksheet, which uses factor tables, might help spark a discussion on how many factors a given number has, in terms of its prime factorization. Note that question 3(b) involves numbers with [exactly] three factors, which must be squares of primes.

Logic and Statistics: This worksheet provides simple illustrations of how logic comes into play analyzing results of data-gathering questionnaires.

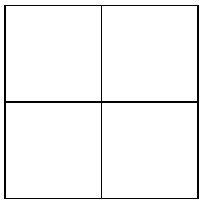
Number Theory Bingo: This activity gets students to practice their number theory skills by having students design their own bingo cards from directions given. In Game I, students place the ten set names given on the five rows and five columns of the card, and the caller calls out numbers randomly. (One way to generate numbers randomly is to roll two six-sided dice and take the result as a base six numeral which you can then convert to base ten to get the numbers 1-36. Of course, if you have some 20-sided dice, there's no conversion necessary.) You can also add to the list of properties available — for example, variations like **4,12** or *3,8*, though if the students have already seen the Stamps Problem, you should be careful not to allow pairs which will generate most of the numbers you'll call.

In Games II and III, students line the rows and columns with whole numbers under 40, and the caller calls out properties of numbers (or pairs of numbers). For these games, lists of ten properties are provided for the caller to call in random order. It is quite possible that bingo will be scored before the list is exhausted, although the caller should also be prepared with some extra properties — perhaps variations on the ones listed.

What may be of more use than playing each game once is playing one game (e.g., Game I) several times, and allowing students to change their cards between games. They will discover during the course of play that, for example, O (odd numbers) and E (even numbers) should both be in the same group (row labels or column labels), as should even and prime. However, odd and prime should be in different groups, since they almost always go together. MPF and prime should go in the same group, since they won't ever go together, etc.

Toothpick Squares

Suppose you have a box of toothpicks, all the same length. How many toothpicks would it take to make a one-by-one square? What about a two-by-two square grid like the one below?



How many toothpicks would it take to make an $n \times n$ grid of 1×1 squares?

Guess the Base

In what base have the following problems been solved?

(a) $$\begin{array}{r} 147 \\ +21 \\ \hline 170 \end{array}$$

(b) $$\begin{array}{r} 1000 \\ -122 \\ \hline 989 \end{array}$$

(c) $$\begin{array}{r} 323 \\ \times\ 103 \\ \hline 2024 \\ 323 \\ \hline 34324 \end{array}$$

Examples of Operations

In all examples, the operation is defined on the set $S = \{1, 2, 3, 4\}$.

$\sim$	1	2	3	4
1	1	2	3	4
2	2	3	4	5
3	3	4	5	6
4	4	5	6	7

The operation $\sim$ is **not** closed. It is commutative and associative.

The identity is 1.

We see that $1^{-1} = 1$, but 2, 3 and 4 do not have inverses.
Thus S does not have inverses with respect to $\sim$.

Δ	1	2	3	4
1	3	4	1	2
2	4	1	2	3
3	1	2	3	4
4	2	3	4	1

The operation Δ is closed, commutative and associative.

The identity is 3.

We see that $1^{-1} = 1$, $2^{-1} = 4$, $3^{-1} = 3$ and $4^{-1} = 2$.
Thus S does have inverses with respect to Δ.

$*$	1	2	3	4
1	2	1	4	3
2	1	2	3	4
3	4	3	2	1
4	3	4	1	2

The operation $*$ is closed, commutative, and associative.

The identity is 2.

We see that $1^{-1} = 1$, $2^{-1} = 2$, $3^{-1} = 3$, $4^{-1} = 4$,
so all elements of S have inverses with respect to $*$.
Thus we say that $*$ has inverses on S.

$\heartsuit$	1	2	3	4
1	1	2	3	4
2	2	4	2	4
3	3	2	1	4
4	4	4	4	4

The operation $\heartsuit$ is closed, commutative, and associative.

The identity is 1.

We see that $1^{-1} = 1$, $3^{-1} = 3$, but 2 and 4 do not have inverses,
so we say that $\heartsuit$ does not have inverses on S.

$\circ$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	2	3	1

Now it's your turn to decide which properties $\circ$ has on S.

Numbers in their Primes

The object of this worksheet is to use the prime factorization of a whole number to find out how many factors it has.

72	3^0	3^1	3^2
2^0			
2^1			
2^2			
2^3			

1. Note $72 = 2^3 \cdot 3^2$. Fill out the table below with the products of the headers of each row and column.

(a) How are the numbers you have placed in the table related to 72? Explain.

(b) How many factors does 72 have?

2. Using the above as a model, fill in the factor tables below for numbers 54 and 225.

54				

225			

(a) How many factors does 54 have? 225? Can you see a quick way to find the number of factors of a whole number without actually finding the factors?

(b) For example, how many factors does the number $200 = 2^3 \cdot 5^2$ have? How about 16200 ($=2^3 \cdot 3^4 \cdot 5^2$)?

3. (a) If you were to look at a list of the number of factors of the first 100 whole numbers, which numbers of factors do you think would appear most often? Least often? Why?

(b) Without going through all of the possibilities, explain why there cannot be a number between 49 and 121 which has exactly three factors.

Logic and Statistics

Below are two simple examples of situations in which some basic logic is required to analyze results of data-gathering questionnaires.

First example: Mary, a statistician, wants to send a questionnaire to a large population and determine how many people have both a college degree and have a mother or father with a college degree. She calls this set 'C'.

The questionnaire looks like this:

```
                                        Yes   No
1. I am a college graduate.             ___   ___

2. My mother is/was a college graduate. ___   ___

3. My father is/was a college graduate. ___   ___
```

Decide which combination(s) would put the respondent into the set C, and which would not.

Can you formulate the requirement to be in the set C using the statements 1, 2 and 3 and logical symbols?

Second Example: John, an experimenter, wants to investigate the following claim: If a person is inoculated for measles before age 18, that person will not contract measles after age 18.

The questionnaire looks like this:

```
                                              Yes   No
1. I was inoculated for measles before age 18. ___   ___

2. I have not had measles since age 18.       ___   ___
```

Decide which combinations of response(s) would be used to establish the *falsity* of the claim.

Can you formulate the claim John is trying to verify using statements 1 and 2 and logical symbols?

Number Theory Bingo

Game I. *Instructions*: Fill in the leftmost column and the top row with the labels below (in boldface). Use each label once. This will be your bingo board. Each time a number is announced, you will fill in all the squares for which the number has both properties (of the column and the row). "Number" when used here refers to whole numbers only.

Let **3N** represent the set of multiples of three.
Let **4F** be the set of numbers with four factors (including 1).
Let **5F** be the set of numbers with five factors (including 1).
Let **9,4** denote the set of numbers that can be obtained by adding nines and fours.
Let **E** be the set of even numbers.
Let **EF** be the set of numbers the sum of whose factors is even.
Let **MPF** be the set of numbers n with more *prime* factors than their neighbors $n-1$, $n+1$.
Let **O** be the set of odd numbers.
Let **P** be the set of prime numbers.
Let **PS** be the set of perfect squares.

Game II. *Instructions*: This time, work the other way around. Pick 10 different whole numbers less than 40 and use them to label the columns and rows. Each time a property is announced, fill in the square(s) whose row and column *both* belong to numbers which have the property. Again, "number" refers to whole numbers only.

Some Properties:

Let **3N** represent the set of multiples of three.
Let **4F** be the set of numbers with four factors.
Let **5F** be the set of numbers with five factors.
Let **9,6** be the set of numbers that can be obtained by adding nines and sixes.
Let **E** be the set of even numbers.
Let **EF** be the set of numbers the sum of whose factors is even.
Let **MPF** be the set of numbers n with more *prime* factors than their neighbors $n-1$, $n+1$.
Let **O** be the set of odd numbers.
Let **P** denote the set of prime numbers.
Let **PS** be the set of perfect squares.

Game III. *Instructions*: Again pick 10 different whole numbers less than 40 and use them to label the rows and columns. Each time a *property of a pair of numbers* is announced, fill in the square(s) which correspond to a pair of numbers for which that property is "true."

Some Properties:

Let **D2** be the set of pairs of numbers which differ by two or less. each other.

Let **GCD>6** be the set of pairs of numbers whose *greatest common divisor* is greater than 6.

Let **LCM<100** be the set of pairs of numbers whose *least common multiple* is less than 100.

Let **RP** be the set of pairs of numbers which are relatively prime.

Let **SD** be the set of pairs of numbers which have at least one digit that matches.

Let **SPF** be the set of pairs of numbers which have the same number of *prime* factors.

Let **STF** be the set of pairs of numbers which have at least two *prime* factors which are the same.

Let **XY>400** be the set of pairs of numbers whose product is greater than 400.

Chapter 3

Teaching in a Cooperative Classroom

This section contains a great deal of information and advice regarding how to manage a cooperative classroom, on all levels. In fact, the amount of reading is rather overwhelming, so we suggest that at first you read just the first two sections, on the basics and on the beginning of the semester. The remaining sections, on small group dynamics, large group dynamics, and organizing yourself, can then be used as reference either as day one nears, or once the class has met a couple of times and you have some more specific questions. As with the rest of this guide, the recommended procedure is to browse first, then come back and zero in on the topics of greatest interest to you.

DISCLAIMER

In training one's self or others to teach a Socratic/Cooperative style class (henceforth SC), it seems no amount of preparation or advice can substitute for a certain kind of on-the-job training. The essential ingredients of on-the-job training are criticism and imitation. Experienced instructors visit the classrooms of new instructors, taking extensive notes on what they see: what went right, what went wrong, what might have worked better, and so on. New instructors also visit the classrooms of experienced instructors, taking equally careful notes on what went well or poorly, what might have worked better, and on what techniques they see used that they would like to use themselves. Some of this type of work can be done beforehand, via visits in the previous semester or videotaped classes, but it tends to be more valuable when it comes after the new instructor has had a chance to try teaching a class or two first. Currently in 130-131-132 we are trying for one visit each direction in the first two weeks, another in the second two weeks, plus at least one more during the term.

If you are not going to adhere to a plan as outlined above, then **don't expect the notes that follow to do much for you**. You can't learn to play the piano by reading books about it or by talking about technique with Rubenstein, so don't try to learn a significant new teaching skill without practice and guidance either. On the plus side, all the interested instructors that have taught SC classes here (admittedly a self-selected sample) have become pretty good at it, so you can be pretty confident that you'll be able to step right in and teach effectively this way even if you're not a virtuoso.

3.1 Basics

3.1.1 The philosophy

Unless you've been on Mars, you're probably aware of the age-old battle between "skills" advocates and "process" advocates. I'm from the "process" camp but I hope to avoid a lot of the partisanship that is prevalent in discussions of pedagogy and stick mainly to points both sides agree on. We want students to come away from (lower level) math classes with certain skills and attitudes. In particular we want them to reason and prove, to translate between words and symbols, to perform algebraic manipulations correctly and with understanding of the justification, and to solve problems other than clones of problems they have been shown how to do. Whether or not you believe specific skills and knowledge to be paramount (arithmetic of negative numbers, solution of quadratic equations, propositional logic, summation of common series), you undoubtedly want them to know these things in the ways mentioned above: verbally and symbolically, with justification and proof, well enough to apply to new situations.

The tenet underlying SC classes is that students need to learn that they can think for themselves, and that they will be able to learn properly if and only if they are forced or enticed to continue thinking things through on their own terms. Realize please that this does not apply to students who already have this skill. I don't think we need SC classes at the advanced undergraduate level, and they become increasingly inefficient at higher levels. If college admissions standards (or high school graduation standards) were what we'd like them to be, we wouldn't need SC classes in college at all. The students who benefit from SC classes are ones I would term remedial: those taking pre-calculus, and those in the Ed program who are required to take what is essentially junior high school mathematics.

Our philosophy with these students is to do anything we can to get them to think and speak mathematics, and then once we have them going, to exact from them a quality of mathematical reasoning that is higher than anything ever asked of them in a traditional course, thus ensuring that they learn the course content in a useful and permanent way. The meta-skill we emphasize is for the students to **know when they know something**, versus when they are just guessing or are confused. In the time-span of the course or sequence of courses we move from a "process is everything, choice of content matters little" approach to a stage where we cover the traditional content and expect students to focus on these topics and skills while applying the critical thinking they have learned in the first phase. The first phase is the harder phase for most instructors, since we have to be psychologists — and sometimes mind-readers — in addition to being mathematicians. These notes concentrate on this phase, though they apply to the other as well.

Perhaps the most controversial part of this approach is our unwillingness to tell students the answer. Some skills can only be acquired this way, and one can be overly dogmatic on this point. The basis for this is that much of this material is accessible to them, with a little help from us, and that our habitually providing answers will cause their problem-solving ability to atrophy, though it may increase their rate of skill acquisition (though we argue probably it won't). Thus we make every attempt when discussing an attempted solution in class not to tip off whether it is right or wrong until the whole class has had a chance to criticize it or register comprehension and agreement. Depending on the context, we do or do not in the end provide model solutions.

3.1.2 Typical classroom mechanics

A usual 50 minute class consists of two kinds of time: some time when the students are working in groups of 3 or 4 on a problem or worksheet (set of problems) and some time when the entire class is discussing the problem set. Some instructors enjoy keeping to a familiar rhythm, spending the initial 25 to 30 minutes each day working in small groups, then spending the latter part of the lesson in a large group discussion detailing what the various small groups found, where they got stuck, and so forth. Often there are parts of the worksheet that are not covered in this phase; some of these are assigned for homework and some are discarded. Other instructors prefer to go back and forth a little more, starting in a small group, then convening the large group when most small groups are done with the first problem, discussing it a bit, then remanding the class into small groups, and so on. When a class meets for 75 minutes twice a week instead of 50 minutes three times a week, it is usually necessary to go back and forth this way, and it is also often convenient to continue a large group discussion from the end of one class at the beginning of the next.

During the small group working time, the instructor circulates among the groups, answering questions when necessary, doling out encouragement when necessary, challenging the students to justify what they claim to have figured out or to explain their half-baked ideas. Often the mere presence of the instructor encourages a renewed attack on a problem.

The large group discussion begins with the students explaining what they have done. Other students are required to listen carefully and to register agreement, disagreement or incomprehension. Once the explanation is comprehensible, those in disagreement are encouraged to justify their disagreement, with the aim of a resolution or synthesis. The instructor plays moderator as long as fruitful ideas are being produced, but slips into the role of leader when needed. For example, if no one challenges something wrong, or if there is a disagreement but it is too inarticulate to produce a good synthesis, then the instructor may rephrase what has been said so as to sharpen the contradiction or caricature a wrong approach, in a way that forces a light to dawn for at least some students.

Note also that in order for SC discussions to work, the class size must be limited, to no more than twenty or thirty maximum. The point is to have the class participate as a whole in the same conversation, but if you think back to social gatherings you've attended, even if everyone knows each other, it is difficult to keep everyone involved in the same conversation once you get more than a dozen or so people involved.

3.1.3 Highly recommended procedures

Before getting into specifics of classroom technique, here are a few simple procedures that make a large difference.

- *Nametags.* Have the students wear nametags each day until you know their names (in my case 2 or 3 weeks). When calling on students, call on them by name and in general attempt to use their names frequently. This serves two purposes. First, by attempting to learn their names, you create a separate mental category for each student, which helps you pay attention to how each student is doing and to their individual needs. Secondly, there seems to be a psychological advantage to students hearing their names. Coming around to a small group and asking "Amy, can you tell me what progress your group has made?" or asking in a large group discussion whether "Sam's objection to

Cindy's idea" holds water elicits more of a response than the same questions without the names identified. Somehow, students are more prone to take their own beliefs seriously when names are attached.

- *Randomize groups.* I usually assign groups using playing cards randomly dealt: all the aces for a group, the twos form another, and so on. I re-form groups twice or thrice during the semester. Preventing students from choosing their buddies for a group helps them form connections, subject their ideas to the intellectual marketplace, and treat all the others in their groups fairly. The playing cards themselves lend an air of intrigue as students await the results of the lottery. If you decide not to use playing cards to change groups, you can assign the students the task of creating new groups such that no two "new" teammates have worked with each other before. (This can be done easily once, and with considerably more effort a second time.)

- *Get enough sleep.* Alertness is required on the part of the instructor. You'd be surprised what a difference this makes. You can fake it when you're lecturing, but try playing moderator when you can't concentrate or respond quickly, and you'll see what I mean.

- *Start the semester with a bang.* That is, don't spend the first day on administrative stuff and the second on some kind of review. Jump into an absorbing problem on day 1, preferably a tried-and-true chestnut, and fill in the administrative details later, when they've gotten the idea of what the class will be about. The tone of the first day's discussion sets an example that's hard to erase, so make sure it is as lively as possible.

- *Minimize the time you spend talking.* In particular, never talk for more than five minutes at a time — monologues shift students back into passive learning mode, and monologues of announcements tune students out. Of course, you may have to do a lot of talking in leading large-group discussions, but if you're talking by yourself for more than five minutes at a time without at least one student speaking up, you're not doing SC — you're lecturing.

- *Be yourself* at least up to a point. If you're the goofy type, be goofy; if you're serious and intense, let them feel the intensity; if you're understated and direct, be that way. You are on stage, and want to use your charisma, but don't pretend to be someone you're not.

3.2 Beginning the semester

3.2.1 The first day

In many ways, the first day can be one of the most important of the semester. It sets tone and precedent for the days that will follow, and lets the students know what to expect. Students will be anxious to know how their grades will be determined, and perhaps anxious in general to be in a math classroom. Your job that first day is to sell them on the class, to help them start to get comfortable. This job has two main parts: dealing quickly with administrative details and getting the students engaged in problem solving. Suggestions follow to help you with both of these parts.

You should already know all the administrative details about the class when you walk in the door the first day, and the quickest fair treatment for communicating these details is in the form of a handout. You may want to write your name and the class name, number, section, etc. on the blackboard before class starts, to make sure people are in the right place, but don't spend time in class writing all the details on the board. Rather, have them collected on a handout, so that you can skim the most important details (components of the grade, exam dates) and leave the students to peruse the rest at their leisure, *outside of class.*

A sample handout is given on the next page. A few notes regarding it follow.

Math 130

Section 1, 1:20-2:10 PM MWF, B139 Van Vleck

Instructor: Christopher Kribs
Office: 822 Van Vleck
Phone: 262-0537
email: kribs@math.wisc.edu
WWW: http://math.wisc.edu/~kribs
Office Hours: after class and TBA
Text materials: available at Bob's Copy Shop in University Square ($8)
Midterm date: Tuesday, March 26, 5:15-7:00 PM
Final exam: Wednesday, May 15, 2:45-4:45 PM

This course is the first in a sequence of three designed to help you develop your intuitive reasoning and problem-solving skills. We will spend most of our time in class working in small groups on various problems, usually from the course packet. The problems are designed to be both interesting and nontrivial, so you should be prepared to spend some serious time on them (in class, for the most part). You will almost certainly get stuck on a problem at some point in the semester, but don't be discouraged — if you persevere, you'll get through! We will usually discuss the problems in a large group after most groups have finished them. Sometimes you will be asked to write up your ideas and solutions, but *always* you are expected to think about the problems, participate in solving them, and communicate your ideas with others. Communicating your ideas to others is as important as developing them in the first place.

Note that this is a math *content* course, and not a pedagogy course. We hope that taking this course will help you be a better teacher, but more by setting an example rather than teaching you math methods.

GRADES: Your grade for the course will be determined by two exams (20% each), by attendance and participation (20%), and in large part by written work you will turn in (40%).

The exams will be similar in nature to the problems we work in class, but short enough that you should be able to complete them in the time given. A sample exam will be distributed before the actual exam in order to give you a closer feel for it, though you should *not* expect it to serve as an exact blueprint for the real thing. There will be a midterm and a final exam (locations TBA); the dates are given above. Please mark these dates and times on your calendar now so as to avoid conflicts. In the event that a conflict arises, *please* see me as soon as possible so that we can resolve it.

Attendance and participation are a significant part of your grade because this course is more an experience than a set of material to be learned. Most of what I hope will happen for you in Math 130 will take place inside the classroom, working in groups and talking with others. You may miss up to 3 days for reasons of health, religion, etc. without penalty. Arriving late or leaving early counts as half an absence. If you have special needs, please see me in the first two weeks of the term.

It is also in your interest to participate in the group problem solving sessions since active learning is better than passive learning. Participation includes both small and large group work. If you don't feel comfortable answering questions, *ask* some of your own: that spurs discussion as much as an answer, and you won't be the only one with that question.

THE WRITTEN WORK will have two components: write-ups (also called problem reports) and reflections. A write-up is a detailed solution to a problem we discussed in class. These write-ups should be readable independently of any worksheet on which they are based, in good English and either legibly handwritten in ink or word-processed. They should always include the following: 1. a statement of the problem at hand, 2. any strategies you used to attack the problem, 3. the solution you obtained, with an explanation of how you got it (and how you know it is complete), and 4. a conclusion that says what we can take with us from the problem. Communication of what you understand (even if it's not a complete understanding) is at least as much the point as finding the solution.

I will also sometimes ask you to write a reflection on a rather less concrete issue, like "What does it mean to get stuck?" These essays, usually a page or two in length, will be graded more loosely, more on how much thought went into it than on organization and content.

I will let you know at the time I assign written work when it is due, but typically it will be due in class a week from the time it is assigned, and you will have roughly one assignment due per week.

Optional reference text: Billstein, Libeskind and Lott, *A problem solving approach to mathematics for elementary school teachers.* Second edition (1984). Menlo Park, CA: The Benjamin/Cummings Publishing Company.

These are *your* blackboards. Own the classroom.

Notes:

- Even though this course is designed to be run entirely from the activity set and class discussions, some students will feel lost without a textbook, for definitions, examples, and the like. The text listed in the sample handout, by Billstein, Libeskind and Lott, covers much of the same material as both Math 130 and Math 131 (the geometry course). Be careful, however, not to list the text alongside the coursepack, nor to give the impression that students who do not consult a text will miss anything. If the whole class seems lost on something, better to spend ten minutes in class explaining it or distribute a handout with examples rather than refer everyone to a textbook.

- As noted in the handout (and numerous times in this guide), Math 130 is a math content course, not a math methods course. Make sure the students understand that.

- See Chapter 6 of this guide for more discussion of how to determine course grades.

Different instructors will have different ways of building the atmosphere they desire. Some instructors do best by jumping right into the first activity ("Poison") in Unit One. They then reserve a fraction of the second day's class to discuss the nature of the course, and why we do things the way we do, using the Poison lesson as an example. Other instructors are more comfortable devoting some time to "warm fuzzy" introductory activities. They will use a little class time to introduce everyone. To take care of any reticence on students' part to talk about themselves, try having everyone pair up with someone s/he doesn't know, and spend five minutes getting to know one another. (This also gets them used to talking in class!) Then go around the room having each student introduce his/her partner. Some instructors also have students fill out index cards with a few facts about the student (name, major, hometown, feelings about math, something unusual about the person, and perhaps suggested office hours); these can serve as backup in case someone forgets his/her partner's name. You can also add a couple of icebreakers such as "favorite cartoon character" and "least favorite vegetable".

If you spend time on introductions, then you will want to choose a short warm-up problem-solving activity that can be completed in the remaining time. Here's one (Pascal's Triangle) that has worked for us. First, make sure everyone has a nametag, and then put them into small groups. Put up on the board, or on a handout if you prefer, the first five or six rows:

$$\begin{array}{ccccccccccc} &&&&& 1 &&&&& \\ &&&& 1 && 1 &&&& \\ &&& 1 && 2 && 1 &&& \\ && 1 && 3 && 3 && 1 && \\ & 1 && 4 && 6 && 4 && 1 & \\ 1 && 5 && 10 && 10 && 5 && 1 \\ &&& . && . && . &&& \end{array}$$

and ask the following questions: Can you find the rule used to generate Pascal's Triangle? What do the next two rows look like? What patterns do you recognize in it?

The groups will discover the rule quickly enough, and the last question is open-ended enough to allow discussion to continue until the bell rings.

3.2.2 Encouraging Good Habits from Day One

Encourage good habits early, to set precedents for the entire semester: Encourage people to talk, to offer ideas without worrying about whether they're "right" or not, to talk to each other (instead of to you) as much as possible, to write on the blackboard, to be precise about what they mean, and to get used to giving explanations and justifications to back up things they really believe. One way to encourage the latter three things is to "play dumb" and keep saying you're not sure what they mean — even if you do — as long as you think someone else in the class might not (which is usually true). Of course, you should give a disclaimer the first day of class explaining that you're not going to be giving them straight answers or telling them whether or not a solution is correct (though you should never let the book be definitively closed on a subject without making sure that there is closure, and the everyone understands that the solution on which the class has agreed is indeed correct). Part of this disclaimer will include the fact that you reserve the right to "play dumb" as long as you think something might not be clear to someone in the class.

Another habit to encourage from the beginning is to keep a running list of different problem solving strategies and techniques they have used. By the time you finish Unit One, they should already have a half dozen or more, and it might be worth taking a few minutes in class at that point to have them compare lists. Chapter 2 mentions another discussion to have around the end of Unit One, regarding group dynamics.

3.2.3 Resistance and "Why are we studying this?"

One of the questions which you as a teacher in the Math 130 sequence will hear over and over again is "Why are we studying this?" Why are we studying these topics which we will never be teaching? Why are we taking this course? These questions deserve an answer, and again this is something best discussed openly and early (although no doubt the question will recur). Among other things, elementary school teachers must lay the groundwork for their students' later experiences. In math, arithmetic is the foundation of algebra, and teachers need to know what's coming up in the years ahead, so that they can teach in a way that will later allow their students to look back and see the connections. Mathematics is all connected, and the sequence of K-12 math classes students take should be a natural progression, not an unrelated set of topics. Teachers also have to be able to anticipate precocious questions. Equally importantly, they must be comfortable and familiar enough with every subject they teach that their students will not all be turned off. Many, if not most, of the students in the Math 130 sequence have had bad experiences with math in the past, and very often this can be traced to a teacher or teachers who passed on their apathy or disinterest for the subject.

These past experiences will, as noted above and to varying extents, lead to resistance by the students to the way Math 130 forces them to shoulder responsibility for their mathematics. Here again, you as their instructor should bring this subject out in the open, early and as often as needed (although it can, at times, threaten to take up a lot of in-class time — try not to let this happen). Point out the first day that the reason for their doing this is not only to provide a model for them to use later is getting their own students to think for themselves, but that someday soon they will have to be the mathematical authority for a classroom full of inquisitive children, and that before they can be someone else's mathematical authority, they must be confident in being their own authority. There will invariably be one or two students in a section who do not "buy into" this philosophy, and in the end there may be nothing more you can do for them but to tell them that the class is designed this way because people who have had a lot of experience teaching math believe in it, and they'll have to play along for now and see what results come of it in the end.

Some students, especially the poorer (for want of a better term) ones, will be vocally and continually concerned for their grades. The most you can do is try, from the beginning, to make grade assignment rules as explicit as possible (but don't feel any obligation to explain down to the point level your grading of particular reports, any more than a composition teacher would), and try to reassure them. Don't let their angst transfer to you.

3.3 Class Composition and Small Group Dynamics

3.3.1 Doing the rounds

When you first assign a problem to work on in small groups, there may not be much you have to do. No one is stuck yet; no one needs your help. There is a lot you can accomplish in this time. First, you can quickly take attendance group by group — after you know the names this takes less than half a minute. After this, you will want to quickly "do the rounds". Visit each group once just to check that they have gotten down to work. On the first round, look for any trouble with the wording of the problem that may be holding people up. If it's part of their job to decipher it, encourage them to do so. If it's a mistake, or if you need to supply a definition, then make a quick announcement. On your second round you can linger longer. This is a good time to make a mental note of which groups are going faster than the others. It helps, during the subsequent large group discussion, to have a good idea of who has gotten how far. If a group has quickly and incorrectly or incompletely answered a problem and gone on to another, this is a good time to ask (innocently) for one of them to summarize for you what they found. The correct question on your part can cause them to re-examine what they've done without feeling that you've invalidated their answer (point of philosophy: you want them to be able to criticize their own work, realizing that mathematics will determine whether they are right, and that what they discover about this cannot be overruled by the teacher).

Example: A group has used an incorrect manipulation: $(a + b)^2 = a^2 + b^2$. You can "explain for those who are rusty on this", that this means "when you compute, e.g., $(7+3)^2$, in case you can't tell at a glance that $7 + 3 = 10$, squared $= 100$, you can use this nifty rule, getting $7^2 = 49$, plus $3^2 = 9$, which by the laws of algebra must also give you 100." If this doesn't elicit an objection, you can line up the sum as you speak:

$$\begin{array}{rr} & 49 \\ + & 9 \\ = & 100 \end{array}$$

When you do get an objection, ask them to pinpoint what went wrong, leading to, "Oh, you mean $(a + b)^2 = a^2 + b^2$ is not a universal rule of algebra?"

A reminder: Lurk early, lurk often. While students are in small groups, eavesdrop constantly. The students should get used to having you listen in to their conversations. This keeps you abreast of both (a) how far the class in general is getting and (b) which groups in particular are having difficulties, and where. If one group seems too conscious of your presence, then stand near another group nearby and appear to be watching them when in reality you're paying attention to the discussion in the other group. This part of the class is an important one for you as instructor and (presently) discussion moderator.

3.3.2 "Help, we're stuck"

The twin dangers here are that lazy students will say they're stuck so you do the work for them, while students who are truly stuck will lose morale if they have to sit idly during class. Asking if they have any ideas on what they might try will prove embarrassing if the answer is no. Sometimes I do this anyway. Sometimes I replace the problem with a smaller one: if you knew that $A = 15$ could you do the problem? Can you do the problem if you aren't required to make the number of cows and chickens the same? Sometimes I guess why they're stuck: So the problem is you don't really know the definition of average speed? An example of what might happen here is that they did know this but didn't think of going back to definitions as a way to proceed. Now when they say no that's not the problem, we know the definition of average speed, I say, "Oh, then you must be saying you don't have any way of determining the quantities defining speed, such as the time or the distance." They then say how they will proceed on this and I can smile and leave. In other cases, they are stuck because they don't really understand what's being asked. You can ask them to rephrase it, or ask them how they would check if someone else's answer (here you specify it) was right. It helps to have snooped enough so you have a good guess of where they're stuck. If not, you can ask them but won't always get reliable information.

3.3.3 Getting Groups to Work Together

The pre-service teachers tend to work well together. The pre-calculus population is less accustomed to this, and groups will degenerate into a lot of individuals ignoring each other, or occasionally explaining to each other. You may at times need to tell them explicitly "Janet has found what she thinks is an answer but Steve and Brenda apparently don't understand what she did, so Janet, you're going to have to explain it and see if you can convince Steve and Brenda." However, you should try other things first, before being this explicit. Ask Brenda what her group has found so far, and don't let anyone else answer for her. If she says she's stuck, ask if her whole group is stuck, and if not, tell her you'll come back in 3 minutes and ask her again for a summary of what her group has done. **Make sure groups are sitting in a tight circle**, not a line or a disarrayed cluster.

Sometimes you can give them specific tasks: Jason, finish the calculation you're doing; meanwhile Ann will add up Rob's numbers and will then check to see if they agree with yours; if not, it's up to all of you to judge which method if either was correct and why. If a group really has bad chemistry, change it. I've had students say to me: I just can't work with Judy — she won't listen and hogs the discussion. In that case, put Judy in with someone smarter or more aggressive than she. When re-forming the groups, I usually randomize again, but if there are trouble students, sometimes I stack the deck so that they get put with students who can handle them.

3.3.4 Free Riders

There are always lazy students who are content to let others do the work. If it's just laziness, I don't hesitate to reprimand them explicitly: "Adam — you're just staring into space and letting the other three figure out the problem; if the problems aren't challenging enough, then I can let you work faster in a group by yourself, but judging from your homework that's not the issue." On the other hand, if it's a student with a confidence problem who needs some nurturing, it's probably better to make a note of it and to continue to ask that student to explain what their group has done whenever you come around (you have to ask other students sometimes, or it gets too obvious). Involving the student as much as possible, with questions that are at their level but not patronizing, will often cure this. Also impress upon the others that it's their job to make sure the free rider is keeping up with the group since that person (you've just decided) will be in charge of the first group writeup.

3.3.5 Staying on Task

The less you have to reprimand here, the better. Make sure that when you tell them to get into groups they know exactly what they're supposed to start working on. Make the rounds quickly at the beginning so they don't start chatting, and keep an ear out for it later on. If a group continues to be bad about this, you can watch them, visit them more, chastise them humorously, make sure they don't get grouped together next time, etc., but if the whole class is bad, you should examine what you're doing that promotes doing something other than the math. You could be leaving them in small groups too long, while the slow groups finish. You could be joking with them too much during class time. I do also use reprimands, but sparingly — on day 3 this year I was in an impatient mood so I reprimanded the two groups that took more than a minute to form their groups and start working. It was something like "Sounds interesting whatever you're talking about, but you've got to get started on problem 1 — time's short this week!"

3.3.6 Students who are behind

First, it's a good idea to know what you do for them and what you can't. You can do a lot for these students in office hours, but it's not realistic to be able to spend more than (or even as much as) half an hour a week outside of class with any one student. So if they need help on a regular basis, suggest that they arrange for tutoring. You should find out before the semester what such resources are available. If they're doing passing work, but still underconfident, point out to them that if they continue to work at this level, they'll pass the class. Don't, however, make promises you can't keep about their grades; it's best not to prognosticate about their grades before they've taken an exam.

All that being said, you have to make sure they get the most out of their group work and don't drag down the group (they're as much afraid of this as you are). Here are some ways to build their confidence and make them more likely to participate to advantage. Spend some time around their group and be ready to pounce on those times when the lagging student, "Lenny", comes up with a good idea. Assign credit: if Jocelyn figures out how to do something with Lenny's idea, then it's "Lenny & Jocelyn's method". Try to give them constructive comments on their homework (I try to do this for everyone, but when time does not permit this, concentrate on students who need it). Making sure they are keeping up with their group is delicate — asking them to explain to you where their group is will help if they're not too far lost. Once they are far behind, consider placing them next time with a group of students that talk a lot, don't go all that fast, and are as kind as possible. At worst, you may have to settle for Lenny participating minimally during class and trying to catch up on his own at home.

Select Lenny to give a presentation, either solo or on behalf of his group. On a one-time basis you can invest a little extra time to help him with this to make sure it goes over well (have him practice it on you till he's confident enough to field questions).

Avoid asking Lenny really easy questions. This will make him feel like he must be dumb. These questions are scary in general since the reward for a correct answer is almost zero and the penalty for an incorrect answer is large. At your discretion, you might choose Lenny to answer questions that are not black and white, asked of the class at large: which of these problems was the hardest?, what kinds of thinking did you think were necessary on this problem?, and so on.

3.3.7 Students who are ahead

Having such a student can be a real boon if they are gifted teachers as well. If they have a good feel for how to explain things and help others, they will make your class run more smoothly than you can, on their own. Even in this case, **avoid treating them in front of the class as a reliable source for right answers**. You don't want to create a situation where calling on them is tantamount to telling the class something yourself. It is OK though, to treat them as a reliable source for intelligent commentary.

If a student is obviously head and shoulders above the rest from day 1, you may consider exempting them from the course. That is, if they can do the worksheets on their own then they can probably pass the exemption exam, go on to the next course, and leave you with a more homogeneous class. Recommend that they see the course coordinator for an exemption exam; everyone will benefit from this. Later in the semester this is less of a good idea, though I've done it.

Assuming "Einstein" stays in the class and is not self-policing, you need to keep an eye on Einstein's group to make sure that Einstein is not explaining things to the others before they have a chance to figure it out themselves. Let Einstein explain things at the board in situations where you know there will be some wrong or unclear stuff in the explanation. Make sure though, that you give Einstein as much encouragement for what was right and clear as you would another student. If Einstein is a loner and tends to work fast but not share with the others, that will probably work out fine. You can explicitly designate Einstein as a group of 1 next time, or simply allow a *de facto* group of 1 to form. Sometimes, you can try asking Einstein explicitly to figure out a hint to give the rest of the group as to how to proceed but that won't completely solve the problem for them. It will make Einstein summon up teaching skills that are worthwhile in general, so it's worth a try, but be aware that Einstein may not be capable of this. In any case, don't let students disparage themselves in comparison to Einstein. You can say, "I see, because Einstein solved this problem in 5 minutes and you can't, you're going to give up?"

3.4 Managing Socratic Discussions

This is the hardest part, and the part of teaching SC classes that improves the most with practice. When observing someone else's Socratic discussions, try to imagine what would have happened had they made different choices (told or not told the students something, came up with a good counter-question, decided to pursue or not to pursue a student's line of reasoning).

3.4.1 Dead Ends

When an idea is proposed, the instructor will usually know right away whether it will lead anywhere. If it won't, there is a strong temptation to discourage the students from pursuing the idea. This may be a mistake. Probably the best thing that can happen in a Socratic discussion is a *flaming dead end*, meaning that an incorrect line of reasoning leads to a consequence so patently false that the students are compelled to re-examine the road that got them there. If you see your students headed for one of these, then all you really need to do is encourage them to get there without undue delay. Some things you might want to do are: get them to explicitly reaffirm the wrong assumption, so that they will remember it later and be able to pinpoint it; shut down any further sidetracks that branch off of this one (e.g., "OK, that's a good idea, but first we'll finish pursuing this one"); hasten their demise by keeping the pace brisk, perhaps doing some of the arithmetic for them or providing clarifying paraphrases.

You can influence how flaming a dead end will be by making the issue more concrete: ask them to illustrate their result with actual numbers (e.g., "so if the initial weight was 250 grams, then we see the final weight of $400 - 2w$ comes out to be what? Oh, I see, -100 grams... rather on the light side.").

Perhaps you will need to summarize their findings, juxtaposing two findings that are contradictory, or in the case that they have contradicted some of the given information, you may need to restate the givens by saying, e.g., "so you have now proved that the only whole number between 100 and 500 having no two digits the same and satisfying blah blah blah is 337." To further rub it in, it often works well to insist that you believed their method and there must be some other mistake: "Ah, you've proved that the long division we did was wrong — can anyone find where?" (of course it is actually correct), or "Ah, you've proved that when you use variables with subscripts, the usual method of solving linear equations doesn't work" (if you can trust them to fight back on this one).

Other kinds of dead end are less useful. Perhaps they are following a reasonable line of inquiry but it doesn't get them anywhere: looking for a nonexistent pattern, introducing too many variables, classifying according to an ineffective scheme. A reasonable goal in this case is to get them to figure out that they're stuck. If you tell them (or indicate in any of 1000 nonverbal ways) that their idea won't work, they will learn to look to you for validation of their ideas, whereas if they reach a dead end themselves and consciously decide to look for another approach, they have learned something valuable about problem-solving. That being said, there are ways to reduce the amount of time spent following a dead end. One trick is to decide after hearing a suggested approach whether to follow it immediately or whether to treat it as one of many to be written on the board before the class decides which to follow. If Jenny reports finding a pattern starting $4, 6, 8, 12$ and reports her reasoning as to what is likely to come next, you get to choose between (a) getting the whole class involved in speculating about the next number or (b) writing on the board "Idea: look for a pattern". The key feature of this trick is that you're not giving anything away. Approach (a) is reasonable in some contexts, where the discussion of pattern promises to have some depth, and more importantly, approach (b) is something you sometimes use when the approach offered is correct. In fact you should make sure to use (b) on occasions where there were multiple interesting approaches but the first one offered happens to be the best: you catalogue every group's approach before asking the class to pick one and follow it.

Another way to expedite matters is to insist that the goals be well defined. Often when a bad approach is put into words, it comes out sounding discouraging: we thought we'd name as many variables as possible and then hope that inspiration struck; we decided that if n was equal to 5 the solution was obvious but we don't know how to do it for any other value. Sometimes mild discouragement doesn't work. I remember a worksheet designed to get them to invent the binary number system by asking them to come up with a scheme for representing all numbers with 1's and 0's. This cost a full day of discussion of the relative merits of various schemes, none of which had anything to do with binary. The instructor was very successful that semester, and in my opinion the investment of days such as that one paid off when students continued to work hard throughout the course because they didn't feel that the instructor was going to provide the answer for them. This takes guts, and doesn't work too well if the instructor conveys a growing uneasiness about the whole project. So if you're not up to following the wind, you're probably better off treating it the same as the situation in the next paragraph.

The least promising dead end is a total lack of ideas. Probably it's best not to convene a discussion at this point but to continue working in small groups where you can ask questions that elicit further work and break the impasse. But suppose a class discussion on a certain problem fizzles out midway. This might be a good time to drop it. If it's not essential that they end up knowing how to do the problem, and they don't have a realistic shot at finishing it for homework, perhaps make it into an extra credit assignment. If it is essential, consider dropping it for now and writing a worksheet for the next class that will lead them to it in more manageable steps. You probably need more time to solve this problem than you have on the spot.

3.4.2 How to Listen

You need to listen to students and they need to listen to each other. Tom Lester once told me of a study showing that the average amount of time between when a teacher asks a student a question and when the teacher prompts the student or gives up on them is 2 seconds. Two seconds is longer than it sounds, but nowhere near long enough to formulate a coherent thought unless you were already thinking it before the question was asked. There may not be anything you can do about the sound-bite trend in TV reporting, but there's a lot you can do about it in your classroom. The first thing to try is waiting. Don't nod yes or no, or say uh huh, or give the student any feedback at all until they have finished saying what they wanted to say. Then wait five or ten more seconds. The odds are that the student will, after pausing for breath, realize that they are not finished and continue. If not, at least the other students will have had a chance to think about what they just heard. If you're uncomfortable with this long a pause, try pacing or holding eye contact with the respondent as if you expect them to continue, or act as if you're trying to digest what they've just told you. In fact often you really will need time to think. If they said something that was wrong in a puzzling way, see if you can figure out what they really meant. Students will only listen to each other if you set an example, so make sure you don't respond without having really heard.

Students are also more likely to listen to each other if they feel that they are responsible for having understood it. In small groups they are more likely to feel this, but at a ratio of twenty or thirty to one, many feel that they can just take notes and sort it out later or not at all. They may also feel they have no right to interrupt since everyone else obviously understands. You can counter this by demonstrating an expectation that each student understands what each other student has said. After one student says something the slightest bit unclear, ask another to repeat it in her own words. This is a good time to pick on students rather than have them raise their hand to volunteer a paraphrase. If student B can't paraphrase what student A said, it's not necessarily student B's fault. Student B can ask student A to clarify if necessary, or ask for volunteers for someone else to clarify. Make sure to go back and find out whether student C's clarification of student A's remark did in fact help student B. After a little experience you'll know better when to go through this routine. If student B simply wasn't listening, they might feel reprimanded, but that's OK. It doesn't really work when the remark was clear in the first place, although it doesn't hurt to get a quick affirmation from the whole class that it is clear so far. The basic standard you are setting is that the discussion involves the whole class and is not a collection of one-on-one dialogues between the teacher and individual students.

3.4.3 Staging

Your expectations of the nature of a class discussion are conveyed nonverbally as much or more than verbally. Most instructors ask the students to rearrange their desks into a large U shape for any but the briefest class discussions. A subtler but important technique is to put as much of the class as possible between you and the respondent. If you call on a student on the left side of the room, walk over to the right side as you're doing so. As the words flow between you and the respondent, the almost physical presence of a stream flowing between the two of you will wash over the students in between. You and the respondent also keep eye contact with the rest of the class this way.

It is often a good idea to get students to come up to the blackboard. Students will give longer monologues at the blackboard, so be prepared to be a more active moderator if the student is losing the rest of the class. Be careful not to make having the right solution a prerequisite for coming to the board, lest the students stop thinking critically and accept any blackboard demonstration as a surrogate for your telling them something. When a student is at the board, I try to take up a position in the back or on the perimeter of the room. Sometimes I sit in the student's seat. This has the effect of including the rest of the class, as above, and also gives me a new vantage — you'd be surprised at what you see this way.

Other body language to be aware of is whether you are passing judgment on what you hear. Do your eyes flit impatiently with wrong answers? Do you gesture in agreement with right answers? Do you angle your body to the board as if to write down something correct, then pause if it's not what you wanted? If you've chosen to teach SC style classes, it's because you want the students to develop their own judgment, so avoid this kind of tip-off.

A related topic is the use of intentional errors. These are a hit with kids, in a slapstick sort of way, but adult students tend to feel patronized. Instead, I substitute the mischievous lie. If a student tells me the found all five regular polyhedra I may say, "Ah, so you haven't found the other two then?" They can often sense that I'm putting them on, but will still take the bait and try to prove that there aren't any more. On a problem best solved by assigning a variable to a certain quantity, I once told a group of frustrated students that I'd tell them the value of one quantity for free if they could decide which quantity they wanted to know. I planned to lie and tell them it was 10 when it was really 6. In fact I had tried this previously with success: the students figured out that the assumption of 10 led to a contradiction and were able to figure out the unique value that didn't lead to a contradiction. This time it was even more successful. By the time I came back to give them their free question, they had figured out what quantity they wanted to be told, had put in x for this, and had gotten the solution (well ahead of the rest of the class).

3.4.4 Asking the Right Questions

When my teaching is evaluated by my students, they often say that I never answer questions, or answer them with another question. I take it as a compliment even though it isn't meant as one. The most common such interchanges are

Student: Is this right?
Instructor: I don't know. Does it sound right to you? Can you elaborate?

Student: What should we do from here?
Instructor: What do you think? Does anyone have any ideas?

Student: Can we say blah blah blah?
Instructor: I don't know, can you?

There is a certain amount of this you can get away with, depending on your personality and theirs. If you start sounding like "Eliza"[1], you won't get good results. Instead, try to ask them useful questions related to the specifics of what they've said. In the three above scenarios, try respectively

> Are you asking if your computation is correct, or if it will prove useful?
>
> Is there a problem-solving strategy that you know that might work here? Why don't these equations tell you what x is?
>
> If you're wondering whether you can assign the variable z to be the average of all the prices, the answer is yes, but you haven't yet said whether we know anything about z.

When observing other classes, this is where you should let your imagination run free. Imagine what questions they might have asked. Your hindsight now will be your foresight tomorrow.

The question "do you understand?" is the most often abused. (Notice that this is virtually the only question in the repertoire of the conventional lecturer and rarely elicits an honest response.) Some better variants are: can you say that in your own words? could you do what John just said with different numbers? do you agree or disagree? in what way is this similar to what so-and-so did? These are all comprehension questions, testing whether the respondent comprehends instead of asking point blank for a Yes/No as to whether the respondent comprehends.

Good questioning can help to reach flaming dead ends. Ask what happens when $x = 5$, or whether their purported method works for all starting data and not just what was given. If a student gives a vague definition, find a borderline case and ask how their definition applies in that case. Try also questions that goad by disingenuity. If their method is more general than they realize, ask how they got lucky enough to try their method on a square rather than a pentagon or hexagon for which it "probably wouldn't have worked".

3.4.5 Order versus Chaos

Ideally your students will be eager to answer your questions and discuss their ideas, but will listen patiently and attentively to each other and to you. If students are not willing to speak up and discuss their ideas, you need to loosen them up. It is a bad sign, for example, if the students are not happily chatting away when you enter the room five minutes before class, and are sitting in silence or whispering. In this case, you have probably done too well at eliminating chaos. Try assigning an activity in small groups where different groups are doing different things and they need to walk across the room to share information with each other. For example, there's a worksheet in Math 112 on infinite series where they approximate

[1] The computer program imitating a nondirectional therapist, an early (and crude) approximation to something that could pass the Turing test.

numerically some infinite sums and try to form conjectures based on each other's conclusions. Assigning a group project where they have to work outside of class together can tighten the bonds and make people feel more comfortable talking. When leading class discussions, be freer and more willing to follow the students' ideas wherever they lead. Dispensing with hand-raising and having students just call out can quicken the pace.

A classroom that's too chaotic is a problem also. If you have to call the students to attention more than once before they pipe down and listen, or if there is crosstalk during class discussions, you probably ought to do something about it. You can address this explicitly, asking the students to pay attention to you and to each other; you have to be consistent about this or they won't believe you mean it. Indirect methods of dealing with this are, however, usually more effective and should be tried first or at least in parallel. Insist on an orderly formation of desks into a U shape before a class discussion, rather than having them minimally perturb the small-group seating arrangements. It takes an extra minute, but it's worth it. In fact tell them you're pressed for time so they have to rearrange the desks in 30 seconds. A snappy set change will set the tone for what follows. When you observe crosstalk, try to get one of the crosstalkers up to the board to explain something, or to comment on what's just been said. By maximally involving that student in the lesson, you'll eliminate most of the off-task crosstalk, and the on-task crosstalk can probably be lived with. Another chaos reduction technique is to give them a more rigid idea of the structure of each class. Say you're going to spend 16 minutes in groups before convening a class discussion and then stick to it with absurd precision. The more aware they are of the structure of the lesson, the more they will stick to the tasks at hand.

The main point of this section is that you should make a conscious effort to optimize your position on the order-chaos axis, and that increasing order or chaos in the physical arrangements or chronological structure or types of assigned activities can help you change the balance in your class discussions.

3.5 Organization (yours)

SC style classes have more inherent disorganization than traditional classes, which is why you need to pay particular attention to organization.

3.5.1 Your Records

Students, at Wisconsin more than at other places I've been, react extremely positively to the appearance of organization. Probably the pre-service teachers are particularly impressed by this since they are consciously judging you on your pedagogical techniques. **Be a compulsive record-keeper.** I find it very helpful to reserve the 20 minutes after each class for writing a short summary of what happened in class that day. That way, if I've told a student I'd find them an extra-credit assignment, or if I've promised to bring something to class next time or promised that the next class would begin with a discussion of something, I can write this down along with other notes as to what I have in mind for the next class. In an SC style class there is a greater opportunity for unexpected things to happen, and therefore a greater need to write down what did happen.

Keeping date records of the homework you've assigned, both due dates and the date it was assigned, is essential. Whether or not you accept late homework is up to you, but it is certainly better to have students ask solicitously in advance for you to accept their late papers, which you will probably grant, than for students to assume it's OK and be upset if you don't grant them an extension after the fact. They are more likely to do the latter if they get the idea that you yourself don't remember when the homework was due. In fact, if I arrive at class early, I often take the opportunity to write up a reminder of what is due when.

Organization (and the appearance thereof) also helps students perceive you as self-confident, a quality which is crucial for an instructor in such a comparatively free-form classroom. Of course, one goal of the course is for them to assume that self-confidence themselves by the time they leave your classroom.

3.5.2 Grading

The message here is the same as in the previous section. Include in your course packet a clearly defined grading policy, specifying what portions of the grade are from homework, exams, quizzes, projects, group work, attendance, or whatever else you grade on. Give it enough thought so that you remember it easily, and can answer their questions immediately. If historical grade distribution data is available, you would be well advised to stick to it, since this will help to allay fears that the unusual pedagogical style will adversely affect their grade.

3.5.3 The Bell

When the bell rings at the end of class, everything becomes exponentially harder. My advice is to watch the clock like a hawk, so that you can make sure to wrap up the discussion at 2 minutes before the end. The discussion usually leaks over an extra minute, giving you one minute to say any summary comments or give instructions on homework, etc. This can be an important routine even when you have little to say: it makes you seem organized and on top of things.

If you can tell you are going to want to go overtime, because of a red-hot discussion you want to complete or something that's necessary so they can do their homework, announce to them 5 or 10 minutes ahead of time that you will probably be going overtime. It's best if you've let them go a minute or two early once before and have mentioned at the time that you're banking those minutes for such an occasion as this. If they're working in small groups at the end of class, it's less crucial but you still may want to halt them 1 or 2 minutes before the bell for closing remarks — it's even harder to get their attention after the bell when you don't already have the stage.

In short: **don't ever be surprised by the bell**.

Chapter 4

Write-ups

The write-ups (also called problem reports) are an important part of the course, because they force the student to communicate his/her knowledge about the problem. One consequence is that students cannot hide shortcomings in their understanding of the problem; another, more to the point, is that the students will develop the ability to give clear explanations on paper. Theoretically, a write-up should explain a problem clearly enough from beginning to end that a student could hand it to a colleague at that same school, and the colleague would be able to understand the whole problem without consulting anyone or anything else. It may well take a while for some students to develop good written communication skills, but you should be able to convince them that it is well worth the effort (and indeed most students have indicated at course's end that they believe it was) — after all, if a clear written explanation is harder to give than a clear verbal explanation, then they should come out well-prepared to explain to their own students.

This section of the guide discusses how to assign write-ups, which write-ups to assign, and how to grade them — although these are, in the end, only guidelines. At the end are two sample handouts you could give students to help them get used to writing math, and writing problem reports.

4.1 How, and how often?

The latter question is perhaps answered more quickly than the former. In general, you might want to aim for one write-up per week. More than this will cause you either to spend long hours grading (q.v.) or to fall behind in your grading; fewer than one every two weeks will not give the students the practice they need in developing their communication skills, and will also make you skip some important write-ups. Overall, you probably want a pretty even balance among individual write-ups, reflections, and other written work (including group write-ups and other homework). You may want to skip assigning a write-up the week of an exam.

As far as *how* to assign the write-ups:

1. Set some guidelines on the first day of the semester (see also Chapter 3 for this). Among other things, set a length of time between the date an assignment is made and the date it is due. These write-ups take a lot of time for the students, so they should have five to seven days in general (possibly less for the reflections, which are treated in Chapter 5). You probably don't want to set a single official length for write-ups, but between two and six pages is probably the norm. Do set rules for format, e.g., will you accept reports handwritten in pen? in pencil? Of course, also mention what you will be looking for in general, though more detailed specifics will come later, when you make assignments (see 3. below). [Part, if not all, of your guidelines may come in the form of a handout. See the example first day handout in Chapter 3, as well as the one given in the last part of this section.]

2. The write-up should almost always be on a problem that you have just finished discussing in class. It is a bad idea to assign a write-up on an activity on which you ran out of time, saying that the students should finish up on their own for the write-up. If the class discussion stops short of a full analysis of some aspect of the problem and you expect the students to address it in the writeup, it should be something you expected to leave for them to think about, not something you ran out of time for. Some suggested write-ups, such as Squares & Paths at the end of Unit One, are specifically designed as out-of-class problems, but you should make such assignments very sparingly.

3. You should discuss the write-up with the class for five minutes or so when you assign it, to make sure everyone knows what you want from them. This is especially important in the beginning, when students are unsure how to explain themselves, and in what generality they should present the solution. One way to do this is to ask the students, "What do you think are the important things about this problem, which we should include in the write-up?" Then write on the board the suggestions students make, adding your own if necessary once the students finish responding. (Of course, this means you have to decide what you want in the write-up before class!) Be willing to let the students argue for or against including certain items, and be careful not to make them include *too* much — you don't want to read a bunch of ten-page write-ups!

4. Most of the write-ups you will want to assign as individual write-ups: this is one of your big opportunities to evaluate an individual as opposed to his/her small group. However, assigning group write-ups can reduce work on both ends (the students' in writing and yours in grading), as well as being a nice change of pace sometimes. You might want to be especially observant the first time you do this, to see that each group member makes a more or less equal contribution. Again, use these in moderation.

4.2 Which ones?

Problems on which write-ups are assigned should be significant ones, where there is some complexity to the solution, and consequently a story to tell, both in the finding and in the explanation of what was found. You will often have spent more than one full day on the problem in class.

Here is a list of the problems on which write-ups are usually assigned:

- Poison (Unit One) — be sure explanations include the general strategy
- Time to Weigh the Hippos (Unit One) — include an interpretation of what it means to have two solutions here
- Squares & Paths (Unit One) — make it clear you want the pattern justified
- Clock Arithmetic (Unit Three) — include a comparison with normal arithmetic
- Explanation of Arithmetic Algorithms (Unit Four) — either have each student choose one algorithm, or else have each group cover one per person
- A Base Four Lesson (Unit Four) — this is a good group write-up

• The Stamps Problem (Unit Seven) — encourage a general explanation based on relative primeness

• The Lockers Problem (Unit Seven) — include an explanations of both why open lockers $\leftrightarrow$ odd number of factors and why odd number of factors $\leftrightarrow$ squares

Of course, you may feel free to add others which generated good in-class discussions, or omit one or two of those mentioned above.

4.3 Grading write-ups

You should make it clear to your students both before and after grading a given assignment what you were looking for, and how you determined the grade in general. However, unlike a calculus exam, you should *not* feel any obligation to give them an explicit point-by-point explanation of their grade, any more than a composition teacher has to explain the accumulation of good and bad points that resulted in giving a paper a B. Instructors in this course have historically used numerical scores rather than letter grades in grading write-ups, but you should do whatever feels comfortable.

Everyone has a different grading style, but one way to go about it is first to make a checklist of things you're looking for, or sections of the write-up, and then read/mark only that one section of all of the papers, to ensure that your comments will be consistent. Then, after you've examined each part of the write-up this way, go back and read each paper individually, skimming where you've made comments, and get an overall impression from which you can assign a total grade. When all papers have been graded, sort them by grade and flip through them to see if any appear to be "out of sequence", in which case you might want to take another look at those. Again, don't get too bogged down at the point level. The time required to grade write-ups properly increases more than linearly with the number of papers you have to grade, so you'll be looking for a middle ground in which each paper gets a fair reading but you don't spend fifteen or twenty hours grading.

Grading criteria: Here again, it's up to you. However, if we follow the ideas expressed in the sample handouts below, you'll be looking, in general, for:

1. A clear paraphrase of the problem description
2. An account of the method(s) used to solve the problem, including any major milestones or blocks, as well as "dead ends" which nevertheless proved enlightening in some way
3. A clear statement of the solution, as complete and general as appropriate, including an interpretation within the problem context
4. A clear explanation of why the solution works (or is the only, or complete, solution)

Since the nature of different problems can be quite different (compare Time to Weigh the Hippos with A Base Four Lesson), these elements may take on different forms.

Problem descriptions should be in the writer's own words, and give a complete enough description that the reader need not consult anything else (like the course materials) to understand it. Also, give credit for creativity, both in problem-solving methods and in writing up the solution.

One important element of the evaluation process is giving your students good feedback when you return their papers, both in written comments on their papers and verbally in class (in more generality). Encourage them; point out where, on the whole, the write-ups were strongest, and where they were weakest. It may help to give a handout back along with the very first problem report, so that they can see some of the elements of a good write-up. If you decide to make one, try to include good (but not bad) elements from specific papers (without naming names). *Don't* hand them a complete ideal problem report, as it tends to give them the impression there is/was a unique ideal which they were all supposed to guess. (It also gives students who take this course in later semesters an easy out for the first assignment.)

Here, for example, is an excerpt from a handout giving feedback on the problem report on "Poison". (This is not the entire handout!)

The winning strategy

> "The second player's strategy for playing with ten pennies is to watch what the first player does and do the opposite. For instance, if the first player chooses 2 pennies, then the second player chooses 1 and vice versa."

It doesn't take a lot of space to do this. Note that the writer makes it clear that this is the ten penny strategy, not the solution for 11 or 12 or 210.

The reason it works

> "If you pick opposites on every move, the total number [of] pennies taken by both of us on one turn is 3. There are 3 sets of 3 in ten. (3x3=9) However, you have to add 1, the poison penny, to make ten. That explains why when you get down to 4, if you pick first, you will have to lose. In 4 there is one set of 3 and the poison 1....
>
> By taking opposites, you take three pennies in each turn. Since 3x3 is 9, plus the poison #1 equals 10. For example, in my Group, if [my opponent] takes 1, I will take two. Now 7 are left. 7 is 3x2 plus the poison #1. [My opponent] takes 1, then I take two. Now there are 4 left. 4 is 3x1 plus the poison #1. She takes one, then I take two. She is left with the poison #1. Therefore, the person moving second, which was me, wins!"

Okay, this could have been better. Neither explanation is completely clear (and this writer did lose some points for clarity), although together they tell the story well. Again, writing out the example game helps.

The general solution

> "In order to win the 210 penny game, you would want to move first and take out two pennies. This will leave you with 208 pennies still on the desk. 208 divided by 3 gives you 69 with a remainder of 1, the poison penny. Just as 10 divided by 3 gives you 3 with a remainder of one. From here on, the team that moved first would take the opposite of the second team to win the game."

This was from a different paper than the previous, so the explanation of why the 10 penny strategy was a winner was somewhat different. But both of these students, by focusing on why the "opposites" strategy was a winner in the 10 penny game, figured out how to win the big game. I don't think it's coincidence that the two best explanations of why the "opposites" strategy works accompanied the two best general solutions.

4.4 Sample handouts

On this and the following pages are a couple of handouts one might give one's students on writing mathematics in general, and problem reports in particular.

Problem Report Tips

- Speak out during large group discussion to get other groups' ideas. Make them explain their ideas to you, so you can explain them clearly to me in your problem report.
- Address every question: even if you do not know the answer, at least acknowledge that and perhaps guess at it or suggest a strategy for finding it.
- Problem reports will ideally all have one or two key sentences which hit upon exactly why the answer you provide is true. For example, in a write-up analyzing the game Tic Tac Toe, a sentence or two which says something like, "Go first and choose the center. On your next two turns choose two adjacent squares (a corner and the center of a side), so that you have two ways to win — diagonally and straight across or down — and your opponent can do no better than a tie," hits the nail on the head.
- Sometimes you may not be able to explain the complete solution. In this case, try to pinpoint the difficulties you are having, and even try to solve a simpler version of the problem. You get partial credit for this, and it may even give you insight that helps you make the leap to solve the original problem. If there is a problem with your solution, point it out clearly, and, if possible, give special conditions under which your solution still works.
- It's better to take a stab at an answer (i.e., to guess intelligently) than to throw up your hands and say, "I don't know." Try to eke out as much of an answer (again, maybe only a partial answer) as possible, and say why you believe it.
- If something relevant comes up in small or large group that is interesting, you might want to tell about it in your report, even if it is not necessary for the solution (however, it should at least be relevant). Report ideas that did not turn out to be correct as well as those which did, if they help you to understand the pitfalls in solving a tricky problem.
- Write a conclusion that actually draws a conclusion, rather than simply rewriting your introduction. The overall flow should be "here's what I'm going to tell you about, here I'm telling you about it, and this is what we can come away with", not "here's what I'm going to tell you, here I'm telling you about it, and here's what I just told you".
- Use paragraphs — one for each idea. It's not a rule, but you usually should have more than one paragraph per side of a page.

- Is order important? There is no unique formula for writing a problem report. Use your judgement to decide the best order in which to present your report of what your group did and your argument for your solution. They may even be mixed together, if you find that useful.
- Have a friend read the report. Can s/he understand it with no help from you?
- A good problem report is exhaustive yet not verbose.

A Guide to Writing in Mathematics Classes

Dr. Annalisa Crannell[1]
Franklin & Marshall College

Adapted for Math 130 at UW-Madison by Brad Franklin and Christopher Kribs, 1997.

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§ 1. Why Should You Have To Write Papers In A Math Class?

For most of your life so far, the only kind of writing you've done in math classes has been on homeworks and tests, and for most of your life you've explained your work to people that know more mathematics than you do (that is, to your teachers). But soon, this will change.

Now that you are taking college mathematics, you will soon know more mathematics than many Americans remember. With each additional mathematics course you take, you further distance yourself from the average person on the street. You may feel like the mathematics you can do is simple and obvious, but you can be sure that other people find it bewilderingly complex. It becomes increasingly important, therefore, that you can explain what you're doing to others that might be interested: your parents, your boss, the media, your students.

Nor are mathematics and writing far-removed from one another. Professional mathematicians spend most of their time writing: communicating with colleagues, applying for grants, publishing papers, writing memos and syllabi. Writing well is extremely important to mathematicians, since poor writers have a hard time getting published, getting attention from the Deans, and obtaining funding. It is ironic but true that most mathematicians spend more time writing than they spend doing math.

But most of all, one of the simplest reasons for writing in a math class is that writing helps you to learn mathematics better. By explaining a difficult concept to other people, you end up explaining it to yourself.

Every year, we buy ten cases of paper at $35 each; and every year we sell them for about $1 million each. Writing well is very important to us.

— Bill Browning, President of Applied Mathematics, Inc.

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§ 2. How is Mathematical Writing Different from What You've Done So Far?

A good mathematical essay has a fairly standard format. We tend to start solving a problem by first explaining what the problem is, often trying to convince others that it's an interesting or worthwhile problem to solve. On your homeworks, you've usually just said, "9(a)" and then plunged ahead; but in your formal writing, you'll have to take much greater pains.

[1] a_crannell@acad.FandM.edu

After stating what the problem is, we usually then state the answer, even before we show how we got it. Sometimes we even state the answer right along with the problem. It's uncommon, although not so uncommon as to be exceptional, to read a math paper in which the answer is left for the very end. Explaining the solution and then the answer is usually reserved for cases where the solution technique is even more interesting than the answer, or when the writers want to leave the readers in suspense. But if the solution is messy or boring, then it's typically best to hook the readers with the answer before they get bogged down in details.

Another difference is that when you do your homework, it is important to show exactly how you got your answer. However, when you write to a non-mathematician, sometimes it's better to show why your answer works, with just a brief explanation as to how you got it. For example, compare:

Homework Mathematics:

To solve for x when $3x^2 - 21x + 30 = 0$, we use the quadratic formula:

$$\begin{aligned} x &= \frac{21 \pm \sqrt{21^2 - 4 \cdot 3 \cdot 30}}{2 \cdot 3} \\ x &= \frac{21 \pm \sqrt{441 - 360}}{6} \\ x &= \frac{21 \pm \sqrt{441 - 360}}{6} \\ x &= \frac{30}{6} \text{ or } \frac{12}{6} \\ x &= 5 \text{ or } 2 \end{aligned}$$

and so either $x = 5$ or $x = 2$.

More Formal Mathematics:

To solve for x when $3x^2 - 21x + 30 = 0$, we used the quadratic formula and found that either $x = 5$ or $x = 2$. It's easy to see that these are the right answers, because

$$(3 \times 5^2) - (21 \times 5) + 30 = 75 - 105 + 30 = 0,$$

and also

$$(3 \times 2^2) - (21 \times 2) + 30 = 12 - 42 + 30 = 0.$$

The difference is that, in the first example, you're trying to convince someone who knows a lot of math that you, too, know what you're doing (and if you don't, to get partial credit). In the second example, you're trying to show someone who may or may not be good at math that you got the right answer.

Math is difficult enough that the writing around it should be simple. 'Beautiful' math papers are the ones that are the easiest to read: clear explanations, uncluttered expositions on the page, well-organized presentation. For that reason, mathematical writing is not a creative endeavor the same way that, say, poetry is: you shouldn't be spending a lot of time looking for the perfect word, but rather should be developing the most clear exposition. Unlike humanities students, mathematicians don't have to worry about over-using 'trite' phrases in mathematics. In fact, at the end of this booklet are a list of trite but useful phrases that you may want to use in your papers, either in this class or in the future.

This guide, together with your checklist, should serve as a reference while you write. If you can master these basic areas, your writing may not be spectacular, but it should be clear and easy to read — which is the goal of mathematical writing, after all.

§ 3. A Checklist

Use this checklist as a guide for yourself while writing a problem report. These are the sorts of things on which the write-ups are graded.

Does this paper:

1. clearly (re)state the problem to be solved?
2. state the answer in a complete sentence which stands on its own?
3. provide a paragraph which explains how the problem will be approached?
4. define all variables, terminology, and notation used?
5. clearly state the assumptions which underlie the formulas and theorems?
6. explain how each formula or theorem is derived, or where it can be found?
7. clearly label diagrams, tables, graphs, or other visual representations of the math?
8. give a precise and well-organized explanation of how the answer was found?
9. give acknowledgment where it is due?
10. use correct spelling, grammar, and punctuation?
11. contain correct mathematics?
12. solve the question that was originally asked?

Here is a more detailed explanation of the criteria.

1. Clearly restate the problem to be solved.

Do not assume that the reader knows what you're talking about. (The person you're writing to might be out on vacation, for example, or have a weak memory). You don't have to restate every detail, but you should explain enough so that someone who's never seen the assignment can read your paper and understand what's going on, without any further explanation from you. Outline the problem carefully.

2. State the answer in a complete sentence or paragraph.

If you can avoid variables in your answer, do so; otherwise, remind the reader what they stand for. If your answer is at the end of the paper and you've made any significant assumptions, restate them, too. Do not assume that the reader has actually read every word and remembers it all (do you?).

3. Provide a paragraph which explains how you and others approached the problem.

It's not polite to plunge into mathematics without first warning your reader. Carefully outline the steps you're going to take, giving some explanation of why you're taking that approach. It's nice to refer back to this paragraph once you're deep in the thick of your calculations.

This is where you give some details about what happened in small and large group. Don't go overboard — just highlight a couple of key moments and ideas (even ones that didn't work). You may want to explain how you chose to interpret the problem if issues arose in this.

4. Define all variables used.

(a) Even if you label your diagram (and you should), you should still explain in words what your variables are.

(b) If there's a quantity you use only a few times, see if you can get away with not assigning it a variable. As examples:

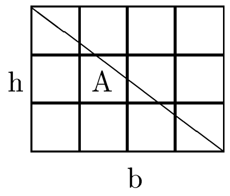
<table>
<tr>
<td rowspan="2">

Figure 1: *Diagram of the triangle* (each square is 1" by 1")</td>
<td>We see that the area of the triangle will be one-half of the product of its height and base — that is, the area of the triangle is $(1/2) \times 3 \times 4 = 6$ square inches.</td>
<td>$\surd$</td>
</tr>
<tr>
<td>We see that $A = (1/2) \times b \times h$, where A stands for the area of the triangle, b stands for thebase of the triangle, and h stands for the height of the triangle, and so $A = (1/2) \times 3 \times 4 = 6$ square inches.</td>
<td>$\times$</td>
</tr>
</table>

Elementary physics tells us that the velocity of a falling body is proportional to the amount of time it has already spent falling. Therefore, the longer it falls, the faster it goes.	$\surd$
Elementary physics tells us that $v_t = g(t - t_0)$, where v_t is the velocity of the falling object at time t, g is gravity, and t_0 is the time at which the object is released. Therefore as t increases, so does v_t: i.e., as time increases, so does velocity.	$\times$

I hope that you'll agree that the first example of each pair is much easier to read.

(c) The more specific you are, the better. State the units of measurement. When you can use words like "of", "from", "above", etc., do so. For example:

We get the equation $d = rt$, where d is the distance, r is the rate, and t is the time.	$\times$
We get the equation $d = rt$, where d is the distance from Sam's car to her home (in miles), r is the speed at which she's traveling (measured in miles per hour), and t is the number of hours she's been on the road.	$\surd$

Avoid words like "position" (height above ground? sitting down? political situation?) and "time" (5 o'clock? January? 3 minutes since the experiment started?). Never mind that your instructor uses these words freely; you can too when you get a Ph.D.

(d) Variables in text are italicized to tell them apart from regular letters.

5. Clearly state the assumptions which underlie the formulas.

For example, what physical assumptions do you have to make? (No friction, no air resistance? That something is lying on its side, or far away from everything else?) Sometimes things are so straightforward that there are no assumptions, but not often.

6. Explain how each formula is derived, or where it can be found.

Don't pull formulas out of a hat, and don't use variables which you don't define. Either derive the formula yourself in the paper, or explain exactly where you found it, so other people can find it, too. Put important or long formulas on a line of their own, and then center them; it makes them much easier to read:

<table>
<tr><td>The total number of infected cells in a honeycomb with n layers is

$$1 + 2 + ... + n = n(n + 1)/2.$$

Therefore, there are $100(101)/2 = 5,050$ infected cells in a honeycomb with 100 layers.</td><td>√</td></tr>
<tr><td>The total number of infected cells in a honeycomb with n layers is $1+2+\ldots+n = n(n+1)/2$. Therefore, there are $100(101)/2 = 5,050$ infected cells in a honeycomb with 100 layers.</td><td>×</td></tr>
</table>

Sufficiently advanced copies of Microsoft Word have an equation editor: pull down the *Insert* menu, select *Object*, and then *Equation*. If you don't have an equation editor, you may either try formulas with tabs and fancy fooling, you may use Excel for matrices if you know how, or you may wish to write the mathematics in by hand. All of these are fine options.

A caution: Mathematical formulas are hard to do on a standard word processor. If you've got something that's really complicated, feel free to leave space and then write it out by hand later– but then don't forget to do so! (Like I often do).

7. Clearly label diagrams, tables, graphs, or other visual representations of the math (if these are indeed used).

In math, even more than in literature, a picture is worth a thousand words, especially if it's well labeled. Label all axes, with words, if you use a graph. Give diagrams a title describing what they represent. It should be clear from the picture what any variables in the diagram should represent. The whole idea is to make everything as clear and self-explanatory as possible. If you decide to draw pictures in MacDraw or SuperPaint (which you can then copy to Write Now, Microsoft Word, or MacWrite), you will (a) have a lot of fun, and (b) waste a lot of time. If you're going to draw diagrams on the computer, I recommend doing so in the early stages of the project, and then saving the document so you can change it later — because you'll have to change it later, and you can't do it once it's in Write Now or whatever.

8. Give a precise and well-organized explanation of how the answer is found.

This is the single most important step.

9. Give acknowledgment where it is due.

Plagiarism is almost certainly the greatest sin in academia — some fiction writers make plagiarism a motive for murder. It's extremely important to acknowldge where your inspiration, your proofreading, and your support came from. In particular, you should cite: any book you look at, any computational or graphical software which helped you understand or solve the problem, any student you talk to (whether in this class or not), any professor you talk to (including and especially me, because I'll catch you if you leave me out). The more specific you are, the better.

10. In this paper, are the spelling, grammar, and punctuation correct?

(a) It may surprise you that it is on spelling and grammar that people tend to lose most of their points on their mathematics papers. Please spell-check and proofread your work for grammar mistakes. Better yet, ask a friend to read your paper. Mathematicians are generally not petty, but neither are we amused by sloppy or careless writing.

(b) Mathematical formulas are like clauses or sentences: they need proper punctuation, too. Put periods at the end of a computation if the computation ends the sentence; use commas if it doesn't. An example follows.

If Dr. Crannell's caffeine level varies proportionally with time, we see that

$$C_t = kt,$$

where C_t is her caffeine level t minutes after 7:35 a.m., and k is a constant of proportionality. We can solve to show $k = 202$, and therefore her caffeine level by 11:02 (t=207) is

$$\begin{aligned} C_{207} &= 202 \times 207, \\ &= 41,814. \end{aligned}$$

In other words, she's mightily buzzed.

√

(c) Do not confuse mathematical symbols for English words (= and # are especially common examples of this). The symbol "=" is used only in mathematical formulas — not in sentences:

We let V stand for the volume of a single mug and n represent the number of mugs. Then the formula for the total amount of root beer we can pour, R, is $R = nV$.	$\surd$
We let V = volume of a single mug and n = the # of mugs. Then the formula for the total amount of root beer $R = nV$.	$\times$
We let V stand for the volume of the mug and n represent the number of mugs. Then the formula for the total amount of root beer we can pour, R, is R is nV.	$\times$

(d) Do, however, use equal signs when you state formulas or equations, because mathematical sentences need subjects and verbs, too.

Then the formula for the total amount of root beer we can pour is $R = nV$.	$\surd$
Then the formula for the total amount of root beer we can pour is nV.	$\times$

11. In this paper, is the mathematics correct?

This is self-explanatory.

12. In this paper, did the writer solve the question that was originally asked?

So is this.

§ 4. Good Phrases to Use in Math Papers:

- Therefore (*also*: so, hence, accordingly, thus, it follows that, we see that, from this we get, then)
- I am assuming that (*also*: assuming, where, M stands for; *in more formal mathematics*: let, given, M represents)
- show (*also*: demonstrate, prove, explain why, find)
- (see the formula above). (*also*: (see *), this tells us that ...)
- if (*also*: whenever, provided that, when)
- notice that (*also*: note that, notice, recall)
- since (*also*: because)
- Cite sources, e.g., This formula can be found on page 234 of *Discovering Algebra* (1999), by Levine and Rosenstein.

For more information, see Crannell, Annalisa, "How to Grade 300 Mathematical Essays and Survive to tell the Tale," PRIMUS **4**, 3 (1994): 193-201.

Chapter 5

Reflections

More or less regularly throughout the semester (once every two or three weeks), you should assign reflections to your students; several possible topics are suggested below. The details, of course, are up to you, but typically reflections are about a page long, and graded loosely, basically just to see that some thought went into the essay, rather than looking for correctness of grammar and organization of any kind of coherent argument. Ideally, a reflection will include both an analysis of the situation (an answer to the question), and then a response (what the student would do about it). For example, in answer to the first question, one might say (briefly) that mathematics is a way of making sense of things in a quantifiable way, but that for many people mathematics is a set of formulæ to memorize, or a bad memory from high school. In order to change the latter impression, we need a systemic change in the way math is presented in our schools.

Many of these topics will arise naturally during the course of the semester (in class discussions), and you may wish to assign a reflection accordingly, and then devote some time in class to discussing the matter as a large group when you hand the papers back. Having a full page describing their thoughts on the matter in front of them will ensure that everyone has something to say. Make it clear in each case that you want them to put some thought into the matter and say what they really believe, not what they think you want to hear. Communication is the whole point.

Some of the reflections below were assigned in conjunction with handouts. These topics are marked with an asterisk, and the accompanying handout follows the list. Other sections of this guide suggest possible points at which to assign (some) particular reflections, but of course the decision is up to you. The numbering below is not intended as a canonical sequence.

1.* What is mathematics? [The handout lists several questions to consider. They can be for the initial reflection or for an in-class discussion; be sure to specify which if you use them. If you don't use them for the initial reflection, and quite possibly even if you do, you will get some rather poor definitions, but keep at it. I ended up assigning this twice, once without the extra questions, and once with them, after an hour's in-class large group discussion of it. Some of the individual questions on the handout could serve as entire reflections in and of themselves.]

2.* How well does this course address the NCTM *Standards*? [The handout gives excerpts from some of the most salient of the Curriculum Standards. This may be a useful reflection to discuss if the question "Why are we doing things this way?" starts to recur. This reflection should not be assigned until at least a few weeks into the term, however, so that students will have the basis for an answer.]

3. Compare the four basic arithmetic operations with the more abstract operations given in Unit Three.

4. Hard problems: Are they of value? (Why or why not?) Are they harmful? Will you assign them? (What makes a problem hard?)

5.* Respond to the quote "Students may not be expected to learn anything they have not been explicitly told to learn." [This can be assigned as stated. The handout gives excerpts from some student responses, and I actually told my students to respond, in addition, to one or two of the most interesting (to them) statements from this handout, in order to set the stage for an in-class discussion. Questions to spark an in-class discussion are also given.]

6. What does it mean to get stuck?

7. What qualities does a good group member have? What does it take for a small group to work well together? Is it important for all group members to have the same qualities? What are your strengths and weaknesses as a group member?

8. Fermat's Last Theorem, famous in the mathematical world, states that the equation $x^n+y^n = z^n$, where x, y and z are integers, has no solutions if n is larger than 2. Read up on the background of this theorem and recent developments regarding it (the history, not the details of the proof), and describe what you think is important about the theorem from a historical and cultural context.

9. What do you think explains the perception that men are somehow better at mathematics than women? Is there anything to do about this?

Reflection Handouts

Handout 1. What is mathematics?

Here are some questions to consider:

• Is math invented or discovered?
Do things already have a certain structure, or do we impose structure on them?
Would math be different if it had been done (or dominated) by a different set of people (or a different culture)?

• Is new mathematics still being done/created/discovered? (If so, is any of it really useful?)

• What do mathematicians do?

• What do you want your students to know about it — what kind of impression do you want them to get?

• Mathematics can be involved in several daily activities, like measuring, estimating, and finances. Do any of these activities require more (or other) math than arithmetic and maybe basic algebra? Apart from scientists and engineers, do we really need to know any other math than that?

• Why do so many people have memorably bad experiences with math (as opposed to history or other subjects)? Is it something inherent in the subject matter? How would you change things if you had the power to do so?

• Is it worthwhile (or useful) to be able to apply algorithms that one doesn't really understand?

• How far should we extend our definition of math? With recognizing patterns, for instance: would we include recognizing the pattern of phases of the moon? What about recognizing things like personality types, handwriting, or æsthetics (where to hang a picture or place furniture)?

• How has technology changed the face of math? (Are computer proofs legitimate?)

• Is math fair? Are some people naturally gifted at math and others not?

• Is there anything after calculus?

• Is math value-free? Before you answer, consider the following story.
A few years ago, it was found that inner-city kids were answering a question on bus fares incorrectly on a national assessment test. The question was something like this:

> You can buy a one-way bus fare for fifty cents, or a bus pass that is good for a month but costs thirty dollars. If you take the bus to school and back 20 times a month, should you buy the monthly pass or the daily fares?

The kids mostly wanted to buy the pass, even though it appears to be the more expensive option. When questioned, they said that you should buy a pass because then other members of the family can use it at night. So it's clear that in this case, there were underlying cultural assumptions which entered into the problem.

Handout 2.

Read the excerpts below from the NCTM *Curriculum Standards* and *Teaching Standards* and the MAA Report *A Call For Change*. How well does this course address these standards? What works? What doesn't? Respond generally, and also in particular as regards Standards 1 and 3 from *A Call For Change* (for which I have reprinted some of the accompanying discussion).

This should not be considered a formal composition, but you should state your position (opinion) clearly and back it up with specifics from the readings and your experiences in class. Be honest, and include any suggestions for improvement you may have (and say why or how you think they would help, e.g., would they help address these standards better? Do they address something the standards don't?).

From the Introduction to the NCTM *Curriculum Standards* (pp.5-6)

New Goals for Students. Educational goals for students must reflect the importance of mathematical literacy. Toward this end, the K-12 standards articulate five general goals for all students: (1) that they learn to value mathematics, (2) that they become confident in their ability to do mathematics, (3) that they become mathematical problem solvers, (4) that they learn to communicate mathematically, and (5) that they learn to reason mathematically. These goals imply that students should be exposed to numerous and varied interrelated experiences that encourage them to value the mathematical enterprise, to develop mathematical habits of mind, and to understand and appreciate the role of mathematics in human affairs; that they should be encouraged to explore, to guess, and even to make and correct errors so that they gain confidence in their ability to solve complex problems; that they should read, write, and discuss mathematics; and that they should conjecture, text, and build arguments about a conjecture's validity.

The opportunity for all students to experience these components of mathematical training is at the heart of our vision of a quality mathematics program. The curriculum should be permeated with these goals and experiences so that they become commonplace in the lives of students. We are convinced that if students are exposed to the kinds of experiences outlined in the *Standards*, they will gain *mathematical power*. This term denotes an individual's abilities to explore, conjecture, and reason logically, as well as the ability to use a variety of mathematical methods effectively to solve nonroutine problems. This notion is based on the recognition of mathematics as more than a collection of concepts and skills to be mastered; it includes methods of investigating and reasoning, means of communication, and notions of context. In addition, for each individual, mathematical power involves the development of personal self-confidence.

1. *Learning to value mathematics.* Students should have numerous and varied experiences related to the cultural, historical, and scientific evolution of mathematics so that they can appreciate the role of mathematics in the development of our contemporary society and explore relationships among mathematics and the disciplines it serves: the physical and life sciences, and social sciences, and the humanities.

Throughout the history of mathematics, practical problems and theoretical pursuits have stimulated one another to such an extent that is is impossible to disentangle them. Even today, as theoretical mathematics has burgeoned in its diversity and deepened in its complexity and abstraction, it has become more concrete and vital to our technologically oriented society. It is the intent of this goal – learning to value mathematics – to focus attention on the need for student awareness of the interaction between mathematics and the historical situations from which it has developed and the impact that interaction has on our culture and our lives.

2. *Becoming confident in one's own ability.* As a result of studying mathematics, students need to view themselves as capable of using their growing mathematical power to make sense of new problem situations in the world around them. To some extent, everybody is a mathematician and does mathematics consciously. To buy at the market, to measure a strip of wallpaper, or to decorate a ceramic pot with a regular pattern is doing mathematics. School mathematics must endow all students with a realization that doing mathematics is a common human activity. Having numerous and varied experiences allows students to trust their own mathematical thinking.

3. *Becoming a mathematical problem solver.* The development of each student's ability to solve problems is essential if he or she is to be a productive citizen. We strongly endorse the first recommendation of *An Agenda for Action* (National Council of Teachers of Mathematics 1980): "Problem solving must be the focus of school mathematics" (p.2). To develop such abilities, students need to work on problems that may take hours, days, and even weeks to solve. Although some may be relatively simple exercises to be accomplished independently, others should involve small groups or an entire class working cooperatively. Some problems also should be open-ended with no right answer, and others need to be formulated.

4. *Learning to communicate mathematically.* The development of a student's power to use mathematics involves learning the signs, symbols, and terms of mathematics. This is best accomplished in problem situations in which students have an opportunity to read, write, and discuss ideas in which the use of the language of mathematics becomes natural. As students communicate their ideas, they learn to clarify, refine, and consolidate their thinking.

5. *Learning to reason mathematically.* Making conjectures, gathering evidence, and building an argument to support such notions are fundamental to doing mathematics. In fact, a demonstration of good reasoning should be rewarded even more than students' ability to find correct answers.

In summary, the intent of these goals is that students will become mathematically literate. This term denotes an individual's ability to explore, to conjecture, and to reason logically, as well as to use a variety of mathematical methods effectively to solve problems. By becoming literate, their mathematical power should develop.

From the MAA Report: *A Call for Change*

Standards Common to the Preparation of Mathematical Teachers at all Levels.

Standards in this section encompass the preparation recommended for mathematics teachers so that they (i) view mathematics as a system of interrelated principles, (ii) are able to communicate mathematics accurately, both orally and in writing, (iii) understand the elements of mathematical modeling, (iv) understand and use calculators and computers appropriately in the teaching and learning of mathematics, and (v) appreciate the development of mathematics both historically and culturally. The recommendations in this section will be assumed to influence, and be a part of, all mathematics courses taken by prospective and practicing teachers of mathematics.

In developing programs for teachers of mathematics that incorporate these standards, we assume that appropriate attention, interpretation, and emphasis will be given to the level of school mathematics for which the preparation is intended.

The standards in this section are:

1. Learning Mathematical Ideas
2. Connecting Mathematical Ideas
3. Communicating Mathematical Ideas
4. Building Mathematical Models
5. Using Technology
6. Developing Perspectives

Standard 1: Learning Mathematical Ideas
The mathematical preparation of teachers must enable them to:

- *become independent learners, capable of doing and learning mathematics on their own;*
- *develop their own processes, concepts, and techniques for solving problems;*
- *exercise mathematical reasoning through recognizing patterns, making and refining conjectures and definitions, and constructing logical arguments, both formal and heuristic, to justify results.*

Learning Mathematical Ideas

Mathematics is learned by doing. In the process of exploring problems, designing the conducting experiments, making, testing, and refining conjectures, and proving theorems, learners begin to understand the value of inductive and deductive reasoning in mathematics. They develop skills in recognizing, extending, and verifying patterns, identifying assumptions, building systems, and using results. They learn not only to answer questions and solve someone else's problems, but to pose their own questions and problems. Through a variety of these learning experiences, mathematics teachers develop a sense of self-confidence and a willingness to explore and learn new mathematics on their own. To make this possible, collegiate mathematics classrooms must become a place where students actively do mathematics rather than simply learn about it.

In collegiate classes, students must be active participants in the learning process rather than passive recipients of information. Classroom activities should include critical thinking and questioning, searching for patterns and contradictions, discussion, debate, and persuasive argument. Students should read and discuss text material and alternative sources, including expository and historical articles. They must experience the use of resource materials, including physical models and technology to enhance their understanding of ideas. They should explore new (to them) areas of mathematics independently and through cooperative teamwork. In such a classroom, the role of collegiate mathematics instructors will change from one of disseminating information to one of guiding the learning of mathematics. This change may not be easy, and for many of us it will take time and effort. *A Source Book for College Mathematics Teaching* (MAA Notes and Reports, 1990), prepared by the MAA's Committee on the Teaching of Undergraduate Mathematics, provides suggestions and resources to help expand a repertoire of instructional techniques.

Standard 2: Connecting Mathematical Ideas
The mathematical preparation of teachers must provide experiences in which they:

- *develop an understanding of the interrelationships within mathematics and an appreciation of its unity;*
- *explore the connections that exist between mathematics and other disciplines;*
- *apply mathematics learned in one context to the solution of problems in other contexts.*

Standard 3: Communicating Mathematical Ideas
The mathematical preparation of teachers must enable them to:

- *develop skills in both written and oral communication of mathematical concepts and technical information;*
- *learn to communicate effectively at various levels of formality and with people who have differing levels of mathematical insight;*
- *understand and appreciate the power of mathematical language and symbolism in the development of mathematical concepts.*

Communicating Mathematical Ideas
Mathematics teachers at all levels must be able to communicate mathematical ideas with ease and clarity. Teachers must discuss concepts, ask and answer questions, guide explorations, pose problems, present logical arguments, and critique the work of their students. Communicating mathematics is also a necessary part of their own learning experience. As they struggle to phrase a definition or justify a conjecture or describe a problem situation, students deepen their own understanding of the underlying mathematics.

Teachers will attain this standard as they are routinely expected to talk and write about mathematics in their college mathematics courses. They need opportunities to work in groups to construct models or solve problems, to draw figures to clarify problem situations, and to design graphs to represent sets of data. We should regularly ask them to respond to questions which promote discussion or require considerable explanation. (For example, How do you know this solution is unique? Why are all measurements approximations? Why do you believe this conjecture is correct? To verify that p is a prime, why is it sufficient to consider divisibility of p only by primes less than or equal to $\sqrt{p}$?) We should require them to present written and oral arguments and explanations, justify their own reasoning, critique the reasoning of others, and answer discussion questions on tests. To facilitate the development of communication skills, we must pay attention not only to the mathematical correctness of our students' written and oral work, but also to the quality of such exposition, and provide, on a regular basis, the kind of feedback which will help students improve.

Standard 4: Building Mathematical Models
The mathematical preparation of teachers must include experiences that enable, motivate, and encourage them to analyze real-world situations through the use of whatever mathematical ideas or quantitative strategies are available. In particular, they should be able to:

- *work with a given model;*
- *recognize constraints inherent in a given model;*
- *construct models to analyze real-world settings and use symbols and reasoning in analysis;*
- *convert among representations (graphical, numerical, symbolic, verbal) that reflect quantitative constraints in a given real-world setting.*

Standard 5: Using Technology
The mathematical preparation of teachers must include experiences in which they use calculators and computers:

- *as tools to represent mathematical ideas and construct different representations of mathematical concepts;*
- *to engender a broad array of mathematical modes of thinking through use of powerful computing tools (including functions graphers, curve fitters, and symbolic manipulators);*
- *to develop and use alternate strategies for solving problems.*

Standard 6: Developing Perspectives
The mathematical preparation of teachers must include experiences in which they:

- *explore the dynamic nature of mathematics and its increasingly significant role in social, cultural, and economic development;*
- *develop an appreciation of the contributions made by various cultures to the growth and development of mathematical ideas;*
- *investigate the contributions made by individuals, both female and male, and from a variety of cultures, in the development of ancient, modern, and current mathematical topics;*
- *gain an understanding of the historical development of major school mathematics concepts.*

From the NCTM *Professional Teaching Standards*/Teaching (pp. 35, 45):

Standard 2: The Teacher's Role in Discourse
The teacher of mathematics should orchestrate discourse by —

- posing questions and tasks that elicit, engage, and challenge each student's thinking;
- listening carefully to students' ideas;

• asking students to clarify and justify their ideas orally and in writing;

• deciding what to pursue in depth from among the ideas that students bring up during a discussion;

• deciding when and how to attach mathematical notation and language to students' ideas;

• deciding when to provide information, when to clarify an issue, when to model, when to lead, and when to let a student struggle with a difficulty;

• monitoring students' participation in discussion and deciding when and how to encourage each student to participate.

Elaboration

Like a piece of music, the classroom discourse has themes that pull together to create a while that has meaning. The teacher has a central role in orchestrating the oral and written discourse in ways that contribute to students' understanding of mathematics.

The kind of mathematical discourse described above does not occur spontaneously in most classrooms. It requires an environment in which everyone's thinking is respected and in which reasoning and arguing about mathematical meanings is the norm. Students, use to the teacher doing most of the talking while they remain passive, need guidance and encouragement in order to participate actively in the discourse of a collaborative community. Some students, particularly those who have been successful in more traditional mathematics classrooms, may be resistant to taking, writing, and reasoning together about mathematics.

One aspect of the teacher's role is to provoke students' reasoning about mathematics. Teachers must do this through the tasks they provide and the questions they ask. For example, teachers should regularly follows students' statements with, "Why?" or by asking them to explain. Doing this consistently, irrespective of the correctness of students' statements, is an important part of establishing a discourse centered on mathematical reasoning. Cultivating a tone of interest when asking a student to explain or elaborate on an idea helps to establish norms of civility and respect rather than criticism and doubt. Teachers also stimulate discourse by asking students to write explanation for their solutions and provide justifications for their ideas.

Standard 3: Students' Role in Discourse

The teacher of mathematics should promote classroom discourse in which students —

• listen to, respond to, and question the teacher and one another;

• use a variety of tools to reason, make connections, solve problems, and communicate;

• initiate problems and questions;

• make conjectures and present solutions;

• explore examples and counterexamples to investigate a conjecture;

• try to convince themselves and one another of the validity of particular representations, solutions, conjectures, and answers;

• rely on mathematical evidence and argument to determine validity.

Elaboration

The nature of classroom discourse is a major influence on what students learn about mathematics. Students should engage in making conjectures, proposing approaches and solutions to problems, and arguing about the validity of particular claims. They should learn to verify, revise, and discard claims on the basis of mathematical evidence and use a variety of mathematical tools. Whether working in small or large groups, they should be the audience for one another's comments – that is, they should speak to one another, aiming to convince or to question their peers. Above all, the discourse should be focused on making sense of mathematical ideas, on using mathematical ideas sensibly in setting up and solving problems.

Handout 5. Respond to the quote "Students may not be expected to learn anything they have not been explicitly told to learn." (Some student responses are excerpted on a separate page.)

Some questions for discussion:

• Some students suggested ["hard"] projects for extra credit as opposed to tests. So is it fair to make going beyond the given information a part of the grade as long as it's not on a (timed, closed-book) test? If not, what sort of encouragement *is* allowable to motivate students to think critically.

• How do you keep from getting to the point where students' only question is "Will this be on the test?"

• Can younger kids synthesize ideas and think creatively and/or critically? (Might it show up differently with younger kids?)

• In this age of political correctness and accommodating disabilities, should we hold different students to different levels of accountability for critical thinking?

Chapter 6

Exams

6.1 What should they be like?

Historically, Math 130 has had more or less traditional exams, in that students work individually in class on a set of problems, sometimes with access to their notes and sometimes without, sometimes with a longer take-home problem and sometimes without. However, some instructors have also used portfolios to assess their students, sometimes in place of an exam; we will discuss this option more below.

An obvious question to ask, given the cooperative nature of the sequence, is: Can the exams be given in groups? In fact, students will most likely have become comfortable working in groups by the time of the first exam, and you can be certain that a chorus will rise up to ask this question. My reason for not giving group exams is that I want to make sure that a reasonable portion of the student's grade is based on work they did without help. There is some research showing that students perform better in cooperative learning classrooms when exams are individual[1], but given the group nature of most of the rest of the work, I think individual accountability is a much stronger reason for sticking to individual exams. You may wish to point out that the students will not have their groupmates with them during their careers as teachers.

Exams, of course, invariably cause anxiety in many students, and you may want to make at least one of your exams open-notes (perhaps the final, if the midterm is closed-notes). This option also allows you to ask more detailed questions than you might be able to otherwise: questions about particular activities, or variations on problems assigned as write-ups. In either case, do be careful to write an exam that is at the level of the students: On one hand, it's easy to get carried away and overestimate what your students can handle. On the other, sometimes your students will surprise you (pleasantly).

One more note: You should probably not give more than three exams counting the final, as time is precious, and using cooperative learning means a sacrifice on the content-versus-time scale. Exams can disrupt the flow of a class, and it's already difficult to cover as much as you'd like.

You may want to consider a portfolio assignment. We don't recommend for or against this, but in case you haven't seen these before and want to know how they work, here is a brief discussion. If you're not interested in portfolios, skip this section.

Portfolios typically involve the student collecting, and revising, a selection of his/her best work in the class, and sometimes adding something new as well: a cover letter explaining the portfolio

[1] Slavin, R.E. Research for the future: Research on Cooperative learning and achievement, what we know, what we don't know. *Contemporary Educational Psychology* **21** (1996): 43-49.

and how its contents interrelate, an extra problem report, or a short paper involving an issue relevant to the class. This can be a good way for students to have a look back at the course, and to sum up what they can take away with them from it. Be sure not to overload them with work by piling a portfolio on top of an exam or a bunch of problem reports. If you decide to use one — and not all instructors have — give them the assignment at the beginning of the term (even though most of them will procrastinate until the week it's due).

A sample handout follows which was used by one instructor in assigning a portfolio. See the next section for suggestions on how to include a portfolio in the course grade.

Project — A Portfolio

Purpose: The main point of this project is to address the fourth of the goals for this course that I gave at the beginning of this semester (look it up). I hope that it also provides you with an opportunity to bring together much of what you have learned during the semester and to highlight what this course has made possible both for you and for your future students. As a bonus, you may end up with something you want to show to prospective employers!

Contents:

1. A cover letter summarizing the contents of the portfolio and stating the purpose of it in your own words.

2. Four of your strongest problem write-ups, along with an explanation of why you chose each one: either because you solved a hard problem, you were creative or had an original solution or nice explanation, or whatever. You may also mention how you might still improve on the report.

3. All of your "reflections" from throughout the semester, along with any further thoughts you have on the topics.

   Also, write a reflection on: what do you *personally* find valuable about co-operative learning as a pedagogical method as it was used in Math 130, and what do you find problematic about it (be critical and frank)?

4. Write a short paper in response to an article in math education research. I have suggested some below. It should be at least 2 but no more than 4 pages. The object here is to bring your own experience to bear to assess the claims made by fellow researchers in math education.

   You have something valuable to contribute to other teachers' understanding of how math is learned and how it can be taught effectively! Explain yourself with clear, well-formulated arguments, specific examples and careful explanations (and pictures, even), just like in your problem reports.

   Important: once you have chosen the article you want to respond to, you must get my approval. Only five people can write about each paper, and it's first come, first served!

The grade on your project will be three-pronged:

- the thoughtfulness and thoroughness of the work.
- The clarity and overall effectiveness of the arguments that you make (remember, specific examples used to illustrate all general statements).
- The overall presentation of the portfolio.

If you want to do something in place of this portfolio which fulfills the same purpose, such as creating and using a cooperative learning lesson plan and writing up a report on your experience, or observing a cooperative learning elementary/secondary classroom, you are highly encouraged to pursue this!!! See me and I will help you to make it happen.

Some math education research papers

JRME is the Journal of Research in Mathematics Education.

Students' Use of Part-Whole and Direct Comparison Strategies for Comparing Partitioned Rectangles. Barbara E. Armstrong & Carol Novillis Larson, pp. 2-19 JRME 1995

Children's Concepts of Average and Representativeness. Jan Mokros & Susan Jo Russell, pp. 20-39 JRME 1995

Reconstructing Mathematics Pedagogy From a Constructivist Perspective. Martin A. Simon, pp. 114-145. JRME 1995

Motivation and Ability as Factors in Mathematics Experience and Achievement. Ulrich Schiefele and Mihaly Csikszentmihalyi, pp. 163-181 JRME 1995

The Relevance of Practice: A Response to Orton. Paul Cobb, pp. 230-253. JRME 1995

A Study of Intrinsic Motivation in the Mathematics Classroom: A Personal Constructs Approach. James A. Middleton, p. 254-279. JRME 1995

Mental Computation Performance and Strategy Use of Japanese Students in Grades 2, 4, 6, and 8. Robert E. Reys, Barbara J. Reys, Nobuhiko Nohda, and Hideyo Emori. pp. 304-326. JRME 1995

Factors Associated With Types of Mathematics Anxiety in College Students. Kenneth C. Bessant, p. 327-345. JRME 1995

Numeracy as Cultural Practice: An Examination of Numbers in Magazines for Children, Teenagers, and Adults. Elana Joram, Lauren B. Resnick, and Anthony J. Gabriele, pp. 346-361 JRME 1995

Cultural Tools and Mathematical Learning: a case study. Paul Cobb, pp. 362-385 JRME 1995

The Frequency of Arithmetic Facts in Elementary Texts: Addition and Multiplication in Grades 1-6. Mark H. Ashcraft and Kelly S. Christy, pp. 396-421. JRME 1995

Confounding Whole-Number and Fraction Concepts When Building on Informal Knowledge. Nancy K. Mack, pp. 422-441. JRME 1995

The Tensions and Contradictions of the School Mathematics Tradition. Jeff Gregg, p. 442-466. JRME 1995

Learning to Solve Addition and Subtraction Word Problems Through a Structure-Plus-Writing Approach. Al Rudnitsky, Susan Etheredge, Sue J. M. Freeman, and Therese Gilbert, pp. 467-486 JRME 1995

Talking About Rates Conceptually, Part II: Mathematical Knowledge for Teaching. Alba G. Thompson and Patrick W. Thompson, pp. 2-24. JRME 1996

Identification of Multiplicative Thinking in Children in Grades 1-5. Faye B. Clark and Constance Kamii, pp. 41-51. JRME 1996

Intention and Significance in the Teaching and Learning of Mathematics. Tony Brown, pp. 52-66. JRME 1996

Inhibiting Factors in Generating Examples by Mathematics Teachers and Student Teachers: The Case of Binary Operation. Orit Zaslavsky and Irit Peled, pp. 67-78. JRME 1996

The development of unitizing: its role in children's partitioning strategies. Susan J. Lamon, pp. 170-193.

Gender-related differences in self-referenced cognitions in relation to mathematics. Gerard Seegers and Monique Boekaerts, pp. 215-240. JRME 1996

Students' understanding of three-dimensional rectangular arrays of cubes. Michael T. Battista and Douglas H. Clements, pp. 258-292. JRME 1996

Multidigit number sense: a framework for instruction and assessment. Graham A. Jones, Carol A. Thornton, Ian J. Putt, Kevin M. Hill, A. Timothy Mogill, Beverly S. Rich and Laura R. Van Zoest, pp. 310-336. JRME 1996

Assessment of a problem-centered mathematics program: third grade. Terry Wood and Patricia Sellers, pp. 337-353. JRME 1996

Efficacy and teaching mathematics by telling: a challenge for reform. John P. Smith III, pp. 387-402. JRME 1996

A longitudinal study of learning to use children's thinking in mathematics instruction. Elizabeth Fennema, Thomas P. Carpenter, Megan L. Franke, Linda Levi, Victoria R. Jacobs, and Susan B. Empson, pp. 403-434. JRME 1996

Exploring the origins, uses, and interactions of student intuitions: comparing the lengths of paths. Ming Ming Chiu, pp. 478-504. JRME 1996

An Analysis of Arithmetic Problem Posing by Middle School Students. Edward A. Silver and Jinfa Cai, p. 521 JRME 1996

Divisibility and Multiplicative Structure of Natural Numbers: Preservice Teachers' Understanding. Rina Zazkis and Stephen Campbell, p. 540 JRME 1996

Elementary Students' Construction and Coordination of Units in an Area Setting. Anne Reynolds and Grayson H. Wheatley, p. 564 JRME 1996

Stipek, Deborah, Granlinski, Heidi, "Gender Differences in Children's Achievement-Related Beliefs and Emotional Responses to Success and Young Children's Perceptions of Mathematics in Problem-Solving Environments." Megan Loef Franke and Deborah A. Carey, pp. 8-25 JRME 1997

Assessing the Relationship Between Attitude Toward Mathematics and Achievement in Mathematics: A Meta-Analysis. Xin Ma and Nand Kishor, pp. 26-47 JRME 1997

Conceptual Units Analysis of Preservice Elementary School Teachers' Strategies on a Rational-Number-as-Operator Task. Merlyn J. Behr, Helen A. Khoury, Guershon Harel, Thomas Post, and Richard Lesh, pp. 48-69 JRME 1997

Students' Development of Length Concepts in a Logo-Based Unit on Geometric Paths. Douglas H. Clements, Michael T. Battista, Julie Sarama, Sudha Swaminathan, Sue McMillen, pp. 70-95 JRME 1997

6.2 Final grades

The course grade will typically have at least one component — attendance and participation — which has historically not been present in traditional math classes. It is important to include this, however, as the process and experiences that take place inside the classroom are the most significant part of the course, and a student cannot really learn what we want him/her to learn without being present. Furthermore, students in the 130-131-132 sequence respond to attendance grading with a more than 90helps keep the groups stable. Typically you should take attendance every day, which can be done unobtrusively, by counting heads five minutes or so into the period, when everyone is working in small groups. After a couple of weeks you'll be able to tell more or less at a glance who's missing. You should set clear guidelines for attendance at the beginning of the term and stick by them (see, for example, the sample first-day handout in Chapter 3).

Measuring class participation is more subjective. By halfway through the semester, you should be able to tell who is pulling his/her weight in the small groups, and should encourage those who are not to participate more. For large group participation, you may want to keep track of who says something, and mark it down after class (don't do it in class — students don't like the idea that you're taking notes on what they do or say). This method requires some good memory on your part, but after a few weeks you'll see who's speaking up and who's not, and can encourage those who don't to participate more. One way of doing so without putting pressure on to know the answer to a question is to call a student up to the blackboard to serve as scribe in writing down others' suggestions during a brainstorming type of discussion. However you decide to do it, you really will have to be diligent about it from the first day.

As a help in deciding how to break down the course grade into components, here is a sampling of breakdowns used by previous instructors:

Exams	Written work	Attendance & participation	Portfolio
40% (2 × 20%)	30%	10%	20%
40% (2 × 20%)	40%	20%	
30% (2 × 15%)	50%	20%	
30% (2 × 15%)	55%	15%	
40% (2 × 20%)	45%	15%	
40% (midterm: 10%, quizzes: 10%, final: 20%)	40%	10%	10%

Sample Exams

6.3 Sample exam bank

Notes

• Exams with the same number are a practice exam/real exam pair, e.g., Exam 9a was given as a practice exam to prepare students for Exam 9b, which was given as an actual midterm.

• Unless otherwise specified, each problem was assigned the same weight in grading.

Index of Problem Topics (by unit)

Math 130 Sample Exam Bank Contents (updated 09 June 1997)

Math 130 Sample Exam 1	*Author:* Jen Szydlik	*Date:* 11 October 1993
Math 130 Sample Exam 2	*Author:* Marshall Osborn	*Date:* 18 May 1994
Math 130 Sample Exam 3	*Author:* Marshall Osborn	*Date:* 11 October 1994
Math 130 Sample Exam 4	*Author:* Marshall Osborn	*Date:* 17 December 1994
Math 130 Sample Exam 5	*Author:* Marshall Osborn	*Date:* 03 March 1995
Math 130 Sample Exam 6	*Author:* John Harvey	*Date:* 19 October 1995
Math 130 Sample Exam 7	*Author:* John Harvey	*Date:* 09 November 1995
Math 130 Sample Exam 8	*Author:* John Harvey	*Date:* 16 December 1995
Math 130 Sample Exam 9a	*Author:* Christopher Kribs	*Date:* 27 March 1996
Math 130 Sample Exam 9b	*Author:* Christopher Kribs	*Date:* 27 March 1996
Math 130 Sample Exam 9b THE	*Author:* Christopher Kribs	*Date:* 27 March 1996
Math 130 Sample Exam 10a	*Author:* Christopher Kribs	*Date:* 10 May 1996
Math 130 Sample Exam 10b	*Author:* Christopher Kribs	*Date:* 12 May 1996
Math 130 Sample Exam 10c	*Author:* Christopher Kribs	*Date:* 15 May 1996
Math 130 Sample Exam 11	*Author:* Ben Collins	*Date:* 01 April 1996
Math 130 Sample Exam 12	*Author:* Ben Collins	*Date:* 13 May 1996
Math 130 Sample Exam 13a	*Author:* Mirna Dzamonja	*Date:* 01 March 1996
Math 130 Sample Exam 13b	*Author:* Mirna Dzamonja	*Date:* 01 March 1996
Math 130 Sample Exam 14	*Author:* Mirna Dzamonja	*Date:* 08 April 1996
Math 130 Sample Exam 15	*Author:* Mirna Dzamonja	*Date:* 14 May 1996
Math 130 Sample Exam 16a	*Author:* Peter Orlik	*Date:* 11 October 1996
Math 130 Sample Exam 16b	*Author:* Peter Orlik	*Date:* 11 October 1996
Math 130 Sample Exam 17a	*Author:* Peter Orlik	*Date:* 15 November 1996
Math 130 Sample Exam 17b	*Author:* Peter Orlik	*Date:* 15 November 1996
Math 130 Sample Exam 18a	*Author:* Peter Orlik	*Date:* 18 December 1996
Math 130 Sample Exam 18b	*Author:* Peter Orlik	*Date:* 18 December 1996
Math 130 Sample Exam 19	*Author:* Heather Lewis	*Date:* 08 November 1996
Math 130 Sample Exam 19 THE	*Author:* Heather Lewis	*Date:* 08 November 1996
Math 130 Sample Exam 20a	*Author:* Heather Lewis	*Date:* 15 December 1996
Math 130 Sample Exam 20b	*Author:* Heather Lewis	*Date:* 15 December 1996
Math 130 Sample Exam 21	*Author:* Brad Franklin	*Date:* Spring 1997
Math 130 Sample Exam 22	*Author:* Brad Franklin	*Date:* 15 May 1997

Math 130 Sample Exam 1
Author: Jen Szydlik
Topics: Problem Solving (Unit One) [1], Operations (Unit Three) [2], Quantifiers (Unit Five) [3], Number Theory (Unit Seven) [4], Number Systems (Unit Seven) [3]
Date: 11 October 1993
Time Allotted: 1 hour

1. List three problem solving strategies that you used to work on problems in this course. After each one, write a few sentences about a problem you used the strategy for and specifically how you used it.

2. "#" is an operation on the natural numbers. Here is how # works: $a\#b = (a \cdot b) + 1$. (Here a and b are natural numbers, and "$\cdot$" and "+" are familiar multiplication and addition respectively.)

3. For which number systems ($\mathbb{N}$, $\mathbb{Z}, \mathbb{Q}$, $\mathbb{R}$) are the following true? Explain each answer! (If the statement doesn't hold for any of the above systems, write "none".)
(a) $\forall x, x \cdot x \geq x$
(b) $\exists u, \forall v, u = v$
(c) $\forall m, \exists n, m = n \cdot n$

4. Find a number bigger than 20 that has exactly five factors. Write a paragraph explaining how you solved (or tried to solve) the problem. (Make sure you explain your solution and convey any insight you had into the problem.)

Math 130 Sample Exam 2
Author: Marshall Osborn
Topics: Operations (Unit Three) [6, 7], Fractions (Unit Six) [2, 4],
Number Theory (Unit Seven) [8], Number Systems (Unit Seven) [5, 9],
Algebra (Unit Eight) [11], Exponents (Unit Eight) [1, 3], Probability (deleted) [10]
Date: 18 May 1994
Time Allotted: 2 hours

1. Evaluate the following expressions in scientific notation, expressing your answer to 3 significant places.

(a) (4 points) $\dfrac{2.3 \times 10^{-6}}{6.9 \times 10^{4}}$ (b) (4 points) $3.65 \times 10^{-9} + 2.78 \times 10^{-4}$

2. (8 points) Express the sum $.\overline{037} + .\overline{03}$ of repeating decimals as a fraction.

3. (8 points) Simplify the following expression and express your answer as a fraction:

$$\frac{2^{-5/7} \times 27^{1/3} \times 5^{-2}}{2^{-12/7} \times 7^{0}}$$

4. (8 points) Simplify the following expression and express your answer as a fraction: $\frac{\frac{7}{3}}{\frac{3}{7}} - \frac{\frac{4}{3}}{.25}$

5. (a) (7 points) What are the rational numbers?
(b) (7 points) What are the real numbers?
(c) (6 points) Is every rational number a real number? Explain.

6. (a) (4 points) Write a statement which asserts that the associative property holds for the set of integers under the operation of subtraction.
(b) (4 points) Is your statement in part (a) true? How do you know?

7. (6 points) If $4 \leq x \leq 5$ and $-.5 \leq y \leq -.4$, what can you say about how big xy is?

8. (a) (4 points) What is the greatest common divisor of $2^2 \cdot 3^5 \cdot 7^2$ and $2^3 \cdot 3^2 \cdot 5^3$?
(b) (4 points) What is the least common multiple of these same two numbers?

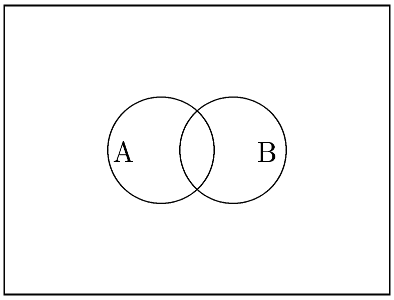
9. (6 points) Is $\sqrt{12}$ a rational or irrational number? How do you know?

10. (a) (4 points) If you flip a coin four times, what is the probability of getting heads the first time, tails the second time, heads the third time, and tails the fourth time?
(b) If you flip a coin four times, what is the probability that you will get heads exactly two of the four times?

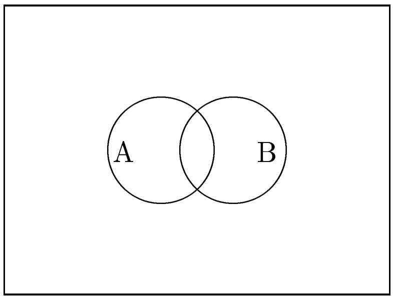
11. (12 points) The number of students who want to take Math 101 next semester will fill all of the sections listed in the timetable to a limit of 19 and there will be 5 students left over. If one of the sections is cancelled and the other sections filled to a limit of 20, there will be 1 student left over. How many sections are listed in the timetable? How many students want to take Math 101 next semester?

Math 130 Sample Exam 3
Author: Marshall Osborn
Topics: Problem Solving (Unit One) [6], Sets (Unit Two) [1], Operations (Unit Three) [3, 4], Logic (Unit Five) [5], Quantifiers (Unit Five) [2]
Date: 11 October 1994
Time Allotted: 1 hour

1. In the following two Venn diagrams, shade in the indicated set. (If you make a mistake, cross out the diagram and draw a new one which is correctly shaded.)



$\overline{A \cup B}$



$\overline{A} \cap \overline{B}$

2. Which of the following statements are true and which are false for the rational numbers $\mathbb{Q}$? Write a sentence in each case explaining how you know.

(a) $\forall x,\ \exists y,\ x = y^2$ (b) $\exists x,\ \forall y,\ x + y = 0$

3. Let $S = \{a, b, c\}$ and define the operation "*" on S using the table at right.
(a) Is "*" associative? How do you know?
(b) Does S have an identity under "*"? Explain.

*	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

4. Define the operation "#" on the integers $\mathbb{Z}$ by $x\#y = x + y - 10$, where "+" and "−" are the usual operations of addition and subtraction, respectively.

(a) Is this operation commutative? How do you know?

(b) Does $\mathbb{Z}$ have an identity under "#"? Explain.

5. (a) Let p and q be statements. Write a statement which is true exactly when p and q are both false.

(b) If p is true and q is false, what can you say about the truth or falsity of $(q \vee s) \rightarrow (q \rightarrow p)$? Explain

6. You are going to cut a circular cake with five straight cuts. How can you make your five cuts so as to end up with as many distinct pieces as possible? How many pieces will you end up with? You cannot move any of the pieces until you have made your five cuts. The pieces that you end up with do not have to be the same size or shape.

Math 130 Sample Exam 4
Author: Marshall Osborn
Topics: Operations (Unit Three) [7], Fractions (Unit Six) [1], Number Systems (Unit Seven) [5], Algebra (Unit Eight) [4, 6, 8], Exponents (Unit Eight) [2, 3], Combinatorics (deleted) [9], Probability (deleted) [10]
Date: 17 December 1994
Time Allotted: 2 hours

1. (a) (3 points) Will the decimal expansion of $\dfrac{23}{117}$ have finitely or infinitely many nonzero digits?
(b) (3 points) Will this be a repeating decimal or not?
(c) (3 points) If it is a repeating decimal, how many digits do you think that one might have to calculate before it starts to repeat?

2. (each part 4 points) Evaluate the following expressions in scientific notation, expressing your answer to 3 significant places:

(a) $\dfrac{1.2 \times 10^8}{2.4 \times 10^{-3}}$ (b) $3.24 \times 10^{-4} + 5.78 \times 10^{-8}$

3. (10 points) Simplify the following expression and express your answer as a fraction:

$$\frac{2^{-3/7} \times 5^0 \times 9^{1/2}}{2^{11/7} \times 7} + \frac{\frac{5}{2}}{3}$$

4. (5 points) Compute the following product of polynomials: $(2x^2 - x)(3x^4 + 4x^2 - 5)$

5. (a) (8 points) What are the rational numbers?
(b) (10 points) What are the real numbers?
(c) (5 points) Is every rational number a real number? Explain.

6. (8 points) Let m and n be two positive integers, and suppose that when you divide m into n you get the quotient q and the remainder r. Which of the following are valid equations relating m, n, q, and r? More than one may be valid.
(a) $\frac{m}{n} = q + r$, (b) $m = nq + r$, (c) $\frac{m}{q} = n + r$, (d) $\frac{m-r}{n} = q$, (e) $\frac{m}{n} = q + \frac{q}{n}$.

7. (7 points) If $4 \leq x \leq 5$ and $.4 \leq y \leq .5$, what can you say about how big xy is?

8. (each part 3 points) Let the function f be defined by $f(x) = x^2 - x + 2$.
(a) What is $f(3)$?
(b) For what values of x is $f(x) = 2$?

9. (12 points) An elementary school class has 11 students, 3 of them boys and 8 girls. How many different committees can be formed which consist of 1 boy and 2 girls?

10. (12 points) You roll two dice and get a total of 9. Your opponent sees what you have rolled, and then rolls his two dice. You do not see what he rolled, but you are sure from his expression that he has rolled a higher total than you. Assuming that his total is higher than yours, what is the probability that at least one of his dice is a 6?

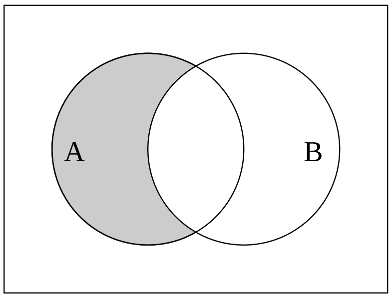
Math 130 Sample Exam 5
Author: Marshall Osborn
Topics: Problem Solving (Unit One) [6], Sets (Unit Two) [4], Operations (Unit Three) [5], Logic (Unit Five) [2, 3] Quantifiers (Unit Five) [1]
Date: 03 March 1995
Time Allotted: 1 hour

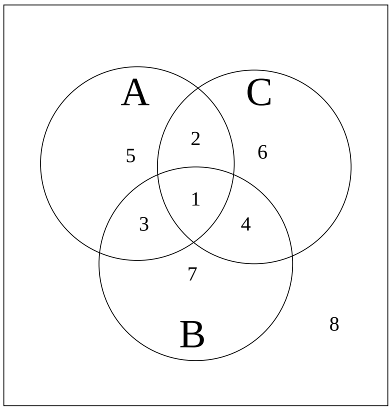
1. (18 points) Which of the following statements are true and which are false for the natural numbers $\mathbb{N}$? Write a sentence in each case explaining how you know.
(a) $\exists x,\ \forall y,\ xy > y$ (b) $\forall p,\ \exists q,\ \forall r,\ pr = q$

2. (11 points) If p is true and q is false, what can you say about the truth of $((\neg p) \wedge q) \rightarrow ((\neg q) \wedge p)$?

3. (11 points) Let p and q be two statements. Is $\neg(p \wedge q) = (\neg p) \vee (\neg q)$? How do you know?

4. (18 points) Find a way to represent each of the following two shaded regions in terms of A, B, and C.





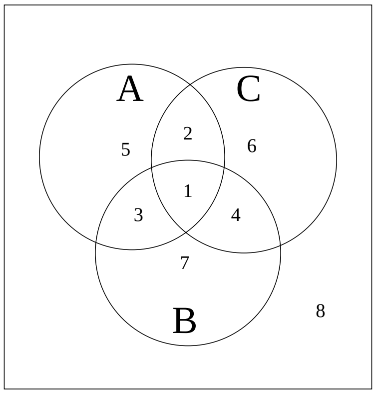
[shade regions 1, 2, 4]

5. (18 points) Let S be the clock system with the operation $\oplus$.
(a) Write down a formal mathematical statement which asserts that S has inverses with respect to $\oplus$.
(b) Is this a true statement? How do you know?

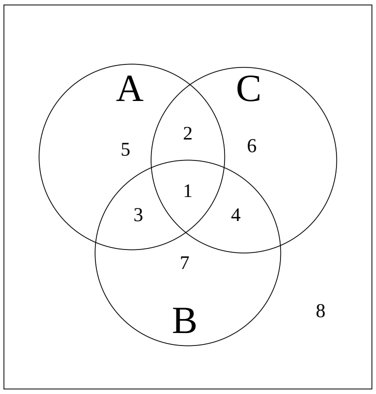
6. (24 points) You are going to cut a large square pizza with five straight cuts (not necessarily parallel to the sides). How can you make your five cuts so as to end up with as many distinct pieces as possible? How many pieces will you end up with? You cannot move any of the pieces until you have made all five cuts. The pieces you end up with do not have to be the same size or shape.

Math 130 Sample Exam 6
Author: John Harvey
Topics: Problem Solving (Unit One) [5], Sets (Unit Two) [1], Operations (Unit Three) [2], Logic (Unit Five) [2], Quantifiers (Unit Five) [3, 4], Number Systems (Unit Seven) [3]
Date: 19 October 1995
Time Allotted: 1 hour

1. Find a way of representing the following shaded regions in terms of A, B, and C using the symbols $\cup$, $\cap$, and $\subset$ and set complement (e.g., $\overline{A}$).

(a) 

[shade regions 1, 2, 3, 5]

(b) 

[shade regions 1, 4, 7]

2. Suppose that you are playing Propositional Logic Bingo against someone else and that the two bingo boards are as shown below. Which board is better, or are they both equally likely to win? Why? (Recall the rules: Letters are chosen in a random order, and eachis declared either true or false at random. When you can prove a square to be true, you get that square. Whoever gets two squares in a row first wins.)

$\neg a$	b
$c \to d$	e

p	$(q \wedge r) \vee (\neg q \wedge \neg r)$
$s \vee t$	$\neg u$

3. Let $\mathbb{N} = \{0, 1, 2, 3, ...\}$, the natural numbers; let $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$, the integers; let $\mathbb{Q} = \{\frac{a}{b}$ where a and b are integers and $b \neq 0\}$, the rational numbers (the fractions). In which of these number systems (that is, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$) is the following statement true? Explain your answer.

$$\forall x \, \exists y, x^2 + 2y = 3$$

4. Let $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$ be the set of integers. Say whether the following statements are true or false in $\mathbb{Z}$ and briefly describe your reasoning for each statement.

(a) $\forall x \, \exists y, x^3 + y = -3$ (b) $\forall y \, \exists x, x^3 + y = -3$

5. One important problem solving strategy is that of finding more than one way of solving a single problem. Briefly describe two different ways that you went about solving one of the problems we have solved in this course.

Math 130 Sample Exam 7
Author: John Harvey
Topics: Sets (Unit Two) [1], Operations (Unit Three) [2], Logic (Unit Five) [3], Quantifiers (Unit Five) [4], Number Systems (Unit Seven) [4]
Date: 09 November 1995
Time Allotted: 2 days (take-home)
Note: calculator but no class notes allowed

1. [included definitions of union, intersection, complement omitted]
Let $\mathbb{N} = \{0, 1, 2, 3, ...\}$, the set of natural numbers.
Let $P = \{p \in \mathbb{N} | p = 3y + 1 \text{ for some } y \in \mathbb{N}\}$.
Let $S = \{s \in \mathbb{N} | s = 2n \text{ for some } n \in \mathbb{N}, n < 20\}$.
Let $T = \{t \in \mathbb{N} | t = m^2 \text{ for some } m \in \mathbb{N}, m \leq 20\}$.
Completely describe each of the following sets:
(a) $S \cap T$ (b) $P \cap S$ (c) $S \cup T$ (d) $S \cap \overline{P}$

2. [five included definitions of operation properties omitted] Let $\mathbb{Q} = \{\frac{a}{b}$ where a and b are integers and $b \neq 0\}$, the set of rational numbers. Define $\oplus$ on $\mathbb{Q}$ as follows: $\forall a, b \in \mathbb{Q},\ a \oplus b = \frac{a+b}{2}$, where $a + b$ is ordinary (rational number) addition.
(a) Prove that $\mathbb{Q}$ is closed with respect to $\oplus$; that is, prove that $\forall a, b \in \mathbb{Q},\ a \oplus b \in \mathbb{Q}$.
(b) Prove that $\oplus$ is commutative; that is, prove that $\forall a, b \in \mathbb{Q},\ a \oplus b = b \oplus a$.
(c) Prove that $\oplus$ is not associative; that is, prove that $\exists a, b, c, \in \mathbb{Q},\ a \oplus (b \oplus c) \neq (a \oplus b) \oplus c$.
(d) Prove that $\oplus$ does not have an identity; that is, prove that $\nexists e \in \mathbb{Q}$ such that $\forall x \in \mathbb{Q},\ x \oplus e = e \oplus x = x$.

3. [included truth tables for *not* ($\neg$), *and* ($\wedge$), *or* ($\vee$), and *implies* ($\rightarrow$) omitted]
(a) Define $f \leftrightarrow g$ as $((f \rightarrow g) \wedge (g \rightarrow f))$. Make a truth table for $f \leftrightarrow g$.
For the remaining parts of this problem, let p and q be statements.
Prove that the following statements *are always true.*
(b) $((p \leftrightarrow q) \wedge \neg p) \rightarrow \neg q$ (c) $(p \vee \neg q) \leftrightarrow \neg(\neg p \wedge q)$ (d) $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$

4. Let $\mathbb{N} = \{0, 1, 2, 3, ...\}$, the natural numbers; let $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$, the integers; let $\mathbb{Q} = \{\frac{a}{b}$ where a and b are integers and $b \neq 0\}$, the rational numbers. Recall that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$. In which of these number systems (that is, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$) are the following statements true? When a statement is true, prove that it is true. When a statement is false, give a counterexample showing that it is false.
(a) $\exists x, y$ such that $x^2 + y^2 = 25$
(b) $\forall y \exists x$ such that $y^2 - 2x = 0$
(c) $\forall$ even integers x, $\exists y$ such that $x + 2y = 12$
(d) $\forall x \forall y,\ x + y \geq x$

Math 130 Sample Exam 8
Author: John Harvey
Topics: Arithmetic, Place Value & Bases (Unit Four) [4], Number Theory (Unit Seven) [1, 2], Fractions (Unit Six) [3], Algebra (Unit Eight) [5], Probability (deleted) [6]
Date: 16 December 1995
Time Allotted: 2 hours
Note: calculator but no class notes allowed

1. [included definitions of factors/divisors and primeness omitted]
(a) Find all the divisors of 1440. (Hint: 1440 has 36 divisors.)
(b) Find a number with exactly twice as many divisors as 1440.

2. Not long ago we studied the problem: Suppose that you have an unlimited supply of 5¢ stamps and an unlimited supply of 13¢ stamps. What amounts *can't* you make with these stamps?

The answers to this question (in cents) are: 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14, 16, 17, 19, 21, 22, 24, 27, 29, 32, 34, 37, 42, 47. That is, the amounts that we *can make* with these stamps are: 5, 10, 13, 15, 18, 20, 23, 25, 26, 28, 30, 31, 33, 35, 36, 38, 39, 40, 41, 42, 43, 44, 45, 46, 48, and all amounts greater than 48.
(a) Show that you *cannot* make a total of 47¢ with 5¢ and 13¢ stamps.
(b) Show that you *can* make a total of 59¢ with 5¢ and 13¢ stamps.

3. (a) Express $\frac{9}{14}$ as a repeating decimal. (b) Express the repeating decimal $0.236\overline{236}$ as a common fraction; that is, as the quotient of two whole numbers.

4. [included base seven addition and multiplication tables omitted] Calculate each of the following in base seven:

(a) $\begin{array}{cccc} & 1 & 2 & 3 \\ + & 4 & 5 & 6 \\ \hline \end{array}$ (b) $\begin{array}{cccc} & 3 & 0 & 3 \\ \times & & 2 & 5 \\ \hline \end{array}$ (c) $\begin{array}{ccccc} & 1 & 2 & 3 & 4 \\ - & & 5 & 6 & 6 \\ \hline \end{array}$ (d) $2\ 4\ \overline{)\ 4\ 5\ 2}$

5. Christopher is now twice as old as Jennifer was four years ago. Next year Robin will be half as old as Jennifer is now.
(a) Choose variables to represent appropriate quantities in this problem, being sure to describe exactly what each variable represents. Represent the problem described above in terms of those variables.
(b) Is it possible that Robin is 7, Jennifer is 16, and Christopher is 24? Explain.
(c) Is there enough information given to determine Christopher's age? Explain.

6. You have been given two numbered cubes. On the faces of one of the cubes are the numbers 1, 2, 3, 4, 5, and 6. On the faces of the other numbered cube are the numbers 1, 2, 3, 5, 7, 11. Both of the numbered cubes are tossed once, and the numbers that show on the tops of the cubes are added.
(a) The sample space for a single toss of these two cubes consists of 36 points. List all of the possible results in a table. The first few entries are filled in as an example.
(b) What is the probability that the sum of the faces showing is even?
(c) What is the sum with the highest probability?

Face showing on the cube with faces 1,2,3,4,5,6	Face showing on the cube with faces 1,2,3,5,7,11	Sum of the numbers on the faces showing
1	1	2
1	2	3
1	3	4
1	5	6
1	7	8
1	11	12
...	...	...

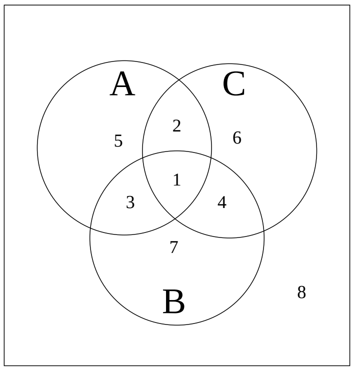
Math 130 Sample Exam 9a
Author: Christopher Kribs
Topics: Problem Solving (Unit One) [4], Sets (Unit Two) [3], Operations (Unit Three) [1], Arithmetic, Place Value & Bases (Unit Four) [2], Quantifiers (Unit Five) [5]
Date: 27 March 1996
Time Allotted: 1 hour (in-class) *plus* 2 days (take-home)

Instructions: Answer all of the following questions, and provide justifications for your answers. You may use no external resources, although you may refer to things that have come up in class discussions if you wish. You have 50 minutes; try to spread your time evenly over the exam rather than spending a lot of time on one problem.

1. Indicate if the sets $\mathbb{N} = \{1, 2, 3, ...\}$ and $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$ are closed under each of the four basic arithmetic operations ($+$, $-$, $\times$, $\div$). Where they are, you need only write yes; where they are not, give a counterexample.

2. (a) Divide 10010_{two} by 11_{two} (in base two). Show your work.
(b) In base ten, an even number (i.e., a multiple of two) always ends in an even digit. Is this true in all bases? If not, in which bases is it true?

3. (a) Use set notation (A, B, C, $\cap$, $\cup$, and $\bar{\ }$ [complement]) to describe the shaded region at right.
(b) In a given class of 25 students, there are a total of 11 boys, and a total of 7 left-handed children. If 9 of the boys are right-handed, how many of the girls are right-handed?



[shade regions 1, 4, 6, 7]

4. In the problem "Weighing the Hippos", we discovered that there were two (physically) possible ways to assign hippo pairs to hippo weights, and therefore two solutions. Did the number of solutions depend at all on the specific pair weights given, or would we have had two solutions no matter what the numbers were? Explain your answer.

5. Say as much as you can about the differences between the following two sentences (on the set of integers): (i) $\exists x\ \forall y,\ x \cdot y = 1$; (ii) $\forall y\ \exists x,\ x \cdot y = 1$.

Take-Home Exam

Instructions: You are to take this problem home and work on it over the next two days. It is due in class two days from now. You may use as resources only your own copy of the course materials and any class notes and papers you have made yourself (no one else's). You should give some explanation of your answer, but you do not need to go into as much detail in justifying the solution as you might in a problem report. You will be asked to turn in the problem sheet (signed below) with your answer.

How many cubes total of all sizes can be made from a three-dimensional n by n by n grid? (Remember a cube has all sides the same length.)

I hereby affirm that in solving the attached problems I have not used any external resources other than my own copy of the course materials and my own class notes and papers. I have not spoken or written about these problems to anyone, whether a student in this class or not, even to the extent of indicating which problem I chose, whether I solved it, or how difficult any or all of it was.

(signature)______________________________

Math 130 Sample Exam 9b
Author: Christopher Kribs
Topics: Sets (Unit Two) [3], Operations (Unit Three) [1], Relations (Unit Three) [4], Arithmetic, Place Value & Bases (Unit Four) [2], Quantifiers (Unit Five) [5]
Date: 27 March 1996
Time Allotted: 1 hour (in-class) *plus* 2 days (take-home)

Instructions: Answer all of the following questions, and provide justifications for your answers. You may use no external resources, although you may refer in your explanations to things that have come up in class discussions if you wish. You have 50 minutes; try to spread your time evenly over the exam rather than spending a lot of time on one problem.

1. Define the operation "$*$" on the natural numbers by $a * b = a^b$. Indicate whether or not $*$ has the following properties. Where it does, you need only write yes; where it does not, give a counterexample.

closure of $\mathbb{N}$ under $*$	
commutativity	
associativity	
identity element (if so, name it)	

2. (a) Divide seventeen by three in base six; express any remainder in heximal. Show your work.
(b) If a (whole) number x's representation in base N ends in 0 (zero), what can we say about x, relative to N?

3. (a) Is it always true that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$? Explain.
(b) In a special ed classroom where there are to be 20 children, 9 of the children will need new desks high enough to accommodate wheelchairs, and seven of the students in the classroom will need desks fitted with electrical power strips for special equipment. If eight of the children can use already existing (lower) desks without power strips, how many of the new, higher desks need to be fitted with power strips?

4. Suppose that the relation $\bowtie$ is symmetric and transitive, and that $7 \bowtie Bob$. Prove that $7 \bowtie 7$.

5. Is the following sentence true or false on the set $\mathbb{Z}$ of integers? If false, can it be corrected with a minor adjustment?

$\exists m \, \forall n, \; m = n \times n.$

Math 130 Sample Exam 9b – Take-Home Exam
Author: Christopher Kribs
Topics: Problem Solving (Unit One) [2A, 2B], Operations (Unit Three) [1]
Date: 27 March 1996
Time Allotted: 2 days (take-home portion)

I hereby affirm that in solving the attached problems I have not used any external resources other than my own copy of the course materials and my own class notes and papers. I have not spoken or written about these problems to anyone, whether a student in this class or not, even to the extent of indicating which problem I chose, whether I solved it, or how difficult any or all of it was. (signature)______________________________

Instructions: You are to take these two problems home and choose *one* on which to work over the next two days. It is due in class on Friday, two days from now. You may use as resources only your own copy of the course materials and any class notes and papers you have made yourself (no one else's). You should give some explanation of your answer, but you do not need to go into as much detail in justifying the solution as you might in a problem report (primarily because both problems are extensions of problem reports in which you should already have given full justifications, to which you can refer as necessary).

Turn in this problem sheet (signed above) along with your answer to *one* of the two problems, A or B. The sheets should be stapled or clipped together.

A. The pirate Black Bart has forced you to play the game of Poison with him, but, knowing the winning strategy, he took the choice of whether to go first or second, and you ended up poisoned. You have one chance left: there is an antidote to the poison, disguised as one of the counters. Black Bart says it will be the last counter taken, and allows you the choice of going first or second. What is your winning strategy?

In other words, we have a new game called "Antidote", which is like Poison, except that this time, whoever takes the last counter *wins*. Devise a winning strategy for this game (hint: it is similar in nature to the one we developed for Poison); be sure to say when you want to go first or second (it will, of course, depend on the initial number of counters), and how many counters you take away each turn.

B. In view of our discussion on clock arithmetic, let's try to extend $\oplus$ and $\ominus$ to $\otimes$ and $\oplus\!\!\!\!\div$. By extend, we mean that the new operations should agree with normal arithmetic when the normal results fall within our set (so $2 \otimes 4 = 8 = 2 \times 4$), and follow the lead of $\oplus$ and $\ominus$ when the results would fall outside the set $\{1, 2, 3, ..., 12\}$.

1. Try to come up with a definition of $\otimes$ that obeys the distributive law "for all a, b, c: $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$ and $(b \oplus c) \otimes a = (b \otimes a) \oplus (c \otimes a)$". Give either a table or formula for finding $a \otimes b$ given a and b (if you want to define it without using either of these, check with me). Does your definition of $\otimes$ have the same properties as normal multiplication on the integers?

2. We can define normal subtraction in terms of addition: $a - b = a + (-b)$, and likewise define $\ominus$ in terms of $\oplus$: $a \ominus b = a \oplus (\dashv b)$, where $\dashv b$ is the inverse of b under clock addition (remember $\oplus$ has inverses, so we can say, for example, $5 \ominus 8 = 5 \oplus (\dashv 8) = 5 \oplus 4 = 9$). Can we define $\oplus\!\!\!\!\div$ similarly in terms of $\otimes$? (If not, do you think it's possible to define $\oplus\!\!\!\!\div$ at all [as a division]?)

Math 130 Sample Exam 10a
Author: Christopher Kribs
Topics: Number Theory (Unit Seven) [2, 6], Number Systems (Unit Seven) [1],
Fractions (Unit Six) [3, 4], Combinatorics (deleted) [5]
Date: 10 May 1996
Time Allotted: 2 hours

Instructions: Answer all of the following questions, and provide justifications for your answers. You may use your class material and class notes, and a calculator if you wish, but no other external resources, and you must do your own work. You have 2 hours; try to spread your time evenly over the exam rather than spending a lot of time on one problem.

1. If I tell you I have found the smallest positive (real) number there is, how can you prove me wrong?

2. Find a number greater than 20 that has exactly five factors. Explain your solution and how you got it.

3. (a) Write $9/14$ as a repeating decimal.
(b) Write $0.0\bar{3}$ as a common fraction.

4. An economist claims that $1/10$ of the population holds $9/10$ of the money. If we believe this claim and make the simple-minded assumption that all of this rich $1/10$ have the same amount of money as each other, while the poor $9/10$ have the same amount of money as each other, how much richer is a rich person than a poor person? (Think in terms of money per person.)

5. At the beginning of the semester, out of a class of 20 students, you and three others were assigned to a small group. Suppose that the first time groups were changed, they were reassigned *completely* randomly (so, for example, there's an extremely remote chance you could end up with the same partners).
(a) How many different sets of partners might you have in your new group?
(b) How many of these sets include none of your original 3 partners?
(c) Given your answers to (a) and (b), is it more likely that you will or will not end up with three new partners (and no old ones)? Explain.

6. At McDonald's, you can order Chicken McNuggets in packages of 6, 9 or 20. What is the largest number of Chicken McNuggets that you *can't* order?

Math 130 Sample Exam 10b
Author: Christopher Kribs
Topics: Problem Solving (Unit One) [1], Number Theory (Unit Seven) [2, 6],
Fractions (Unit Six) [3, 4], Combinatorics (deleted) [5]
Date: 12 May 1996
Time Allotted: 2 hours

Instructions: Answer all of the following questions, and provide justifications for your answers. You may use your class material and class notes, and a calculator if you wish, but no other external resources, and you must do your own work. You have 2 hours; try to spread your time evenly over the exam rather than spending a lot of time on one problem.

1. An important part of problem-solving is knowing how to solve a problem in more than one way. Describe a problem *other than the chickens and rabbits problem* that you solved in more than one way this semester, and tell how you solved it.

2. What kind(s) of numbers have exactly three factors? Be as specific as you can.

3. (a) Write $1/35$ as a repeating decimal.
(b) Write $7.\overline{07}$ as an improper fraction (i.e., as a fraction; the numerator will obviously be larger than the denominator).

4. Suppose that in Lecture 1 of Math 130, precisely $2/5$ of the students plan to take Math 131 next semester. Of the students who *don't* plan to do so, the number who plan never to take it is exactly twice the number of those who are taking 131 now.
(a) Is there anything you can say with certainty about the total number of people in the class? (If so, what?)
(b) Is there anything you can say with certainty about the proportion (fraction) of students who will have taken Math 131 by the end of *next* semester? (If so, what?)

5. In a given Math 130 class, there are 2 students named Dawn, 2 students named Jenny, and 3 students named Amy. (There are 18 students total.)
(a) How many different 3-person groups can be formed (*not* simultaneously) which include one person with each of the three names?
(b) How many different 3-person groups can be formed at all (regardless of names)?
(c) Given your answers to (a) and (b), what is the probability that a 3-person group chosen at random will contain a Dawn, an Amy, and a Jenny?

6. If you have a supply of 4¢ stamps and 7¢ stamps, which of the following amounts can you make? Explain.
(a) 13 ¢ (b) 18 ¢ (c) \$1.01

Math 130 Sample Exam 10c
Author: Christopher Kribs
Topics: Problem Solving (Unit One) [1], Number Theory (Unit Seven) [2, 6], Fractions (Unit Six) [3, 4], Combinatorics (deleted) [5]
Date: 15 May 1996
Time Allotted: 2 hours

Instructions: Answer all of the following questions, and provide justifications for your answers. You may use your class material and class notes, and a calculator if you wish, but no other external resources, and you must do your own work. You have 2 hours; try to spread your time evenly over the exam rather than spending a lot of time on one problem.

1. List three specific problem-solving strategies that you used to work on problems in this course. After each one, write a few sentences about how you used the strategy on a particular problem (mention a given problem no more than once).

2. Among the numbers 1 through 1000, do you think there are more numbers with exactly two factors or exactly three factors? Explain.

3. (a) Write $1/78$ as a repeating decimal.
   (b) Write $0.1\bar{3}$ as a common fraction.

4. Suppose that in Lecture 1 of Math 130, $2/5$ of the students plan to take Math 131 next semester. In Lecture 2 of Math 130, $1/2$ of the students plan to do so, but the *number* of students in each class who plan to do so is the same!
   (a) Can you determine from this information how many students total in these two classes will be taking Math 131? If so, how many? If not, why not?
   (b) Can you determine what proportion (fraction) of the total students will be taking Math 131? If so, what is it? If not, why not?

5. In a given Math 130 class, there are 2 students named Jessica, 2 students named Stacy, and 3 students named Amy. (There are 20 students total.)
   (a) How many different 4-person groups can be formed (*not* simultaneously) which include *exactly* one person with each of the three names?
   (b) How many different 4-person groups can be formed at all (regardless of names)?
   (c) Given your answers to (a) and (b), what is the probability that a 4-person group chosen at random will contain precisely one Jessica, one Amy, and one Stacy?

6. If you have a supply of 6¢ stamps and 10¢ stamps, which of the following amounts can you make? Explain.
   (a) 14¢   (b) \$2.02   (c) \$2.07

Math 130 Sample Exam 11
Author: Ben Collins
Topics: Sets (Unit Two) [1], Operations (Unit Three) [3], Logic (Unit Five) [4],
Arithmetic, Place Value & Bases (Unit Four) [2, 5]
Date: 01 April 1996
Time Allotted: 1 hour
Note: Open book and calculator

1. The universal set in this problem is the set of capital letters:

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

Set 1 = {letters that are drawn only with straight lines}
Set 2 = {consonants}
Set 3 = {letters that appear in MISSISSIPPI}

(a) List the members of $\overline{\text{Set 1}} \cap \text{Set 2} \cap \text{Set 3}$.
(b) Describe the set {E, F, K, L, N, P, S, Z} using intersection, union, and/or complement. [Note, it's not necessarily a combination of Sets 1, 2, and 3. You may have to invent Sets 4, 5, 6.]

2. The famous Floobian yuzzball player, Rabe Booth, scored 10 goals in the year 1927. That record was broken by Dodger Harris in the year 1961, when she scored 111 goals. [Recall that the Floobians count in base twelve, using our familiar digits 0 through 9, and # and for ten and eleven, respectively.]
(a) How many more goals did Harris score than Booth?
(b) For how many years did the record stand?

3. Let $\mathbb{Z}$ denote the set of integers: $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$. Define an operation $\star$ on $\mathbb{Z}$ by $x \star y = 2(x + y)$.
(a) Is $\mathbb{Z}$ closed under $\star$? Explain.
(b) Is $\star$ associative? Explain.
(c) Is there an identity? Explain.

4. If a number is prime and greater than 10, then it's a Glork. Classify each of the following as True, False, or Can't Tell, and explain.
(a) 13 is a Glork.
(b) 25 is a Glork.
(c) There are less than five Glorks.

5. Suppose that a base six number ends in 0. What numbers is it divisible by? Explain.

Math 130 Sample Exam 12
Author: Ben Collins
Topics: Quantifiers (Unit Five) [2], Number Theory (Unit Seven) [3, 6], Number Systems (Unit Seven) [2], Fractions (Unit Six) [1, 5], Algebra (Unit Eight) [4]
Date: 13 May 1996
Time Allotted: 2 hours
Note: Open book and calculator

1. (a) Find the common fraction that equals 3.2424242424....
(b) Explain why the rational number $\frac{23}{191}$ must be a repeating decimal. (Warning: do not try to divide this out.)

2. Consider the statement: $\forall x, \exists y$ such that $x^2 + y = 0$.
(a) Is the statement true for the set of natural numbers? Explain.
(b) Is the statement true for the set of real numbers? Explain.

3. Describe the numbers which have exactly three factors.

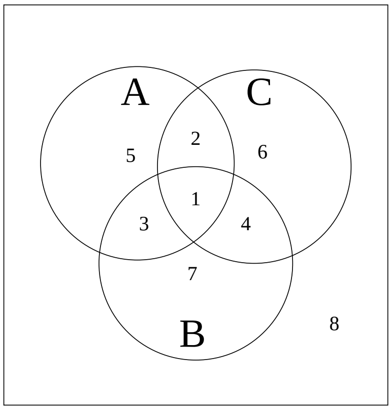
4. Estimate the number of times your heart will beat in your lifetime.

5. In a certain math class, half the left-handers are getting A's, and one-third of the right-handers are getting A's. Twice as many righties are getting A's as lefties.
(a) What proportion of all the students are getting A's?
(b) Can you tell me how many students there are? Explain.

6. You have an unlimited supply of 20¢ stamps and of 32¢ stamps, which of the following totals can you make? Be sure to explain your reasoning.
(a) 74¢ (b) 92¢ (c) \$4.13

Math 130 Sample Exam 13a
Author: Mirna Dzamonja
Topics: Sets (Unit Two) [1, 3, 4], Operations (Unit Three) [6], Relations (Unit Three) [5], Logic (Unit Five) [2]
Date: 01 March 1996
Time Allotted: 1 hour
Instructions: Justify all your answers.

1. Describe the area shaded at right in terms of A, B and C.



[shade regions 2, 7]

2. Is the following statement a tautology?

$$(p \vee q) \vee (\neg p \wedge \neg q)$$

3. At a certain party, there were 28 people over the age of 21. There were 22 people who major in education. The total number of people at the party was 43, and everybody at the party either majors in education or is over 21, or both. How many people at the party both major in education and are over 21 years of age?

4. Is the following true for all sets A, B and C?

$$\overline{A \cap B} \cap C = (\overline{A} \cap B) \cup (\overline{B} \cap C)$$

5. Let us define the relation $\lhd$ among natural numbers by the following: $a \lhd b$ means that a is a divisor of b. Is $\lhd$ reflexive? Is $\lhd$ symmetric? Is $\lhd$ transitive?

6. Let us define the operation σ among real numbers by $a \;\sigma\; b \equiv 2a + 2b$.
   (a) How much is $0.5 \;\sigma\; \frac{3}{2}$?
   (b) Is σ commutative?
   (c) Is there an identity for σ? If so, what is it?
   (d) What is the inverse of 5 under σ?

Math 130 Sample Exam 13b
Author: Mirna Dzamonja
Topics: Sets (Unit Two) [3, 5, 6], Operations (Unit Three) [4], Relations (Unit Three) [2], Logic (Unit Five) [1]
Date: 01 March 1996
Time Allotted: 1 hour
Instructions: Justify all your answers.

1. (10 points) Is the following a tautology?

$$(\neg p \vee \neg q) \vee (p \wedge q)$$

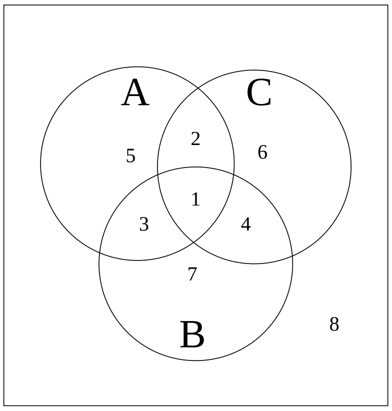
2. (20 points) Let us define the relation $\propto$ among the whole numbers by the following: $a \propto b$ means that a is a multiple of b. Is $\propto$ reflexive? Is $\propto$ symmetric? Is $\propto$ transitive?

3. (20 points) Is the following true for all sets A, B and C?

$$\overline{A \cup B} \cap \overline{C} = (\overline{A} \cap \overline{C}) \cap (\overline{B} \cap \overline{C})$$

4. (20 points) We define operation τ among the rational numbers by $a \; \tau \; b \equiv 2a^2 - 3b$.
(a) Is the set of rational numbers closed under τ?
(b) How much is $0.5 \; \tau \; \frac{1}{3}$?
(c) Is τ commutative?
(d) Is τ associative?
Reminder: rational numbers = fractions (including the integers)

5. (10 points) Describe the shaded area at right in terms of A, B and C, and symbols $\cap$, $\cup$ and $^{-}$.



[shade regions 1 and 8]

6. (20 points) In a section of Math 130 there are 25 women. In the same section there are 17 people who have children. All the men in the section have children. All together, there are 36 people in the section.
How many women from this section do not have children?

Math 130 Sample Exam 14
Author: Mirna Dzamonja
Topics: Arithmetic, Place Value & Bases (Unit Four) [1, 2], Number Theory (Unit Seven) [6, 7], Algebra (Unit Eight) [3, 4, 5]
Date: 08 April 1996
Time Allotted: 1 hour
Instructions: Justify all your answers.

1. (13 points) (a) Convert 135 from base six into base two.
   (b) Convert 121 from base three into base ten.

2. (12 points) (a) Evaluate $131 - 33$ in base four.
   (b) Evaluate $162 + 845$ in base nine.

3. (15 points) A store holding a clearance sale advertises that all prices have been discounted 20%. If a certain article is on sale for $28, what was its price before the sale?

4. (15 points) A student has test scores of 46 and 78. What score in a third test will give the student an average of 80?

5. (25 points) A man has $15,000 to invest. He plans to invest a part of it in a savings account paying 5% simple interest and the remainder in an investment fund yielding 8% simple interest. How much should he invest in each to obtain a 7% return on his money after one year?

6. (15 points) Jane and Raymond went to the same high school in Littletown. Now they both live away from Littletown, and only go back there for class reunions. Class reunions are held every year on May 15th. Raymond goes to every 8th reunion, and Jane to every 12th. They last met in Littletown for a reunion in 1995. When will they meet in Littletown again?

7. (5 points) Find the greatest common divisor of 256 and 240.

Math 130 Sample Exam 15
Author: Mirna Dzamonja
Topics: Sets (Unit Two) [12], Logic (Unit Five) [4],
Arithmetic, Place Value & Bases (Unit Four) [6, 10], Number Theory (Unit Seven) [5],
Fractions (Unit Six) [2], Algebra (Unit Eight) [1, 3, 8], Probability (deleted) [7, 9, 11]
Date: 14 May 1996
Time Allotted: 2 hours
Instructions: Justify all your answers.

1. (9 points) Simplify the following expressions:

(a) $\dfrac{4b^6 - 9}{2b^3 + 3}$ (b) $\dfrac{3a^5b^4 + 15a^3b^2}{30b^3}$ (c) $\dfrac{xy^2z - yz}{x^4z + y^2z}$

2. (8 points) Write $4.\overline{15}$ as a fraction.

3. (10 points) In a certain factory, machine A produces three times as many bolts as machine B. Machine C produces 13 more bolts than machine A. If the total production is 4997 bolts per day, how many bolts does each machine produce in a day?

4. (7 points) Suppose that p is a true sentene and q is a false one. Is the sentence $(p \wedge q) \vee (q \vee \neg p)$ true or false?

5. (7 points) What is the LCM of 128 and 132?

6. (8 points) Write 356 in base seven.

7. (8 points) If you throw a fair coin 5 times, what is the probability you get 5 heads?

8. (10 points) Solve the following system of equations:

$$\begin{array}{llll} x & -10y & -z & = 2 \\ & 9y & -\frac{z}{10} & = 8 \\ 2x & & -23z & = 1 \end{array}$$

9. (8 points) If we choose 3 students from the set {Amy, Bill, Charlie, David, Ed}, what is the probability Amy and Bill are among the three?

10. (8 points) If a number is written as 458 in base nine, how is it written in base ten?

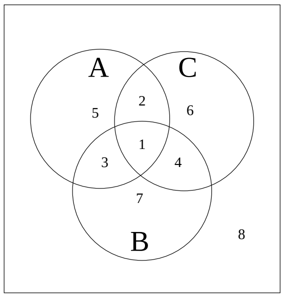
11. (8 points) You and your friend are playing a game. At each inning you pick the name of a business in your city whose telephone number you do not recall at the moment. Your friend takes the telephone book and checks the number. If it ends with a digit divisible by 3, you score a point. If it ends with a digit divisible by 2, your friend scores. If the last digit of the number is divisible neither by 3 nor by 2, you call it a tie.

Assuming you play honestly, who has a better chance of winning? Justify your answer. (0 is divisible by all numbers.)

12. (10 points) In an art class, 20 students used red pigment; 11 used red but no blue pigment; 27 used blue or yellow pigment; 6 used red and yellow, but no blue pigment; 3 used all three pigments; and 2 used both blue and yellow, but not red pigment. Each student used at least one of the three pigments. How many students were there in the class?

Math 130 Sample Exam 16a
Author: Peter Orlik
Topics: Problem Solving (Unit One) [7], Sets (Unit Two) [1, 2, 3, 4], Logic (Unit Five) [5, 6], Algebra (Unit Eight) [8, 9, 10]
Date: 11 October 1996
Time Allotted: 1 hour

1. Let $U = \{1, 3, 5, 7, 9, 11\}$, $A = \{1, 3, 7, 11\}$, $B = \{1, 5, 9, 11\}$, $C = \{3, 5, 7\}$. Find (a) $(A \cup B) \cap C$, (b) $(A \cap B) \cap \overline{C}$, (c) $A \cap \overline{B}$, (d) $\overline{A} \cup \overline{C}$, (e) $A \cup \overline{B} \cup C$. [$U$ is the universe of elements.]



[shade regions 4 and 5]

2. Find the truth table for the statement: $p \wedge [q \to (\neg p \wedge \neg q)]$.

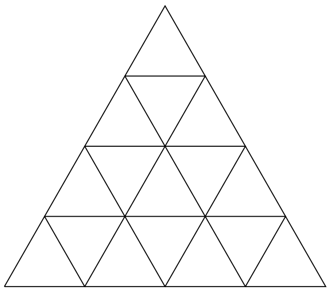
3. Make a Venn diagram and shade the region $\overline{A} \cap (B \cup C)$.

4. Describe the shaded region in the Venn diagram at right.

5. Let $N = \{0, 1, 2, 3, ...\}$ be the set of natural numbers. Write a sentence equivalent to the statement: $\forall x \in N, \forall y \in N, \exists z \in N, z = 3x - 2y$.
Is this statement true?

6. If a number is *pretty*, then it is divisible by 6 and is less than 100. Determine for each sentence below if it is true, false, or undecidable.

T	F	U	13 is pretty.
T	F	U	42 is pretty.
T	F	U	There is no pretty number ≥ 100.
T	F	U	60 is not pretty.
T	F	U	Every pretty number is even.



7. Assume that paths in the figure at right must always go down the page along edges (strictly down, not sideways).

(a) How many paths lead from A to B?
(b) How many paths lead from A to any point in the bottom row?

8. Fred has a collection of dimes and quarters worth \$2.50. Which of the following cannot be the total number of coins in the collection?

10 13 16 18 19 22 25

9. Jefferson School has 1400 students. The teacher-pupil ratio is 1:35. How many additional teachers will have to be hired to reduce the ratio to 1:20?

10. Pat can mow a lawn in 4 hours. Meg can mow the lawn in 6 hours. How long does it take for the two of them if they work together?

Alternate: In 1, make (a) $A \cup \overline{B}$, (b) $(A \cap B) \cup C$, (c) $(\overline{A} \cap \overline{B}) \cup \overline{C}$, (d) $\overline{(A \cap B)} \cup \overline{C}$. Or let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 3, 6, 8\}$, $B = \{1, 2, 4, 7\}$, $C = \{2, 3, 4, 6\}$. Find (a) $(A \cup \overline{B}) \cap C$, (b) $(A \cup B) \cap \overline{C}$, (c), (d), (e) as before. In 2, construct the truth table for $(p \to \neg q) \to (p \vee \neg q)$ instead. In 4, shade regions 1, 5, 6 instead. In 5, change "$z = 3x - 2y$" to "$x + y = 3z + 1$".

Math 130 Sample Exam 16b
Author: Peter Orlik
Topics: Problem Solving (Unit One) [6, 8], Sets (Unit Two) [1, 3, 4, 10],
Logic (Unit Five) [2, 5, 9], Algebra (Unit Eight) [7]
Date: 11 October 1996
Time Allotted: 1 hour

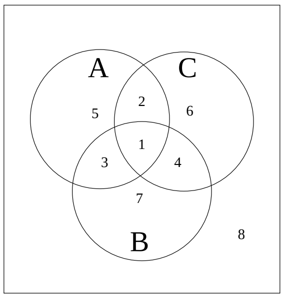
1. Let $U = \{a, b, c, d, e, f\}$, $A = \{a, b, c, d\}$, $B = \{a, c, d, f\}$, $C = \{a, c, e, f\}$. Find (a) $(A \cup B) \cap C$, (b) $(A \cap B) \cap C$, (c) $A \cap B$, (d) $A \cup C$, (e) $A \cup \overline{B} \cup C$. [$U$ is the universe of elements.]

2. Is the following implication a tautology? $\neg(p \wedge q) \rightarrow (\neg p \wedge \neg q)$

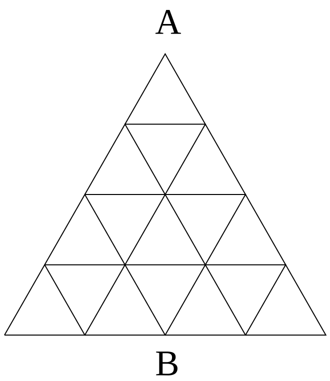
3. Use Venn diagrams to prove or disprove the statement
$A \cup (B \cap \overline{C}) = (A \cup B) \cap \overline{C}$

4. Describe the shaded region in the Venn diagram at right.

5. Let $N = \{0, 1, 2, 3, ...\}$ be the set of natural numbers. Write a sentence equivalent to the statement: $\forall x \in N, \exists y \in N, y = 3x - 2$. Is this statement true?



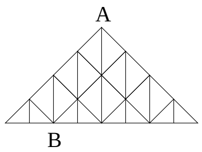
[shade regions 1, 2, 4, 7]



6. How many triangles can you count in the picture at left?

7. Looking out in the backyard one day, I saw an assortment of girls and cats. Counting heads, I got 22. Counting feet, I got 68. How many girls and how many cats were in the yard?

8. Assume that the paths in the figure at right must always go down the page.
(a) How many paths lead from A to B?
(b) How many paths lead from A to any point in the bottom row?



9. Suppose p is true and q is false. Find the following truth values:

T F $\neg p \vee \neg q$
T F $p \wedge q$
T F $p \rightarrow q$
T F $\neg p \rightarrow q$
T F $\neg p \rightarrow \neg q$

10. The soccer team, chess team, and math team are to travel on a bus together. The chess team has 12 members, the math team has 16 members, and the soccer team has 17 members. Three people are on only the math and soccer teams, two people are on only the math and chess teams, and one person is on only the chess and soccer teams. There is on person on all three teams. How many seats will the bus need for members of all three teams?

Alternate: Change the second $\wedge$ in 2 to $\vee$. On the LHS of 3, extend the bar to $\overline{B \cap C}$. In 9, let p also be false, and consider $(p \rightarrow q) \rightarrow \neg q$, $(\neg p \wedge \neg q) \vee (p \rightarrow q)$, $[(\neg p \rightarrow \neg q) \vee q] \rightarrow (p \rightarrow \neg q)$. Or, instead, let r also be true, and consider $(p \vee q) \wedge r$, $(p \wedge q) \vee r$, $\neg p \rightarrow (q \vee r)$, $(\neg p \wedge r) \rightarrow q$, $(\neg p \rightarrow \neg q) \rightarrow r$. In 8, lop off the three leftmost triangles (two small and one large) and all their edges, move B one vertex to the right, and pull it down to make the two small triangles which touch it full size.

Math 130 Sample Exam 17a
Author: Peter Orlik
Topics: Operations (Unit Three) [1], Logic (Unit Five) [6], Arithmetic, Place Value & Bases (Unit Four) [3, 4, 5], Number Theory (Unit Seven) [2, 7], Algebra (Unit Eight) [8, 9, 10]
Date: 15 November 1996
Time Allotted: 1 hour

1. Define the binary operation $\circ$ for elements of $\mathbb{N}$ by $a \circ b = a + b + ab$.
   (i) Is $\mathbb{N}$ closed under $\circ$?
   (ii) Is $\circ$ commutative?
   (iii) Is $\circ$ associative?
   (iv) Does $\circ$ have an identity element?
   (v) Does every $a \in \mathbb{N}$ have an inverse with respect to $\circ$?

2. Which natural numbers can be written as the sum of a multiple of 5 and a multiple of 6?

3. (a) Construct the addition table in base five.
   (b) Construct the multiplication table in base five.

4. Calculate in base five:

(a)
$$\begin{array}{rrrr} & 2 & 4 & 3 \\ & 3 & 3 & 2 \\ + & 3 & 4 & 3 \\ \hline \end{array}$$

(b)
$$\begin{array}{rrrr} & 3 & 2 & 4 \\ \times & 3 & 3 & 2 \\ \hline \end{array}$$

5. Convert the following: (a) $233_{five} = \quad_{ten}$; (b) $566_{ten} = \quad_{five}$

6. If a number is *pretty*, then it is divisible by 6 and is less than 100. Determine for each sentence below if it is true, false, or undecidable.

T F U 23 is pretty.
T F U 24 is pretty.
T F U There is no pretty number ≤ 25.
T F U 40 could be pretty.
T F U There are even pretty numbers.

7. Find all numbers between 1 and 150 with exactly 12 factors (including 1 and the number itself).

8. Fred has a collection of dimes and quarters worth \$2.50. Which of the following cannot be the total number of coins in the collection?

10 13 16 18 19 22 25

9. Jefferson School has 1400 students. The teacher-pupil ratio is 1:35. How many additional teachers will have to be hired to reduce the ratio to 1:20?

10. Pat can mow a lawn in 4 hours. Meg can mow the lawn in 6 hours. How long does it take for the two of them if they work together?

Alternate: In 6, replace "by 6" with "by 5 or 8", "23 is" with "23 is not", "≤ 25" with "≥ 125", and "even" with "odd".

Math 130 Sample Exam 17b
Author: Peter Orlik
Topics: Operations (Unit Three) [1], Logic (Unit Five) [5], Arithmetic, Place Value & Bases (Unit Four) [2, 3, 4], Number Theory (Unit Seven) [6, 7, 8], Algebra (Unit Eight) [9, 10]
Date: 15 November 1996
Time Allotted: 1 hour

1. Define the binary operation $\circ$ for elements of $\mathbb{N}$ by $a \circ b = a + b - ab$.
   (i) Is $\mathbb{N}$ closed under $\circ$?
   (ii) Is $\circ$ commutative?
   (iii) Is $\circ$ associative?
   (iv) Does $\circ$ have an identity element?
   (v) Does every $a \in \mathbb{N}$ have an inverse with respect to $\circ$?

2. (a) Construct the addition table in base six.
   (b) Construct the multiplication table in base six.

3. Calculate in base six:

(a) $\begin{array}{r} 2\ 4\ 5 \\ 3\ 5\ 2 \\ +\ 5\ 4\ 3 \\ \hline \end{array}$ (b) $\begin{array}{r} 5\ 2\ 4 \\ \times\ 5\ 3\ 2 \\ \hline \end{array}$

4. Convert the following: (a) $143_{six} = \quad_{ten}$; (b) $143_{ten} = \quad_{six}$

5. If a number is divisible by 5 or 8, then it is *cute*. Determine for each sentence below if it is true, false, or undecidable.

T	F	U	23 is cute.
T	F	U	24 is cute.
T	F	U	There is no cute number ≤ 25.
T	F	U	4 could be cute.
T	F	U	There are odd cute numbers.

6. Determine the postages that cannot be made with a supply of 4 and 7 cent stamps.

7. A frog and a turtle live in a pond. The frog surfaces every 35 seconds and the turtle every 60 seconds. If they surface together at noon, at what time do they surface together next?

8. Find all numbers between 1 and 100 with exactly eight factors (including 1 and the number itself).

9. A fraction has value 1/2. If we add one to the numerator and subtract 1 from the denominator, the resulting fraction has value 2/3. Find the original fraction.

10. How many children are there in a family where each girl has as many brothers as sisters, but each boy has twice as many sisters as brothers?

Alternate: Substitute base seven for base six in 2, 3, 4. In 6, replace "4 and 7" with "5 and 8". In 8, change "eight" to "nine".

Math 130 Sample Exam 18a
Author: Peter Orlik
Topics: Number Theory (Unit Seven) [7, 8, 9], Number Systems (Unit Seven) [1], Fractions (Unit Six) [2], Algebra (Unit Eight) [5, 10], Combinatorics (deleted from curriculum) [3, 4, 6]
Date: 18 December 1996
Time Allotted: 1 hour

1. Circle true or false:

T F The product of two irrational numbers can be rational.
T F If x is a positive real number, then $x^2 > x$.
T F The rational numbers are closed under addition.
T F If the sum of two numbers is irrational then their product is also irrational.
T F If x and y are positive rational numbers, then x^y is a positive rational number.

2. Find a fraction equal to 1.14232323...

3. How many different arrangements of the letters ALABAMA are possible?

4. Jackson School wants to form a committee consisting of four teachers, three parents, and two students. Eleven teachers, fourteen parents, and sixteen students agreed to serve. How many different committees could be formed? [not simultaneously]

5. Four gallons of 50% alcohol solution was supposed to be diluted with one gallon of water. By mistake, one gallon of alcohol was added instead. What solution strength were we aiming for? What solution strength did we actually get?

6. There are eight swimmers in a race. How many different outcomes are possible for gold, silver, and bronze medals?

7. Find the number of zeroes at the end of 10! (ten factorial)

8. How many factors does 4400 have?

9. Find GCF(72, 144, 216) and LCM(72, 144, 216).

10. The surface of Big Lake is 31 feet above the surface of Long Lake. Long Lake is half as deep as Big Lake, and the bottom of Long Lake is 8 feet below the bottom of Big Lake. How deep is each lake?

Math 130 Sample Exam 18b
Author: Peter Orlik
Topics: Sets (Unit Two) [2], Number Theory (Unit Seven) [3], Number Systems (Unit Seven) [9], Fractions (Unit Six) [1], Algebra (Unit Eight) [4, 5, 6, 8, 10], Combinatorics (deleted) [7]
Date: 18 December 1996
Time Allotted: 1 hour

1. (a) Find a fraction equal to 0.7423423423...
(b) Write down a decimal expression which does not represent a rational number and explain why it is not rational.

2. A high school has 477 seniors. Of these, 211 are enrolled in some math course, 257 are enrolled in some science course, and 39 are taking both math and science. How many seniors are taking neither math nor science?

3. (a) Find the prime trees (prime factorizations) for 21, 36, and 45.
(b) Use this information to find their GCD and LCM.

4. How much 60% solution should be mixed with how much 20% solution to get 6 gallons of 45% solution?

5. A collection of nickels and dimes is worth $2.75. There are 39 coins. How many of them are nickels?

6. Three years ago Jenny was three times as old as Mary. In two years she will be twice as old. How old are they now?

7. How many different arrangements of the letters BOOKKEEPER are possible?

8. A plane can fly at 150 mph if there is no wind. On a given day the wind is blowing at 30 mph from A to B. If the distance between A and B is 300 miles, how much longer does it take to fly from B to A than from A to B?

9. Circle true or false:

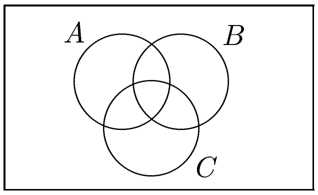
T F The product of two irrational numbers can be irrational.
T F If $x < y$, then $x^2 < y^2$.
T F The rational numbers are closed under addition.
T F The irrational numbers are closed under addition.
T F If x and y are rational numbers, then x^y is a rational number.

10. Fencing is sold in 8 foot sections at $15 per section. Fence posts cost $4 each. How much will it cost in materials to build a fence around a square field which is 240 feet on each side?

Alternate: In 6, change "Three years" to "Six years" and "In two years" to "In four years".

Math 130 Sample Exam 19
Author: Heather Lewis
Topics: Sets (Unit Two) [2], Operations (Unit Three) [4], Relations (Unit Three) [3], Quantifiers (Unit Five) [5], Arithmetic, Place Value & Bases (Unit Four) [6], Number Theory (Unit Seven) [1]
Date: 08 November 1996
Time Allotted: 1 hour (in-class) *plus* 2 days (take-home)

1. (5 points) List all of the (positive) factors of 10.



2. (10 points) In the Venn diagram at right, shade in the set $(A \cup C) \cap \overline{(A \cap B)}$. Everything with a mark will be assumed to be shaded, so be sure to **only** shade in that set.

3. On the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2 \ldots\}$, the relation $a \sim b$ means that $a + b \leq 0$.

(a) (5 points) Is $\sim$ reflexive on $\mathbb{Z}$? If it is, you may just write *yes*; if it is not, write *no* and give a counterexample.

(b) (5 points) Is $\sim$ symmetric on $\mathbb{Z}$? If it is, you may just write *yes*; if it is not, write *no* and give a counterexample.

(c) (5 points) Is $\sim$ transitive on $\mathbb{Z}$? If it is, you may just write *yes*; if it is not, write *no* and give a counterexample.

4. On the set $S = \{1, 2, 3, 4\}$, the operation $\diamondsuit$ is defined by the chart at right.

$\diamondsuit$	1	2	3	4
1	3	4	1	2
2	4	2	2	1
3	1	2	3	4
4	2	1	4	2

(a) (5 points) Is the set S closed under $\diamondsuit$? If it is, you may just write *yes*; if it is not, write *no* and give a counterexample.

(b) (5 points) Is $\diamondsuit$ commutative? If it is, you may just write *yes*; if it is not, write *no* and give a counterexample.

(c) (5 points) Is $\diamondsuit$ associative? If it is, you may just write *yes*; if it is not, write *no* and give a counterexample.

(d) (5 points) Is there an identity with respect to $\diamondsuit$? If there is, what is it? If there is not, you may just write *no*.

(e) (5 points) Fill in each of the following, or write "no inverse" if the element does not have an inverse:

The inverse of 1 is ____________. The inverse of 2 is ____________.
The inverse of 3 is ____________. The inverse of 4 is ____________.

5. On the set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, say whether each of the following statements is true or false. Explain your answer. (5 points each)

(a) $\forall x\ \exists y$ such that $x^2 = y$ (b) $\forall y\ \exists x$ such that $x^2 = y$ (c) $\exists x\ \forall y$ such that $x^2 = y$

6. Do the following calculations in the bases indicated. Leave your answers in the given bases, and remember that in bases bigger than ten, the letter A represents ten, B represents eleven, C represents twelve, and so on. (5 points each)

(a)
$$\begin{array}{r} 3402_{five} \\ 4221_{five} \\ +\ 1431_{five} \\ \hline \end{array}$$

(b)
$$\begin{array}{r} 1122_{sixteen} \\ \times\ \ A_{sixteen} \\ \hline \end{array}$$

Math 130 Sample Exam 19 — Take-Home Exam
Author: Heather Lewis
Topics: Problem Solving (Unit One) [2A, 2B], Operations (Unit Three) [1]
Date: 08 November 1996
Time Allotted: 2 days (take-home portion)

I hereby affirm that in solving the attached problems I have not used any external resources other than my own copy of the course materials and my own class notes and papers. I have not spoken or written about these problems to anyone, whether a student in this class or not, even to the extent of indicating which problem I chose, whether I solved it, or how difficult any or all of it was.

(signature)______________________________

Instructions: You are to take these problems home. You must do Problem 1. In addition, you must do *either* Problem 2A *or* Problem 2B. They are due at the beginning of class on Monday, November 11, 1996. You may use as resources only your own copy of the course material and any class notes and papers you have made yourself (no one else's).

Turn in this problem sheet (signed above) along with your answer to Problem 1 and your answer to *either* Problem 2A *or* Problem 2B. The sheets should be stapled or clipped together. You do **not** have to type your work, but it should be easy to read.

If you have any questions, send me e-mail or call me at home. **Good luck!**

Problem 1: (Short answer, on a separate piece of paper). On the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots, \}$, the operation $\otimes$ is defined by $a \otimes b = a^b + b^a - 1$.

(a) Is $\mathbb{N}$ closed under the operation $\otimes$? If it is, just write *yes*; if it is not, give a counterexample.
(b) Is $\otimes$ commutative? If it is, just write *yes*; if it is not, give a counterexample.
(c) Is $\otimes$ associative? If it is, just write *yes*; if it is not, give a counterexample.
(d) Does $\otimes$ have an identity in $\mathbb{N}$? Explain why or why not.
(e) Does the set $\mathbb{N}$ have inverses under $\otimes$? Explain why or why not.

Do **one** of the following problems. Your write-up should include a description of the problem in your own words, your solution, and an explanation of why your solution works.

Problem 2A: The Pirate Black Bart has forced you to play the game of Poison with him, but, knowing the winning strategy, he took the choice of whether to go first or second, and you ended up poisoned. You have one chance left: there is an antidote to the poison, disguised as one of the counters. Black Bart says it will be the last counter taken, and allows you the choice of going first or second. What is your winning strategy?

In other words, we have a new game called "Antidote," which is like Poison, except that this time whoever takes the last counter *wins*. Devise a winning strategy for this game. Be sure to say when you want to go first or second (it will, of course, depend on the initial number of counters) and how many counters you take away each turn.

Problem 2B: Weird Table Arrangements! A large banquet hall contains 427 tables. All of the tables are exactly alike and have square tops. You are to move all of the tables together, with no tables against the walls, to make an arrangement that will seat the *fewest* number of people. The manager of the banquet hall provides you with a set of four rules for joining tables:

(a) Joined tables must share at least one complete side.
(b) Table tops must form a continuous region.
(c) Exactly one person must be seated at each exposed side of a table.
(d) If the table arrangement surrounds an open area, people must be seated in the interior, one at each exposed side of the table.

Be sure to include a diagram of your table arrangement. Also say how many people can be seated with that arrangement.

Math 130 Sample Exam 20a
Author: Heather Lewis
Topics: Sets (Unit Two) [8], Operations (Unit Three) [3], Relations (Unit Three) [2], Arithmetic, Place Value & Bases (Unit Four) [1], Fractions (Unit Six) [4, 6], Algebra (Unit Eight) [5, 7]
Date: 15 December 1996
Time Allotted: 2 hours

1. (a) (4 points) Write twenty-one in base two: $21_{ten} = \underline{\hspace{4cm}}_{two}$
(b) (4 points) Write 112_{six} in base ten: $112_{six} = \underline{\hspace{4cm}}_{ten}$
(c) (5 points) Do the following subtraction in base six (leave your answer in base six):

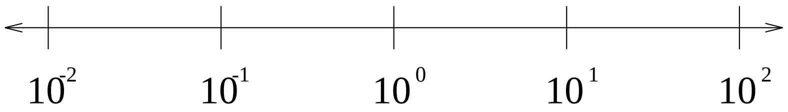
$$\begin{array}{r} 30312_{six} \\ -\ 14321_{six} \\ \hline \end{array}$$

2. Relations! For this problem, our set is the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and our relation is the happy, familiar relation $\geq$. (4 points each) (a) Is $\geq$ reflexive on $\mathbb{Z}$? (b) Is $\geq$ symmetric on $\mathbb{Z}$? (c) Is $\geq$ transitive on $\mathbb{Z}$? Explain each answer.

3. Operations! For this problem, our set is the set of **positive** rational numbers, and our operation is the happy, familiar operation $\div$ (division). (3 points each) (a) Is the set of positive rational numbers closed under $\div$? (b) Is $\div$ commutative? (c) Is $\div$ associative? (d) Is there an identity with respect to $\div$? (e) Does the set of positive rational numbers have inverses under $\div$? Explain each answer.

4. (a) (8 points) Can $1.2091209120912091209\ldots$ be written as a common fraction? If it can't, explain why not. If it can, write it as a fraction (it should have integers in the numerator and denominator, but it does **not** have to be reduced). Be sure to show all your work.
(b) (6 points) Can $\sqrt{2}$ be written as a repeating decimal? Explain your answer.

5. (a) (4 points) An amoeba is about 10^{-2} centimeters in diameter. How many amoebas, laid next to each other, would fit in a line one kilometer long? Show your work.
(b) (4 points) Write 2^{-2} as a decimal.
(c) (6 points) Show where the numbers 0.09, 1, 4, and 103 are on the multiplicative number line below.



6. While attending your favorite Badger Game, you notice that half of the men in the audience are wearing red and two-thirds of the women in the audience are wearing red. The number of women wearing red is twice the number of men wearing red.
(a) (12 points) What proportion of the audience is wearing red? Explain your answer.
(b) (4 points) Can you tell how many women and men are at the game? Explain your answer.
(c) (4 points) Can you tell whether there are more men or more women at the game? Explain.

7. (17 points) A certain instructor, who shall remain nameless, recently went on a trip to Montréal. While there, her husband phoned every evening. When they got their phone bill last week, they discovered that one-third of the minutes were billed at ten cents a minute, and two-thirds of the minutes were billed at a whopping fifty cents a minute. The total cost of the calls to Montréal (ignoring tax) was \$110.

How many minutes did they talk on the phone? (Write your answer in simplest form).

Moral: Always read the fine print about phone rates.

[see next page for Problem 8]

8. You have just been hired as a teacher, and are getting ready to prepare for your first day of class. Unfortunately, you don't know how many students will be in your class! You decide to figure it out from the following information, which the principal sent to you:

(i) fifteen of your students arrive by bus

(ii) thirteen of your students eat in the cafeteria

(iii) ten of your students go to after-school programs

(iv) eight of your students arrive by bus and also eat in the cafeteria

(v) six of your students arrive by bus and also go to after-school programs

(vi) five of your students eat in the cafeteria and also go to after-school programs

(vii) three students arrive by bus, eat in the cafeteria, and also go to after-school programs

(viii) two of your students don't do *any* of these (they don't ride the bus, don't eat in the cafeteria, and don't go to after-school programs.)

How many students do you have? Explain.

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8. You have just been hired as a teacher, and are getting ready to prepare for your first day of class. You have twenty-six students. The principal sent you the following information:

(i) sixteen of your students arrive by bus

(ii) fourteen of your students eat in the cafeteria

(iii) eleven of your students go to after-school programs

(iv) nine of your students arrive by bus and also eat in the cafeteria

(v) seven of your students arrive by bus and also go to after-school programs

(vi) six of your students eat in the cafeteria and also go to after-school programs

(vii) four students arrive by bus, eat in the cafeteria, and also go to after-school programs

How many students don't do *any* of those things (don't ride the bus, don't eat in the cafeteria, and also don't go to after-school programs)? Explain.

Math 130 Sample Exam 20b
Author: Heather Lewis
Topics: Sets (Unit Two) [8], Operations (Unit Three) [3], Relations (Unit Three) [2], Arithmetic, Place Value & Bases (Unit Four) [1], Fractions (Unit Six) [4, 6], Algebra (Unit Eight) [5, 7]
Date: 15 December 1996
Time Allotted: 2 hours *Note:* Make-up version of Sample Exam 20a

1. (a) (4 points) Write thirty in base three: $30_{ten} =$ ____________$_{three}$
(b) (4 points) Write 112_{five} in base ten: $112_{five} =$ ____________$_{ten}$
(c) (5 points) Do the following subtraction in base five (leave your answer in base five):

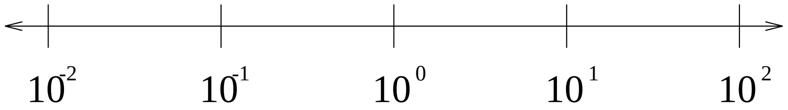
$$\begin{array}{rr} & 30312_{five} \\ - & 14321_{five} \\ \hline \end{array}$$

2. Relations! For this problem, our set is the set of rational numbers $\mathbb{Q}$ and our relation is the happy, familiar relation $\neq$ (remember, rational numbers can be written as a fraction with an integer in the numerator and a nonzero integer in the denominator). (4 points each)
(a) Is $\neq$ reflexive on $\mathbb{Q}$? (b) Is $\neq$ symmetric on $\mathbb{Q}$? (c) Is $\neq$ transitive on $\mathbb{Q}$? Explain each answer.

3. Operations! For this problem, our set is the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, and our operation is the happy, familiar operation $-$ (subtraction). (3 points each) (a) Is the set of integers closed under $-$? (b) Is $-$ commutative? (c) Is $-$ associative? (d) Is there an identity with respect to $-$? (e) Does the set of integers have inverses under $-$? Explain each answer.

4. (a) (8 points) Can $32.07320732073207\ldots$ be written as a common fraction? If it can't, explain why not. If it can, write it as a fraction (it should have integers in the numerator and denominator, but it does **not** have to be reduced). Be sure to show all your work.
(b) (6 points) Can $\sqrt{2}$ be written as a repeating decimal? Explain your answer.

5. (a) (4 points) A molecule is about 10^{-3} centimeters in diameter. How many molecules, laid next to each other, would fit in a line one kilometer long? Show your work.
(b) (4 points) Write 2^{-1} as a decimal.
(c) (6 points) Show where the numbers 0.09, 1, 4, and 103 are on the multiplicative number line below.

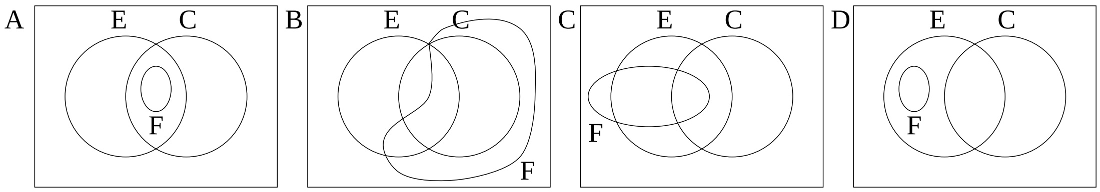


6. While attending your favorite Badger Game, you notice that half of the men in the audience are wearing red and three-fourths of the women in the audience are wearing red. The number of women wearing red is three times the number of men wearing red.
(a) (12 points) What proportion of the audience is wearing red? Explain your answer.
(b) (4 points) Can you tell how many women and men are at the game? Explain your answer.
(c) (4 points) Can you tell whether there are more men or more women at the game? Explain.

7. (17 points) A certain instructor, who shall remain nameless, recently went on a trip to Montréal. While there, her husband phoned every evening. When they got their phone bill last week, they discovered that one-fourth of the minutes were billed at ten cents a minute, and three-fourths of the minutes were billed at a whopping fifty cents a minute. The total cost of the calls to Montréal (ignoring tax) was \$160.
How many minutes did they talk on the phone? (Write your answer in simplest form).
Moral: Always read the fine print about phone rates.

[see previous page for Problem 8]

Math 130 Sample Exam 21
Author: Brad Franklin
Topics: Sets (Unit Three) [1, 2, 4, 5], Operations (Unit Three) [6, 7], Arithmetic, Place Value & Bases (Unit Four) [8, 9, 10, 12], Logic (Unit Five) [3], Quantifiers (Unit Five) [11]
Date: Spring 1997
Time Allotted: 2 hours



1. Next to each sentence, give the letter of the above Venn diagram which best describes it.
   i. All Floobians like espresso, but none play chess. ____
   ii. All chess players who don't like espresso are Floobian. ____
   iii. Every Floobian plays chess and likes espresso. ____
   iv. No chess player who dislikes espresso is Floobian. ____

2. For each of the following, tell whether they are true for *all* sets A, B, and C. Show your reasoning.
   (a) $A \cup (B \cap C) \subset \overline{A \cup \overline{B}}$.
   (b) $(\overline{A} \cup B) \cap C \subset \overline{A \cap \overline{C}}$

3. Make a truth table for the statement

$$(p \to q) \vee (q \to p).$$

A tautology is a statement which is always true. Is this a tautology?

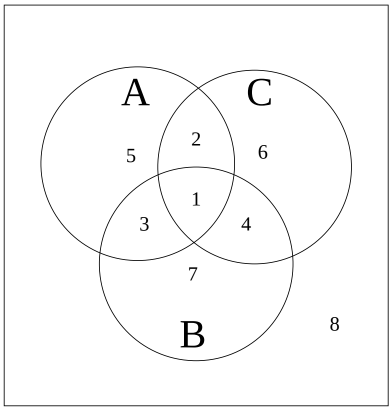
4. *i.* Let E stand for the set of even numbers, F stand for the set of numbers which are divisible by five, and let H stand for the set of Hoopahs. Make a venn diagram for the following sentence.

If a number is even and it is a Hoopah, then it is divisible by 5.

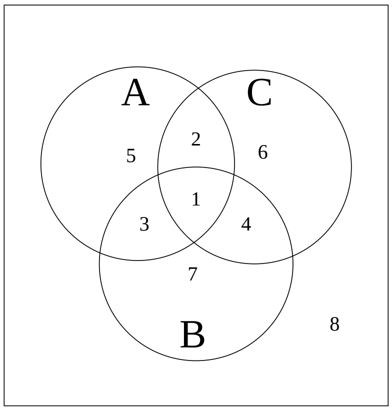
   ii. Now what can we conclude from this sentence? Say, 'true', 'false', or 'can't tell'.
   (a) 10 is a Hoopah. ____
   (b) All numbers that are Hoopahs and odd are not divisible by 5. ____
   (c) All numbers that are even and divisible by 5 are Hoopahs. ____
   (d) If a number is a Hoopah and it is not divisible by five, then it is odd. ____

(continued on next page)

5. Shade the regions indicated, and then express the shaded regions using set notation.

(a) 

[shade regions 3, 4, 6]

(b) 

[shade regions 1, 3, 4, 8]

6. Fill out the operator table for the operation $\circ$ on the set $\{a, b, c\}$ to define it such that it has the properties identity and inverses, but not commutative.

$\circ$	a	b	c
a			
b			
c			

7. Let the operation \$ be defined on the whole numbers by $a\$b = ab^2$. What properties does \$ have? For each property that it *doesn't* have, give a counterexample.

8. Convert to base ten: $1A8_{twelve}$. (A represents ten.)

9. Find the sum of these base five numbers. Give your answer in base five.

$$\begin{array}{r} 342_{five} \\ +\ 123_{five} \\ \hline \end{array}$$

10. Find the product of these base two numbers. Give your answer in base two.

$$\begin{array}{r} 101_{two} \\ \times\ 110_{two} \\ \hline \end{array}$$

11. Determine which of the following statements are true on the set of whole numbers. Explain your answers.

(a) $\forall x\, \forall y \quad x^2 + y^2 > x$.
(b) $\exists x\, \forall y \quad x^2 + y^2 > x$.
(c) $\forall y\, \exists x \quad x^2 + y^2 > x$.
(d) $\exists x\, \exists y \quad x^2 + y^2 > x$.

12. Convert the base ten number 94 to base 8.

Math 130 Sample Exam 22
Author: Brad Franklin
Topics: Sets (Unit Two) [7], Operations (Unit Three) [1, 5], Arithmetic, Place Value & Bases (Unit Four) [5c, 8, 9], Logic (Unit Five) [6], Quantifiers (Unit Five) [10], Fractions (Unit Six) [2, 3], Number Theory (Unit Seven) [4], Algebra (Unit Eight) [11]
Date: 15 May 1997
Time Allotted: 2 hours
Instructions: Answer all of the following questions, and provide justifications for you answers. You may use a calculator. Try to spread your time evenly over the exam rather than spending a lot of time on one problem.

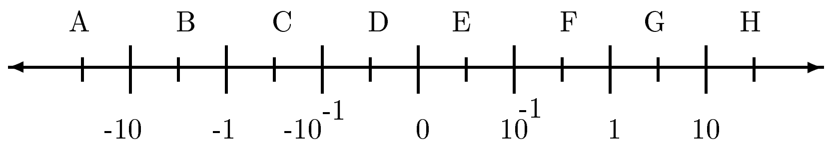
1. (a) (3 points) Which of these properties does addition have on the set of rational numbers: closure, commutative, associative, identity, inverses?
(b) (3 points) Which of the above properties does division have on the set of rational numbers?

2. Suppose that, in a class with 24 students, exactly 3/4 of students have brothers and exactly 2/3 have sisters. Suppose further that two-thirds of the students with brothers also have sisters.
(a) (8 points) Can you determine from the information given how many of the students are only children? If so, how many? If not, what further information would you need to find this quantity?
(b) (8 points) Can you determine from the information given what proportion of students with sisters also have brothers? If so, what's the answer? If not, what further information would you need to find this quantity?

3. (a) (4 points) Write 23/33 as a repeating decimal.
(b) (4 points) Write $.1\overline{2}$ as a common fraction.

4. (5 points) Find a number with exactly 126 factors.

5. (16 points) The following is a base *ten* number line. Use it to answer parts a–h. Some parts may have more than one acceptable answer.



a. Which letter best represents the result of $A \div B$?
b. Which letter best represents the result of $A \times D$?
c. Which letter best represents the location of 323/21 (base *five*)?
d. Which letter best represents the result of C/B?
e. Which letter best represents the result of D/E?
f. Which letter best represents the result of $B - D$?
g. Which letter best represents the result of B^2?
h. Which letter best represents the location of 10^{-3}?

(continued on next page)

6. Consider the truth table at right:

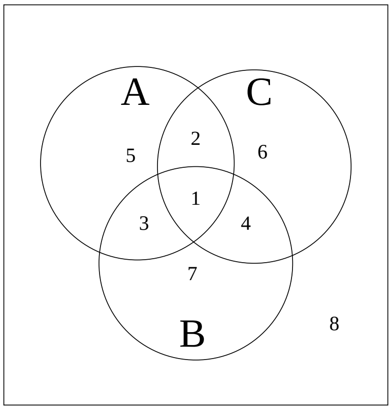
(a) (5 points) How many different truth tables are possible for p and q, allowing for all possible combinations of T's and F's in the last column?

p	q	
T	T	$\boxed{?}$
T	F	$\boxed{?}$
F	T	$\boxed{?}$
F	F	$\boxed{?}$

(b) (5 points) Suppose that you wanted the adjoining truth table. Place a statement at the top of the last column which agrees with the pattern of T's and F's in the table.

p	q	$\boxed{? \qquad ?}$
T	T	$\boxed{\text{T}}$
T	F	$\boxed{\text{T}}$
F	T	$\boxed{\text{F}}$
F	F	$\boxed{\text{T}}$

7. (6 points) Shade the regions indicated, and then express the shaded region using set notation.



[shade regions 2 and 7]

8. (6 points) Convert 162_{ten} to base seven.

9. (6 points) Find the sum of these base five numbers. Give your answer in base five.

$$\begin{array}{r} 343_5 \\ +\ \ 124_5 \\ \hline \end{array}$$

10. (9 points) Determine which of the following statements are true on the whole numbers. Explain your answers.

(a) $\forall x\, \forall y \quad xy > x + y$.
(b) $\exists x\, \forall y \quad xy > x + y$.
(c) $\forall y\, \exists x \quad xy > x + y$.

11. (a) (8 points) Jane drives to work each morning at an average of 35 miles per hour. She returns home at an average of 40 miles per hour. If it takes her 10 minutes longer to go to work than to return home, how far does she live from work?

(b) (4 points) Suppose we wanted to change the number 35 in the above problem. Give a [whole] number that we could *not* replace 35 by and still have a workable problem.