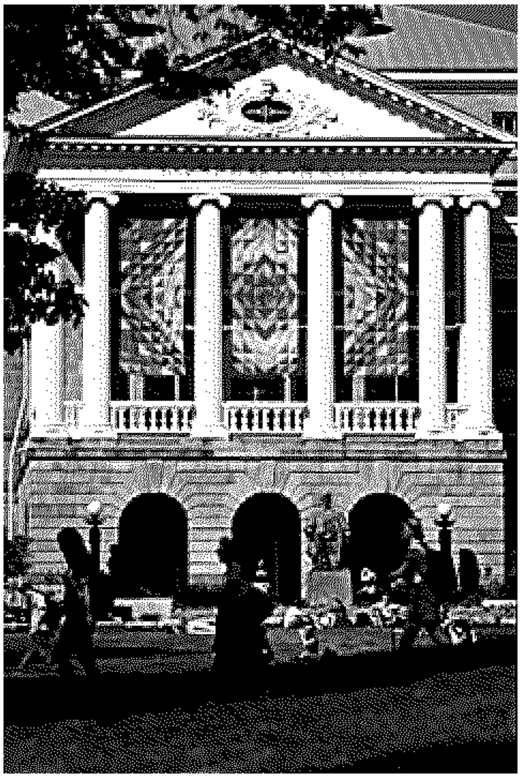
Mathematics 130

Arithmetical
Problem Solving



University of Wisconsin

Madison, WI

Math 130 Course Materials

Revised Summer 1999

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Math 130
UNIT ONE
Problem Solving

POISON

A Game

Number of Players: Two teams of one or more players

Equipment: Counters

Rules:
1. Begin with 10 counters.
2. Teams take turns playing.
3. When it is your team's turn, take away 1 or 2 counters.
4. The team that takes the last (POISONED) counter loses.
5. Toss a coin to decide which team goes first the first time you play POISON; after that, alternate which team begins the game.

Instructions: Play the game a few times. As you play, remember or keep a record of how you played the game; that is, how many counters did you pick up? Why? Who won the game?

After you have played the game a few times, reflect on your game records and see if your group can devise a winning strategy. Once you think you have one, first test it out (against other players), and then see if you can make it work when the number of initial counters is changed from ten.

Question: Suppose a game of POISON begins with 431 counters. Do you want to play first or second? What should be your strategy (that is, how should you play)?

RAT POISON

To win you have to be as sneaky as a rat!

RAT POISON has the same rules as POISON except that in RAT POISON you may take 1, 3, or 4 counters; you *cannot* take 2 counters. The team that takes the last counter is still the loser.

Questions: 1. If RAT POISON starts with 10 counters, is there a winning strategy? What is it?

2. Can you answer question 1 when the game starts with 20 counters? With any larger number of counters?

BOOKS and PAGES

Folios. Books are sometimes made up of sections of pages called folios. In a folio each sheet of paper is printed as shown on both the front and the back.

Page #	Page #
and then he ran into a big dog who came after him, saying, "Woof woof woof woof woof woof woof woof woof woof woof woof!"	by which time all seven hundred members of the graduating class had come and chased off all of the ferocious animals

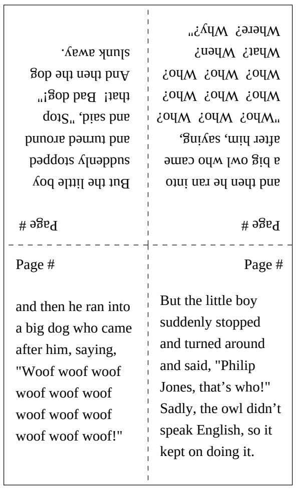
After printing, the folio pages are stacked up, folded (in half) along the dashed line shown above, and sewn together on that fold with nylon cord.

1. A short novel, *The Golden Gate*, has 156 pages. How many folio sheets were needed? How did you figure this out?

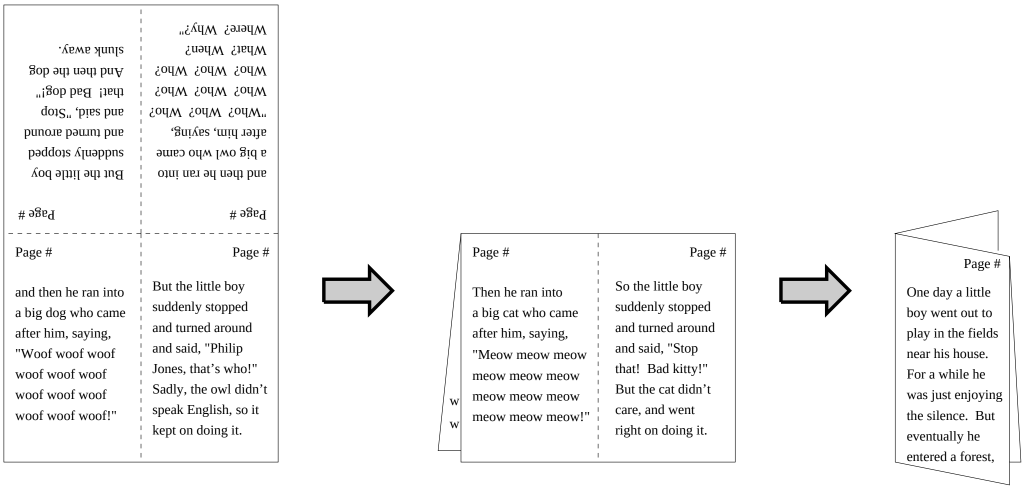
2. The folio page shown below (from *The Golden Gate*) fell on the floor. What is the number on the other corner (that is, where ??? appears)? What are the two numbers on the other side of this sheet of paper?

104	???
and then he ran into a big cat who came after him, saying, "Meow meow meow meow meow meow meow meow meow meow meow meow!"	the way," said Inspector Columbo, "I hope you don't mind that I brought some of the boys along. I'm afraid I'm going to have

Quartos. In quarto publishing, four book pages are printed on each side of the sheets of paper used. One side of a quarto sheet is shown at right.



Once the sheets are printed on both sides, they are stacked up, and the stack is folded twice: first from top to bottom on the dashed horizontal line and then from side to side on the dashed vertical line. The tops of the pages are cut apart. Thus, every quarto sheet produces eight book pages.



1. How many quarto sheets are needed for a book of 128 pages?

2. When the pages of this book are stacked and ready to be folded, what does the top sheet look like?

3. Draw a picture that shows where on the sheet the page numbers appear and which are upside down.

4. What other page numbers are on the the same quarto sheet with page 13?

Time to Weigh the Hippos

Martha is the chief hippopotamus caretaker at the Wild Animal Park in San Diego, California. She has just arrived at the cargo dock in order to pick up four members of the endangered species *hippopotamus mathematicus* that were recently rescued from African poachers. Before the animals are released by the harbormaster, Martha must weigh them. BUT the only scale big enough to weigh a hippo is the truck scale that doesn t weigh anything lighter than 300 kilograms (kg); this is more than each of the hippos weighs. Martha is puzzled for a few minutes, then gets the idea of weighing the hippos in pairs, thinking that if she gets the mass of every possible pair, she can later figure out the masses of the individual hippos. She weighs the hippos pair-by-pair and gets 312 kg, 356 kg, 378 kg, 444 kg, and 466 kg. When she tries to weigh the heaviest pair of hippos, the scale breaks. Alas!

1. What was the mass of the last pair of hippos who broke the scale?

2. What are the masses of the individual hippos?

Note: Be sure to write down explicitly any assumptions you make, and be able to explain your reasoning at each step.

A Frame-Up

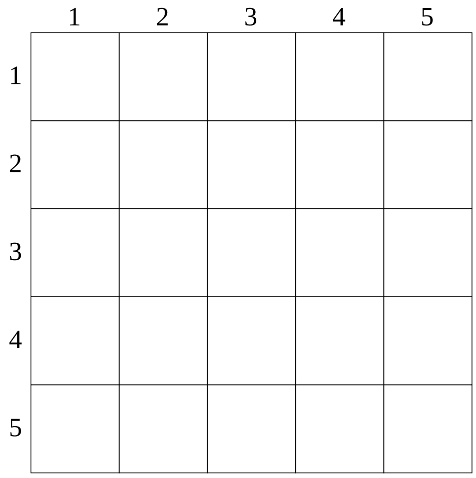
I painted a picture that is 4 inches longer than it is wide. If I put a 3-inch frame around my picture, the area increases by 48 square inches. What are the dimensions of my picture?

Four Cuts

If you were to take a block of cheese and make a cut all the way through, you would end up with two pieces. If you were then to make another cut without moving either of those pieces, you should be able to do it in such a way that you end up with four pieces. What is the maximum number of pieces you can have after three cuts (remember you can't move any of the pieces)? What about after four cuts? You should be ready to explain how to make the cuts so as to obtain the maximum possible number of pieces.

Squares & Paths

At home, work on each of the following two problems (Squares and Paths) for half an hour each. Write down how far you get on each, and then choose one of the two to finish solving.



Squares

1. How many squares of all sizes are contained in the 5×5 grid shown at left?

2. How many squares of all sizes are contained in an $n \times n$ grid? (n can be any whole number: 1, 2, 3, 4, ...)

Paths

How many paths are there that start at some A and end at Z, that contain each letter of the alphabet exactly once, and traverse the letters in alphabetical order? (Diagonal moves are not allowed.)

```
                         A
                        ABA
                       ABCBA
                      ABCDCBA
                     ABCDEDCBA
                    ABCDEFEDCBA
                   ABCDEFGFEDCBA
                  ABCDEFGHGFEDCBA
                 ABCDEFGHIHGFEDCBA
                ABCDEFGHIJIHGFEDCBA
               ABCDEFGHIJKJIHGFEDCBA
              ABCDEFGHIJKLKJIHGFEDCBA
             ABCDEFGHIJKLMLKJIHGFEDCBA
            ABCDEFGHIJKLMNMLKJIHGFEDCBA
           ABCDEFGHIJKLMNONMLKJIHGFEDCBA
          ABCDEFGHIJKLMNOPONMLKJIHGFEDCBA
         ABCDEFGHIJKLMNOPQPONMLKJIHGFEDCBA
        ABCDEFGHIJKLMNOPQRQPONMLKJIHGFEDCBA
       ABCDEFGHIJKLMNOPQRSRQPONMLKJIHGFEDCBA
      ABCDEFGHIJKLMNOPQRSTSRQPONMLKJIHGFEDCBA
     ABCDEFGHIJKLMNOPQRSTUTSRQPONMLKJIHGFEDCBA
    ABCDEFGHIJKLMNOPQRSTUVUTSRQPONMLKJIHGFEDCBA
   ABCDEFGHIJKLMNOPQRSTUVWVUTSRQPONMLKJIHGFEDCBA
  ABCDEFGHIJKLMNOPQRSTUVWXWVUTSRQPONMLKJIHGFEDCBA
 ABCDEFGHIJKLMNOPQRSTUVWXYXWVUTSRQPONMLKJIHGFEDCBA
ABCDEFGHIJKLMNOPQRSTUVWXYZYXWVUTSRQPONMLKJIHGFEDCBA
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Math 130
UNIT TWO
Sets

Banned Book Survey

As part of the Banned Books Week commemorations, the Madison Public Library survey their patrons on what previously banned books they had read. Fifty-three percent had read *Huckleberry Finn*; 42% had read the Bobbsey twins books; and 36% had read *Lady Chatterly's Lover*. To get more detailed information, the library staff did a follow-up survey and found that 25% of their patrons had read both *Huckleberry Finn* and the Bobbsey twins books, while 14% had read the Bobbsey twins books and *Lady Chatterly's Lover*, and only 10% had read *Huckleberry Finn* and *Lady Chatterly's Lover*. A reporter tried to figure out from these data how many of the people surveyed had read all three books. It turned out that 12% of those surveyed had not read any of the books. Did the reporter need this last piece of information to answer her question? How many had people had read all of the books?

Introduction to Sets

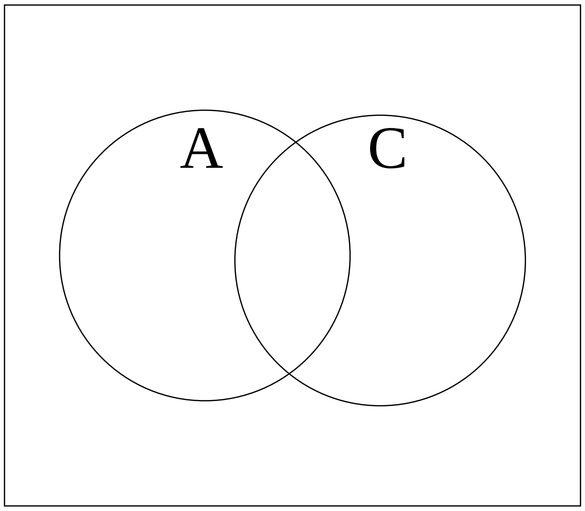
Remember that a *set* is a collection of objects, like colors, numbers, people, pizza toppings, etc. These objects are called *elements* of the set, which is sometimes written like this: $A =$ {red, orange, yellow, pepperoni}. Here are a few bits of set notation you need to be able to recognize and use:

• $\bigcup$ UNION: a joining together of two sets to form a larger set. For example, if $A =$ {red, orange, yellow, pepperoni} and $B =$ {1, 2, pepperoni}, then $A \cup B =$ {red, orange, yellow, pepperoni, 1, 2}. A union is a logical OR: in order to be a member of $A \cup B$, you must be a member of either A OR B (or both).

• $\bigcap$ INTERSECTION: an extraction of the elements that two sets have in common. For example, with sets A and B as above, the set $A \cap B$ would contain the single element pepperoni. An intersection is a logical AND: in order to be a member of $A \cap B$, you must be a member of both A AND B.

• $\overline{\ }$ COMPLEMENT: the set which contains everything which is *not* in a given set. $\overline{A}$ "complements" A because together they contain every possible element — $A \cup \overline{A} =$ the universe. Here we have to be careful to specify what "the universe" is. For example, taking our set A as above, if our universe is the set of all words in the dictionary, then $\overline{A}$ is huge. But if we are only considering the "universe" of primary colors and pizza toppings, then we might say $\overline{A} =$ { green, blue, indigo, violet, cheese, hamburger, mushrooms, onions }. A complement is a logical NOT: in order to be a member of $\overline{A}$, you must NOT be a member of A.

1. At right is what is called a *Venn diagram.* This is a drawing which allows us to visualize how elements relate to sets. The circle on the left, labelled "A", represents set A; the one on the right represents set C. The circles are enclosed in a rectangle which represents the universe of things we are considering. The way we use it is to put (i.e., write) every element where it belongs with regard to which sets it is a member of. An object which belongs to neither set should go outside the circles, inside the rectangle; an object which belongs to both sets should go in the area which is inside both circles.



Now try putting all the objects in the following universe in their place in the diagram, using the set definitions below: blue, cheese, green, hamburger, indigo, mushrooms, onions, orange, pepperoni, red, violet, yellow.

$A =$ { red, orange, yellow, pepperoni }

$C =$ { orange, onions, blue, cheese, yellow, mushrooms }

2. Now write down the members of the following sets, in the way A and C are defined above.

(a) $A \cup C$

(b) $A \cap C$

(c) $\overline{C}$

(d) $A \cap \overline{C}$

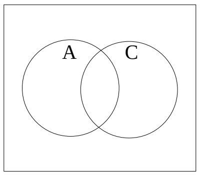
(e) $\overline{A \cup C}$

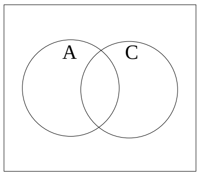
(f) $\overline{A} \cup \overline{C}$

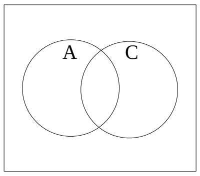
(g) $A \cap \overline{A}$

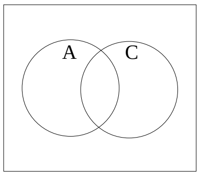
(h) $\overline{A} \cap \overline{C}$

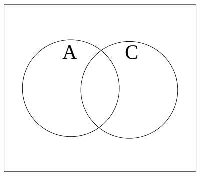
3. Now shade the regions of the small Venn diagrams below which correspond to the sets in question 2.

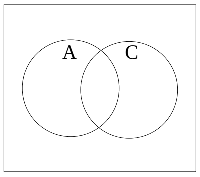
(a) 

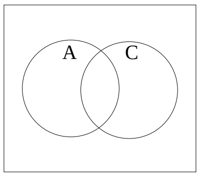
(b) 

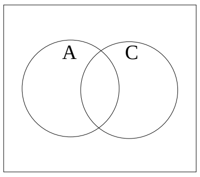
(c) 

(d) 

(e) 

(f) 

(g) 

(h) 

4. Two of the eight sets above should be identical. This is no coincidence — it is in fact a law, one of two called De Morgan's Laws. Write an equation saying that these two sets are equal. Then obtain the other of De Morgan's Laws by making another copy of the first equation in which all unions ($\cup$) become intersections ($\cap$), and vice versa.

Sets of People

1. Write each of the following sets as some combination (union, intersection, and/or complement) of simpler sets. Be sure to state very carefully which sets you're using, particularly when you are using complements.

   (a) the set of left-handed, near-sighted mathematicians with beards

   (b) the set of education students who have not taken Math 130, Math 131, [n]or Math 132

   (c) the set of students at this university who are named Amy or Dawn

   (d) the set of adults who are under five feet tall or over six feet tall

   (e) the set of people who have been President but not Vice President

2. Now let us restrict our attention to a math class with the following roster of students:

   Jennifer Ahrendt
   Barbara Allen
   Carrie Ann Baerman
   Elise Black
   Allison Camp
   Bill Cooper
   Chris Danes
   Joy Smith

Let A be the set of people in the class with a name that starts with A, and likewise let B be the set of people in the class with a name that starts with B.

(a) Who is in each of the following sets? (A Venn diagram may help.) Can you give a description in words of each of these sets without mentioning any specific names?

$\overline{A} \cup B$ $\qquad\qquad$ $\overline{(A \cap B)}$ $\qquad\qquad$ $\overline{A} \cap \overline{B}$ $\qquad\qquad$ $\overline{(\overline{A} \cup \overline{B})}$

(b) How would you use set notation to describe the following sets in terms of A and B?

people with a name that starts with A or a name that starts with B, but not both

people with a name that starts with A but without a name that starts with B

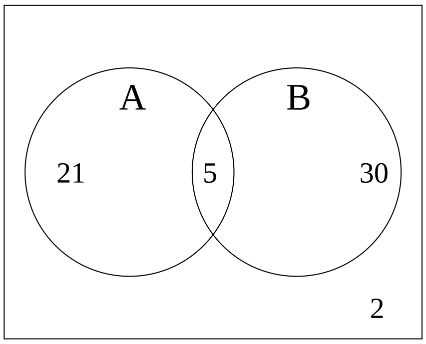
(c) Now let C be the set of class members who have a name that starts with C. Try to create a Venn diagram showing how the class members are placed with respect to sets A, B and C.

Describing Sets of Numbers

In each of the diagrams, identify the sets A and B by selecting the choices that best describe the sets.

Choices for sets A and B
(a) Multiples of 2
(b) Odd numbers
(c) Prime numbers
(d) Divisors of 24
(e) Smaller than 10
(f) Multiples of 3
(g) Larger than 10
(h) Multiples of 6
(i) Multiples of 5
(j) Divisors of 15

1.



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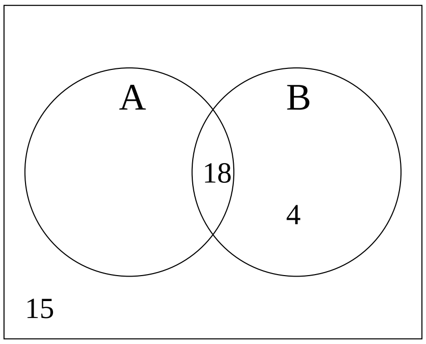

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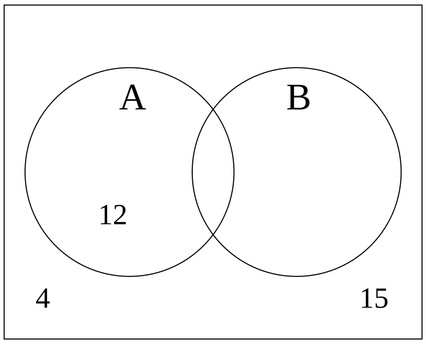

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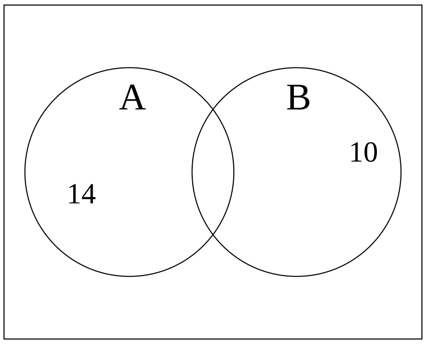

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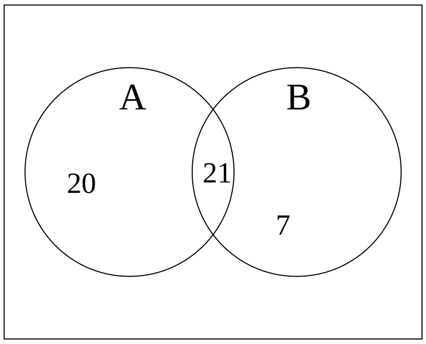

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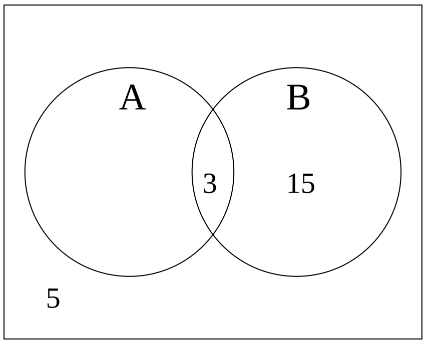

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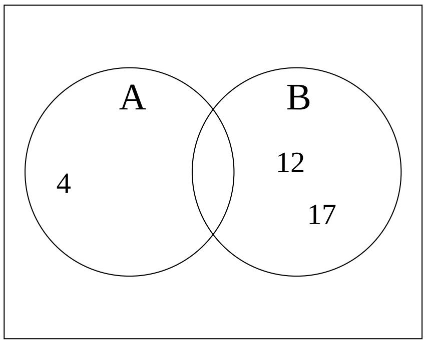

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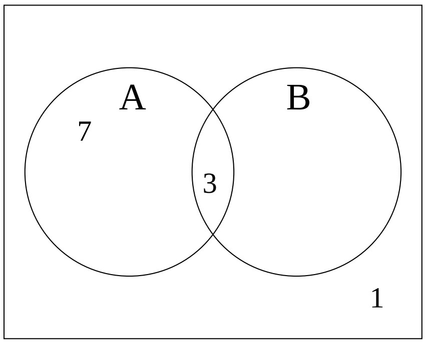
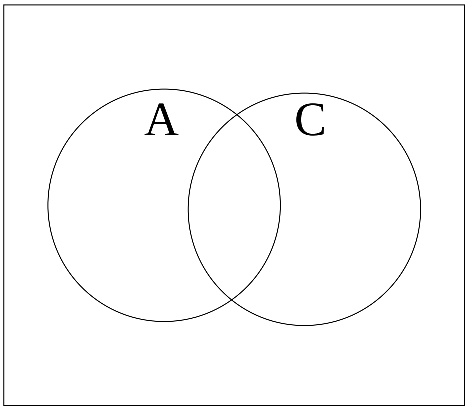


9. Now choose two different sets from the list above (a different pair from the eight already used), and place three or four numbers in appropriate places in the Venn diagram below so that they uniquely define your sets A and C among the choices given.



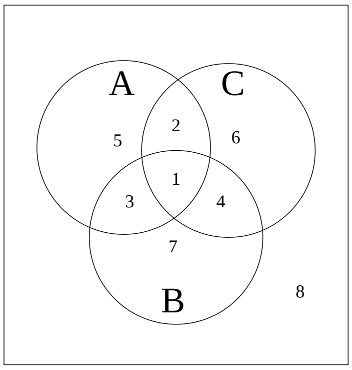
Extra Problems: Sets

1. In a special ed classroom where there are to be 20 children, 9 of the children will need new desks high enough to accommodate wheelchairs, and seven of the students in the classroom will need desks fitted with electrical power strips for special equipment. If eight of the children can use already existing (lower) desks without power strips, how many of the new, higher desks need to be fitted with power strips?

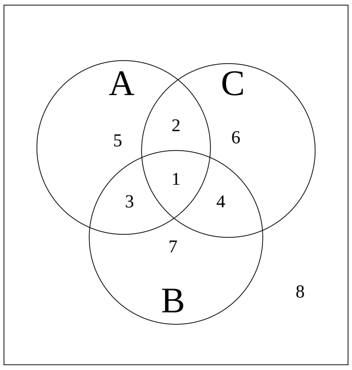
2. In a given class of 25 students, there are a total of 11 boys, and a total of 7 left-handed children. If 9 of the boys are right-handed, how many of the girls are right-handed?

3. In a class of 35 students, 20 have a last name beginning with a consonant. 25 students have a last name with more than one syllable. What can you say about the number of students with a multisyllabic last name that begins with a vowel? What about the number of students with a multisyllabic last name that begins with a consonant?

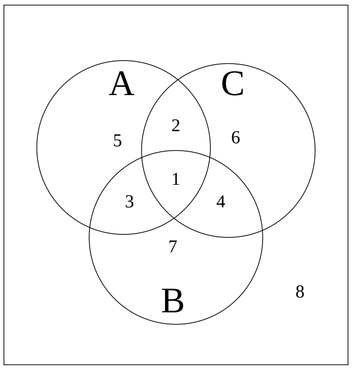
4. For each Venn diagram below, shade the numbered regions indicated, and find a way to express the region in terms of A, B and C using unions, intersections, and complements.



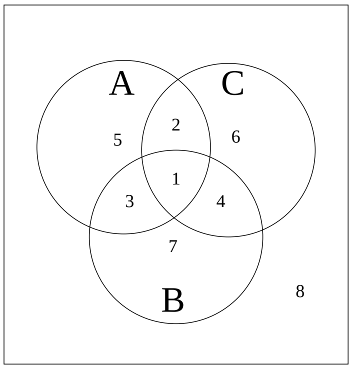
(a) 1, 2, 3



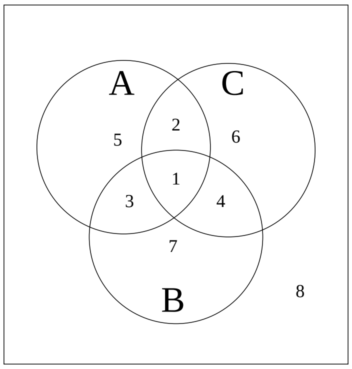
(b) 1, 2



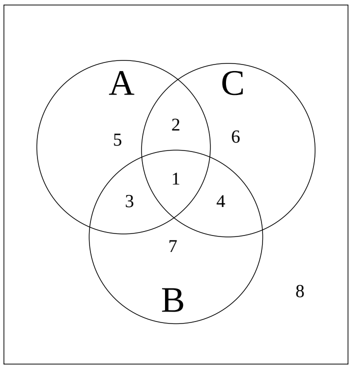
(c) 2, 3, 4



(d) 1, 2, 3, 4



(e) 5, 6, 7

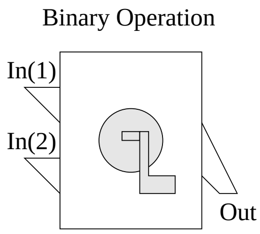


(f) 1, 8

Math 130
UNIT THREE
Operations

Introduction to Operations

An *operation* is a way to combine some elements of a set to get another element of the set. You are already familiar with many operations: addition, subtraction, multiplication and division are all operations. They are called *binary operations* because they operate on two elements of a set. Taking a square root is called a *unary* operation because you only operate on one number to get the square root. You can think of an operation like a machine, with a certain number of slots for inputs and a slot for the output. You put the inputs into the appropriate slots, turn a crank, and out comes the output.



Operations can be defined on any kind of set, not just numbers. Let's consider the following set S: $\{\Box, \bigcirc, \triangle, \star, \heartsuit\}$. Now let's define an operation $\bowtie$ on S by making the output of $a \bowtie b$ equal to whichever of the symbols a and b in S has more (sharp) corners. Thus $\Box \bowtie \bigcirc = \Box$ since $\Box$ has four sharp corners while $\bigcirc$ has none.

Another way to represent a binary operation on a set of finite size is to make a table showing all the possible outcomes. The elements of S run down the left side and across the top. The fact that $\Box \bowtie \bigcirc = \Box$ has been filled in on the table. Complete the table by filling in the rest of the spaces.

$\bowtie$	$\bigcirc$	$\heartsuit$	$\triangle$	$\Box$	$\star$
$\bigcirc$					
$\heartsuit$					
$\triangle$					
$\Box$	$\Box$				
$\star$					

Now see if you can work backward to figure out the algebraic rule for the operation $\odot$ defined on the set $\{0, 1, 2, 3, 4\}$ by the table at right.

$\odot$	0	1	2	3	4
0	0	-2	-4	-6	-8
1	3	1	-1	-3	-5
2	6	4	2	0	-2
3	9	7	5	3	1
4	12	10	8	6	4

Properties of Operations

• CLOSURE: A set S is *closed under* an operation iff (if and only if) all the operation's outputs are elements of S. The set S on the last page is closed under $\bowtie$ because $\bowtie$ing two elements of S always produces an element of S. If, however, we define the operation $\uplus$ on the digits 0 through 9 by $a \uplus b = \max(a, b) + 1$, the set of digits 0 through 9 is not closed under $\uplus$ because $0 \uplus 9 = 9 + 1$ is not in that set.

∘ Is the set $\{0, 1, 2, 3, 4\}$ closed under the $\odot$ operation defined on the previous page? If not, can you think of a set which *is* closed under $\odot$?

• COMMUTATIVE: An operation is *commutative* iff the order of the two things being operated on doesn't matter. For example, $\bowtie$ above is commutative because $a \bowtie b = b \bowtie a$ no matter what a and b are.

∘ Is there a visual clue in the $\bowtie$ table that lets you know it's commutative?
∘ Is $\uplus$ as defined in the previous paragraph commutative?
∘ Can you think of an operation that is not commutative?

• ASSOCIATIVE: An operation is *associative* iff when you apply the operation repeatedly, it doesn't matter where you put the parentheses. For example, addition on the natural numbers is associative, which is illustrated by the fact that $1 + (9 + 7) = (1 + 9) + 7$.

∘ Is the $\bowtie$ operation defined above associative?
∘ Is associativity something that you can see from looking at a table?
∘ Can you make up an operation which is associative but *not* commutative? (It doesn't have to involve numbers.)

• IDENTITY: An operation has an *identity* element if one of the elements in the set makes the operation leave all other elements alone. This must be true regardless of whether the identity is the first or second of the numbers being operated on. Addition on the natural numbers has an identity element, namely 0, because $0 + a = a$ and $a + 0 = a$, no matter what a is. However, the $\odot$ operation defined on the previous page has no identity element, no single element which makes $\odot$ leave the other operand (thing being operated on) alone, no matter what it is.

∘ Does the $\bowtie$ operation defined on the last page have an identity element? If so, what is it?
∘ Does the $\uplus$ operation defined above have an identity element? If so, what is it?
∘ Can you tell from looking at a table whether an operation has an identity element?
∘ Is it possible for an operation to have *two* (or more) identity elements?

• INVERSES: If an operation has an identity, then it may also have *inverses*. The inverse of an element a is something which gets operated on along with a in order to get the identity element. Addition on the integers has inverses, e.g., the inverse of 42 is -42, because adding 42 and -42 gets you 0, the additive identity. Multiplication on the natural numbers does not have inverses, because the multiplicative inverse of 2 is $\frac{1}{2}$, which is not a natural number. Of course, if an operation does not have an identity element, it cannot have inverses.

∘ Does the $\bowtie$ operation have inverses?
∘ Can you think of a set on which the $\odot$ operation has not only an identity element but inverses?

Analyzing Operations

Try to see which of the five properties defined on the previous page the operations listed below have. If there is an identity or inverse elements, say what they are.

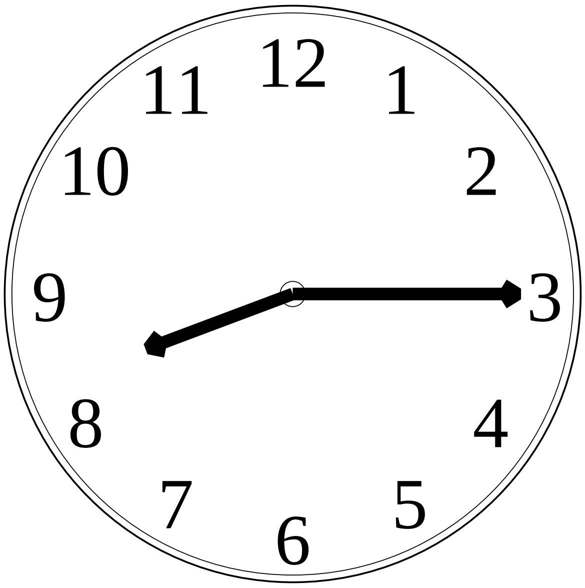
1. The four arithmetic operations $+$, $-$, $\times$ and $\div$, on the set of integers $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$

2. The operation $\smile$ defined on the set $\{\heartsuit, \diamondsuit, \spadesuit, \clubsuit\}$ by the table below:

$\smile$	$\heartsuit$	$\diamondsuit$	$\spadesuit$	$\clubsuit$
$\heartsuit$	$\heartsuit$	$\diamondsuit$	$\spadesuit$	$\clubsuit$
$\diamondsuit$	$\diamondsuit$	$\spadesuit$	$\clubsuit$	$\heartsuit$
$\spadesuit$	$\spadesuit$	$\clubsuit$	$\heartsuit$	$\diamondsuit$
$\clubsuit$	$\clubsuit$	$\heartsuit$	$\diamondsuit$	$\spadesuit$

3. The exponentiation operation $\lhd$ defined on the whole numbers $\{1, 2, 3, ...\}$ by $a \lhd b = a^b$.

Clock Arithmetic



1. Come up with a definition for a new operation $\oplus$ which will be addition in "clock arithmetic", under which the set $\{1, 2, 3, ..., 12\}$ is *closed*.

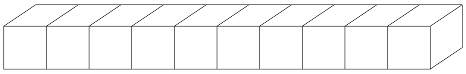
2. Explain how to calculate $3 \oplus 5$ and $7 \oplus 10$ using your definition.

3. Now write a careful definition of your operation, and analyze the structure of this new "clock arithmetic" system (i.e., describe the properties the operation and the set it operates on have).

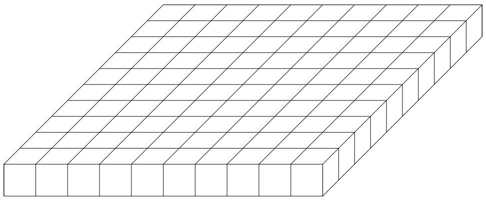
Math 130
UNIT FOUR
Arithmetic,
Place Value,
& Bases

Blocks, Rods, Flats, and Cubes

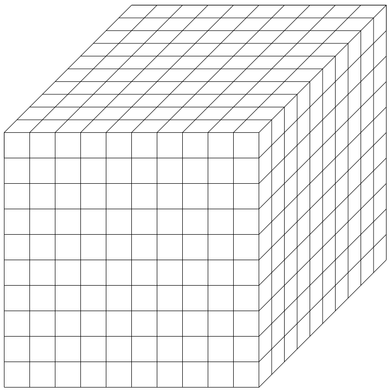
Our system for writing down numbers can be described using blocks, rods, flats, and cubes. For any number of blocks less than ten, we simply write down the familiar symbol for that number of blocks. Every group of ten blocks is connected to form a rod, like so:



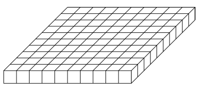
Every group of ten rods is connected to form a flat, like so:

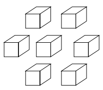


Every group of ten flats is connected to form a cube, like so:



To represent a (small) number, we simply write down (the symbol for) the number of cubes, followed by (on its right) the number of flats, followed by the number of rods, followed by the number of blocks. For instance,





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Self-check Questions

1. If I have six hundred eighty two blocks and I start assembling them into rods, assembling rods into flats when I can, and so on, how many blocks, rods, flats, and cubes will I have in the end, and how will I represent that as a written number?

2. (a) Suppose I have a rod of cubes. What does this mean? How many blocks do I have? What if I have a flat of cubes? A cube of cubes?

   (b) How many rods are there in a cube of cubes? How many flats in a rod of cubes?

(c) What is a rod of flats more commonly known as?

(d) Suppose a cube represents one unit. How many units does a rod represent? A block?

Problems

1. In an effort to promote its new cereal "Frosted Honeycombs", Mills General has included a prize consisting of 7 blocks, 8 rods, and 5 flats in every box. These are specially designed out of interlocking plastic so that ten blocks can be assembled to form a rod, etc. Suppose you buy one box of Frosted Honeycombs every day starting January 1, and you always reassemble your prize pieces into the biggest possible units (after cubes come rods-of-cubes, flats-of-cubes, etc.). One day you notice that you are out of blocks and rods; everything has been assembled into larger units. What day is it? How many assembled pieces do you have? (An assembled piece is a block or rod or flat or cube or rod-of-cubes, etc.)

2. Your best friend also buys a box of Frosted Honeycombs every day starting January 1, but is sometimes too lazy to assemble the smaller pieces into bigger pieces. On the day you find you are out of blocks and rods, how many assembled pieces might your friend have? List or describe all possiblities.

Introduction to Bases

As you may know, our normal way of representing numbers is called "base ten", because it uses ten digits. We count from 0 to 9, and then we write 10, representing 1 group of ten and 0 leftover units. The next number, 11, represents 1 group of ten and 1 unit. This can continue indefinitely; the number 43,507 represents 4 groups of ten thousand, 3 groups of one thousand, 5 groups of one hundred, 0 groups of ten, and 7 units.

1. Perhaps the best way to get a feel for other bases is to practice grouping. Take fifty three chips. Those can be grouped into 5 groups of ten and 3 units. That's why this number is written 53 (base ten). But they could also be grouped into 1 group of thirty six (which is six times six), 2 groups of six, and 5 units. So we could write this number as 125 (base six), or 125_{six}. By grouping your counters, write this number in the bases two through twelve:

Base two:	Base six: 125	Base ten: 53
Base three:	Base seven:	Base eleven:
Base four:	Base eight:	Base twelve:
Base five:	Base nine:	

2. Now that you are good at grouping, it's fairly easy to add numbers. Take thirty-seven and fifteen, and put both in base ten groupings (we'll start by adding in base ten). Now, start by adding the units. There are 7 units and 5 units, for a total of twelve. This is too many for our base, so we take a group of ten and move them over. Now we add the tens. There are $3 + 1 + 1$ groups of tens, for a total of 5. This number is less than our base, so we leave it. We now have 5 groups of ten and 2 units, so $37 + 15 = 52$ in base ten.

Once you understand this idea, do the same (adding thirty-seven and fifteen) in the following bases:

(a) base eight (b) base four (c) base two

Driving in Septobasiland

The king of Septobasiland has proclaimed that the digits 7, 8 and 9 are never to be used in Septobasiland upon pain of death. As a result, the little wheels on the odometers of all the cars in Septobasiland have just the digits 0, 1, 2, 3, 4, 5, 6 on them. Thus, if the odometer registers 0006 and you drive one more mile, the odometer will register 0010, and if the odometer registers 0016 and you drive one more mile, it will register 0020.

1. If the odometer registers 0066 and you drive one more mile, what will the odometer register?

2. If the odometer registers 0325, how many miles has the car gone?

3. How many miles will the car have gone when the odometer turns over to all zeroes again?

4. After a car has gone nine hundred miles, what will its odometer register?

5. You and your companions go on a trip in Septobasiland, travelling together in two cars. The first car, which is new, starts with its odometer reading 0000, and the second starts with the odometer reading 1435. At the end of the trip, the odometer of the first car registers 0324. What will the odometer of the second car register?

6. Suppose again that the odometers of the two cars register 0000 and 1435 at the beginning of the trip. But this time at the end of the trip the odometer of the second car registers 2153. What will the odometer of the first car register?

Addition and Subtraction Problems

One of your students, Jeff, is having trouble with addition with regrouping. When Jeff adds numbers, he doesn't carry the ones; instead, he does the following.

$$\begin{array}{cccc} & 3 & 8 & 4 \\ + & 4 & 7 & 9 \\ \hline & 7 & 15 & 13 \end{array}$$

On two of his homework problems, Jeff obtained the answers $3|7|11$ and $9|19|23$; here the vertical bars are intended to separate columns.

- If Jeff had used the standard algorithm for addition with regrouping, what would he have "carried" into the tens column in each case? How about into the hundreds column?
- What answers should Jeff have obtained, had he used the standard algorithm?
- Suppose Jeff got $a|b|c$ as the answer to an addition problem. What answer should he have obtained using the standard algorithm? (This should be an expression involving a, b, and c.)

Another of your students, Alicin, is having trouble with subtraction. When she subtracts numbers, she doesn't regroup. Instead, she simply subtracts the smaller number in each column from the larger, as in the following.

$$\begin{array}{cccc} & 3 & 1 & 5 \\ - & 2 & 6 & 3 \\ \hline & 1 & 5 & 2 \end{array}$$

On two of her homework problems, Alicin obtained the answers $4\ \underline{9}\ \underline{5}\ 4$ and $2\ \underline{8}\ 1$, where the underlined digits are those she obtained by subtracting backwards.

- If Alicin had used the standard algorithm for subtracting with regrouping, then what answers should she have obtained?
- Suppose Alicin had obtained $a\ \underline{b}\ c$ as her second answer. What should she have obtained using the standard algorithm?
- Alicin needed to find the sum of the two answers above, so she added

$$\begin{array}{ccccc} & 4 & \underline{9} & \underline{5} & 4 \\ + & & 2 & \underline{8} & 1 \\ \hline & 4 & & \underline{3} & 5 \end{array}$$

and then got stuck. What belongs in the blank space?

Addition and Subtraction with Regrouping

In this activity, we will consider the standard algorithms for addition and subtraction, which are sometimes called addition and subtraction with regrouping.

1. To start, spend twenty to thirty minutes writing down as clear an answer as you can to the following question.

    > How do you know that when you add numbers like 187 and 246, addition with regrouping (or "carrying the ones") will give you the right answer? In other words, how can you justify addition with regrouping using what you know about addition and place value?

2. Now that you have tried your hand at answering the question above, read our answer to this question, which begins on the next page. (We should admit that we spent much more than thirty minutes writing our answer.)

3. Now that you understand addition with regrouping, answer (in two to four page write-up) the same question for subtraction with regrouping. Can you justify this algorithm in more than one way, as we have done for addition with regrouping?

In order to justify the algorithm, we have to understand how to perform it. So we start with a (somewhat dry) description of how to perform addition with regrouping.

1. Write the summands in a vertical list with ones places in the right most column, tens places in the next column to the left, etc. Write a horizontal line below the list.

2. Beginning with the ones column, do the following for each column.

   (a) Add the digits in the column.
   (b) Write the last digit of the sum below the line in the current column.
   (c) Write the number formed by the remaining digits above the next column to the left and move to that column.

3. The process ends when the current column contains no numbers. The sum is the number written below the line.

Here is an example in which we use addition with regrouping to compute 187 + 246.

$$\begin{array}{ccc} & & \\ 1 & 8 & 7 \\ 2 & 4 & 6 \\ \hline & & \end{array} \xrightarrow{7+6=13} \begin{array}{ccc} & 1 & \\ 1 & 8 & 7 \\ 2 & 4 & 6 \\ \hline & & 3 \end{array} \xrightarrow{1+8+4=13} \begin{array}{ccc} 1 & 1 & \\ 1 & 8 & 7 \\ 2 & 4 & 6 \\ \hline & 3 & 3 \end{array} \xrightarrow{1+1+2=4} \begin{array}{ccc} 1 & 1 & \\ 1 & 8 & 7 \\ 2 & 4 & 6 \\ \hline 4 & 3 & 3 \end{array}$$

Therefore, $187 + 246 = 433$. Of course, it is not good enough to just explain how to execute the algorithm. We also need to understand why the algorithm is correct. Below we give two different justifications for addition with regrouping. (Can you think of others?) Don't be discouraged if you have to read them several times before understanding them.

Justification by Laws of Arithmetic

Our first explanation uses the laws of arithmetic. Namely, we use the following facts which are true for any numbers a, b, and c.

1. Commutativity of Addition: $a + b = b + a$

2. Associativity of Addition: $a + (b + c) = (a + b) + c$

3. Distributivity of Multiplication over Addition: $(b + c) \cdot a = b \cdot a + c \cdot a$

We start by using ideas of place value to write out what each number is in powers of ten.

$$187 + 246 = (1 \cdot 10^2 + 8 \cdot 10 + 7) \quad + \quad (2 \cdot 10^2 + 4 \cdot 10 + 6)$$

Next we use the associative and commutative laws to regroup the terms according to their powers of ten. This corresponds to writing the summands in a vertical list and lining up the ones columns, tens columns, etc.

$$187 + 246 = (1 \cdot 10^2 + 2 \cdot 10^2) + (8 \cdot 10 + 4 \cdot 10) + (7 + 6)$$

Now we add $7 + 6 = 13$ in the right most term and we think of $13 = 10 + 3$. Using the associative law, we can group the 10 with the other tens. This corresponds to adding the ones column and carrying the one in the sum into the tens column. We end up with

$$187 + 246 = (1 \cdot 10^2 + 2 \cdot 10^2) + (8 \cdot 10 + 4 \cdot 10 + 1 \cdot 10) + 3.$$

Now we use the distributive law to pull out the common factor of 10 in the second to last term. We obtain

$$187 + 246 = (1 \cdot 10^2 + 2 \cdot 10^2) + (8 + 4 + 1) \cdot 10 + 3.$$

Now we add $8 + 4 + 1 = 13$ and we think of $13 = 10 + 3$ again. Using the associative law, we can group the $10 \cdot 10$ with the other 10^2's. This corresponds to adding the tens column and carrying the one in the sum into the hundreds column. We end up with

$$187 + 246 = (1 \cdot 10^2 + 2 \cdot 10^2 + 1 \cdot 10^2) + 3 \cdot 10 + 3.$$

As before, we use the distributive law to pull out the common factor of 10^2 and add the terms in the resulting coefficient, obtaining

$$187 + 246 = 4 \cdot 10^2 + 3 \cdot 10 + 3,$$

so the sum is 433. Strictly speaking, we have only shown that the algorithm works for this specific example, but it should be clear that the same justification will show that the algorithm works for any example.

Justification Using Manipulatives

Imagine that each of the summands stands for a certain number of blocks. According to our base ten system of writing numbers, every group of ten blocks is put together to form a rod, and every group of ten rods is put together to form a flat. So we may think of 187 as 1 flat plus 8 rods plus 7 blocks, and we may think of 246 as 2 flats plus 4 rods plus 6 blocks. The sum 187 + 246 corresponds to the number of flats, rods, and blocks we have when we can make no more legal combinations. That is, when we don't have enough blocks to make any more rods and we don't have enough rods to make any more flats.

Starting with the blocks, we see that we have 7 + 6 = 13 blocks total. We regroup ten of these blocks to form a rod, which we put with the rest of the rods, and we have three blocks left over. This corresponds to writing a 3 in the ones (blocks) column of our answer and carrying a 1 into the tens (rods) column.

Now we consider rods: we have 8 + 4 + 1 = 13 of them. We regroup ten of these rods to form a flat, which we put with the rest of the flats, and we have three rods left over. This corresponds to writing a 3 in the tens (rods) column of our answer and carrying a 1 into the hundreds (flats) column.

Finally we consider flats: we have 1 + 2 + 1 = 4 of them. We don't have enough to form a cube, so must be content with just having four flats left over. This corresponds to writing a 4 in the hundres (flats) column of our answer.

Putting all of this together, we see that we have 4 flats, 3 rods, and 3 blocks, so 187 + 246 = 433.

A Base Four Lesson

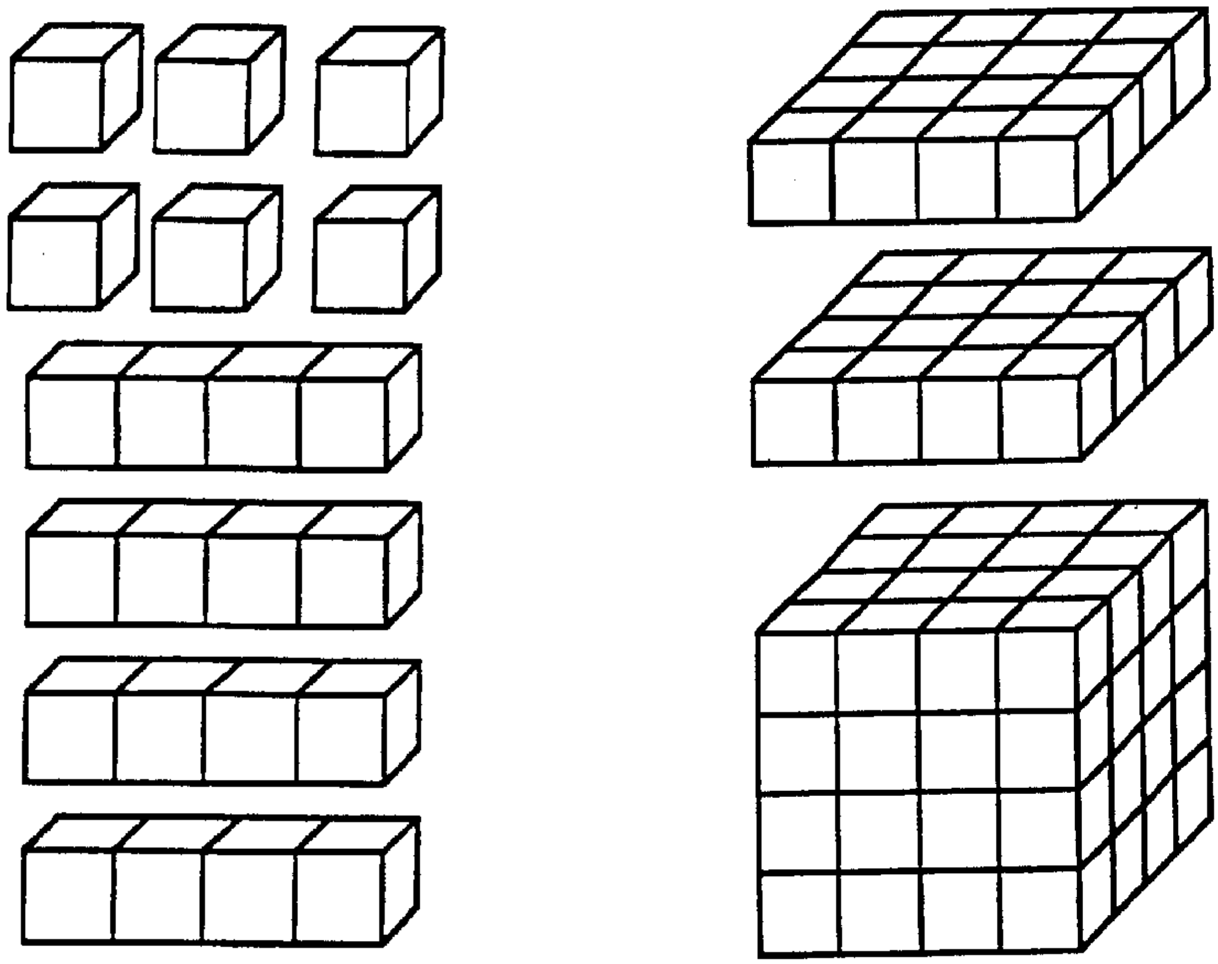
Representing numbers in different number bases and operating on (for example, adding) those numbers in different number bases can help your students to understand better ways of using different representations, and to see that (correct) algorithms for addition, subtraction, multiplication, and division work the same no matter what number base is being used.

You are to plan a lesson (not necessarily completable in one day) that makes use of base 4 blocks like those shown below (or other appropriate hands-on material). Your goals are:

1. Students will be able to represent numbers in base 4 and base ten and to translate from one of these number bases to the other one.

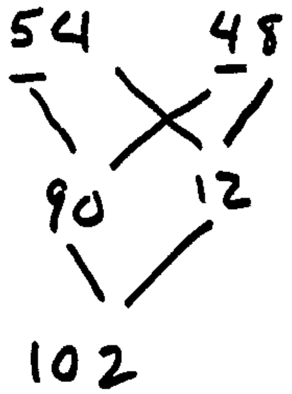
2. Students will be able to add two base 4 numerals correctly.

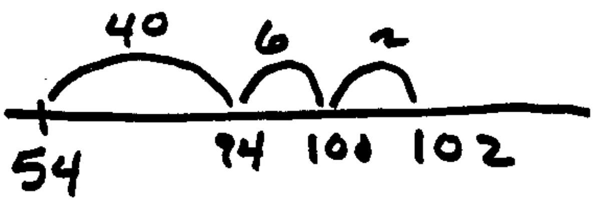
You may assume that students already know how to add in base ten. Be creative! Remember your target audience. Also be sure to specify grade level.



Ben and Kris's Arithmetic

The work below on the left is Kris's work to solve a certain arithmetic problem and the work below on the right is Ben's work on the same problem.





- What arithmetic problem are Ben and Kris solving?
- Is their work correct? If so, then justify it. If not, then correct it.
- Write a word problem which illustrates this arithmetic problem. Does your word problem still make sense if 54 and 48 are replaced by $53\frac{1}{2}$ and $48\frac{2}{5}$ respectively? If not, then write a word problem that does.

The Waupun License Plate Factory

Suppose there are two license plate factories at Waupun, each of which prints one thousand plates per day, including Saturdays, Sundays, and all holidays. The plant which opened first prints regular style Wisconsin license plates and the plant which opened second prints commemorative Wisconsin plates. The numbers on both styles of plate consist of three letters followed by three digits. Each factory prints their plates in numerical order, so that the first plate printed is AAA 000, the tenth printed is AAA 009, etc.

Pat was born on June 2, 1972 and her birthday is on her license plate, which is a regular style plate and reads JUN 272. Shanon's birthday is May 6, but he doesn't want to say what year. Suppose the plant printing commemorative plates began operation on the day Pat's plate was printed. Use addition and subtraction of license plates to answer the following questions.

- What commemorative style plates will be printed on the same day as Shanon's regular style birthday plate?
- What regular style plates will be printed on the same day as Shanon's commemorative style birthday plate?

Recently there has been a substantial increase in the available work force for license plate factories in Wisconsin. To take advantage of this, Tommy Senior and Tommy Junior have each decided to open their own license plate factory. They will print plates with alternating numbers and letters, such as A7B3C0D, and let the states who buy them print the state names on them. Tommy Junior plans to print plates in order, starting with A0A0A0A. Tommy Senior also plans to print plates in order, but he wants to print his birthday plate (S9E6P0T) exactly when Tommy Junior prints his (J5U8L3Y). If the factories start printing simultaneously, and always remain synchronized, then what plate should Tommy Senior print first? Be sure to justify your answer, including your calculations.

A Multiplication Problem

Some sixth grade teachers noticed that several of their students were making the same mistake in multiplying large numbers. In trying to compute

$$\begin{array}{rrrr} & 1 & 2 & 3 \\ \times & 6 & 4 & 5 \\ \hline \end{array}$$

the students seemed to be forgetting to “move the numbers” (i.e., the partial products) over on each line. They were doing this:

$$\begin{array}{rrrr} & 1 & 2 & 3 \\ \times & 6 & 4 & 5 \\ \hline & 6 & 1 & 5 \\ & 4 & 9 & 2 \\ & 7 & 3 & 8 \\ \hline 1 & 8 & 4 & 5 \end{array}$$

instead of this:

$$\begin{array}{rrrrr} & & 1 & 2 & 3 \\ & \times & 6 & 4 & 5 \\ \hline & & 6 & 1 & 5 \\ & 4 & 9 & 2 & \\ 7 & 3 & 8 & & \\ \hline 7 & 9 & 3 & 3 & 5 \end{array}$$

While these teachers agreed that this was a problem, they did not agree on what to do about it. What would you do if you were teaching sixth grade and you noticed that several of your students were doing this?

Lynna's Arithmetic

The following is Lynna's work to solve a certain arithmetic problem.

$$
\begin{array}{rcr}
544 & & \\
\underline{170} & & 10 \\
374 & & \\
\underline{170} & & 10 \\
204 & & \\
\underline{170} & & 10 \\
34 & & \\
\underline{34} & & \underline{2} \\
0 & & 32
\end{array}
$$

- What arithmetic problem is Lynna solving?
- Is her work correct? If so, then justify it. If not, then correct it.
- Write a word problem which illustrates this arithmetic problem. Does your word problem still make sense if 544 is replaced by $544\frac{2}{3}$ and 17 is replaced by $16\frac{4}{7}$? If not, then write one that does.

Arithmetic in Other Bases

If you haven't done so already, discuss whether (and how) the four basic arithmetic algorithms with which you're familiar extend to other bases. Then try to work out the problems below, *without* converting to base ten and back. Work directly in the given bases, and think about what you're doing.

1. (a) base six

$$\begin{array}{r} 4\;0\;4\;3 \\ +\;\;3\;1\;3 \\ \hline \end{array}$$

(b) binary (base two)

$$\begin{array}{r} 1\;1\;0\;0\;1\;1 \\ +\;\;1\;1\;1\;0\;1 \\ \hline \end{array}$$

(c) hexadecimal (base sixteen)

$$\begin{array}{r} 3\;\mathrm{A}\;\mathrm{E} \\ +\;\mathrm{B}\;0\;5 \\ \hline \end{array}$$

2. (a) base eight

$$\begin{array}{r} 2\;6\;1\;3 \\ -\;\;7\;0\;4 \\ \hline \end{array}$$

(b) binary (base two)

$$\begin{array}{r} 1\;1\;0\;0\;1\;1 \\ -\;\;1\;1\;1\;0\;1 \\ \hline \end{array}$$

(c) base eleven

$$\begin{array}{r} 2\;3\;\mathrm{X}\;5 \\ -\;\;6\;0\;6 \\ \hline \end{array}$$

3. (a) ternary (base three)

$$\begin{array}{r} 2\;0\;2\;1 \\ \times\quad 1\;2 \\ \hline \end{array}$$

(b) base nine

$$\begin{array}{r} 8\;8 \\ \times\;4\;6 \\ \hline \end{array}$$

4. (a) heximal (base six)

$$1\;3\;\overline{)\;5\;4\;3\;0}$$

(b) binary (base two)

$$1\;1\;0\;\overline{)\;1\;0\;1\;1\;0\;1}$$

Note: The character X represents ten in base eleven; in hexadecimal, the characters A, B, C, D, E, F represent the numbers ten through fifteen.

Double Negatives

Part of the point of this course is to examine some of the ideas we (and most other elementary school graduates) take for granted. One of these is the fact that the product of two negative numbers is a positive number. Of course, this fact was not just arbitrarily decided — it follows from the way we set up our number system.

Try to come up with a simple proof that $-1 \times -1 = +1$, using basic facts like the definition of -1 (you may have to think a minute about this), and additive and multiplicative identities.

Math 130
UNIT FIVE
Logic

Introduction to Propositional Logic

What does math have to do with logic, and why should we care about using math to speak about logic, instead of plain English? Have a look at the following two quotes:

Nothing is better than eternal happiness. A ham sandwich is better than nothing. Conclusion: A ham sandwich is better than eternal happiness.[1]	Every dog is an animal. An animal is in my front yard. Conclusion: Every dog is in my front yard.

The point of formal logic is to make mathematically precise the meanings of the terms *and*, *or*, *not* and *if*[2]. The symbols that stand for these terms are:

$$\wedge = \text{“and”}, \quad \vee = \text{“or”}, \quad \neg = \text{“not”}, \quad \ldots \to \ldots = \text{“if} \ldots \text{, then} \ldots \text{”}.$$

Here are a couple of examples of these terms in common language; for each example say what you think the answer is to the question that follows it, and whether it is ambiguous.

(A) You will send in your tax return by April 15 or pay a $50 late fee.

• Is it possible that both will happen: send in your tax bill by April 15 and pay a late fee?

(B) You will turn in your homework on time or receive an F on the assignment.

• Is it possible that both will happen: turn in your homework on time and receive an F on the assignment?

(C) Your lunch comes with soup and cole slaw or fries.

• Do you have to choose between getting fries and getting soup & cole slaw, or do you get soup for sure and a choice between fries and cole slaw?

Traditionally the letters $p, q, r, s, \ldots$ are used for the names of variables representing *propositions* — a proposition is a statement like “you received an F on the assignment” or “your lunch comes with soup”, which can have one of two truth values: True or False. Let's say for example that

$p =$ “You turn in your homework on time”
$q =$ “You receive an F on the assignment”

The proposition in example (B) is a compound proposition which we write as

$$p \vee q.$$

• Sometimes you need parentheses to make a compound proposition unambiguous. In the space below, write the compound proposition in example (C) by assigning $p =$ “you get soup”, $q =$ “you get cole slaw”, $r =$ “you get fries” and then using only the symbols $p, q, r, \wedge, \vee, \neg, \to,$ (and).

[1] Raymond Smullyan
[2] Later on we'll consider more potentially ambiguous terms like “nothing” and “an”.

Now try each of these. Make sure you state which simple propositions you assign to be p, q and r.

(D) Either I'm going crazy or the bank made a mistake and they owe me ten dollars.

(E) Either Dole or Forbes will get the Republican nomination, and Clinton will get the Democratic nomination.

(F) I'm not going home for Thanksgiving or for Christmas.

(G) If I study hard and the test is fair, then I'll pass the course.

(H) I'm going to take either Math 221 or Physics 201 but not both.

(I) It's not true that if you work 60 hours a week you'll get a raise.

Whether a compound proposition is true or false depends on whether each of the simple propositions it is built from is true or false. If you don't know whether each of the simple propositions is true or false, you can list all possible cases; this is called making a *truth table*. Let's take the compound proposition $p \vee q$ from example (B). How many possibilities are there for whether p and q are true or false? Make a table with a row for each possibility. The first possibility, namely that they're both true, has already been filled in.

p	q
T	T

Now put another column with $p \vee q$ at the top, and fill in each row with the correct truth value for the compound proposition $p \vee q$. The top row is filled in for you, indicating that when p and q are both true then $p \vee q$ is *by definition* true. The definitions may be intuitive to you, but if not, look them up on the definition page.

p	q	$p \vee q$
T	T	T

Now let's make a truth table for example (D). Let's say p = "I'm going crazy", q = "The bank made a mistake" and r = "The bank owes me ten dollars". The compound statment is then $p \vee (q \wedge r)$. There are going to be quite a few rows in the truth table, since there are quite a few possibilities for whether p, q and r are true or false. To help keep track of things, we'll make a column for the proposition $q \wedge r$ as well as for the whole proposition $p \vee (q \wedge r)$.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Does the truth table match the intuitive interpretation? Perhaps I am going crazy and the bank made no mistake and owes me no money. In this case we would say the sentence is true. This corresponds to the third line of the table. How about if I'm not crazy, the bank made a mistake, and it owes me ten dollars: this corresponds to the fifth line of the table and again we'd say the sentence was true. How about if I'm not crazy but the bank made no mistake and doesn't owe me money: then we'd say the sentence was false (last line of the truth table).

- Make a truth table for each of the propositions (E) – (I).

Definition Page

The proposition $p \wedge q$ is true only if p and q are both true. The proposition $p \vee q$ is true as long as at least one of p or q is true; the only case it is false is when p and q are both false. The proposition $\neg p$ (pronounced "not p") is true when p is false and false when p is true. The proposition $p \to q$ (pronounced "if p then q" or "p implies q") is true unless p is true and q is false. The truth tables for these four terms are as follows.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	$\neg p$
T	F
F	T

Whether or not these make intuitive sense to you, these are the definitions! To most people, the definitions of $\wedge$ and $\neg$ make sense, as does the definition of $\vee$ once you accept the principle that $p \vee q$ always allows both p and q to be true. The definition of $\to$ seems counterintuitive to some people. For example, if p = "Madison is in Minnesota" and q = "There are 105 senators", most people would probably argue that the statement $p \to q$ "If Madison is in Minnesota then there are 105 senators" is either false or nonsense. Mathematically, however, it is defined to be true, since both p and q are false. If you want all the definitions to seem more intuitive, try thinking of a contract. If you promise someone you will do $p \vee q$ and you do both, they certainly can't sue you for breach of contract. How about if you promise someone "if you do X then I will do Y". Under what conditions can they sue you for breach of contract?

Glicks, Glucks, Glacks, and Glocks

Read each of the following statements (1-4) and indicate TRUE, FALSE, or CAN'T TELL for each of the conclusions listed.

1. All **Glick** numbers greater than 20 are even.

Statement	True?	False?	Can't tell?
a. No odd number greater than 20 is a Glick.			
b. If an even number is greater than 20, then it is a Glick.			
c. No odd number is a Glick.			
d. No number less than 20 can be a Glick.			
e. All even numbers are Glicks.			
f. 33 is not a Glick.			

2. If a number is a multiple of 6 or is divisible by 7, then it is a **Gluck**.

Statement	True?	False?	Can't tell?
a. 17 is a Gluck.			
b. 42 is a Gluck.			
c. 38 is not a Gluck.			
d. No even number is a Gluck.			
e. No positive number less than 5 is a Gluck.			
f. All Glucks are divisible by 42.			

3. A **Glack** is an odd number that either is less than 29 or is a divisor of 48.

STATEMENT	TRUE?	FALSE?	CAN'T TELL?
a. All Glack numbers less than 15 are odd.			
b. 14 is a Glack.			
c. All odd numbers less than 21 are Glacks.			
d. 16 is a Glack.			
e. No odd number greater than 27 is a Glack.			
f. Nine Glacks are prime.			

4. All **Glock** numbers are divisible by 2 and are multiples of 13.

STATEMENT	TRUE?	FALSE?	CAN'T TELL?
a. 169 is a Glock.			
b. No prime number is a Glock.			
c. 52 is a Glock.			
d. All Glocks are even.			
e. All Glocks are divisible by 26.			
f. All numbers divisible by 26 are Glocks.			

5. EXTRA PRACTICE. Ben and Nancy have a daughter Tessa. They read to Tessa a lot, and
   - All of the books by Dr. Seuss which they own are among Tessa's favorites.
   - If they read a book to Tessa yesterday, then it was one of her favorites.
   - They own *Green Eggs and Ham* by Dr. Seuss and *Jamberry* by Bruce Degen.

Classify each of the following as TRUE, FALSE or CAN'T TELL:

(a) *Green Eggs and Ham* is one of Tessa's favorites.
(b) *Jamberry* is one of Tessa's favorites.
(c) They read *Green Eggs and Ham* to Tessa yesterday.
(d) They own all of Tessa's favorites.
(e) They own all the books by Dr. Seuss.

Propositional Logic Bingo

On the next seven pages are the playing cards for *Propositional Logic Bingo*. Each group should play a different card and get some counters to cover the squares on those cards. You can cover a square as soon as you have enough information to prove that the statement in the square is true.

Propositional Logic Bingo
Card One

$e \vee e$	$\neg z \to \neg w$	$(q \vee r) \to a$	$\neg t \vee p$	$\neg\neg h$
$\neg(\neg a \wedge p)$	$j \to \neg k$	$\neg(\neg\neg y)$	$\neg u$	$t \wedge (y \vee s)$
$\neg h \vee \neg x$	$f \to (r \to d)$	$p \vee \neg p$	$m \wedge (n \to m)$	$\neg g \to \neg g$
$z \to (v \wedge \neg b)$	$u \to v$	$(e \to q) \to q$	$d \vee u \vee \neg a$	z
$s \vee \neg\neg c$	$\neg a$	$w \wedge \neg g$	$\neg(b \wedge x)$	$i \to (t \wedge f)$

Propositional Logic Bingo
Card Two

$a \vee a$	$\neg w \rightarrow \neg f$	$(z \vee t) \rightarrow x$	$\neg i \vee v$	$\neg\neg q$
$\neg(\neg x \wedge v)$	$c \rightarrow \neg k$	$\neg(\neg\neg b)$	$\neg d$	$i \wedge (b \vee j)$
$\neg q \vee \neg s$	$h \rightarrow (t \rightarrow u)$	$v \vee \neg v$	$r \wedge (e \rightarrow r)$	$f \wedge \neg n$
$w \rightarrow (p \wedge \neg g)$	$d \rightarrow p$	$(a \rightarrow z) \rightarrow z$	$u \vee d \vee \neg x$	w
$j \vee \neg\neg m$	$\neg x$	$\neg n \rightarrow \neg n$	$\neg(g \wedge s)$	$y \rightarrow (i \wedge h)$

Propositional Logic Bingo
Card Three

$p \vee p$	$\neg g \rightarrow \neg x$	$(i \vee w) \rightarrow n$	$\neg(\neg\neg m)$	$\neg b \vee d$
$\neg(\neg x \wedge v)$	$f \rightarrow \neg r$	$\neg\neg h$	$\neg u$	$b \wedge (m \vee a)$
$\neg h \vee \neg v$	$c \rightarrow (w \rightarrow j)$	$\neg d \vee d$	$e \wedge (y \rightarrow e)$	$x \wedge \neg q$
$g \rightarrow (t \wedge \neg z)$	$u \rightarrow t$	$(p \rightarrow i) \rightarrow i$	g	$j \vee u \vee \neg n$
$\neg q \rightarrow \neg q$	$a \vee \neg\neg s$	$\neg n$	$\neg(z \wedge v)$	$k \rightarrow (b \wedge c)$

Propositional Logic Bingo
Card Four

$\neg x$	$\neg(\neg\neg b)$	$(z \vee t) \rightarrow x$	$\neg w \rightarrow \neg f$	$\neg\neg q$
$\neg(\neg x \wedge v)$	$c \rightarrow \neg k$	$\neg i \vee v$	$\neg d$	$i \vee (b \wedge j)$
$w \rightarrow (p \wedge \neg g)$	$h \rightarrow (t \rightarrow u)$	$t \vee \neg t$	$f \wedge \neg n$	$r \wedge (e \rightarrow r)$
$j \vee \neg\neg m$	$d \rightarrow p$	$(a \rightarrow z) \rightarrow z$	$\neg q \vee \neg s$	$\neg(g \wedge s)$
$u \vee d \vee \neg x$	$a \vee a$	$\neg n \rightarrow \neg n$	w	$y \rightarrow (i \wedge h)$

Propositional Logic Bingo
Card Five

$\neg s$	$\neg(\neg\neg g)$	$(w \vee f) \rightarrow s$	$\neg f \rightarrow \neg h$	$\neg\neg z$
$\neg(\neg s \wedge p)$	$m \rightarrow \neg d$	$\neg y \vee p$	$\neg u$	$y \vee (g \wedge c)$
$f \rightarrow (v \wedge \neg n)$	$q \rightarrow (i \rightarrow k)$	$i \vee \neg i$	$h \wedge \neg e$	$t \wedge (a \rightarrow t)$
$c \vee \neg\neg r$	$u \rightarrow v$	$(x \rightarrow w) \rightarrow w$	$\neg z \vee \neg j$	$\neg(n \wedge j)$
$k \vee u \vee \neg s$	$x \vee x$	$\neg e \rightarrow \neg e$	f	$b \rightarrow (y \wedge q)$

Propositional Logic Bingo
Card Six

$\neg v$	$\neg(\neg\neg z)$	$(g \vee x) \to v$	$\neg x \to \neg c$	$\neg i$
$\neg(\neg v \wedge t)$	$s \to \neg u$	$\neg k \vee t$	$\neg j$	$k \vee (z \wedge f)$
$x \to (d \wedge \neg q)$	$h \to (b \to r)$	$b \vee \neg b$	$c \wedge \neg y$	$w \wedge (p \to w)$
$f \vee \neg\neg e$	$j \to d$	$(n \to g) \to g$	$\neg i \vee \neg a$	$\neg(q \wedge a)$
$r \vee j \vee \neg v$	$n \vee n$	$\neg y \to \neg y$	x	$m \to (k \wedge h)$

Propositional Logic Bingo
Card Seven

$v \rightarrow \neg j$	$\neg(\neg\neg i)$	$(z \vee n) \rightarrow d$	$\neg n \rightarrow \neg f$	$\neg b \vee \neg p$
$\neg(\neg d \wedge w)$	$\neg d$	$\neg r \vee w$	$c \rightarrow (m \rightarrow e)$	$r \vee (i \wedge x)$
$n \rightarrow (u \wedge \neg h)$	$\neg a$	$m \vee \neg m$	$f \wedge \neg k$	$g \wedge (t \rightarrow g)$
$x \vee \neg\neg y$	$a \rightarrow u$	$(q \rightarrow z) \rightarrow z$	$\neg\neg b$	$\neg(h \wedge p)$
$e \vee a \vee \neg d$	$q \vee q$	n	$\neg k \rightarrow \neg k$	$s \rightarrow (r \wedge c)$

What's True About the Whole Numbers?

For this assignment, number means whole number; that is, a number is an element of the set $\mathbb{W} = \{1, 2, 3, 4, 5, ...\}$. Determine whether each of the following statements is true or false for these numbers. Explain your answers.

1. Every number is one less than some number.

2. There is a number that is not equal to the square of any number.

3. The product of any two numbers is a number.

4. For any two numbers there is a third number such that the first of the two numbers is equal to the second number less the third number.

Using Quantifiers

The quantifier $\forall$ means "for all"; it is called the *universal quantifier* because it claims that something is universally true. The quantifier $\exists$ means "there is"; it is called the *existential quantifier* because it claims that at least one example of something exists.

Examples:

i. The statement $\forall x,\ x + 27 > x$ claims that for every number x, if you add 27 to it, you get something bigger than the original x. This is true (say, on the whole numbers).

ii. The statement $\forall x,\ x > 14$ claims that all numbers x are greater than 14. This statement is false on the whole numbers because 1 through 13 are less than 14.

iii. The statement $\exists x,\ x > 14$ claims that there is some number x which is greater than 14. This statement is true on the whole numbers because there is at least one such x (in fact, there are many).

1. Write a sentence using quantifiers that states the associative law for multiplication. Is this sentence true on the set of whole numbers?

2. Write a sentence using quantifiers that states that every number has an additive inverse (i.e., an inverse for the operation $+$). Is this sentence true on the set of whole numbers?

3. Decide whether each of the following statements is true or false on the set of whole numbers. Explain your answers.

a. $\forall x,\ x + x > x$

b. $\exists c$ so that $c \times c = c + 42$

c. $\forall w,\ 1 + 2 = 3$

d. $\exists x$ so that $x + 1 < x$

e. $\forall x\, \exists y,\ x > y$

f. $\exists y\, \forall x,\ x > y$

g. $\exists s\, \forall t,\ s \times t = t$

h. $\exists m\, \forall n,\ m = n \times n$

i. $\forall u\, \forall z,\ u + z \geq u$

j. $\exists a\, \forall b,\ a \times b = a$

k. $\forall x\, \exists y\, \forall z,\ x \times z \neq y$

l. $\forall q\, \exists r\, \exists s,\ q = 3 \times r - 5 \times s$

4. Can you now go back and use quantifiers to rewrite the two quotes at the beginning of this unit (in the introduction) in a way that shows up the fallacies?

Math 130
UNIT SIX
Fractions

Law and Order

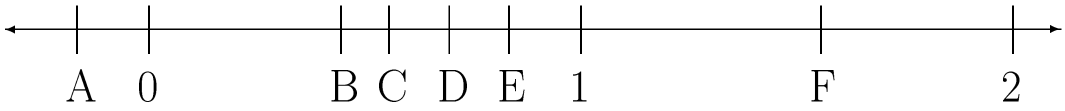
1. Suppose that $a > 1$, $0 < b < 1$, and $0 < c < 2$. Fill in each box with $<$, $=$, $>$, or NMI (Need More Information).

(a) $a \cdot b \; \square \; a$ (b) $b \cdot c \; \square \; a$ (c) $a \cdot b \cdot c \; \square \; b$

(d) $a + b \; \square \; a$ (e) $a + c \; \square \; a$ (f) $a + c \; \square \; b$

(g) $\frac{c}{c} \; \square \; c$ (h) $b^2 \; \square \; b$

2. Pictured below is a number line with some identified points on it. Use this number line to answer the questions in (a)–(e).



(a) If the numbers represented by the points D and E are multiplied, what point on the number line best represents this product?

(b) If the numbers represented by the points C and D are divided, what point on the number line best represents the quotient?

(c) If the numbers represented by the points B and F are multiplied, what point on the number line best represents the product?

(d) Suppose 20 is multiplied by the number represented by E on the number line. Estimate the product.

(e) Suppose 20 is divided by the number represented by E on the number line. Estimate the quotient.

The Condominium Problem

In an adult condominium complex, $\frac{2}{3}$ of the men are married to $\frac{3}{5}$ of the women. What portion of the residents are married? (Assume men are married only to women, and vice versa, and that married residents' spouses are also residents.)

Ratio and Proportion Problems

Below are a bunch of ratio and proportion problems suitable for elementary school kids. For each one, solve it in at least two different ways. You should be sure to clearly explain each method, why it works and why it makes sense. Avoid using standard algebraic methods (e.g. cross-multiply and divide), unless you can make sense of them. Focus on the *reasoning* necessary to solve each problem.

1. You're having a party, and have ordered pizza for everyone to share. There are 12 people at the party, and you ordered 4 pizzas. Three more people show up a little late, but before the pizza arrives. How much less pizza does each person get than you originally planned, now that the extra people are there?

2. Joan used exactly 15 cans of paint to paint 18 chairs. How many chairs can she paint with 25 cans?

3. John and Mary are making lemonade. John used 10 cups of lemon juice and 5 teaspoons of sugar. Mary used 7 cups of lemon juice and 3 teaspoons of sugar. Whose lemonade will be sweeter, or will they taste the same? If they don't taste the same, how can John adjust his recipe to make it taste the same as Mary's?

4. Your residence hall association is conducting a lottery as a fund-raiser. Being a loyal resident, and having some money to burn, you bought 15 tickets. Not everyone was so loyal, and only 88 tickets were sold. Your friend Sandy, who lives across campus, bought 12 tickets out of the 62 tickets that her dorm association sold. Who has a higher probability of winning? Why? What does probability tell you?

5. You've been looking at that new tape deck in the Happy Medium window for months. It's usually \$179, but today you saw it marked down to \$143. How much of a savings does that represent?

   If you have to pay 5.5% sales tax, would you prefer to have it calculated on the discounted price, or to have it added to the original price before the discount is taken? What if the discount was based on a percentage of the price, instead of a set dollar amount off?

6. Sally and Samantha share an apartment. Sally leaves for long weekends (she works at home to pay for school), so she is in the apartment only 4 days each week. Samantha leaves for one day each week. The heating bill comes to \$75 each week. How should they split the bill?

Look back over your work. How did your problem solving strategies change from problem to problem? Why did they change? How does this inform how you will teach fractions?

Fractions!

In this worksheet contains several manipulations involving fractions. You should concentrate not just on *how* to complete each task but *why* you do it that way. Each step should make sense, and you should be prepared to explain it to kids who have little or no experience with fractions.

1. Explain how to convert mixed numbers to fractions, and improper fractions to mixed numbers. Explain what "lowest terms" are, and how to reduce fractions to that state.

2. Expain how to add any two fractions (including mixed numbers and improper fractions). Include an explanation of *why* each step makes sense. You may refer to #1 in your explanation.

3. Expain how to subtract any two fractions (including mixed numbers and improper fractions). Include an explanation of *why* each step makes sense. You may refer to #1 and #2 in your explanation.

4. Expain how to multiply any two fractions (including mixed numbers and improper fractions). Include an explanation of *why* each step makes sense. You may refer to #1,#2, and #3 in your explanation.

5. Expain how to divide any two fractions (including mixed numbers and improper fractions). Include an explanation of *why* each step makes sense. You may refer to #1 - #4 in your explanation.

$\frac{1}{3}$ and $\frac{2}{5}$

The following problems all involve the fractions $\frac{1}{3}$ and $\frac{2}{5}$; however, each has different answer. Why?

1. In your fourth grade class, $\frac{1}{3}$ of the students voted in the class elections. $\frac{2}{5}$ of the votes in your class were for Tammy. What proportion of the class voted for Tammy?

2. In the same class, $\frac{1}{3}$ of the boys get free lunch and $\frac{2}{5}$ of the girls do. What proportion of the class gets free lunch?

3. In the same class, $\frac{1}{3}$ of the students have a TI 82 and $\frac{2}{5}$ of them have a TI 86. Nobody has both, and nobody has any other graphing calculator. What proportion of the class has a graphing calculator?

4. At a regional math contest, your top student won a TI 92. Since she didn't previously own a graphing calculator, the proportion of the class which has one has changed. What is it now?

5. In your class, $\frac{1}{3}$ of the students remembered to turn in their field trip permission slips. In the other fourth grade class, $\frac{2}{5}$ returned theirs. If both classes have the same number of students, what proportion of the fourth graders remembered to turn in their permission slips?

Common Fractions & Repeating Decimals

1. Write down any non-integer rational number both as a common fraction (i.e., in a/b form) and as a decimal. How can you prove that these two representations of the rational number are the same?

2. Can $\frac{1}{7}$ be written as a repeating decimal? How do you know? How about $\frac{5}{17}$?

3. Now try to write each of the following decimals as a common fraction.

   (a) 0.432

   (b) $0.\overline{123} = 0.123123123...$

   (c) $0.\overline{1}$

   (d) $0.0\overline{1}$

   (e) $2.2\overline{43}$

   (f) $0.939339333933339...$

4. Can all common fractions be written as repeating decimals? Can all repeating decimals be written as common fractions? Explain how you know in each case, and tell how to do so if it's possible.

Going to Extremes

1. Given your knowledge about the correspondence between common fractions and repeating decimals, can you write $0.\overline{9} = 0.99999...$ as a common fraction?

2. Is there a largest...

   (a) integer?

   (b) real number?

   (c) number less than 1?

3. Is there a smallest positive number (greater than zero)?

Math 130
UNIT SEVEN
Number Theory
& Number Systems

Factors & Prime Numbers

1. You may have heard the terms *prime* and *composite* before. These words talk about how numbers can be broken down in terms of *factors.* Factors are numbers (usually whole numbers) which divide evenly into a given number — or, to put it another way, numbers which are multiplicative building blocks of larger numbers. For example, the factors of 6 are 1, 2, 3 and 6, because we can write $1 \times 6 = 6$ and $2 \times 3 = 6$. The factors of 49 are 1, 7 and 49 (we don't list 7 twice). A number is *prime* if its only factors are 1 and itself. The first prime number is 2 (we don't count 1). A number is *composite* if it is not prime (again, we usually do not classify 1 in either category).

- What is the first composite number?
- Are there any other even primes besides 2?
- List the first ten prime numbers.
- If p is a prime number, what are the factors of p^4?

2. The *greatest common factor*, or GCF, of two numbers, is the largest factor they share in common. GCF(12,18)=6. The *least common multiple*, or LCM, of two numbers is the smallest number which is a multiple of both numbers. LCM(12,18)=36. Finally, two numbers are said to be *relatively prime* if their GCF is 1 — in other words, they have no common factors (except 1), so relative to each other, they appear to be prime. 12 and 18 are clearly not relatively prime, but 10 and 21 are relatively prime even though neither number individually is prime.

- Can you find two numbers whose GCF is 10? Can you find two numbers whose LCM is 10?
- Can you find a relationship between two numbers a and b and GCF(a, b) and LCM(a, b)?
- Can you find two numbers, each larger than 100, which are relatively prime?
- These definitions can be extended to larger sets of numbers. What would GCF(4,8,10) be? GCF(4,8,11)? What about LCM(4,8,10)? LCM(4,8,11)? Is either of these sets of numbers relatively prime?

Related Problems

1. Can you find a number smaller than 100 with exactly one factor? Exactly two factors? three factors? What is the smallest number n for which no number smaller than 100 has exactly n factors?

2. What number less than 1,000 has the most factors?

3. Jupiter orbits the sun once every 12 years. Saturn orbits the sun once every 30 years. They were in the same position in the sky in 1978. When will they next be in the same position?

4. The U.S. Census Clock in Washington, D.C. has signs with flashing lights to indicate gains and losses in population. Here are the time periods of these flashes in seconds: birth, 10; death, 16; immigrant, 81; emigrant, 900. In other words, every 10 seconds there is a birth, and so on.

(a) If the immigrant and emigrant signs lit up at the same time, how long will it be before they light up simultaneously again?

(b) What is the increase in population during a one-hour period?

The Stamps Problem

Suppose you have an unlimited supply of 4 cent stamps and 9 cent stamps. What amounts *CAN'T* you make with these stamps? What amounts can't you make with a supply of 9 cent stamps and 21 cent stamps? Can you make any generalizations about what can and cannot be made with a supply of a cent stamps and b cent stamps?

The Locker Problem

The lockers at Martin Luther King Middle School are numbered from 1 to 100. (It's a small school.) On Sunday morning the janitor opens all of the lockers. Then another janitor comes through and closes every second locker (that is, those numbered 2, 4, 6, ...). Then the first janitor goes through and changes every third locker: if it is open s/he closes it and if it is closed s/he opens it. Then the other janitor changes every fourth locker, and so on. At the last stage, one of the janitors changes every 100th locker (just the last locker, in other words). Which lockers are open at the end of all of this opening and closing of locker doors?

Patterns That Aren't

One of the most important problem-solving skills in mathematics is the ability to recognize a pattern. It is also important, however, to be able to prove that a pattern is true once one has recognized it. Showing that the pattern works for a few different values of the variable in question is *not* the same as showing that the pattern always holds, and this is an important difference. To show that a pattern holds for all values of a variable n, one either proves that the pattern holds using n in place of a particular number, or using a technique called induction.

Here is an example of a pattern that appears to hold for a while, but fails further down the line than one might normally think to check. It involves what are called *cyclotomic polynomials.* These are polynomials of the form $x^n - 1$. Let's see what happens when we factor these polynomials over the integers (or the rationals):

$$\begin{aligned}
x^1-1 &= (-1+x) \\
x^2-1 &= (-1+x)\,(1+x) \\
x^3-1 &= (-1+x)\,(1+x+x^2) \\
x^4-1 &= (-1+x)\,(1+x)\,(1+x^2) \\
x^5-1 &= (-1+x)\,(1+x+x^2+x^3+x^4) \\
x^6-1 &= (-1+x)\,(1+x)\,(1-x+x^2)\,(1+x+x^2) \\
\ldots & \\
x^{104}-1 &= (-1+x)\,(1+x)\,(1+x^2)\,(1+x^4)\times \\
&\quad (1-x+x^2-x^3+x^4-x^5+x^6-x^7+x^8-x^9+x^{10}-x^{11}+x^{12})\times \\
&\quad (1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}+x^{11}+x^{12})\times \\
&\quad (1-x^2+x^4-x^6+x^8-x^{10}+x^{12}-x^{14}+x^{16}-x^{18}+x^{20}-x^{22}+x^{24})\times \\
&\quad (1-x^4+x^8-x^{12}+x^{16}-x^{20}+x^{24}-x^{28}+x^{32}-x^{36}+x^{40}-x^{44}+x^{48})
\end{aligned}$$

It appears to be true that if we factor a cyclotomic polynomial, we get factors in which all the coefficients (of powers of x) are either 0 or ± 1. You can check this for any of the polynomials left out above (where n is between 6 and 104). You'd think that if you checked the pattern over 100 times, surely any anomalies would have shown up by now. But you'd be wrong! Take a look at the coefficients of x^7 and x^{41} in the last factor of $x^{105} - 1$:

$$\begin{aligned}
x^{105}-1 = {} & (-1+x)\,(1+x+x^2)\,(1+x+x^2+x^3+x^4)\,(1+x+x^2+x^3+x^4+x^5+x^6)\times \\
& (1-x+x^3-x^4+x^5-x^7+x^8)\,(1-x+x^3-x^4+x^6-x^8+x^9-x^{11}+x^{12})\times \\
& (1-x+x^5-x^6+x^7-x^8+x^{10}-x^{11}+x^{12}-x^{13}+x^{14}-x^{16}+x^{17}-x^{18}+x^{19}-x^{23}+x^{24})\times \\
& (1+x+x^2-x^5-x^6-\mathbf{2}x^7-x^8-x^9+x^{12}+x^{13}+x^{14}+x^{15}+x^{16}+x^{17}-x^{20}-x^{22}-x^{24} \\
& -x^{26}-x^{28}+x^{31}+x^{32}+x^{33}+x^{34}+x^{35}+x^{36}-x^{39}-x^{40}-\mathbf{2}x^{41}-x^{42}-x^{43}+x^{46}+x^{47}+x^{48})
\end{aligned}$$

Those -2's ruin a perfectly good pattern, and in fact has some consequences in abstract algebra. So you see that a pattern that appears to be, may in fact be a pattern that isn't. Only a real proof using either a general case for n or induction can tell for sure.

For the following problem, try first to find the pattern, and then to decide whether or not the pattern really continues *ad infinitum.*

Definition. A *Mersenne prime* is a prime number which can be written as $2^p - 1$, where p is a prime. The first Mersenne prime is $2^2 - 1 = 3$. However, not all numbers $2^n - 1$ are primes; if we let $n = 4$, we see that $2^4 - 1 = 15 = 3 \times 5$ is composite rather than prime.

Explore the relationship between the primeness of n and the primeness of $2^n - 1$.

Counting Blocks

The blocks in your classroom come in packages of 72, and are of three colors: red, green, and blue. (Not all of the blocks in each package are the same color.) One day you had your students count how many of each color of block you have and find the total. They wrote the result on the board, but later in the day some of the digits were erased. Here is what remains; the stars stand for missing digits.

red		2	*	0
blue		7	2	*
green	*	*	*	*
total	*	7	5	*

After counting, the children arranged the red blocks in a cube with a square sitting on top of it. They also arranged the blue blocks into one large square. How many of each color of blocks do you have? What if the blocks came in packages of 22 instead of 72?

$\sqrt{2}$

This activity presents a proof that $\sqrt{2}$, the square root of 2, is not a rational number. This is important because it proves that *irrational* numbers exist — that is, real numbers which are not rational numbers. ($\pi \approx 3.1415926535$ is another example of an irrational real number.)

First recall the following:

Definitions. Let m and n be whole numbers. We say that m is a *factor* of n if there is a whole number x such that $n = mx$. When m is a factor of n, we also say that n is a *multiple* of m.

Examples. 2 is a factor of 312. 3 is a factor of 39. 7 is a factor of 91. 124 is a multiple of 4. 65 is not a multiple of 4. 365 is not a multiple of 9.

Now answer the two following diagnostic true-false questions:

1. If 2 is a factor of r^2, then 2 is a factor of r. TRUE FALSE

2. If 4 is a factor of r^2, then 4 is a factor of r. TRUE FALSE

Here is the proof. Your job is to try to understand it. Keep in mind that in a proof there are generally two things to keep in mind: (a) the overall form of the argument, and (b) how each step follows from the ones before it.

Theorem. $\sqrt{2}$ is irrational.

PROOF. Suppose that $\sqrt{2}$ is not irrational; that is, suppose that $\sqrt{2}$ is a rational number.

Then we can write $\sqrt{2} = \frac{a}{b}$, where are a and b are whole numbers, $b \neq 0$, and $\frac{a}{b}$ is a reduced fraction — that is, in lowest terms (a and b are relatively prime).

When we multiply both sides of the equation by b, we get $\sqrt{2}\,b = a$, and thus $2b^2 = a^2$.

Therefore, 2 is a factor of a^2. (Why?)

This implies that 2 is a factor of a (i.e., that a is even). (Why?)

Since 2 is a factor of a, we can write a as $2c$ where c is a whole number. Now we have that

$$2b^2 = a^2 = (2c)^2 = 4c^2.$$

Thus, $b^2 = 2c^2$, and so 2 is a factor of b^2. As a result, 2 is a factor of b. (Why?)

But this is a contradiction since we assumed that the fraction was in lowest terms and, thus, that a and b had no common factors. So our original assumption that $\sqrt{2}$ is rational must be incorrect; hence $\sqrt{2}$ is irrational.

Questions for Reflection

1. Give explanations for the two diagnostic questions on the previous page.

2. Write down reasons for those steps in the proof where "Why?" is asked.

3. The overall form of this proof is called *proof by contradiction*. How does a proof by contradiction go?

4. Which steps of this proof are hard to follow? Be specific.

5. Could you adapt this proof to show that $\sqrt{3}$ is irrational? What about $\sqrt{4}$? If not, where would the proof fall down?

Life in Hell

1. You are in hell. The Devil challenges you to a game. When you arrived in hell, he randomly chose a whole number and assigned it to you. Once each day you will have a chance to guess your assigned number. If you guess your number, the Devil will let you out of hell. Will you get out? (Remember you have an eternity!) Explain why you will or will not get out.

2. What if, instead, the Devil randomly picks an integer and assigns it to you when you arrive. Will you get out of hell? Explain.

3. What if the Devil randomly chose a rational number?

4. What if the Devil randomly chose a real number?

Comparing Number Systems

Let $\mathbb{W} = \{1,\ 2,\ 3,\ ...\}$ be the set of whole numbers, $\mathbb{N} = \{0,\ 1,\ 2,\ 3,\ ...\}$ be the set of natural numbers, $\mathbb{Z} = \{...,\ 3,\ 2,\ 1,\ 0,\ 1,\ 2,\ 3,\ ...\}$ be the set of integers, $\mathbb{Q}$ be the set of rational numbers, and $\mathbb{R}$ be the set of real numbers. For which of these number systems are the following statements true?

STATEMENT	$\mathbb{W}$	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$
1. $\forall x,\ \exists y$ such that $x + y = 0$					
2. $\forall x \neq 0,\ \exists y$ such that $x \cdot y = 1$					
3. $\exists y\ \forall x,\ x \cdot y = 0$					
4. $\exists x,\ x \cdot x = 2$					
5. $\exists y,\ y \cdot y = -2$					
6. $\forall x,\ \forall y > x,\ \exists z$ such that $x < y < z$					
7. $\forall w,\ w + 12 = w$					
8. $\exists z$ such that $\forall y,\ z = y$					
9. $\forall x,\ x \cdot x \geq x$					
10. $\exists x\ \forall y,\ x + y = x$					

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