GUIDELINES FOR DISCOVERY TEACHING

A Handbook for Teachers of General Mathematics and Prealgebra in Grades 9-12.

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Helen Smiler, Project Director
Project SEED, Inc.
2336A McKinley
Berkeley CA 94703

644-3422

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FOREWORD

Since 1963, mathematicians and scientists trained by Project SEED have been using a Socratic, group discovery method of instruction to teach abstract, conceptually-oriented mathematics (topics from high school and college level algebra) to elementary school children from educationally disadvantaged backgrounds. Not only were the students successful at learning algebra, but their basic arithmetic scores improved dramatically.

The basic Project SEED group discovery techniques were initiated by William F. Johntz, the founder and National Director of Project SEED. Over the years they have been refined by SEED specialists throughout the country into a powerful tool for teaching mathematics. Although the methods have been used primarily with elementary school students, they have implications for teaching students at all levels.

Project SEED techniques are designed to overcome the low motivation and feeling of academic inferiority that are common among students from low socioeconomic backgrounds. They are designed to maximize student involvement in the development of the mathematics being taught. Project SEED mathematicians concentrate on making students feel good about their ability to do mathematics and want to participate in each lesson. There also is an emphasis on developing the students' critical thinking and problem solving abilities.

During the school years 1980-81 and 1981-82, under contract with the U.S. Department of Education, Basic Skills Improvement Program, Project SEED staff in Atlanta, Boston, Chicago, Detroit, Los Angeles, and Oakland worked with ninth through twelfth grade students and their teachers in Title I eligible areas in an attempt to modify the SEED methods for use with secondary school students. Our work primarily focused on general mathematics[*] and prealgebra classes. Students in these classes usually have fallen behind the norm and seem to be trapped by their own feelings of academic inferiority. On the whole, our efforts were quite successful in providing these students with a successful learning experience. We remain, however, firmly convinced that it would be far more effective to work with these same students in elementary school, before their attitudes about their academic abilities have set so firmly.

The present manual, "Guidelines for Discovery Teaching," is the result of this experience with secondary school students. It is written for teachers of general mathematics and prealgebra classes, although it contains many suggestions that apply to other levels and other subject areas. It is designed to capture the essential elements of discovery teaching which we found successful with students in grades nine through twelve.

Usually mathematicians who are trained to teach by the Project SEED method learn their techniques through a procedure of observation, discussion, and supervised teaching over an introductory period of several weeks and continuously throughout their tenure as SEED specialists. Each mathematician who teaches SEED group discovery classes has learned the process through modeling and oral tradition. Experienced SEED specialists are successful because they have imitated a successful SEED specialist.

In using these notes to adopt discovery teaching in your classroom, you will not have this luxury. It is unlikely that there will be an experienced SEED specialist to observe or to give you feedback on your teaching. We recommend, therefore,

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[*] There are a variety of titles, such as "Math Workshop," "Consumer Math," and "General Mathematics," given to classes for secondary students who have not yet mastered the basic arithmetic skills to enroll in prealgebra or algebra. General Math is used throughout to refer to these classes.

that you begin by reading the entire manual quickly. This will give you an overall picture of the method that would have been provided by observation.

Begin discovery teaching by picking your own topic or using one of the Curriculum Modules developed by Project SEED. Go back through this manual and pick out one or two key techniques from each chapter to incorporate into your lesson. As they become part of your permanent repertoire, experiment with additional techniques. Use the checklist in the Appendix as a reminder to keep your methods varied.

We hope this manual will become a guidebook leading both you and your students to experience greater success.

CHAPTER 1

INTRODUCTION AND OVERVIEW

Introduction

If you were to observe a successful group discovery class, you would witness both the instructor and the students actively involved in the learning process. Students discover mathematical concepts through answering the instructor's questions. There is a positive atmosphere in which there are no "wrong" answers, and students who ask questions often receive more recognition than students who answer them. Students are confident of their ability to learn and frequently debate with the instructor or their peers.

To help orient the reader, we begin with an illustration. The following lesson is typical of hundreds of SEED lessons over the past nineteen years with students traditionally labeled "educationally disadvantaged" because of their socioeconomic and achievement levels.

The lesson began with some review material, which was designed to prepare a receptive mood for the central question "$2E{}^{-}1=?$" ("E" here is used for the operation of exponentiation so that $2E3=2x2x2=2^3$.) The class had never considered questions on negative exponents before although, as shown by the review questions, they were familiar with the additive law for E and also with addition of integers.

When the instructor invited conjectures on $2E{}^{-}1$ he received a number of answers, but most favored $2E{}^{-}1=1$. The students' arguments in favor of this answer were various and interesting. The one which seemed to have the most support turned about the similarity between "$2+{}^{-}1$" and "$2E{}^{-}1$". Then one student suggested using "$2E{}^{-}1$ in a sentence: $(2E{}^{-}1)$ x $(2E2)=2E({}^{-}1+2)=2E1$. From this the class derived, in the usual way, that $2E{}^{-}1$ was acting like $\frac{1}{2}$: $\begin{array}{ccccc} (2E{}^{-}1) & \times & \underline{(2E2)} & = & \underline{2E1} \\ \downarrow & & \downarrow & & \downarrow \\ \boxed{\frac{1}{2}} & \times & 4 & = & 2 \end{array}$

This result provoked a truly excellent debate, with some students arguing for the previous answer ($2E{}^{-}1=1$), and other students for $2E{}^{-}1=\frac{1}{2}$. One student devised a further argument for $2E{}^{-}1=\frac{1}{2}$ by pointing out the pattern

$$\begin{array}{l} 2E3=8 \\ 2E2=4 \\ 2E1=2 \\ 2E0=1 \\ 2E{}^{-}1=\frac{1}{2} \end{array}$$

Throughout the lesson there was a great deal of student dialogue, which the instructor handled in a purely non-legislative fashion--being careful to keep open the status of $2E{}^{-}1$.

At one point in the discussion, a student was insisting that "2 and $^{-}1$ make 1, not $\frac{1}{2}$. And $\frac{1}{2}$ doesn't make sense." Whereupon another student replied that "2 and $^{-}1$ are 1 in addition, but not in exponentiation.

In the above lesson, the students were thinking critically about mathematics, exchanging ideas and listening to each others' opinions. There are numerous strategies and techniques which are used to establish this classroom situation.

The techniques fall roughly into four major categories which are described briefly below. The remaining chapters of these guidelines elaborate more fully on each of the categories, and contain detailed suggestions on how to successfully incorporate discovery teaching into the teaching of general mathematics and other subjects. Specific techniques often accomplish more than one purpose; however, to avoid repetition, we have tried to place each where it is used most frequently.

Key Elements of Discovery Teaching

There are four main elements of discovery teaching as used by Project SEED.

1. Questioning - Virtually all the dialogue in SEED classes consists of instructor questions and student responses, except, of course, when the instructor responds to a student's questions with another question. Structured sequences of questions lead the class to an understanding new mathematical concepts. Other questions review and reinforce previously learned material. Chapter 2 is devoted to general strategies for presenting material without lecturing.

2. Advanced, conceptual mathematics - Students often view mathematics as a sequence of arbitrary, meaningless rules to be memorized. It's no wonder they often don't retain what they learn. Conceptual understanding is an essential component of discovery teaching. Students who thoroughly investigate and understand key examples are better able to retain what they've learned and apply it to new situations.

   Frequent repetition of material leads to boredom, resentment, and perhaps discipline problems, even when students have not mastered the material. This is particularly true of general mathematics students, who know that the material they are covering is normally taught in 6th or 7th grade. SEED instructors choose advanced topics which embed and reinforce elementary ones (or teach elementary concepts from an abstract, conceptual point of view). Because the material is fresh, they are able to capture the students' interest, while at the same time reinforcing basic skills.

Chapter 3 contains techniques for structuring questions to develop conceptual understanding. The reader is also referred to the curriculum modules which are a companion to these guidelines. The modules contain outlines illustrating in detail the Socratic development of selected topics.

3. Feedback and involvement - The instructor constantly uses a variety of techniques to monitor student understanding and to maintain a high level of participation. The teacher uses information about the class level of understanding in formulating the next question. Feedback and involvement techniques are found in Chapter 4.

4. Positive, supportive atmosphere - The previous three characteristics are combined with additional techniques for supporting and motivating students to present an atmosphere in which there is a strong individual and group sense of achievement and success. There are no wrong answers. Students feel that they can learn and they do. Chapter 5 describes techniques for building student confidence.

A Note on Large Group Instruction

The techniques contained in this manual deal primarily with whole group instruction. While individual practice (seatwork and homework) are clearly necessary to reinforce skills, we have found that the discovery process for learning mathematics works most effectively with an entire class, particularly in courses like general mathematics, where the students lack confidence in their own academic abilities.

Some reasons for this somewhat surprising phenomenon are:

1. Frees the student to think.

   A group of sufficient size (20-30 students) can create an illusion of anonymity for the student, freeing him or her* to think creatively. In a one-to-one relationship with the teacher, or in a small group of three or four, the student is put on the spot when he is asked a question that calls upon him to think or reason. Most people tend to freeze--or at least to become anxious--when asked questions that require them to think. To be singled out can only intensify the student's anxiety. The large group enables him to feel less exposed, less vulnerable. The sensitive teacher can help an insecure student relax in a large group: He can supportively "ignore" him for a few minutes by turning his attention to other students giving him the time he needs to unfreeze and think through a question, a procedure that would obviously be impossible in a one-to-one or small group situation, since ignoring the student under those circumstances, however benign it might be, would be obvious to the point of seeming accusatory.

2. Students more likely to perform.

   By the same token, the group situation, by providing the student with a sense of anonymity, tends to destroy in him any lingering passivity. Students can be asked to respond in chorus and to use hand symbols and gestures in large groups. Such requests would seem foolish in a small group or the one-to-one relationship, merely

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*There is no acceptable convention for dealing with the third person singular. In these notes, we occasionally use the awkward "he or she," but generally arbitrarily choose one gender or the other.

causing the student to feel embarrassed. Moreover, there are ways in which the skillful teacher can draw out a truly reluctant student in a group situation that would be impossible if she were alone with him. (See Chapter 5)

3. <u>Student's self-concept enhanced</u>.

   When she responds in a group situation, the student's self-concept is much more likely to be enhanced than when she responds individually to the teacher. First of all, the group situation makes it possible for the teacher to use subtle, indirect methods of involving the student in the learning process. Once involved, the student feels better about herself, and her self-concept inevitably is enhanced as a result.

   Second, students are far more eager for peer approval than for teacher approval. The skillful teacher who provides students with an exciting group learning experience, one in which becoming involved is "the thing to do," creates in the reluctant learner a need to participate in order to seek peer approval. Once he knows he has earned peer approval by performing well, his self-concept has been even further enhanced and he is motivated to perform even better.

4. <u>Group discussions are more productive</u>.

   The richness of a discussion among students in a large group is far more productive than what is ever possible in a small group or between the teacher and one student, simply because there are more people contributing their ideas. When students discuss particularly difficult subjects, such as mathematics, a group can often solve together a problem that no single one of them could solve alone. Such a shared experience of success by a group adds a dimension to learning impossible to duplicate in the small group or one-to-one setting.

5. <u>Students like group learning</u>.

   Watch students in a well-taught group learning situation. They learn and interact with zest and enthusiasm impossible to duplicate in an individualized setting. Students <u>like</u> to compete with their peers; they become truly involved in the content of the discussion and will often spend time outside of class "researching" a particular point in order to contribute to a later discussion.

CHAPTER 2

GENERAL SOCRATIC STRATEGIES

Effective Questioning Techniques

Good questions are at the heart of the discovery method. Carefully sequenced sets of questions enable students to understand extremely sophisticated material. The most successful discovery teachers only ask questions and rarely or never make declaratory statements in class about anything more serious than the weather.

Questions allow the instructor to gain immediate feedback from the students and, thus, to pace the introduction of new material appropriately. Students become active rather than passive participants in the learning process. They are focused on the topic at hand and begin to develop a framework for problem solving on their own.

The following suggestions will help you design question sequences that stimulate true discovery and critical thinking in a group setting.

1. Write out your sequence of questions in preparation for the class. This will help you find some of the areas of difficulty. You can then write a sub-routine of questions that will help ease the students over the difficult spots. A useful exercise is to consider what questions you would need to ask if you, yourself, were learning the topic.

2. Keep a log. Because discovery teaching moves with the response of the class, it is imperative to keep a record of the day's lesson. Many instructors have a student take notes. Others spend a few minutes annotating the day's plan after class and use that as a springboard for planning the next day's lesson.

3. <u>Vary the difficulty of questions</u>. In order to keep students of different abilities constantly involved in the lesson, you must ask a mixture of hard and easy questions. Challenging questions keep the faster students interested, while routine questions give the slower students a chance to respond and build confidence.

Examples:

Problem	Routine Questions	Challenging Questions
$\frac{1}{2} + \frac{1}{4} = ?$	Who can find an equivalent fraction for $\frac{1}{2}$ that we can add to $\frac{1}{4}$?	Who can find two different fractions whose sum is $\frac{1}{2}$?
	What's $\frac{2}{4} + \frac{1}{4}$?	(Is $\frac{3}{6} + \frac{2}{8}$ a legitimate answer?)
	Who can draw a picture to illustrate this problem?	
$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$	What's the next question?	What problem would we solve to find S_{10}?
$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$	What problem do I have to solve to find S_4? (All the steps to computing $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$)	Find a formula for S_n.
	Who can predict S_5?	

3. <u>Vary the question pace</u>. A varied pace helps avoid monotony and maintains student interest. Pace will also depend on the degree of difficulty of the question being asked.

Alternate fast-paced series of straightforward questions with slower conceptual discussions. Most discovery classes begin and end with a quick pace. Fast-paced questions are also useful when reviewing, building up to a generalization through examples, or practicing an idea the class has just verbalized. Have the class chorus their answers when you really want to move fast. Using quick consecutive chorus responses is a very effective device for focusing a class. Appropriate times for a more relaxed pace include: when the class is discovering a key concept, when students are attempting to articulate thoughts, when they are checking a result on their papers, and when a student is teaching.

5. Vary the mode of response. This again will avoid monotony while involving more students. In addition to calling on an individual, you can: ask the whole class to say an answer, ask a particular group of students to say it, have the students write the answer on their papers, or have them use their hands to express an answer or to signal agreement or disagreement with someone else's answer. In order to avoid confusion and called out answers, it is advisable to make it clear to the class, when you ask the question, how you want them to respond.

   A typical pattern is to ask four or five similar questions in a row. Call on individuals for the first two. Then have the whole class answer a question in chorus. Finally, have each student work a problem on paper. You might also have a student work the problem on the board. Ask about half your questions so that they are answered by the whole class together in some way, and the other half of your questions so that they can be answered by individuals. Make sure that each member of your class has a chance to answer several questions each day.

   (See also Chapter 4 on Feedback and Involvement)

6. Ask a variety of questions on the same concept. Asking a variety of questions on the same concept gives more students a chance to participate, reinforces skills, and may give students new insights and understanding. Often students who didn't understand a concept the first time find it makes sense when presented from a different angle.

7. Set up patterns. Patterns help the class move smoothly from simple to more complex examples. Patterns can also be used to provide a challenge (discovering the pattern) to students who already understand the concept. Less complex patterns can be used to provide questions (predicting the next problem) that less advanced students can answer successfully. Since patterns can sometimes be misleading, it is wise not to always be predictable. (For a more thorough description of patterns, see Chapter 3.)

Examples:

$(\frac{1}{2} + \frac{1}{2}) \div \frac{1}{2} = ?$, $(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}) \div \frac{1}{2} = ?$,

$(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}) \div \frac{1}{2} = ?, \ldots ;$ $\frac{1}{3} + \frac{1}{3} = ?$, $\frac{1}{3} + \frac{1}{6} = ?$,

$\frac{1}{3} + \frac{1}{9} = ?$, $\frac{1}{3} + \frac{1}{12} = ?$, . . .

$\frac{1}{2} + \frac{1}{3} = ?$, $\frac{1}{3} + \frac{1}{4} = ?$, $\frac{1}{4} + \frac{1}{5} = ?$ . . .

$\frac{1}{2} \times 4 = ?$, $\frac{1}{2} \times 6 = ?$, $\frac{1}{2} \times 8 = ?$, . . .

$2^1 = ?$, $2^2 = ?$, $2^3 = ?$, . . .

Note that most of the sequences just given admit of generalizations:

$$\sum_{i=1}^{2n}\left(\frac{i}{i} \cdot \frac{1}{2}\right) \div \frac{1}{2} = n, \quad \frac{1}{3} + \frac{1}{3 \cdot n} = \frac{n+1}{3n}, \quad \frac{1}{n} + \frac{1}{n+1} = \frac{2n+1}{n(n+1)}, \quad \frac{1}{2} \times 2n = n.$$

In teaching such patterns, it is often a good idea to arrive eventually at the general form, provided it can be done, which it usually can, without undue strain. Hence one might ask "$\frac{1}{3} + \frac{1}{3.\lambda} = ?$" or--more difficult--who can generalize this?

8. Use parallel problems for contrast.

$\frac{1}{2} + \frac{1}{3} = ?$ versus $\frac{1}{2} \times \frac{1}{3} = ?$

$10 \times \frac{1}{2} = ?$ versus $10 \div \frac{1}{2} = ?$

$\frac{1}{10} \div \frac{1}{10} = ?$ versus $\frac{1}{10} - \frac{1}{10} = ?$

$5 + ? = 0$ versus $5 \times ? = 1$

These kinds of problems are also helpful in reviewing and consolidating. For instance, one might start out the class by writing the following sentences on the board:

$\frac{1}{2} + \frac{1}{3} = \underline{\ 5\ }$ $\qquad$ $\frac{1}{2} \times \frac{1}{3} = \underline{\ \frac{1}{5}\ }$

$10 \times \frac{1}{2} = \underline{\ ^{-}5\ }$ $\qquad$ $10 \div \frac{1}{2} = \underline{\ 1\ }$

$\frac{1}{10} \div \frac{1}{10} = \underline{\ \frac{5}{6}\ }$ $\qquad$ $\frac{1}{10} - \frac{1}{10} = \underline{\ \frac{1}{6}\ }$

$5 + \underline{\ 0\ } = 0$ $\qquad$ $5 \times \underline{\ 20\ } = 1$

Then the class is asked to permute the solutions so as to make all sentences true.

9. Be specific. Unless you want to generate an open discussion, make your questions specific. A question that can be answered quickly frees you to move on to

the next question. Specific questions also avoid correct answers that slow you down or lead in a different direction from what you had planned.

Examples:

.7 x .05 = .035	If you want to review the relationship of fractions and decimals, ask "How can we check this answer using fractions?", not "Why is this the right answer?"
(2 x 3) + (2 x 4) = ?	If you are trying to teach the distributive law, ask for an answer that uses the numbers in the problem.

10. Leave some questions open until the next class or even longer. It creates an intellectual challenge which students may think about on their own. It also gives you a chance to present more tools or a different approach for solving the problem so that more students will grasp it. When the class eventually solves the problem, there is a greater sense of accomplishment--"This problem was so hard it took us three days to solve it."

    Many instructors have students keep a page of "Unsolved Problems" in their notebooks. These are frequently questions like 3 - 5 = ☐ that arise during class and will be covered later in the year. Special recognition should be given to students who solve problems from the list or who bring them to the class's attention when it is ready to solve them.

11. Don't push for results the class isn't ready for. If the students are forced to a conclusion they don't quite understand, they lose confidence and are likely to forget what you've taught. It is preferable to return to the subject at the next lesson, perhaps approaching it from a different direction.

The same principle holds true for verbalizations and generalizations. When a new concept is introduced, students frequently discover how to solve problems without being able to articulate how they got their answers. Forcing them to verbalize prematurely may lead to confusion. If only a few members of the class can provide an explanation or generalization, the rest of the class tends to rely on rote memorization of their classmate's rules, rather than their own conceptual understanding.

12. When a student asks a question, don't answer it yourself. Ask for another student to volunteer to answer it, or ask a sequence of questions that will lead the class to answer the question. Praise students who ask questions to encourage more questions.

13. Don't kid yourself. Making a declarative statement and following it with the question, "Right?" is not using the discovery method. Such questions are really lectures in disguise, since the students will give the obvious answer, "Yes," without understanding the concept at all.

Review

Review is a key factor in successful discovery teaching. Review doesn't have to be done at the beginning of the lesson, although frequently the instructor who begins a class where it left off at the last lesson finds the students getting bogged down in material they previously handled easily. A brief conceptual review should be part of each class. This reinforces the concepts and allows the students to build on them later. It also provides a success experience for the students. Review enables new students, absentees, and slower students to catch up to the rest of the class, and more advanced students to gain new insights and deeper conceptual understanding. Review provides a springboard for new material.

In doing review, the overall strategy is to disguise it and keep it fast paced and surprising. In general, review questions should be skew-directional, the next question in the sequence being unpredictable. Occasionally, however, predictability itself can be a virtue. A question which you can then use with the class is, "What do you think I'm going to ask next?" And, of course, the spiral method should be employed regularly--asking questions on the same material but at a different level.

Review should focus both on conceptual understanding and fast-paced drill. During the course of a two-week period, the entire year's work should be touched on to enable the students to maintain the mastery of a considerable body of mathematics. This gives them a sense of intellectual strength and increases their willingness to risk the unknown. On the average, 20%-60% of any lesson will be review.

There are a number of ways of making the review portion of the lesson as interesting as the discovery of new concepts. Some suggestions follow:

1. Cover familiar ground rapidly. For example, you might lead up to a generalization with only one or two examples, or ask a series of rapid oral questions.

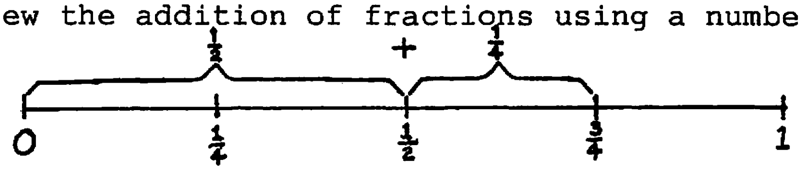
2. Approach a review concept from a new direction.

Examples:

Approach $2^0 = 1$ via division $2^0 = \frac{2^3}{2^3} = 1$

or by the pattern
$$\begin{array}{lll} 2^3 = 8 & & \\ 2^2 = 4 & \downarrow & \div 2 \\ 2^1 = 2 & \downarrow & \div 2 \\ 2^0 = 1 & \downarrow & \div 2 \end{array}$$

Review the addition of fractions using a number line.



3. Try to ask more provocative questions that lead the students deeper into the mathematical concepts.

Examples:

<u>Inverses</u> - What are the similarities between $^-2 + 6 = \square$ and $\frac{1}{2} \times 6 = \triangle$?

<u>Decimals</u> - If $23296 \div 728 = 32$, what's $23.296 \div 3.2$?
(Have students do the second problem in their heads to reinforce the rule for placing the decimal point when multiplying decimals.)

<u>Open sentences</u> - Who can make up an open sentence that has 2 in its truth set?

Who can make up an open sentence that has an empty truth set?

4. Choose new material that presupposes previous concepts so that review is an essential part of the lesson.

Example: Review the additive law of exponents ($a^n \times b^m = a^{(n+m)}$) and multiplicative inverses ($a \times \frac{1}{a} = 1$ for $a \neq 0$) in showing that $2^{-1} = \frac{1}{2}$.

5. Embed review material in new concepts.

Example: $(\frac{1}{4})^3 = ?$

$2^{3.16} \times 2^{4.7} = 2^?$

6. Introduce new notation or terminology.

   Example: In reviewing multiplication of fractions, introduce the words "multiplicative inverse" and "reciprocal."

   Change the notation for exponentiation from 2E3 = 2 x 2 x 2 to the standard $2^3 = 8$. Introduce the words "base" and "exponent."

7. Have students make up review questions. Be specific about the directions, such as: "Who can make up a problem using two to the zero power?" so that the review will keep the focus you want.

8. Put on the board a number of mathematical statements containing errors. Ignore the students' disagreement until you've finished writing. Feign disbelief in your mistakes, reluctantly admitting the existence of further errors only after the class corrects you.

9. Mid-lesson reviews can provide challenge and variety if you "mistakenly" or deliberately erase all or part of the board and then ask the class to help you put it up again. This is useful when you want to use previous results but need to display them differently.

10. As you are winding up a lesson, have students make up review questions for you to use the next day. Place students in charge of remembering particular results. An interesting variation is to ask the students to scan the whole blackboard and suggest which sentences can be erased and which should be kept for tomorrow, or which best capsulize what the day covered.

11. Use worksheets, letting students ask each other for help or refer to their notebooks.

12. Occasionally inject a review question into the middle of a lesson as a change of pace.

13. Use any of the various techniques listed elsewhere which increase student involvement and participation.

Vocabulary

In discovery teaching, vocabulary is introduced in context.

Examples: 1) $2^3 = 2 \times 2 \times 2 = 8$

$2^4 =$ "What if I change the exponent to 4?"

2) "Who can give me a division sentence that undoes $4 \times 8 = 32$? Who can give me a division sentence equivalent to $4 \times 8 = 32$?"

Answer: $32 \div 4 = 8$

$32 \div 8 = 4$

New words are written on the board either off to one side or linked to an appropriate example.

Example: $2^{\overset{\text{exponent}}{\textcircled{3}}} = 8$

Students should keep a vocabulary page in their notebooks and should be reminded to record new words as they are encountered. The list should occasionally include nonmathematical words unfamiliar to the class that arise during the course of a lesson.

Vocabulary should be part of the ongoing review. The instructor should ask questions using the vocabulary to verify student understanding. Students might also be asked to provide or circle on the board an illustration of the word being reviewed. Key vocabulary words should be written on the board frequently, and students might be asked to read or spell the more difficult ones.

Boardwork

A clear, well-arranged board can facilitate the smooth flow of mathematical discovery. A useful planning device is to take a page and sketch exactly how you imagine the board will look at the end of the lesson. This helps to organize the space so that important hints and prerequisites will be left up and ideas don't run into each other haphazardly.

For example, the chart

$$\begin{aligned} 2^4 &= 2 \text{ x } 2 \text{ x } 2 \text{ x } 2 = 16 \\ 2^3 &= 2 \text{ x } 2 \text{ x } 2 = 8 \\ 2^2 &= 2 \text{ x } 2 = 4 \\ 2^1 &= 2 \end{aligned}$$

leads to the obvious question, "What's 2^0?"

A useful technique for focusing students on important intermediate results is to enlist their help in deciding what can be erased and what should be saved. You might ask the class to read certain important results on the board while you write them off to one side to make space for new work.

Another helpful technique is "circling" for emphasis or focus.

Examples:

a) $^{-}2 + 2 + 5 = \square$ "Who can draw a closed curve around the part that adds up to zero?"

b) (2E3)E4 = (2E3) x (2E3) x (2E3) x (2E3)
= 2E(3 + 3 + 3 + 3) = 2E(3 x 4)

Who can read the part of the sentence in the closed curve?"

c) 2E3 = 2 x 2 x 2 "Who can circle the base?"

d) $\boxed{\sum}_{i=1}^{5} \textcircled{i}$ $\boxed{\text{The summation}}$ from i=1 up to 5 of $\textcircled{i}$, etc.

[illegible]

CHAPTER 3

MATHEMATICS DEVELOPMENT STRUCTURES

The goal in discovery teaching is for students to feel that the mathematics they learn belongs to them. They understand it. They developed its rules and formulas. They can explain it to someone else. They are confident about their ability to extend their knowledge.

The basic process to accomplish this goal begins with problems the students understand or can solve readily, and builds more and more complex concepts upon them. The trick is to structure the sequence of questions so that the mathematical concept being taught unfolds as the students answer them. There are several recurring question patterns or structures which we have found useful in leading students to mathematical discoveries with conceptual understanding. They also help develop the students' critical thinking skill and provide them with a framework for solving problems on their own.

Gradual Escalation to a Generalization

Generalizations are central to mathematics and to the development of mathematical concepts in discovery classes. Students can discover many general mathematical principles or laws by investigating a number of particular examples. Variables can then be used to write a formula which expresses a true statement for all members of a given set. For example, the statement "zero times any number is zero" becomes "$\forall \alpha, 0 \times \alpha = 0$."

The most basic method for arriving at a generalization and understanding its power is usually referred to as "erase and replace." A description of the process, along with an example, follows:

(a) Begin with several particular examples the class can solve.

$3 + \boxed{{}^{-}3} = 0$

$5 + \triangle = 0$

(b) Change the term you want to generalize--"If I change the 5 to a 17, what else has to change? (Note that there will be several correct ansers. Acknowledge each and continue until you get 17 + ${}^{-}17$ = 0.)

$17 + \triangle\,{}^{-}5 = 0$

(c) Continue to change the same term until the class realizes that any number can be used to make a true statement.

> "Who can give me another number to put in this sentence?"
>
> "Can I make a true sentence using 117?"
>
> "Can I use a large number?"
>
> "Can I use a small number?"
>
> "Can I use a fraction?"
>
> "How long can we go on putting different numbers here?
>
> "How many numbers can we use?"

(d) Use variables to stand for arbitrary numbers. (Students seem to enjoy using Greek letters to stand for universal variables.) (Note: Technically, there should be a quantifier, "for every number γ " written " $\forall\gamma$ ". It is frequently omitted, which rarely causes confusion.)

$\gamma + {}^{-}\gamma = 0$

(e) Check by substituting particular values.

"Will I get a true sentence if I substitute a number for γ ?"

"Who knows what a substitute is?"

"What would it mean to substitute a number for γ ?"

"Who has a number to substitute for γ ?" Let $\gamma = 10$

"Who can read the sentence substituting 10 for γ ?" $10 + {}^{-}10 = 0$

"Is the sentence true when we substitute 10 for γ ?"

Another way to introduce a generalization is by doing several examples which illustrate the principle. Then ask a similar question involving variables, e.g.:

$$2 + \square = 0, \ 7 + \square = 0, \ 17 + \square = 0, \ \alpha + \square = 0$$

Note that this approach does not lead to an understanding of what α represents. However, it is an efficient method for arriving at generalizations once the class understands what they are. To reinforce the concept, it is important to ask review questions such as "What does α stand for?", and "Will I get a true sentence if I substitute numbers for α?"

Generalizations are often a good way to help demystify mathematical symbolism and encourage students to try problems, whether they think they know how to do them or not.

Examples: $f'(x) + \square = 0$

$$\int_0^5 x^2 dx + {}^{-}\int_0^5 x^2 dx = \square$$

Introduce the words "generalization" and "generalize" in context at some point by asking, "How would a mathematician show that we can use any number; how would she generalize this statement?" As the students become more familiar with the process, you can leave out more and more steps, giving only a few examples or saying, simply, "Who can generalize this statement?"

Be careful, however, about pushing the class to arrive at a generalization through a mechanical process before they thoroughly understand the concept. A class which has not seen enough examples may be perfectly willing to accept $(a \times b) + (a \times c) = a + (b \times c)$ or $a^n + a^m = a^{(n+m)}$.

When you have one or more generalizations on the board, you can demonstrate their power by asking the class to tell you what to erase so that you only keep the statements which tell you the most information or which summarize the day's lesson.

Here are several more examples of generalizations.

(a) The Distributive Law $a \times (b + c) = (a \times b) + (a \times c)$

(b) $k \times \frac{1}{k} = 1$ for $k \neq 0$.

(Note that the question of whether $k \times \frac{1}{k} = 1$ for all numbers k leads to an interesting discussion about multiplication by 0.)

(c) $\alpha \times 1 = \alpha$

(d) $a^n \times a^m = a^{(n+m)}$

It is important for students to realize that not all statements and patterns lead to generalizations. This can be shown by examples such as:

(a) $2^3 + 2^3 = 2^4$ but $3^3 + 3^3 \neq 3^4$

(b) $2^4 = 4^2$ but $3^4 \neq 4^3$.

Another useful activity is for students to make conjectures about possible generalizations and attempt to verify or disprove them. For example, a class which has studied the distributive law $\alpha \times (\beta + \gamma) = (\alpha \times \beta) + (\alpha \times \gamma)$ might conjecture that $\alpha + (\beta \times \gamma) = (\alpha + \beta) \times (\alpha + \gamma)$.

An excellent exercise is to present several true statements and ask which ones can be generalized and how.

Example: $5 + 0 = 5$ $\quad 2 + 3 = 3 + 2$ $\quad 2^4 = 4^2$
$5 + 3 = 8$ $\quad (2 \times 3) + 2 = 2 \times 4$ $\quad \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

Equivalent Sentences

Replacing a mathematical expression by an equivalent one is often an invaluable aid to solving a problem, understanding a new concept, or arriving at a generalization.

$\frac{1}{2} + \frac{1}{4} = ?$ becomes $\frac{2}{4} + \frac{1}{4} = ?$

$^-2 + 5 = ?$ becomes $^-2 + 2 + 3 = ?$

$2^3 \times 2^4 = ?$ becomes $(2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2) = ?$

$(2 \times 3) + (2 \times 4) = 2 \times 7$ is easier to generalize in the form $(2 \times 3) + (2 \times 4) = 2 \times (3 + 4)$.

The following process helps the students discover the concept of equivalent mathematical sentences in a comfortable situation:

Begin with an open sentence at an appropriate level for the class, like

$$18 + \square = 32.$$

Quickly give the class additional problems like

$(6 + 12) + \square = 32$

$(3 \times 6) + \square = 32$

$(36 \div 2) + \square = 32$

$(9 \times 2) + \square = 32$

$(6 + 6 + 6) + \square = 32$

$(\frac{1}{2} \times 36) + \square = 32$, etc.

Tease the students about being stuck on "14" since they keep giving the same number to different problems. The class will quickly tell you that the sentences are all the same. Introduce the terminology "another name for 18," "another way of writing 18," and "an equivalent expression for 18." Reinforce the concept by having the students make up additional equivalent sentences or repeat the process with another problem.

We have found vertical arrows to be useful to indicate equivalent expressions.

Examples:

$$\begin{tikzcd}
\frac{1}{2} \arrow[d] & + \arrow[d] & \frac{1}{4} \arrow[d] & = \arrow[d] & \square \arrow[d] \\
\frac{2}{4} & + & \frac{1}{4} & = & \frac{3}{4}
\end{tikzcd}$$

$$\begin{tikzcd}
(2 \times 2 \times 2) \arrow[d] & \times \arrow[d] & (2 \times 2 \times 2 \times 2) \arrow[d] & = \arrow[d] & 2^7 \arrow[d] \\
2^3 & \times & 2^4 & = & \square
\end{tikzcd}$$

$$\begin{tikzcd}
{}^{-}5 \arrow[d] & + & 8 \arrow[d] & = & \square \arrow[dr] & \\
{}^{-}5 & + & \overbrace{5+3} & & = & \square
\end{tikzcd}$$

Don't allow arrows to outlive their usefulness and become a fetish to the point where a problem is "wrong" in the students' minds if it doesn't have all the arrows. Arrows for operational and relational symbols can be omitted, although they are helpful, at first, in focusing on the renaming process.

The many steps involved in writing an equivalent sentence provide opportunities to involve students with a wide range of skills in the solution of the same problem. Questions such as "How can I rewrite the 8 using a 5?", "What do I bring down for +?", "Where did this $\frac{2}{4}$ come from?", etc., often involve the least active participants in the class.

Frequently an open sentence is equivalent to (asks the same question as, checks, has the same truth set as) another sentence which is easier to solve, e.g.:

(a) $2^5 \times 2^y = 2^0 \Longleftrightarrow 5 + y = 0$

(b) $9L3 = \square \Longleftrightarrow 3E\square = 9$

(c) $10 - 6 = \triangle \Longleftrightarrow 6 + \triangle = 10$

$5 - {}^-2 = \bigtriangledown \Longleftrightarrow {}^-2 + \bigtriangledown = 5$

(d) $4 \div \frac{1}{2} =$ ⬡ $\Longleftrightarrow \frac{1}{2} \times$ ⬡ $= 4$

<u>Use of Generalizations to Extend Definitions</u> (Use it in a sentence)

Frequently in mathematics, one is confronted with the problem of defining operations on new sets of numbers or extending definitions to new numbers.

Examples: (a) $\frac{1}{2} \times \frac{1}{3} = ?$

(b) What's a sensible definition of 2^0?

(c) ${}^-2 \times {}^-3 = ?$

A common approach in discovery classes is to make up a "true" sentence containing the unknown expression and then see what it "acts like." For the above cases, the sentences might be

$$\begin{tikzcd}[column sep=small] \text{(a)} & \underbrace{\frac{1}{2} \times \frac{1}{3}} \arrow[d] & \times & \underbrace{3 \times 2} \arrow[d] & = & 1 \arrow[d] \\ & \square & \times & 6 & = & 1 \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] \text{(b)} & 2^{0} \arrow[d] & \times & 2^{3} \arrow[d] & = & 2^{(0+3)} \arrow[d] \\ & \square & \times & 8 & = & 8 \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] \text{(c)} & ({}^{-}2 \times {}^{-}3) \arrow[d] & + & ({}^{-}2 \times 3) \arrow[d] & = & {}^{-}2 \times ({}^{-}3 + 3) \arrow[d] \\ & \square & + & {}^{-}6 & = & 0 \end{tikzcd}$$

Note that each sentence is true because we assume a property derived for positive integers is also valid for the integers and rationals. In the first example, we assume the commutative and associative properties for multiplication; in the second, the formula for multiplying exponential terms; and in the third, the distributive law. These assumptions generally will be tacit at first, although through language such as "If this sentence is true, what does that tell us about $^{-}2 \times {}^{-}3$", the class can gradually appreciate the process by which mathematicians extend definitions to larger number systems.

Patterns

Patterns appear frequently in mathematics. They often suggest a conclusion or make results derived through some other method seem more plausible. Patterns are helpful for reinforcing and reviewing previously learned concepts. Since patterns can be misleading, care should be taken to verify results obtained via patterns with another approach and to introduce the students to some misleading patterns.

Examples:

(a) Reinforce 2E0 = 1 by looking at:

$$\begin{tikzcd}
2E3 = 8 \arrow[d, bend left, "\div 2"] & & 2E3 = 8 \\
2E2 = 4 \arrow[d, bend left, "\div 2"] & & 2E2 = 4 \arrow[u, bend right, "\times 2"'] \\
2E1 = 2 \arrow[d, bend left, "\div 2"] & & 2E1 = 2 \arrow[u, bend right, "\times 2"'] \\
2E0 = ? & & 2E0 = ? \arrow[u, bend right, "\times 2"']
\end{tikzcd}$$

(b) Derive multiplication of negative numbers using the distributive law. Reinforce via

¯3 x 4 = ¯12
¯3 x 3 = ¯9
¯3 x 2 = ¯6
¯3 x 1 = ¯3
¯3 x 0 = 0
¯3 x ¯1 = 3
¯3 x ¯2 = 6

(c) Use the additive law for exponents to establish 2E¯1 = $\frac{1}{2}$ and 2E¯2 = $\frac{1}{4}$. Extend to 2E¯3 etc., via

$$\begin{tikzcd}[row sep=small]
2E4 = 16 \arrow[dddddddd, bend left] \\
2E3 = 8 \arrow[dddddd, bend left] \\
2E2 = 4 \arrow[dddd, bend left] \\
2E1 = 2 \arrow[dd, bend left] \\
2E0 = 1 \\
2E\,{}^{-}1 = \frac{1}{2} \\
2E\,{}^{-}2 = \frac{1}{4} \\
2E\,{}^{-}3 = \\
2E\,{}^{-}4 =
\end{tikzcd}$$

(d) Some misleading patterns:

(i) $2^2 = 2 \times 2$

$2^1 = 2 \times 1$

$2^0 \overset{?}{=} 2 \times 0$

(ii) 3 is prime, 5 is prime, 7 is prime, 9 is prime?

(iii) 6E2 = 36

5E2 = 25

4E2 = 16?

(e) A challenging pattern:

$1^2 + 1 + 41 = 43$, prime

$2^2 + 2 + 41 = 47$, prime

$3^2 + 3 + 41 = 53$, prime

$4^2 + 4 + 41 = 61$, prime

$5^2 + 5 + 41 = 71$, prime

Is $n^2 + n + 41$, prime for all n?

Equality; Transitivity

To many students, "=" is a command which says "do the problem" or "find the answer." Thus they may be uncomfortable in accepting such statements as 6=4+2 or 2x2x2=2E3. They may also have difficulty drawing conclusions from a long chain of equalities. This section discusses things the specialist can do to clarify the meaning of "=" to students.

A useful exercise in helping students understand what equals means is to write on the board:

2 + 4 3 x 2 and ask what symbol could be placed between the two expressions to make a true sentence. Once a student suggests equals and the class agrees, have the students think of other expressions besides 3 x 2 that are equal to 2 + 4.

Now, again ask what "=" means. Probably students will suggest: the same as, or, means the same number. This understanding of equality will be useful when you work with equivalent sentences.

Drawing conclusions from a chain of equalities often does not come naturally to students. They are perfectly comfortable writing a chain of equalities like 2 x 6 = 12 ÷ 3 = 4 x 9 = 36 ÷ 6 = 6 x 3 = 18 without recognizing it is logically absurd.

To clarify meaning of a sentence with more than one equal sign in it, you might want to discuss transitivity in general:

(a) If □ has the same number of candy bars as △ has, and △ has the same number of candy bars as ○ has; does □ have more or fewer candy bars than ○?

(b) If John is the same age as Joe, and Joe is the same age as Stan, what is true about Joe and Stan?

(c) If $\sum_{i=1}^{4} 3 \times i$ = 3 + 6 + 9 + 12 and 3 + 6 + 9 + 12 = 30, what is the numerical value of $\sum_{i=1}^{4} 3xi$? How did you know?

(d) If □ = △ and ○ = △, then what symbol can I put between □ and ○ ? Check by filling in the shapes with numbers or expressions.

(e) If a = b and b = c and c = d and d = e, what can you say about a and e?

Once you have done many problems and examples, you might want to develop a standard expression for drawing conclusions, such as "the conclusion of the chain of equalities is . . ." (COTCOE)

The generalization (aEb) Ec = aE (b x c) is an example of a concept that results from considering a chain of equalities. In trying to develop this concept, you might consider: (2E3) E4 = (2E3) x (2E3) x (2E3) x (2E3) = 2E (3+3+3+3) = 2E (3 x 4); you wish the class to draw the conclusion (2E3) E4 = 2E (3 x 4). Some suggestions for leading the class to this result are:

(a) Who can find another name for (2E3) E4 that uses the least chalk?

(b) Who can find a name for (2E3) E4 that uses only a 2, 3, and 4 once as in the problem?

(c) Who can point to something on the board that's equal to (2E3) E4? Keep having students do this until you get the 2E (3 x 4) form.

(d) Enclose the first and last parts of the sentence in a simple closed curve, and have the class read what is in the curve.

$$(2E3)\ E4 = (2E3)\ x\ (2E3)\ x\ (2E3)\ x\ (2E3) = 2E(3+3+3+3) = 2E(3x4)$$

(e) Rearrange the problem with the help of vertical arrows. Have the students read the arrows as equals. Finally ask:

"What symbol is missing?"

$$\begin{array}{ccc} \underline{(2E3)\ E4} & & \underline{2E\ (3x4)} \\ \downarrow & & \uparrow \\ (2E3)x(2E3)x(2E3)x(2E3) & = & 2E\ (3+3+3+3) \end{array}$$

(f) Discuss transitivity in general (see above).

Embedding the Material in a Conceptual Framework

One of the most frequently used strategies in discovery teaching is to embed elementary material in a more advanced, conceptual framework. Students become motivated to understand topics, which seem to have little inherent interest, when these topics are essential to the solution of conceptual mathematical problems. Several examples involving fractions and decimals are described below.

To a mathematician, fractions and decimals are each ways of looking at the rational numbers, but each leads to different insights. It so happens that decimals are really a kind of infinite series, that is, each decimal is so many tenths, plus so many hundredths, plus so many thousandths, plus etc. Most decimals we run into terminate, such as in .25 or .456, but sometimes they do not, as in .2222222222222, with an unending string of 2's, and since decimals such as this last one are adding up an infinite number of terms, doesn't it seem sensible that they should increase without bound? In fact 2.3 is bigger than 2.22222222222 (with an infinite number of twos). Everybody knows this, but how many can explain why? From the point of view of a discovery teacher, this can be fruitful territory.

Let's examine this example more closely. First it is essential that we as teachers fully understand the concept we are trying to get across, and then it is necessary that we trace the process we went through to arrive at our conclusions, so that we can devise a sequence of questions. So why is 2.222222222222 less than 2.3? Well, 2.3 = 2.30. Why? Because $2\frac{3}{10} = 2\frac{30}{100}$. We can carry this further, however, and see that 2.3 = 2.300000000000, with as many zeros as we want, and now it is easier to see why 2.3000000000000 is greater than 2.222222222222, and hence why 2.3 is greater than 2.22222222222222. But that still leaves us with the fact that you can add up a series like $2 + \frac{2}{10} + \frac{2}{100} + \frac{2}{1000}$ + etc., and that you will be bounded from above by something as small as 2.3.

A simpler but related problem begins by looking at what happens when you add $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$: Do you reach any sort of a limit? Have the students try some examples to find out. They will see a pattern, and you will arrive at the conclusion that you get closer and closer to 1. Now try it for $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$ You will find that the answer gets closer and closer to $\frac{1}{2}$. Now see if the students can see a "pattern in the patterns," that is, for a given series, can they predict the answer.

Discussions such as these will help students to see why they can add zeros behind the decimal point and will provide practice in adding fractions, and this is, after all, one of your practical goals. The point here is that, in the course of their discussions about conceptual matters, students will be continually calling upon the basic arithmetic facts you want them to learn. In this way, they begin to master the facts, but they tend not to experience the drudgery.

Another problem which motivates the study of fractions is the "infinite pie" idea. What happens if you keep cutting a pie in half? Do you every run out of pie, even if you keep cutting forever? You might come to a point in your discussion when students are talking about atoms, and they may realize that they need to consult an encyclopedia after class. This is physics, not math, but they are motivated, and they will know they have been discussing something that is not trivial.

CHAPTER 4

FEEDBACK AND INVOLVEMENT

This chapter contains a number of techniques for getting feedback from a class and for increasing the level of student participation in the lesson. Both are essential to effective discovery teaching. In order to ask appropriate questions, you need to know the class level of understanding. In order to gain from the discovery process and give you feedback, students need to be involved in the lesson. Frequently, the same technique accomplishes both feedback and involvement, hence we have included both in this chapter.

Circulating

Circulating the class is an important way of getting feedback from the students. It also is useful for involving students and giving the teacher a chance to make individual contact with them.

At least four times in the class period, the students should be asked to answer a question or work a problem on their papers. Be sure that each student writes something, even if it is only a question mark. Allow the students to signal you by raising their hands when they are finished. Circulate rapidly, making some contact with each student. In general, do not comment on whether the answer is right or wrong, or provide individual tutoring. (You do not want to single out students with the wrong answer, or spend so much time you lose the class's attention.)

It is not necessary to read every answer each time. You might check only two rows and then make a prediction about the percentage of the class that will have the answer correct.

Circulating gives you accurate feedback on student understanding, helping you gauge when to move on to a more advanced topic. If most of the students were correct, then the answer can be put on the board and the class can move on. If not, you might want to have the class investigate the different ideas about the problem by offering all answers, correct or incorrect.

Circulating makes all of the students active participants, often exposing misunderstandings you and the class were unaware of. It also gives you an opportunity to involve students who understand the concepts but are unsure of themselves or are reluctant to respond. Such students can be reinforced by asking them to put their "absolutely correct" answer on the chalkboard. One way to be certain to remember is to leave the chalk with them when you circulate.

Circulation as it is described here is a technique for gaining accurate feedback from the entire class without disrupting the flow of the discovery lesson. Questions should be relatively straightforward ones that can be answered with a simple number or phrase, or should be problems that can be solved quickly. More complex problems or sequences of practice problems should be saved for a "seatwork" period during which you and students who have completed their work might offer assistance, or students might work together in small groups.

Polling

When a student gives an answer, quickly ask for a show of hands--"How many of you would have said that?", or "How many agree with Jane?" Although the feedback will not be as accurate as from circulating, as some students will go along with the majority, this is rarely a serious problem. You can get a good sense of how many students actually have the answer by how rapidly the hands go up. Because you get the response quickly, there is time to ask similar questions. Often a student who merely copies his neighbor begins to focus on the problem and soon begins answering questions spontaneously.

Another variation of the polling technique is to put several answers on the board and have the students vote on them.

You might also rapidly get answers from a number of students. Again, students who are unsure of themselves may jump on the backwagon and gain confidence as they are recognized for having the right answer.

Finger and Hand Signals

Finger and hand signals enable the teacher to get a rapid reading of the class, and they allow many students to participate at once.

Do not be overly concerned about the problem of copying when answers are shown in fingers. You can get a good sense of how well the class is understanding by noticing how quickly and confidently they respond. If truly accurate feedback is desired, the students can be asked to close their eyes or write their answers on paper. On other occasions, when fingers are used to obtain mass participation, slower students can participate along with the rest of the class. They are more likely to focus on the problem and catch up to the rest of the class. Be careful about putting a student on the spot by asking him to explain an answer he has on his fingers. Volunteering to show an answer is quite different from volunteering to give an explanation.

Examples of signals often used follow:

(a) Numerical values can be shown on the fingers.

(b) Other mathematical symbols such as operational or relational symbols can be shown on the fingers or drawn in the air.

Note: Younger students genuinely enjoy showing answers with their hands. Older students tend to think of it as childish, so it should be used sparingly. It can, however, be inserted occasionally in a lighter moment or as a change of pace.

(c) Signals for agreement and disagreement should be developed. Younger students like waving their arms back and forth for disagreement and pumping both fists for agreement. With older students, head nods and shakes are often the best signals. You can ask the students to indicate whether they agree or disagree every time a question is answered.

Chorus Response

Often, during a lesson, ask the students to tell you their answer together, or to read in chorus something from the board. This creates a break in the normally quiet question-answer pattern. It focuses attention and brings back daydreamers. Reading provides an opportunity for students who don't yet understand the mathematics to participate. Chorus response allows the entire class to move rapidly through a sequence of questions leading to a conclusion.

In order to avoid chaos and called out answers when you don't want them, it is helpful to cue the class when you want a whole group response. "Class, what is . . .," "When I count to three, everyone tell me . . .," and "When I drop my hand, everyone with his hand up tell me . . ." are ways of achieving this. When you want to return to a different mode of response, begin your question with a direction such as "Raise your hand if you know . . ." or "Write on your papers . . . ."

Some situations in which chorus responses are appropriate and effective are:

- (a) Emphasizing a concept or generalization that has just become clear.
- (b) Focusing on a problem that has just been put up on the board.
- (c) Learning new words or symbols.
- (d) In response to short, fast-paced questions, such as simple computations, gradual escalation to a generalization, or review questions.
- (e) To refocus the class after an interruption.
- (f) To involve nonparticipants.
- (g) To summarize the day's work. At the end of the class, have students read the most important words and concepts on the board, while the class makes sure their notes are in order.

Rapid Oral Questions

A round of rapid-fire, oral response questions provides a change of pace, while many students get a turn to respond in a short period of time. The questions should be easy, or follow a pattern, so everyone will feel capable of answering. Some of the best times to use a rapid series of oral questions are when reviewing, when building up to a generalization, or when doing a routine calculation as part of a larger problem.

Examples: What's $\frac{1}{2}$ of 8? $\frac{1}{2}$ of 16? $\frac{1}{2}$ of 32? etc.

In 9,876,543.21, what place is the 4 in?
The 5? The 8? The 2? etc.

Counting, Naming, and Predicting Hands

Counting the number of hands raised or mentioning the names of students with raised hands often increases the number of students who participate. An auctioneering style: "Ten people have it," "This whole row," etc., or naming students: "John has it," Sue's got an idea," etc. are both effective for this purpose. Students want the recognition you are giving, and the time you spend talking about hands gives them extra time to think.

To avoid putting an insecure student on the spot, it's best to call on one of the first hands or the entire class for the actual answer. The same student, however, might be called on to second another student's answer. Once they are focused on the problem and participating in some way, reluctant students frequently are eager to answer the next question.

A variation of these techniques is for you or one of the students to predict the number of students who will be able to answer the question. You might challenge them: "I bet only ten people will be able to solve this."

Counting and predicting provide an excellent opportunity to review percents: "What percent of this row has their hands up?"; "If 75% of the class gets the problem, how many students will that be?"

Chain Answering

Chain answering is an effective way to involve as many students in answering a question as there are steps in the problem. "Cheryl, would you start us off?" "That's good, I know you can do the rest. Call on someone with a raised hand to do some more." Each time you stop a student, she quickly calls on another student.

A variation has students come to the board and hand the chalk to the next student when their turn is finished.

Examples of situations where chain answers are effective are:

(a) Reading a complex generalization such as $(\alpha \times \beta) + (\alpha \times \gamma) = \alpha \times (\beta + \gamma)$

(b) Substituting numerical values in a mathematical formula such as $a^n \times a^m = a^{(n+m)}$

(c) Building a table such as. powers of 2.

(d) Providing the next term or problem in a sequence such as $\frac{1}{3} + \frac{1}{6} = ?, \frac{1}{3} + \frac{1}{9} = ?, \frac{1}{3} + \frac{1}{12} = ?$ etc.

(e) Working each step of a multi-step problem such as

$$5 \times \frac{1}{3} \times 3 \times 2 \times \frac{1}{2} \times \frac{1}{5} \times \pi = ?$$

$$1 \times 1 \times 1 \times \pi = ?$$

$$1 \times \pi = ?$$

$$\pi = ?$$

Deliberate Errors

Successful discovery teachers make deliberate errors at judicious points in the lesson. Deliberate errors serve to increase student participation, to reinforce mathematical understanding, and to build student confidence. They force students to evaluate each response critically and not to rely on the teacher or other students. Students delight in catching the teacher in a mistake. At the same time, the teacher's error removes the stigma of being wrong, and thus makes students more willing to venture an answer. Explaining their disagreement helps students to demonstrate and articulate their understanding.

The simplest errors are clerical, such as transposing two numbers, miscopying a problem, or misspelling a word. They serve primarily to focus the class and keep them alert. Sparing use of this technique, coupled with a warning that there may be some mistakes in your boardwork, avoids the danger of seeming patronizing.

A second category of errors involves literal interpretation of student answers.

Example: $2^3 = 2 \times 2 \times 2$

$2^5 = ?$ Student response: "You add two more 2's."

You write $2^5 = 2 \times 2 \times 2 + 2 + 2$.

When the students are secure in their understanding of a concept, errors of this type help them to communicate their knowledge better.

The final category of errors are conceptual ones, often focusing on the same types of errors the students tend to make. These errors generate enthusiasm, debate, and discussion, but most important of all, reinforce understanding.

Examples:

False Statement	Devious Defense
$2^3 \times 2^4 = 2^{12}$	The "x" tells you to multiply.
$2^0 = 0$	2E3 = 2 x 2 x 2, but 2E0 = . And nothing is zero.
2E¯1 = 1	2 + ¯1 = 1 (The best arguments for false statements are often those originally presented by the students in their innocence.)
5 + ¯5 = 5	Negative numbers do not exist.
$\frac{1}{2} + \frac{1}{4} = \frac{2}{6}$	1 + 1 = 2 and 2 + 4 = 6. And that has to be the way to work the problem, because how do you work $\frac{1}{2} \times \frac{1}{4} = \square$?

Attention and Focus

Student attention is bound to wander. This happens for many reasons: noise distractions, intercom interruptions, people going in or out, earlier incidents. Besides these non-SEED reasons, students who are experiencing some difficulty with the mathematics will tend to "drop out" from class participation. A student who hasn't heard or wasn't listening for whatever reason will not be raising his hand. It is OFTEN necessary first to catch student attention, before you pose your question.

Each technique described below is useful for either the whole class or for individuals. Variety is essential. Any participation techniques, especially chorus responding, also help students focus and pay attention since they involve the students in a response. Other techniques whet the students' appetite and motivate their interest through elements of suspense, challenge and contradictions.

1. ARE YOU READY? CAN YOU SEE THIS?

Before you ask the math question, ask the class or certain students: "Are you ready?"; "Who is ready for a hard problem?"; "Who thinks they will be able to do the next problem?"; "Do you think you are ready to make a quantum leap to the generalization?"; "Do I have everybody's attention--no, someone is looking out the window."

Specify some mode of response to your question: a specific way for them to show you they are. "Raise your hand; say 'Yes,' if you are ready." If not many students respond, repeat your instruction till you get a stronger (faster, more unanimous) response.

2. VOICE EMPHASIS

Any CHANGE in your voice will gather attention. When it is especially desirable to emphasize an important part of a student's idea or of the question, either suddenly or gradually talk more rapidly or talk very de-lib-er-ate-ly; speak much louder or softer; speak with a higher or lower pitch. Stating your question with pauses stimulates anticipation, attention and excited responses: "What is . . . the numerical value . . . of 4E2?"

3. CHALLENGE

Challenging is a form of teasing. Properly timed and with the proper tone, a challenge will energize the students and increase the focus they give to the upcoming problem. Choose problems you believe students can do. With too hard a problem or the wrong tone, students may be discouraged from trying the problem.

Comment before you give the problem: "This one may be too hard."; "I'll catch you on this one."; "This is tricky, only the most alert will get it."; "Last time I gave this problem, no one got it."; "You need to switch into a higher mental gear."; "In fact this is a star problem."; "This is a no hand problem."

When students are secure about their answer, you might challenge them by claiming the answer is wrong and forcing them to defend it.

Obvious contradictions and "double talk" are also effective for revitalizing a class. Tell the students to forget something, then ask them to tell you what they were supposed to forget. "Don't anybody listen to what Cheryl just said because she is absolutely right!" "This is so important, I'm going to erase this before anyone has a chance to write it down."

4. CUED RESPONSES

Occasionally ask the class to do something on a certain predetermined signal. If you stall before giving the cue, the time that passes before the cue or signal is given builds up the class's anticipation towards making the agreed upon response: almost invariably you will have nearly 100% participation at the cue. Some examples of cues include: "When I turn around . . .;" "When I lower the chalk (or the eraser) from over my head;" "When the chalk touches the blackboard;" "When I count to three;" "When I close the door;" "When Tanya raises her hand." You can use this as a daily technique so long as the cue is a different signal each time you use it. Younger students enjoy your trying to catch them on a false cue, such as touching the board with a finger instead of the chalk, counting "1, 2, 4," skipping over "3," or lowering the hand without the chalk.

With younger students, a very powerful technique to refocus their attention after a major disruption has occurred is a series of fast directions which appear to have no point until the end, when you are pointing to the problem that you wish to ask a question about. It goes like this: "Look out the window, OK, now look at the door, now look at the clock, read this word on the board, read this sentence, read this problem."

5. POINT TO AN ANSWER

This involves every student, focuses their attention on some particular mathematics, and usually wakes up any daydreamers, who can then pick up from their classmates what is going on. Ask the students to point, from their seats, to: a sentence on the board that we could use in solving this problem; a place where 3 goes in this sentence; the chart that shows the answer. Tease a student by going back to a student's desk to take a sighting along his/her finger, then follow it to some spot on the board.

6. BOX OR CIRCLE

Box, circle or draw an arrow ---› to make the problem or question currently under investigation stand out. The visual reference will help a non-attender pick up where the class is. Box all important problems which emphasize the conclusions the class arrived at in that lesson. Let students come to the board to circle a hint, or circle what told them to use "X" for the factors.

7. READ

Write a problem, ask someone to read it while everyone else is thinking about it. Sometimes ask the whole class to read a problem as you write it on the board.

To get a class to scan the blackboard for helpful information, make an elaborate show of erasing any clues or hints on the board, telling the students you don't want to give away the answer. You intend, of course, to help them focus on a clue. You might erase something close to the clue or erase the hint itself. Vary this by circling clues, covering them up, saying "Don't look for any hints" (they will then study the board, whether there are any hints or not). On a very difficult problem, you might want to reverse this technique by erasing everything except the clue: "I'm going to erase all of the irrelevant information."

CHAPTER 5

BUILDING STUDENT CONFIDENCE

A major goal of discovery teaching is to build students' confidence in their ability to think critically and to learn mathematics successfully. This is particularly true in general mathematics classes, where the students often have nine or ten years of school experience convincing them that the opposite is true. You want them to be involved in the lessons, building the mathematics themselves. You want them to experience success and to regard what they have accomplished as having value.

An instructor who applies the principles of the preceding chapters is well on the way to achieving this goal. There will be numerous conceptual mathematics questions which students can answer successfully. Frequent review will help the students maintain mastery of an ever-increasing body of mathematics. Feedback and involvement techniques will provide ample opportunities for the class as a whole to participate.

The manner in which you treat student responses will influence their willingness to answer future questions. This chapter presents ways to respond to student answers, and techniques that will establish an atmosphere of intellectual support and respect in which students feel free to participate and are rewarded for it. Coupled with the question and feedback techniques discussed previously, they create a classroom in which students learn mathematics with a sense of accomplishment and achievement.

Success Reinforcement. Generally, if a student responds with a correct answer, he or she will be rewarded by you or the class. It's hard not to show your pleasure (although it often heightens the student's experience of success if you act noncommittal and ask the class its opinion). Some ways of reinforcing and creating success experiences are outlined below:

1) Use names. Identify ideas and formulas by the names of students who developed them. "Who could do this problem using John's system?"; "Let's use Sue's method for finding the LCM," etc.

2) Student agreement. After a response from a student, ask the class if they agree. A roomful of raised hands is definitely reinforcing.

3) Acknowledge other students. Mention the names of students who have their hands raised when someone else is called on to answer, or who are showing agreement or disagreement. "Mary has the answer;" "John has it too." "Joe agrees with Jane;" "How many got 3/4?" etc. Eye contact or a smile can also let a student know that you know that she knows the answer.

4) Students to the board. Ask students to show their work on the board. This technique is especially reinforcing when you call on a normally shy or non-participating student whose correct answer you have just seen while circulating. If several students come to the board at the same time or in rapid succession, have them put their names by their work.

5) I know you know. When you are sure a student understands a concept, interrupt his explanation with "Good, I know you understand this." Then invite him to call on another student to finish the explanation. A variation allows you to bypass a student who is threatening to monopolize the class. "I know you know the answer, Jane; I want to see what Jerry thinks;" or "John, call on someone else you think knows the answer."

6) Experts. Designate students, who have caught on to a concept, as experts for the day. Have them check the correctness of other students' answers, or help you circulate. These students can also be involved as peer teachers or tutors.

7) Star problems. Label occasional challenging questions which are within the grasp of the class as "star problems." Put a star on the board, and next to it put the names of students who solve the problem.

Many instructors also give similar recognition for students who ask good questions.

8) Advanced material. Praising students lavishly for trivial work is patronizing and they recognize it. If you embed remedial work in advanced material, you can give students a real ego boost.

Examples:

$$\begin{array}{ccccc} 2^{-1} & \times & 2^{-2} & = & 2^{-3} \\ \downarrow & & \downarrow & & \downarrow \\ \frac{1}{2} & \times & \frac{1}{4} & = & \frac{1}{8} \end{array}$$

"Normally students don't study this until algebra."

$$\sum_{i=1}^{3} 2^{-i} = \begin{array}{ccccccc} \frac{1}{2} & + & \frac{1}{4} & + & \frac{1}{8} & = & \frac{7}{8} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \frac{4}{8} & + & \frac{2}{8} & + & \frac{1}{8} & = & \frac{7}{8} \end{array}$$

"I didn't study this until I was in college."

9) Who has it now? Give students a chance to indicate their progress in learning an idea, particularly after they have met with initial frustration. Ask: "Who has the answer now ("Who sees how to do this?") Who didn't have it before?"

10) Reward questions. Mathematical research proceeds when mathematicians ask themselves questions. A student in a discovery class who asks a good mathematical question should be praised. Students should be encouraged to ask themselves if alternate approaches can be used for solving a problem, or what happens if the problem is changed. Occasionally, the volume of good questions threatens to slow down the pace of the class because too much time is spent investigating tangents. In this case, ask the students to write their questions on paper and hand them in. The few minutes that it takes to write a response can be richly rewarded in the ensuing mathematical correspondence that will develop.

Student errors. The manner in which a student's incorrect answers are handled will influence his willingness to respond to future questions. Try to avoid telling a student directly that he's wrong, or putting him on the spot.

Some suggestions for handling incorrect answers follow:

1) Deliberate errors. If you make frequent deliberate errors, there will be no stigma attached to being wrong. Students will disagree with you or each other matter-of-factly.

2) Allow revisions or call on someone. Put incorrect answers on the board with a straight face. Let the class disagree politely. Allow the student to revise her answer or to call on another student who she thinks knows the answer. After you've gotten the correct answer, return to the student and ask if she understands now.

3) Explore consequences. Often technically incorrect answers contain good thinking and mathematical creativity. Students often find that by slightly changing the question or one of the assumptions (axioms) of the problem, they can more easily answer the question. Research mathematicians are rewarded for this kind of thinking. Students are usually told they are wrong and made to feel foolish.

Good discovery teachers turn technically incorrect answers into a positive learning experience. If the student has changed the question, such as $\frac{1}{2} + \frac{1}{4} = \frac{1}{8}$, ask "What problem did Sue solve? or "What was Sue thinking of?" Sue gets credit for having solved a problem correctly and the class benefits from focusing on the contrast between Sue's problem and the original problem. If the student has changed the axiom system, say 2E3 = 23, you might give the system his name, and ask the class to do some more problems in "Carl's system." In both cases, the student's response has provoked a fruitful class discussion, and he will be more willing to offer an idea in the future.

If you can see the reasoning behind the student's answer, you might want to present the argument to the class and let them disagree with you, rather than the student. For example, if a student tells you 2x2x2=6, you might argue that "2 times 2 is 4 plus 2 more is 6."

4) Partial answers. If a student has worked some steps of a multistep problem correctly, be sure he gets credit for it. Focus on the correct parts: "How many agree with this step?" and have the class redo only the part that needs correction.

Encouraging insecure students. In addition to the group techniques for encouraging student participation, there are a number of techniques for building confidence in shy or insecure students. These are particularly helpful at the crucial moment when a student you have called on starts to falter in a response or explanation. There are several techniques for helping him out of a potentially embarrassing spot without his losing face. Some suggestions follow:

1) Techniques for handling errors. (See above section.)

2) Hints. Often a student who is hesitating needs only a hint to continue the answer. Have him or her call on someone to give a hint.

3) Rephrase the question. Restate the question so that the answer is more apparent, or ask a subquestion, or ask for a student volunteer to do so. For example,

   Question: "What's the factor form of 2E3?"
   Restatement: "How many times do I use 2 as a factor in 2E3?"

   Question: $\frac{1}{4} + \frac{3}{8} = ?$
   Subquestion: "Can you shade in $\frac{1}{4}$ of this circle?"

4) Call on someone to work with you. Have the student call on another student to work with him. This is particularly effective when a student gets stuck at the board.

5) Can you tell anything about it? Encourage the student to make some contribution or estimate toward solving the problem. "Can you tell me something about part of it?" "Is it bigger than 10?" "Is it negative?"

6) Call on someone. If the student still doesn't want to try the problem, have her call on someone else.

7) Come back with success. Try to come back to the student during the same class period with a positive experience. Call on him for a question he can answer successfully. Acknowledge his hand when it is raised. Give him a special responsibility, like recording an important result for the next day. Use any of the techniques for involving non-participants listed elsewhere.

8) Preteach. Work with insecure students for a few minutes outside of class. Teach them something the class doesn't know, such as a new Greek letter or the next step in a problem the class is working on. When the opportunity arises in class, they have a chance to star.

9) Encourage questions. Frequently praise students who ask questions when they don't understand something. Point out that asking for help when you don't understand is an important step in learning. Often if the student basically understands, asking him another question will enable him to clear up his own confusion. Asking how many other students have the same question often relieves the pressure of not knowing.

Student interactions.

1) Listening to each other. It is important to make certain that students listen to each other and do not rely on you to determine the correctness of an answer. The signals for agreement and disagreement discussed earlier facilitate this. You can also encourage it

by insisting that students speak so that they can be heard. Do not repeat inaudible answers, but say something like "Joe, did you hear Jane's answer?"; "Sue, repeat your answer so John can hear it."; or "Who can repeat what Bill said?"

2) Mathematical debates. Many people recall taking classes in the humanities in which they were encouraged to debate and discuss ideas with their fellow students, but they do not realize that it is possible to have the same experience in a math or science class, because in mathematics and the sciences "everything is so exact." This is emphatically not the case, however, for in mathematics and the sciences there is much that is ambiguous. The classic example in the Project SEED curriculum concerns the notion that any number raised to the zeroth power is one. It turns out that this is a highly ambiguous "fact," and that it can provide much fruitful debate. Just as in a discussion of philosophy or theology, students must carefully weigh the merits of various points of law. They must call upon their powers of persuasion and reasoning to sway other students towards their point of view. In a good debate everyone is participating in some way, from the vocal students who are doing much of the talking, to the quiet students who are being exhorted to take sides.

A good discovery teacher must be adept at keeping a debate lively and focused. Sometimes it is necessary for the teacher to introduce material into the discussion that will either clear up intellectual logjams, or rattle the foundations of student arguments. All the while, the teacher is trying to orchestrate the discussion so that students will arrive at a lucid understanding of the concepts.

Once teachers who use the discovery style have seen the benefits to be derived from debates, they will be on the lookout for likely topics. "Mini-debates" are possible on just about any subject, such as which way to move the decimal point when converting decimals into percents, or whether you need a common denominator when you multiply fractions, or whether there is such a thing as a square root that is not irrational, but not a whole number either. By being constantly called upon to debate and discuss, students will begin to realize that mathematics is not an obstacle course filled with facts that must be memorized by rote, but rather a subject that is fascinating in its form and structure.

APPENDIX

Checklist of Discovery Techniques

I. General Socratic Strategies

A. Effective Questioning Techniques.

1. Write out question sequence.
2. Keep a log
3. Vary difficulty
4. Vary pace
5. Vary response
6. Many questions on same concept
7. Patterns
8. Parallel problems
9. Be specific
10. Open questions
11. Don't push results
12. Students answer students
13. Don't disguise statements

B. Review.

1. Rapid summary
2. New directions
3. Provocative questions
4. Foundation for new material
5. Embed in new material
6. New notation or terminology
7. Student questions
8. Deliberate errors
9. Mid-lesson reviews
10. Plan at end of class
11. Worksheets
12. Change of pace
13. Involvement techniques

C. Vocabulary.

   1. Introduce in context
   2. Write words on board
   3. Vocabulary page

D. Boardwork.

   1. Organize space
   2. Erase all but important results
   3. Circle for focus

II. Mathematics Development Structures

   A. Gradual Escalation to a Generalization
   B. Equivalent Sentences - Vertical arrows
   C. Use generalizations to extend definitions (Use it in a sentence)
   D. Patterns
   E. Equality, transitivity
   F. Embed in conceptual framework

III. Feedback and Involvement

   A. Circulating
   B. Polling
   C. Finger and Hand Signals
   D. Chorus Response
   E. Rapid Oral Questions
   F. Counting, Naming, Predicting Hands
   G. Chain Answering
   H. Deliberate Errors
   I. Attention and Focus
      1. Are you ready?
      2. Change voice
      3. Challenge
      4. Cued responses

5. Point to an answer
6. Box or circle
7. Read

IV. <u>Building Student Confidence</u>

A. <u>Success Reinforcement</u>.

1. Use names
2. Student agreement
3. Acknowledging other students
4. Students to the board
5. I know you know
6. Experts
7. Star problems
8. Advanced material
9. Who has it now?
10. Reward questions

B. <u>Student Errors</u>.

1. Deliberate errors
2. Allow revision/Call on someone
3. Explore consequences
4. Partial answers

C. <u>Encouraging Insecure Students</u>.

1. Handle errors positively
2. Hints
3. Rephrase questions
4. Call on someone to work with you
5. Can you tell anything?
6. Call on someone
7. Come back with success
8. Preteach
9. Encourage questions

D. <u>Student Interactions</u>.

1. Listening to each other
2. Mathematical Debates

DISCOVERY MATHEMATICS MODULES

A Curriculum Guide for Teachers of
General Mathematics and Prealgebra
in Grades 9-12

United States Department of Education
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Helen Smiler, Project Director
Project SEED, Inc.
2336A McKinley
Berkeley, CA 94703

TABLE OF CONTENTS

FOREWORD

"Discovery Mathematics Modules" is designed as a companion to "Guidelines for Discovery Teaching." The "Guidelines" catalog the general techniques and teaching strategies of a successful discovery teacher. The "Modules" take specific topics in mathematics and detail how to teach them through a Socratic, question-and-answer process. We recommend that the reader be familiar with the "Guidelines" before beginning this volume.

These "Modules" were written as part of Project SEED's contract with the U.S. Department of Education, Basic Skills Improvement Program. Project SEED staff in Atlanta, Boston, Chicago, Detroit, Los Angeles, and Oakland have taught the curriculum outlined in these "Modules," and their experiences have been incorporated into this final draft. The "Modules" are targeted for general mathematics and prealgebra students in grades 9-12. However, with suitable modification, many of the topics are appropriate for other grade levels, particularly grades 3-8. In fact, much of the material in these modules is an elaboration and extension of lessons which have been used successfully by SEED-trained mathematicians and scientists in elementary school classrooms throughout the country. The Basic Skills Project represents the initial step in extending the program to the secondary level and the first attempt to capture the curriculum and methodology on paper for use by secondary teachers.

As the "Guidelines" point out, the goals of discovery teaching are student success, involvement, and conceptual understanding. Students who believe they can learn will learn. Students who learn to question and think critically with advanced mathematical topics will begin to approach other subjects in the same manner. These mathematics modules are designed to achieve these goals when used with the techniques and strategies of the "Guidelines."

Because the skills and needs of general math and pre-algebra students vary widely, we have designed these modules as an overall framework for instruction, not as an explicit day-to-day lesson plan. Use them to supplement a standard program or use them as your central focus and provide students with extra practice on the standard curriculum.

There are six modules, each focusing on a different topic although several unifying concepts recur throughout. The topics have been chosen because they are central to further study of mathematics, motivate the acquisition and review of basic skills, lead to conceptually interesting results, develop problem-solving and critical-thinking skills, or illustrate how to teach an arithmetic concept from an algebraic point of view. The rationale for choosing a particular topic is included in the introduction to each module.

Each module contains sequences of problems and suggested questions which lead students to discover the basic concepts of the unit and which help create the positive learning atmosphere described in the "Guidelines." Alternate approaches based on classroom experience, and suggestions for review and related activities, are also included. No one is expected to ask all of the questions or use all of the approaches. They are included to allow you to choose the ones which fit best with your students' needs and abilities and to demonstrate a variety of approaches to developing mathematical concepts that you can apply to teaching other topics. Small, intuitively understood numbers are used in the presentation of new concepts. We have found this the most effective way for students to learn and understand the concepts. Practice with grade level skills (multiplication of three-digit numbers, addition of mixed numbers, division of decimals, etc.) can be woven into classwork and assignments although students should always be asked to begin with a few simple problems to refresh their understanding of the concept involved. As a reminder, we have pointed out opportunities for arithmetic review at several points in the "Modules." However, this can and should be an ongoing activity if your students need it.

Written work, both classwork and homework, is an essential part of the learning process. Because the progress of a discovery class is based on student response, it is necessary to tailor assignments to the needs and skill level of each class. The most appropriate assignments often arise directly out of the lesson such as "Extend the table of powers of 2 up to 2^{20}" or "Find the next four partial sums of $\sum_{i=1}^{n} 2^{-i}$." You will also want to include problems that incorporate practice with the computational skills your students need to build. A set of exercises is included at the end of each chapter. You can assign these problems as the class reaches them or use the entire set as a review at the end of the unit. They can also serve as models for the types of problems you should assign as the unit progresses. You might also want to have students make up assignment sheets for their peers using these problems as models. You can embed computational review in the assignments on algebra or use standard texts and workbooks as supplementary materials.

Each chapter ends with a chapter summary outlining the key concepts of that particular module and a chapter test, a criterion-based test on those concepts. These tests can be reproduced and given as is, although you may also want to add to them based on the work you have done with your class.

We suggest that before teaching the material you read through the entire volume and make a preliminary plan. It is possible, for example, to skip directly from Module 1 to Module 4, weaving in the results from Modules 2 and 3 only as they come up in context.

The topics included here are not the only ones possible. The basic concepts underlying virtually any topic in high school and much college-level mathematics (functions, graphing, integration, etc.) can be taught by discovery with conceptual understanding to students from grade 4 on up. Anyone who has successfully taught the "Discovery Mathematics Modules" should have little difficulty applying the same principles to teaching other topics.

A final note of caution is in order. Teaching by discovery is not easy. It represents a radical departure from the "Here is an example. Now you do one." school which mathematics education often becomes. It requires preparation and planning--the next lesson isn't on the next page of the book. It seems slow and tedious and there's a real possibility of not covering everything. We can only point out that from our experience, the extra effort pays off in students who are more motivated and retain more of what they learn.

MODULE 1: EXPONENTIATION

1.0 Introduction

There are a number of reasons for choosing exponentiation as the subject of the first discovery unit. The goal is to build students' confidence by providing them with success in a topic from algebra, a high status subject. Exponentiation is conceptually accessible to virtually all students regardless of their computational skills. Since it is new to them, it is free from the connotations of previous failure that often cause students to tune out of remedial lessons. At the same time, it can provide a framework for extensive arithmetic review and practice. Working as a group, the class can quickly arrive at both an impressive mathematical generalization and a challenging debate topic. Exponentiation also provides a unifying thread that can be woven throughout an entire year's work.

Cover this material carefully, although do not demand mastery of all of the concepts at this time. Some are intended to provoke debate which should not be closed off prematurely and others are best introduced later in the year as review. While this chapter does introduce mathematical concepts that will be woven throughout the rest of the curriculum, your primary goal is to build in your students the foundation of, approach to, and an attitude toward, mathematics that will support the rest of the year's work.

1.1 Basic Definitions

The best way to introduce exponentiation is by presenting a new binary operation. The usual exponent (power) notation contains an implied binary operation. For example, 6^2 means that there is a relationship between the 6 and the 2. An explicit notation will clarify many mysteries of exponentiation. Before introducing this new operation, the old operations need to be reviewed in the following format:

6 2 =
6 2 =
6 2 =
6 2 =

Ask the class to fill in the familiar operations: addition, subtraction, division, and multiplication along with the answers. A more natural way of coming up with this format is with a short review of odd and even numbers.

> Who has an even number?
>
> How many agree that 12 is an even number?
>
> Who has an odd number?
>
> How many agree?
>
> Who has another even number?
>
> How can you tell if a number is even or odd?
>
> Who has a smaller even number?
>
> Who has a greater even number?

So when 6 is mentioned, write it on the board. After a few questions, the 2 is mentioned and written down.

6 2

Write the relational symbol =

6 2 =

> Who can make a mathematical sentence with the 6, 2, and = ?

Does anyone have an answer?

What operation are we using?

What is a mathematical name for plus?

Write 6 and 2 again.

$$6 + 2 = 8$$

$$6 \quad 2$$

and continue on.

Here are some questions that can be asked of the class as the above chart is being completed.

Who has an operation for the first line?

Who has another?

Who has another name for times?

Who has another name for plus?

Who has another name for take away?

Who has another name for minus?

Who has an easy word for 'subtraction'?

Who has the answer to the addition problem?

Who has the solution to the last problem?

Who can combine 6 and 2 using division?

Who knows the sum of 6 and 2?

Who has the product of 6 and 2?

Which operation goes with the word, 'product'?

What is the difference between 6 and 2?

On this line which is the divisor?

On this line which is the minuend?

On this line which is the subtrahend?

Which number(s) represents the dividend?

Which number(s) represents the quotient?

Which number(s) represents the sum?

Which number(s) represents the difference?

Which number(s) represents the addend(s)?

Which number(s) represents the factors?

Which number(s) represents the product?

Is this mathematical sentence closed or open?

Is this sentence closed?

Is '=' an operation?

Is '6' an operation?

How many factors are there on this line?

Who can spell multiplication?

Who can spell division?

Who can spell subtraction?

Who can spell addition?

How many letters are in multiplication?

How many letters are in division?

How many letters are in subtraction?

How many letters are in addition?

What is 8 x 8, then?

Which operation gives the greatest answer?

Which operation gives the largest answer?

Which operation gives the smallest answer?

How many operations do we have so far?

Are there any more?

On your fingers show me how many operations we have.

Quickly, who can name the operations?

On your paper, write the operations.

Who is sure that we have all the operations?

At this juncture, one can ask the deliberately open-ended question, "Who can think of another operation?" This is an example of the technique discussed below.

Student responses to questions may be handled in several ways. The preferable way is to accept the student's suggestion and have the class analyze the reasons for his/her particular suggestion. In this case, when the class is pressed to come up with a different operation, many ideas will be expressed, some of which are

(a) Fraction

(b) Square root

(c) Percent

(d) Decimal

(e) Equal

Note that some of these responses reflect the students' incomplete understanding of 'operation'. An example of a creative way to handle the response 'fraction' is to write F between 6 and 2

6 F 2 =

and ask the class to make a fraction using 6 and 2. Once $\frac{6}{2}$ or $\frac{2}{6}$ is obtained, one could continue with a very brief chart on simplification of fractions

$$6 \text{ F } 2 = \frac{6}{2} = 3$$

$$\text{or} \quad 6 \text{ F } 2 = \frac{2}{6} = \frac{1}{3}$$

or continue with further examples, such as 7 F 2, 10 F 2. Then tell the class that fractions were not what you had in mind. Handle subsequent responses in a similar fashion. This methodology is preferable, not only because it rewards the student for participating in the class but because it promotes the analytical thinking of students.

Now the class is ready for the 'new' operation. 6 and 2 is written below the existing chart.

$$6 + 2 = 8$$

$$6 - 2 = 4$$

$$6 \times 2 = 12$$

$$6 \div 2 = 3$$

$$\boxed{6 \quad 2 \qquad}$$

Again ask the class if anyone can think of a different operational symbol. After a while write 'E' between 6 and 2.*

---

*6 E 2 is used at this point to emphasize that exponentiation is a binary operation. When the students are familiar with the concept you might want to introduce the notation from various computer languages as well as the standard power notation. For example, in Basic $2 \uparrow 3 = 2^3$ and in

Ask the class to read the open ('open' because it contains an open variable '□') mathematical sentence. Draw a square after the = . Now the last line looks as follows:

$$\boxed{6 \text{ E } 2 = \square}$$

Some combination of the following questions should be asked:

Has anyone seen this operation before?

Who can read the new operation?

Does the new operation look different from the others?

Who can read the last line again?

What answer goes in the square?

At this point, we briefly interrupt the above question-asking sequence to make another comment on methodology. Many students will have no way of knowing what number goes into the square. But the questioning technique does not require a student response to (absolutely) <u>every</u> question asked. Moreover, there are many instances when the question will not be understood by the class. Thus, rephrasing the question becomes necessary. In this particular case, guessing is encouraged. Ask the class how the 6 and 2 were used or combined to obtain, for example, 26, a frequent answer. Some typical responses are:

Fortran 2**3 = 2^3. In Fortran, 2 E 3 is used for scientific notation, 2 x 10^3. We have found that this potential ambiguity is not confusing. In fact, it illustrates the fact that notation is a convention and that you must be certain you know what the underlying rules and assumptions are whenever you do mathematics.

(1) Take the 2 and put it in the tens place and 6 in the ones place.

(2) Put the smaller number in the tens place and the larger number in the ones place.

(3) Write down the smaller number first, and follow with the larger number.

Reward the student by naming his suggested system, e.g., 6 J 2 = 26 and having the class do some additional problems using it. But assert that you have a different system in mind, a system that uses 6 and 2 in such a way as to obtain an answer greater than 26. Repeat in a similar way with subsequent responses until 36 is obtained.

This methodology is highly stimulating for the class. It requires patience and recognition of the fact that students can devise intricate mathematical systems.

New we return to elaborate further the question sequence on exponentiation.

> Is the suggested solution of 6 E 2 = □ different from the others? Who can guess at the answer?
>
> The answer is greater than . . . (student response).
>
> The answer is less than . . . (student response).
>
> Good thinking. Who knows what system he/she used to get that answer?

When a student arrives at the correct number, '36' is written inside the square. Ask the class to read the newly closed mathematical sequence. Erase the chart from the board, and ask the class if anyone remembers the last line. Some student will say "6 E 2 = 36." Write it on the board. Below '6 E 2 = 36' write '7 E 2 = '. Have the class conjecture the solution. Ask questions similar to the sequence above. Repeat with '8 E 2 = ', '9 E 2 = '. By this time, students will want to explain how one arrives at the solution.

Now the chart looks as follows:

6 E 2 = 36
7 E 2 = 49
8 E 2 = 64
9 E 2 = 81

How many of you know how 'E' works?

How did we get '36'?

How did we get '49'?

How did we get '64'?

How did we get '81'?

On your paper write the answer to '10 E 2 = '.

On your paper write the answer to '11 E 2 = '.

Who wants to come to the board to fill in an answer to 10 E 2?

Who wants to come to the board to fill in an answer to 11 E 2?

Now erase 36, 49,..., 121 off the existing chart. You will now replace the numerical forms 36, 49, etc., with factor forms by asking appropriate questions of the students. The completed chart looks as follows:

6E2=6x6=36
7E2=7x7=49
8E2=8x8=64
9E2=9x9=81
10E2=10x10=100
11E2=11x11=121

How did we get 36?

What is the factor form for 36?

What is the factor form for 6E2?

How many factors are in 7E2?

How many factors are in 8E2?

How many 9's are in the factor form 9E2?

How many 10's are in the factor form for 10E2?

Which part of this line is a factor form?

Which part of this line is the numerical form?

Which part of this line is the E form?

Why do we have two 9's in the factor form?

Which part of 10E2 tells us to write two 10's?

Which part of 10E2 tells us to write 'x'?

How do we know that we use 9's instead of 10's as a factor?

Who can come to the board and circle the number that tells us to write 3's instead of 11's?

Who can come to the board and draw a circle around the number that tells how many 3's to write?

That 2 is called an 'exponent'.

Again we interrupt the narrative, to make a comment on technique (in this case, the introduction of terminology).

As a rule, two or three vocabulary words are introduced formally within a lesson. But (surreptitious) use of new words within questions is desirable. For example, once we have 6E2=36 on the board,

How did we get 36? (Response: 6x6.)

How did we get 49? (Response: 7x7.)

What is a factor form of 64? (Response: 8x8.)

The reason the students' response will be 8x8 is because of the sequential format of the questioning. The students know on which concept we have focused, and most of them intuitively know the answers before the question is asked.

Eventually, formal introduction is desirable. For example, when we have

6E2=6x6=36

on the board, we might ask the class:

Which part of the sentence is the factor form?

Which part is the E-form?

Which part is the numerical (number) form?

Again we resume the narrative development.

Who can raise their hand to spell 'exponent'?

On your papers solve 22E2.

Again a digression on methodology, this time on the embedding of arithmetic.

By exploiting the context of exponentiation, questions may be designed for review and for reinforcement of arithmetic skills.

2.5E2=	Multiplication of decimals
$\frac{1}{2}$E2=	Multiplication of fractions

$\frac{3}{4}E2=$	Multiplication of fractions
$2\frac{1}{2}E2=$	Multiplication of mixed numerals

Once the additive property of exponents has been grasped, further reinforcement of arithmetic skills can be carried out.

(2E__)x(2E52)=2E69	Subtraction of integers
(2E4.56)x(2E5.44)=__E__	Addition of decimals
(2E4.567)x(2E7.4)=__E__	Addition of decimals
$(2E\frac{1}{4})x(2E\frac{3}{4})=$__E__	Addition of fractions
$(2E\frac{1}{4})x(2E\frac{1}{2})=$__E__	Addition of fractions
$(2E\frac{3}{4})x(2E\frac{2}{7})=$__E__	Addition of fractions
$(2E5\frac{1}{2})x(2E7\frac{1}{3})=$__E__	Addition of mixed numbers
(2E4.56)x(2E__)=2E7.56	Subtraction of decimals
$(2E\frac{1}{4})x$(2E__)$=2E\frac{1}{2}$	Subtraction of fractions

Returning once again to the sequential development of exponentiation:

On your papers, solve 342E2.

On your papers, solve 3,452E2.

The other number is called a 'base'.

Who can spell base?

Who can make up an 'E' sentence using 5+2?

Who can make up an 'E' sentence using 15 for a base?

Who can make up an 'E' sentence using 2 as an exponent?

Who can make up an 'E' sentence using 5 as an exponent?

Who can make up an 'E' sentence using 3 as an exponent?

Who can make up an 'E'sentence using 5 as a base and 3 as an exponent?

What is the factor form for 5E3?

What is the factor form for 5E4?

What is the factor form for 5E7?

How many see how the exponent works?

What is the E-form for 6x6x6x6x6?

What is the E-form for 2x2x2x2?

What is the E-form for 1x1x1x1x1x1x1x1x1?

What is the answer for 1E9?

What is the answer for 1E50?

What is the answer for 1E100?

What is the answer for 1E0?

Now we set up a chart for base 2. This chart should be displayed on the board daily until the class is familiar with it. A project for some students is to make up a 2E-chart to be displayed permanently on the wall, bulletin board, etc.

2E1=2
2E2=2x2=4
2E3=2x2x2=8
.
.
.
.
.
.
.
2E10=2x2........x2=1,024

In working with the chart, students will begin to notice that you can get each entry in the table by doubling the previous one. Have them come to the board and prove their theory by using parentheses to identify the preceding problem.

> Who can put parentheses around a problem we've already solved?
>
> Who can find a 32 in 2E6?
>
> Who can prove that 2E6 is 32x2?
>
> Who can find 2E5 in the factor form of 2E6?

```
2E5=2x2x2x2x2=32

2E6=(2x2x2x2x2x2)x2=?
         ↓          ↓ ↑
         32   x     2=64
```

With many classes, you may wish to generalize after a few more examples.

> If 2E50=Q, how do you find 2E51?
>
> If 2E100=P, 2E101=?
>
> If 2EK=R, 2E(K+1)=?

A challenging question is to ask the class how long they think it will take to reach one million. How large does n have to be so that 2En>1,000,000? Record the answers (usually much too large) and leave the question open until a student has worked out the answer.

1.2 The Additive Property of Exponentiation

Review is extremely important. Some days the entire development of E should be repeated from the beginning. This review will go rapidly, and some new material can be introduced towards the end of that lesson. About this time, the class is ready for the introduction of a very useful property of 'E'.

Write on the board:

(2E3)x(2E4)=___

Questions:

Any guesses?

Who has a conjecture?

What is the thinking behind that guess?

What is the factor form for (2E3)?

What is the factor form for (2E4)?

What is brought down for the 'x'?

How many two's altogether?

How do we write this in E-form?

How do we write this in exponential form?

On the board:

$$\begin{array}{l} (2E3)x(2E4)=2E7 \\ \quad\downarrow\quad\downarrow\quad\downarrow \\ 2x2x2x2x2x2x2=2E7 \end{array}$$

Write:

(2E4)x(2E5)=

Guesses:

Factor form for 2E4, 2E5.

How many factors altogether?

Who could put that into exponential form?

Erase and change exponent.

What should be changed to make this into a true sentence?

More erase and change:

(2E4)x(2E6)=__E__

What number should the base be?

What number should the exponent be?

Where did the 10 come from?

Did the 10 come from the base?

Did the 10 come from the exponents?

What did we do with the exponents?

Try this on your paper:

(2E643)x(2E742)=__E__.

(2E3)x(2E10)

I would like to erase this 10 and use numbers to show where the 10 came from.

Write:

(2E4)x(2E6)=2E(4+6)

And back to:

(2E4)x(2E5)=2E(__+__)

$$(2E4)x(2E4)=2E(\underline{\quad})+\underline{\quad})$$

Try this on your paper:

$$(2E5)x(2E6)=\underline{\quad}E(\underline{\quad}+\underline{\quad})$$

Erase and change the first exponent '5'.

> What should I change on the right side of the equal sign?
>
> Is the sentence true now?
>
> How many agree?

More erase and change on the first exponent. Pace should become rapid at this point.

Write:

$$(2E\beta)x(2E4)=2E(\quad+\quad)$$

> What should go here? And what should go here? (Pointing to the places indicated.)
>
> Who can read the sentence?
>
> Who else can read the sentence?
>
> Who knows what the new symbol is?
>
> How many of you have seen it before?
>
> Who can spell 'beta'? (Write beta on a side board.) How many have heard of beta before?
>
> Which alphabet do you think it comes from?
>
> What country did it come from?
>
> Who knows what the beta stands for?

The following development will result in a beautiful generalization of the Additive Property of Exponentiation using Greek letters for universal variables.

Can we change the second exponent?

Who has a number for this exponent?

Who has a decimal number for this exponent?

Who has a fractional number for this exponent?

How many agree with the sentence?

How many disagree with the sentence?

The board work looks as follows:

$$(2E\beta)x(2E4)=2E(\beta+4)$$

$$(2E\beta)x(2E4.3)=2E(\beta+4.3)$$

$$(2E\beta)x(2E3\frac{1}{2})=2E(\beta+3\frac{1}{2})$$

The changing of exponents using various numbers should follow a rapid pace to accentuate their interchangeability. Then write 'γ', the third letter of the Greek alphabet.

$$(2E\beta)x(2E\gamma)=2E(\beta+\_\_)$$

And ask the class to complete (close) the sentence.

Who can read this sentence?

How many agree with the reading?

Who can read it quicker?

Erase the sentence.

Who remembers the sentence?

Who else remembers?

Write it on the board as a student says it. Now we are ready to vary the base.

Do we have to use '2' for the base?

What other number can we use?

Who has another number we can use?

Can we use 99? 729? 6? 1?

What about decimal numbers?

How about fractional numbers?

How many different numbers can we use?

Who knows what we are going to use to represent the base?

What can we use to generalize the base?

Can anyone think of another Greek letter?

Who knows the first Greek letter?

Who can spell 'alpha'?

Who knows how to draw the symbol for alpha?

On the board write α.

On your paper draw the symbol for alpha.

$$(E\beta)x(E\gamma)= E(\beta+\gamma)$$

(Place α here .)

If I put alpha here, what other places would I have to put alpha?

How many agree?

Who thinks they can read the entire sentence now?

$$(\alpha E\beta)x(\alpha E\gamma)=\alpha E(\beta+\gamma)$$

How many will remember this mathematical sentence?

How many will forget this mathematical sentence?

How many will remember that this is the generalization for the Additive Property of Exponentiation?

Reinforce this result by substituting small values for the variables and verifying that the result is a true mathematical statement.

1.3 Zero Exponents

The additive property can be used to illustrate an important mathematical principle and to provoke a stimulating mathematical debate.

Review powers of 2:

$$\begin{aligned}
&2E5=2x2x2x2x2 &&=32\\
&2E4=2x2x2x2 &&=16\\
&2E3=2x2x2 &&= 8\\
&2E2=2x2 &&= 4\\
&2E1=2 &&= 2
\end{aligned}$$

What would the next problem be? (Response: 2E0.)

What do you think the numerical value of 2E0 is? Why?

The usual answer is "zero since you don't write any twos down." To set the ground for future discussion, agree that this is a reasonable response. You might have a student notice the pattern of doubling and suggest 'one'. Agree that that makes sense, too. Finally, to create a further sense of ambiguity, you may wish to point out that in the table of powers of two, the last digits of the numerical values repeat the pattern 6, 8, 4, 2, 6, 8, 4, 2,..., so 2E0 ought to be 6.

To help us figure out what 2E0 should be, let's put it in a true mathematical sentence using the additive property.

What would the base be?

What should the first exponent be?

Who's got a number less than 10 for the second exponent?

What operation do I use between the two terms?

Who can tell me what goes on the right side of the equal sign?

Do we know the numerical value of any part of this sentence?

What do I bring down for the 'x'?

On the board, you have something like:

$$\begin{array}{ccccc} (2E0) & x & (2E3) & = & 2E(0+3)^{*} \\ \downarrow & & \downarrow & & \downarrow \\ \square & x & 8 & = & 8 \end{array}$$

---

[*] Technically, our derivation of the additive property makes sense only for positive integral exponents although the question, "How many times is 2 used as a factor in the first quantity? all together?" makes sense in this case. A rigorous treatment of this topic would take one of two approaches: (a) use the above sentence as scratchwork to motivate a definition 2E0=1 and then prove the additive property still holds for zero exponents or (b) assume 2E0 exists and follows the

How many times is 2 used as a factor in 2E0?

How many times is 2 used as a factor in 2E3?

How many times is 2 used as a factor all together?

Is this a true sentence?

What do most people think 2E0 is?

When you look at the '2E' table, 2E0 seems to be what?

Does anybody think we have a problem?

What happens if we use it in a different sentence?

We have a paradox. In the '2E' table 2E0 looks like it should be ____________ and when we use it in a sentence it looks like it should be ____________.

At this point, students may try to resolve the controversy in favor of 2E0=0 by changing □x8=8 to □+8=8 or □x8=0. Credit students for their thinking and explore the origins of □x8=8 to show that it cannot be changed if (2E0)x(2E3)=2E(0+3) is true.

If you feel a need for closure at this point and want to resolve the issue, you can invoke mathematicians' great love of formulas to weight the discussion: "Mathematicians want the Additive Property to be true for all exponents. What does 2E0 have to be?" This course of action closes off a concept which tends to motivate genuine debate and discussion.

---

additive property and then prove that 2E0 is one. In either case, the degree of mathematical sophistication required would mask an opportunity for creative mathematical discovery and would be lost on students at this level. Throughout this document we will continue to use this type of argument without apology.

Many instructors leave the question open for some time, occasionally exploring other lines of reasoning that support the standard conclusion, 2E0=1. As the weight of the evidence accumulates, students gradually move from the 'zero' to the 'one' camp and become more and more zealous in their attempts to convert their classmates.

Here are some alternate ways to approach 2E0. You may wish to look at them all now or you may choose to intersperse them with other work so that the intellectual tension surrounding 2E0 continues to grow.

(a) Parallel problem sets (do the left column first, then the right)

(2E5)x(2E3)=__E__	$\square$x8=8
$\square$x(2E3)= 2E7	$\square$x1982=1982
$\square$x(2E3)=2E29	$\square$xn=n
$\square$x(2E3)=2E101	$\square \mathrm{x} \sum_{1=1}^{5} i = \sum_{1=1}^{5} i$
$\square$x(2E3)=2E3	$\square$x(2E3)=2E3

(b) Patterns

How do you get from one entry to the next going up the '2E' table?

How do you get from one entry to the next going down the '2E' table?

$$\begin{array}{ll@{\qquad\qquad}ll}
2E5=32 & & 2E5=32 & \\
2E4=16 & \uparrow\times 2 & 2E4=16 & \downarrow\div 2 \text{ or } \times\frac{1}{2} \\
2E3=\ 8 & \uparrow\times 2 & 2E3=\ 8 & \downarrow ? \\
2E2=\ 4 & \uparrow ? & 2E2=\ 4 & \downarrow ? \\
2E1=\ 2 & \uparrow ? & 2E1=\ 2 & \downarrow ? \\
2E0\ = & \uparrow ? & 2E0= & \downarrow ?
\end{array}$$

(c) Subtractive Property

Develop the 'subtractive property': $\frac{\alpha E\beta}{\alpha E\gamma} = \alpha E(\beta-\gamma)$.

There are several ways to approach an illustration of this principle:

(1) $\frac{2E7}{2E4} = \frac{128}{16} = 8 = 2E3$

(2) $\frac{128}{16} = 8 \longleftrightarrow 8 \times 16 = 128$

How do you use multiplication to check division?

Who can give me a multiplication sentence equivalent to this division sentence?

$$\frac{2E7}{2E4} = \square \longleftrightarrow \square \times (2E4) = 2E7$$

(3) $\frac{2E7}{2E4} = \frac{\cancel{2}\times\cancel{2}\times\cancel{2}\times\cancel{2}\times 2\times 2\times 2}{\cancel{2}\times\cancel{2}\times\cancel{2}\times\cancel{2}} = 2E3$

Derive several more examples. Then generalize for nonzero bases.

Now look at problems like $\frac{2E3}{2E3}=?$

Once the class has decided 2E0=1, you can look at other problems: 3E0, $\frac{1}{2}$E0, 1.2 E0, etc., and generalize, αE0=1 for $\alpha \neq 0$.*

1.4 Review and Related Activities

(1) Commutativity

Is 2E3=3E2? Is it ever true that aEb=bEa?

If a#b=b#a for all numbers a and b, # is called a commutative operation.

What are examples of commutative operations?

What are examples of noncommutative operations?

(2) Introduce standard notation, 2^3=2x2x2. Review previously learned concepts using power form.

(3) Use exponentiation to review and reinforce basic computational skills appropriate for your class, e.g.:

$(\frac{1}{2})^3 =$

$(.2)^4 =$

$(\frac{1}{2})^2 + (\frac{1}{2})^1 =$

$(2E\frac{1}{2}) \times (2E\frac{3}{2}) =$

etc.

---

*0E0 remains undefined since there is more than one logical answer. For example, $\lim_{x\to 0^+} 0^x = 0$ and $\lim_{x\to 0} x^0 = 1$. Investigation of 0E0 can inspire some excellent debates and an understanding of the fact that not all problems in mathematics have solutions. Unless it arises spontaneously, however, it should only be brought up after the class has developed a strong sense of mathematical confidence.

(4) Use the additive property to multiply large powers of 2:

Review the chart of powers of 2.
Ask the students to multiply:

32x64=

128x16=

256x32= , etc.

Soon some students will begin to get answers from the table. Use them as judges. When more students catch on, have them explain what they are doing. Write each number in exponential form and use the additive property to verify the result.

Multiplying by adding exponents is an example of using a logarithm table to multiply large numbers.

(5) Distributive Law

Change (2E3)=2x2x2 to 2x3=2+2+2 and derive the distributive law (2x3)+(2x4)=2x(3+4).

See Module 5 for a parallel analysis of 'E' and 'x'.

1.5 Exercises

1. Fill in the missing numerals.

Exponential Form		Factor Form		Numerical Form
6E2	=	________	=	____________
10E2	=	________	=	____________
____	=	7x7	=	____________
mE2	=	________		
3E4	=	________	=	____________
____	=	3x3x3x3x3	=	____________
____	=	kxkxk		

2. In 2E3, which number is the base? _____

   In 2E3, which number is the exponent? _____

   What does the exponent tell you? ________________

   ____

3. Complete the table of powers of 2 up to 2E20.

4. Make a table of powers of 10 up to 10E9. Beside each write the name of the number it represents.

5. a. A colony of bacteria doubles in population every hour. If you start with 1 bacterium, how many do you have after

   1 hour? _____

   2 hours? _____

   3 hours? _____

   4 hours? _____

   n hours? _____

b. Repeat above question for the case where you start with 10 bacteria.

What if you started with 100? 200? 300? γ?

c. What if the population tripled every hour?

d. A population of amoebae in a laboratory vial doubles every minute. After 15 minutes, the vial is half full. How long will it take for the vial to be full of amoebae?

6. Fill in the missing numerals:

(3E5)x(3E4)=___E___

(3E0)x(3E2)=___E___

(3E   )x(3E7)=3E9

(2E101)x(2E27)=___E___

(2E   )x(2E   )=2E9   Is there more than one solution to this problem?

How many whole number solutions are there?

(2E3)x(2E4)x(2E5)=___E___

7. If (2E3)x(2E5)=2Ey, y=?

If (2Ez)x(2E6)=2E17, z=?

If (mEq)x(mE9)=mE14, q=?

If (2En)x(2E7)=2E7, n=?

8. Show that (2E5)x(2E3)=2E8 using factor forms.

Show that (3E2)x(3E4)=3E6 using numerical forms.

9. Fill in the missing numerals:

(3E0)x(3E2)=___E△
↓   ↓   ↓
□ x ▽ = 9

10. Find the numerical value of (be careful):

a. (2E1)+(2E2)=

(2E1)+(2E2)+(2E3)=

(2E1)+(2E2)+(2E3)+(2E4)=

b. $(\frac{1}{2}E1)+(\frac{1}{2}E2)=$

$(\frac{1}{2}E1)+(\frac{1}{2}E2)+(\frac{1}{2}E3)=$

c. (3E2) + (5E2) =

1.6 Chapter Summary

Exponential Notation

2E3	= 2x2x2	= 8
Exponential Form	Factor Form	Numerical Form

Additive Property of Exponentiation

$(\alpha E\beta) \times (\alpha E\gamma) = \alpha E(\beta+\gamma)$

Zero Exponents

2E0=1

$\alpha E0=1$ for $\alpha \neq 0$

Optional Topics

Multiplication of powers of 2 using logarithms.

Subtractive property $\frac{\alpha E\beta}{\alpha E\gamma} = \alpha E(\beta-\gamma)$ for $\alpha \neq 0$

Vocabulary

Operations

Exponentiation

Exponent

Base

Exponential form

Factor form

Numerical form

Additive property of exponentiation

Equivalent sentences

Generalization

Variables

Paradox

1.7 Chapter Test

1. Put a circle around the base in 2E3.

   Put a square around the exponent in 6E7.

   Underline the operation in 9E2.

2. Fill in the missing numerals:

Exponential Form		Factor Form		Numerical Form
3E4	=	_____	=	______
_____	=	5x5x5	=	______

3. What is the factor form of αE5? _____

   What is the exponential form of $\gamma x \gamma x \gamma x \gamma$? _____

4. Give the numerical value for:

   10E2=_____

   10E3=_____

   10E4=_____

   10E7=_____

5. (2E5)x(2E4)=___E___

6. (2E□)x(2E7)=2E15

7. (αEβ)x(αEγ)=___E(___+___)

   What do α, β, and γ stand for in this sentence?

8. Show that (5E2)x(5E4)=5E6 using factor forms.

9. Fill in the missing numerals:

$$\begin{array}{ccccc} (4E0) & \times & (4E2) & = & \_\_\_E\_\_\_ \\ \downarrow & & \downarrow & & \downarrow \\ \square & \times & \triangledown & = & 16 \end{array}$$

What is 4E0 acting like in this sentence? _____

10. Find the numerical value:

(2E3)+(2E3)=_____.

MODULE 2: FRACTIONS

2.0 Introduction

The present chapter on fractions is not meant to be an exhaustive treatment of the subject. Fractions are covered comprehensively in every "general mathematics" textbook and such a text should be used to supplement and provide extra practice for what is done here. Our goal in this section is to continue the spirit of mathematical inquiry that was started in Chapter 1 and to show that an arithmetic topic like "fractions" can be approached from an algebraic point of view. To this end, we develop the general result that $a\mathrm{x}\frac{1}{a}=1$ for $a\neq0$ and use it to derive the rule for the multiplication of fractions. Only then do we move to addition of fractions, a much more difficult topic, which is generally introduced first. Our approach to division of fractions is also based on the same basic principle.

2.1 Multiplicative Inverses

There are many approaches to teaching fractions. The approach which we adopt here seeks to find a common thread among the various algorithms for combining fractions. What does '$6\div\frac{1}{2}=12$' have to do with '$\frac{1}{2}+\frac{1}{2}=1$'? What does '$6\div\frac{1}{2}=12$' have to do with '$6\mathrm{x}2=12$'? What does '$3\frac{1}{2}\div\frac{1}{2}=7$' have to do with '$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=\frac{3}{2}=1\frac{1}{2}$'? What does '$\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$' have to do with '$\frac{1}{4}+\frac{1}{4}=\frac{2}{4}=\frac{1}{2}$'? What does '$\frac{1}{2}\mathrm{x}\frac{1}{2}=\frac{1}{4}$' have to do with '$\frac{1}{2}\mathrm{x}1=\frac{1}{2}$' and '$\frac{1}{2}\mathrm{x}0=0$'? In teaching fractions (indeed any mathematics) the trick is to involve the entire class by employing a mix of questions of varying degrees of difficulty and by weaving in a variety of feedback and involvement techniques. The sequence of questions and techniques below should not be followed inflexibly. It is intended only for illustration. Each class requires the teacher to be daily inventive, to constantly revise one's questions and approach.

We begin with a very brief review of open and closed sentences. A sentence such as '$\boxed{}+\boxed{}=8$' is not properly closed in the form (e.g.) '$\boxed{5}+\boxed{3}=8$,' although '5+3=8' is true. On the other hand '$\boxed{5}+\boxed{5}=8$' is false but properly closed. Perhaps this becomes more apparent if the above sentence is written as follows:

$$n + n = 8$$

where the solution of 'n' is unique. It cannot be that the first 'n' is 5 and the second 'n' is 3. As a matter of fact, if the class is competent in solving sentences like '2+n=6', '5xn=25', '27÷n=9', etc., a good approach to review fractions is to use 'n+n=8'. In the following development we shall use geometric variables, but one can safely substitute the 'algebraic' approach without much adjustment.

Some typical questions to be asked are:

Who can read this sentence?

Who can read this open sentence?

What is the name of the open variable?

Why do we call the 'square' an open variable?

Why do we call this sentence an open sentence?

How can we close this sentence to make it true?

Does using 5 and 3 make the sentence true?

Does using 4 and 4 make the sentence true?

Does using 4 and 1 make the sentence true?

Which is the only way to close it properly and have the sentence true?

Can we close it properly and have it false?

Can we close it improperly and have it true?

Can we close it improperly and have it false?

How many agree?

How many disagree?

Who wants to tell me why you disagree?

When the sentence is closed properly, and is true, write

$$\boxed{4}+\boxed{4}=8$$

$$\boxed{}+\boxed{}=10$$

and use similar questioning techniques as above. Repeat with $\boxed{}+\boxed{}=12$, $\boxed{}+\boxed{}=14$. One should increase the pace of questions once the concepts become clearer. Now back to the questions.

Is the number after the '=' odd or even?

Who can make up a similar open sentence using odd numbers?

Write it on the board.

$$\boxed{}+\boxed{}=5$$

How many agree this sentence is open?

How many of you can solve this equation?

Who knows what number goes in the square?

Does $2\frac{1}{2}$ go in the first square or the second square?

How many agree that $2\frac{1}{2}$ goes in both squares?

Is the sentence closed properly?

Who can make up another similar sentence using an odd number?

How many can solve it on your paper?

What is the smallest odd number?

Who can make up a similar sentence using it?

Who can solve it?

How many agree that the sentence is true?

How many agree that the sentence is properly closed?

How many agree that the sentence is true and properly closed?

Erase the board work, and ask the class if anyone remembers the last sentence. Write it on the board when someone gives you

$$\boxed{\frac{1}{2}}+\boxed{\frac{1}{2}}=1$$

What number can we add three times to obtain the sum of 1?

How many agree that $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=\frac{3}{3}=1$?

Will $\frac{1}{3}$ close the sentence properly?

Will $\frac{1}{2}$, $\frac{1}{2}$, and 0 close the sentence properly?

Who knows the solution to $\square+\square+\square+\square=1$?

How about $+\square+\square+\square+\square=1$?

Suppose there were 20 squares; what would go inside each square?

How many agree it is one-twentieth?

How many agree it is 1 over 20?

How many twentieths does it take to make 1?

Who can spell "twentieth"?

How many fifths does it take to make 1?

How many fourths does it take to make 1?

How many thirds does it take to make 1?

How many halves does it take to make 1?

Suppose there were 100 squares in the sentence; what would go in each square variable?

How many agree that 1 over 100 will close the sentence properly and make it true?

How many hundredths does it take to make 1?

How many fiftieths does it take to make 1?

How many twentieths does it take to make 1?

How many thousandths does it take to make 1?

Listen carefully. How many thousandths does it take to make 2?

How many thousandths does it take to make 3?

How many thousandths does it take to make 4?

If it takes four fourths to make 1, how many fourths does it take to make 2? 3? 4?

Suppose that we had one million squares in the sentence; what number would go in each square?

One should keep in mind that many of the above questions are strictly oral, meaning that it is not necessary to write on the board every mathematical sentence associated with a question given to the class. However, a small nucleus of sentences pertinent to the concept discussed should always be visible on the board. It is crucial that the board work is not cluttered with unnecessary sentences. Plan ahead of time which sentences will be left on the board. So far we should have on the board:

$$\boxed{\frac{1}{2}} + \boxed{\frac{1}{2}} = 1$$

$$\boxed{\frac{1}{3}} + \boxed{\frac{1}{3}} + \boxed{\frac{1}{3}} = 1$$

$$\boxed{\frac{1}{4}} + \boxed{\frac{1}{4}} + \boxed{\frac{1}{4}} + \boxed{\frac{1}{4}} = 1$$

$$\boxed{\frac{1}{5}} + \boxed{\frac{1}{5}} + \boxed{\frac{1}{5}} + \boxed{\frac{1}{5}} + \boxed{\frac{1}{5}} = 1$$

Now one could extend the review in several directions. Certain fascinating addition sentences could be probed

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \square$$

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \square$$

with the following questions:

> Who can read this sentence?
>
> Who has a solution to this open sentence?
>
> How many agree this will make the sentence true?
>
> Who can explain the answer?
>
> Who can give us a reasoning for this solution?
>
> Who can prove this solution?
>
> How many of you noticed the "ones" in the sentence?
>
> Who can come to the board and point out one group of fractions that equals 1?
>
> Do we have any more groups that equal one?
>
> How many groups do we have?
>
> So how many agree that this is the correct solution?

Now shuffle the fractions as follows:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} = \square$$

$$\frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{2} = \square$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{3} + \frac{1}{4} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{3} = \square$$

and use questions similar to the ones above.

Also, facts of multiplication can be reinforced. If the class is aware that 2x5=2+2+2+2+2 and that 3+3+3+3=3x4, then structure your question to reveal:

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{1}{3} \times 3$$

$$\frac{1}{4} \times 4 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$

$$\frac{1}{3} \times 6 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$$

and follow up with the following questions:

> Who knows an addition sentence that means the same as $\frac{1}{3}$x3?
>
> Who remembers an addition sentence that means the same as 2x3?
>
> How many agree that 4+2=2x3?
>
> The addition sentence I'm looking for uses the 2 and the 3. Does anyone know which sentence I'm thinking about?
>
> 5+1 is the same as 2x3, but my addition sentence uses two plus signs. Who knows it?
>
> Are the 2 and the 3 used in it?
>
> How many agree that 2+2+2=2x3?
>
> How is the 3 used in 2+2+2?

How is the 2 used in 2+2+2?

So who can make up an addition sentence for $\frac{1}{3}$x3 using $\frac{1}{3}$ and 3?

How is the 3 used in $\frac{1}{3} + \frac{1}{3} + \frac{1}{3}$?

How is the $\frac{1}{3}$ used in $\frac{1}{3} + \frac{1}{3} + \frac{1}{3}$?

Who can give me an addition sentence for $\frac{1}{4}$ x 4?

Do we use the 4? $\frac{1}{4}$?

What is the addend form for $\frac{1}{5}$ x 5?

On your paper, write the addend form for $\frac{1}{2}$ x 10.

How many $\frac{1}{2}$'s in the addend form of $\frac{1}{2}$ x 10?

What is the answer to the addend form?

Who can explain how Mark obtained 5?

How many groups of 1's did you use?

How many $\frac{1}{2}$'s in each group?

What is the solution to $\frac{1}{2}$ x 10?

Who knows the answer to $\frac{1}{2}$ x 12? $\frac{1}{2}$ x 20? $\frac{1}{2}$ x 100?

Who knows the addend form of $\frac{1}{3}$ x 3?

What is the answer to $\frac{1}{3}$ x 3?

Who can explain the answer?

How many agree?

What is the addend form for $\frac{1}{3}$ x 6?

What is the answer to the addend form?

What is the solution to $\frac{1}{3}$ x 6? $\frac{1}{3}$ x 12? $\frac{1}{3}$ x 21?

What is the addend form of $\frac{1}{4}$ x 8?

What is the solution to $\frac{1}{4}$ x 8?

At this time, one might want to review some vocabulary, e.g., reciprocal and multiplicative inverse. Write on the board

$$\square \times 2 = 1$$

This sentence will seem strange to the class, even though they have just solved '$\frac{1}{2} \times 2$.' As a matter of technique, the variable should be moved from one place to another to solidify the skill involved.

$$2 \times \frac{1}{2} = \triangle$$

$$\frac{1}{2} \times \square = 1$$

$$\triangle \times 2 = 1$$

$$\square \times \frac{1}{2} = 1$$

These questions do not have to be asked during the same lesson. Indeed, a simple technique like shifting the positions of variables in a sentence helps to sustain the freshness of the ongoing review which should be part of each lesson. Now for some questions concerning reciprocals or (the equivalent term) multiplicative inverses:

> Something times 2 gives you 1. What is that number?
>
> How many remember seeing a '2' and a '1' in the same sentence?
>
> What was the third number?
>
> How many now remember the $\frac{1}{2}$?
>
> What is the solution to $\square \times 3 = 1$?
>
> What is the solution to $\square \times 4 = 1$?
>
> What is the solution to $5 \times \triangle = 1$?

Ten times what number gives you 1?

$\frac{1}{2}$ times what number gives you 1?

6 times what number gives you 1?

$\frac{1}{8}$ times what number gives you 1?

Who can give a multiplication problem whose answer is 1?

Who else can give me a multiplication problem whose answer is 1?

Anybody else?

Who can tell me the number which is multiplied by 2 to get 1?

Who can give me the reciprocal of 2?

How many of you have heard the word 'reciprocal' before?

Who can spell 'reciprocal'?

What is the reciprocal of 3? 5? 10?

What is the reciprocal of $\frac{1}{2}$? $\frac{1}{3}$? $\frac{1}{9}$?

Who can give a number and its reciprocal?

How many agree?

How can you tell that 2 and $\frac{1}{2}$ are reciprocals of each other?

How many agree $2 \times \frac{1}{2} = 1$?

A number multiplied by its reciprocal always gives what answer?

What is the reciprocal of 'n'?

Why is it $\frac{1}{n}$?

How many agree with $\frac{1}{n} \times n = 1$?

What does 'n' stand for?

Suppose n=5; who could rewrite $\frac{1}{n} \times n = 1$?

Suppose n=10; who could rewrite $\frac{1}{n} \times n = 1$?

So what is the product of a number and its reciprocal?

This sequence of questions gives an idea of how a topic is approached. It is important for the students to realize that mathematical topics are interrelated. One can go from addition to multiplication, or from addition to subtraction, or from multiplication to division, or from division to multiplication, and so on. Knowledge in one area illuminates calculations in another, less familiar topic.

2.2 Multiplication of Fractions

Probing further into multiplication, write on the board

$$(\frac{1}{3} \times 3) \times (\frac{1}{2} \times 2) = \triangle .$$

and ask the following questions:

What number will make the sentence true?

Who can prove that (give a reason, explain why) 2 works?

How many think that 2 does not work?

What number should replace the 2?

Again, what is $\frac{1}{3} \times 3$?

What is $\frac{1}{2} \times 2$?

So 1 x 1 equals what?

What is the answer to $(\frac{1}{4} \times 4) \times (\frac{1}{3} \times 3) \times (\frac{1}{5} \times 5) = ?$

How many think it's 3?

How many think it's 1?

Who can explain the answer?

Who else can give me a proof for it?

How about an answer to $\frac{1}{4} \times 4 \times \frac{1}{3} \times 3 = ?$

How about an answer to $\frac{1}{4} \times 3 \times \frac{1}{3} \times 4 = ?$

How many think it's still the same?

Who can explain why to the class?

How about $\frac{1}{3} \times \frac{1}{4} \times 3 \times 4 = ?$

How many agree?

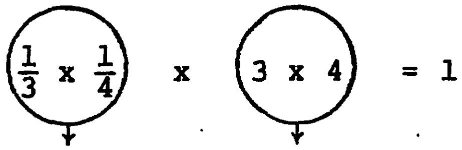
How many are convinced that the answer is 1?

Who can explain why to me again?

So what is $\frac{1}{3} \times \frac{1}{4}$?

Well, let's look at $\frac{1}{3} \times \frac{1}{4} \times 3 \times 4 = 1$.

Circle as follows:



and ask

What do I bring down for 3 x 4?

What do I bring down for "="?

What do I bring down for "x"?

What do I bring down for $\frac{1}{3} \times \frac{1}{4}$?

Is the bottom sentence true if $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$?

How many agree that $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$?

What is the answer to $\frac{1}{5} \times \frac{1}{3} = ?$

On your paper calculate $\frac{1}{5} \times \frac{1}{8}$.

How many agree that the answer is $\frac{1}{40}$?

Could we prove that $\frac{1}{5} \times \frac{1}{8} = \frac{1}{40}$?

Can we use $\frac{1}{5} \times \frac{1}{8}$ in a sentence that equals 1?

How many agree that we need some whole numbers in that sentence?

Who knows those two whole numbers?

Write them on your paper.

Who can tell me those numbers?

How many agree it is going to be 5 and 8?

Who can tell me the whole sentence?

How many agree that $\frac{1}{5} \times \frac{1}{8} \times 5 \times 8 = 1$?

Who can prove that it is true?

Who can prove it using groups of 1's?

Who knows the answer to $\frac{1}{5} \times \frac{1}{10}$?

How many agree with $\frac{1}{50}$?

Who can prove that this is the solution by using $\frac{1}{5} \times \frac{1}{10}$ in a sentence?

To increase the precision and clarity of your questions, constantly rephrase them. Most questions should be quickly rephrased, even repeated. Eventually, this becomes an automatic component of your teaching style, a part of your teaching which neither you nor the class consciously notices but which has a significantly positive impact on student response.

Students will readily say that $\frac{2}{3}\times\frac{4}{5}=\frac{8}{15}$, but they just as readily say that $\frac{2}{3}+\frac{4}{5}=\frac{6}{8}$. Our task in the next sequence of questions is to explicate the algorithm which underlies $\frac{2}{3}\times\frac{4}{5}=\frac{8}{15}$. In the above, we dealt with $\frac{1}{3}\times3=1$. Next we take up $\frac{2}{3}\times3$.

$$
\begin{aligned}
\frac{2}{3} \times 3 &= (\frac{1}{3} + \frac{1}{3}) \times 3 \\
&= (\frac{1}{3} + \frac{1}{3}) + (\frac{1}{3} + \frac{1}{3}) + (\frac{1}{3} + \frac{1}{3}) \quad \text{(repeated addition)} \\
&= (\frac{1}{3} + \frac{1}{3} + \frac{1}{3}) + (\frac{1}{3} + \frac{1}{3} + \frac{1}{3}) \quad \text{(associativity)} \\
&= 1 + 1 \\
&= 2
\end{aligned}
$$

How about $\frac{4}{5} \times 5$?

$$
\begin{aligned}
\frac{4}{5} \times 5 &= (\frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5}) \times 5 \\
&= (\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}) + (\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}) + (\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}) + (\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}) + (\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}) \\
&= (\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}) + (\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}) + (\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}) + (\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}) \\
&= 1 + 1 + 1 + 1
\end{aligned}
$$

Or if the class is capable of doing the following problems

$$
\frac{6}{2} = 3 \qquad \frac{12}{4} = 3 \qquad \frac{20}{5} = 4 \qquad \frac{12}{3} = 4
$$

then look at

$$
\frac{2}{3} \times 3 = \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = \frac{6}{3} = 2
$$

$$
\frac{4}{5} \times 5 = \frac{4}{5} + \frac{4}{5} + \frac{4}{5} + \frac{4}{5} + \frac{4}{5} + \frac{20}{5} = 4
$$

So the class arrives at the general result $\frac{a}{b} \times b = a$. But this will require some reinforcement. Do not forget to ask (for instance)

$$\begin{array}{lll} \frac{4}{7} \times \square = 4 & \qquad & \square \times 3 = 5 \\[1em] \frac{\square}{3} \times 3 = 5 & \qquad & \triangle \times 7 = 9 \\[1em] \frac{9}{\square} \times 7 = 9 & \qquad & 7 \times \triangle = 9 \end{array}$$

as reinforcement for $\frac{a}{b} \times b = a$.

Now we are ready for

$$\underbrace{\frac{2}{3} \times 3}_{\substack{\downarrow \\ 2}} \underset{\times}{\times} \underbrace{\frac{4}{5} \times 5}_{\substack{\downarrow \\ 4}} = \underset{\substack{\uparrow \\ 8}}{\square}$$

and mixing it up

$$\begin{array}{ccccccccc} \frac{2}{3} & \times & \frac{4}{5} & \times & 3 & \times & 5 & = & \square \\ & \downarrow & & & & \downarrow & & & \uparrow \\ & 2 & & \times & & 4 & & & 8 \end{array}$$

But let's look a little closer. We have on the board

$$\frac{2}{3} \times \frac{4}{5} \times 3 \times 5 = 8$$

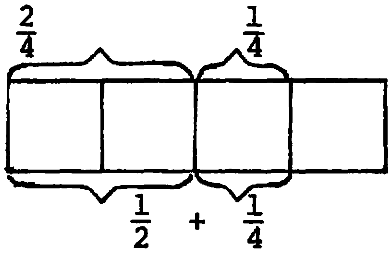
which we analyze in the following fashion

$$\underbrace{\frac{2}{3} \times \frac{4}{5}}_{\substack{\downarrow \\ \square}} \underset{\substack{\downarrow \\ \times}}{\times} \underbrace{3 \times 5}_{\substack{\downarrow \\ 15}} = \underset{\substack{\downarrow \\ 8}}{8}$$

from this it follows that (since $\frac{a}{b} \times b = a$) $\frac{8}{15}$ goes into the $\square$. And it is for this reason that '$\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$' is true.

2.3 Addition of Fractions

The preceding approach is largely algebraic, developing and using such algebraic laws as $\frac{a}{b}xb=a$. It is essential that the students also become familiar with the 'geometric' interpretation of fractions as parts of a whole. For instance, to dispel a misconception such as $\frac{1}{2}+\frac{1}{4}=\frac{2}{6}$, one should have the class shade in pictures such as



What follows is an elaboration of this point of view.

One might begin by asking:

Who can name a fraction?.......$\frac{1}{2}$

Which number represents the numerator?

Which number represents the denominator?

Who can name another common fraction?.......$\frac{1}{4}$

Which number represents the denominator?

Which number is larger?

Which number is smaller?

If we wanted to check these two fractions, which operation would we use?.......+

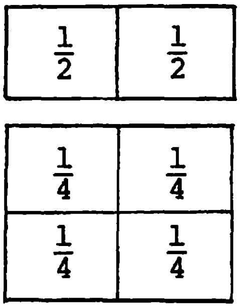
Your board work should now look like

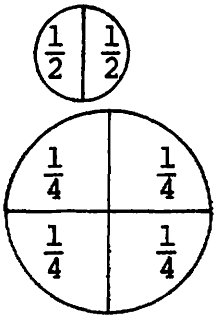
$$\frac{1}{2} + \frac{1}{4} = \square$$

Who can come to the board and circle the largest fraction?

Who can come to the board and circle the smallest fraction?

On the board, you would now have one side displaying the open sentence $\frac{1}{2}+\frac{1}{4}=\square$, while the other wide would contain the following figures:





Who can come to the board and shade in one section of the square?

Who can come to the board and shade in two sections of the square?

Who can come to the board and shade in three sections of the square?

Who can come to the board and shade in four sections of the square?

Who can shade in one section of the circle?

What is the fractional name of the shaded portion? Of the unshaded portion?

Who can shade in two portions of the unshaded portion?

What is the fractional name of the shaded portion?

The unshaded portion?

(Note: This is part of the culminating steps of answering $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$. It needs to be prominently in the sequence.)

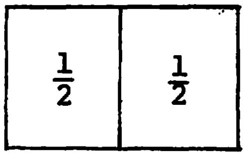
Who can make up a sentence using $\frac{1}{2}$?

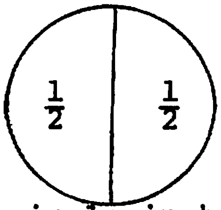
Who can make up a sentence using $\frac{1}{4}$?

Who can come to the board and draw a circle?

Who can come to the board and draw a square?

Board work:



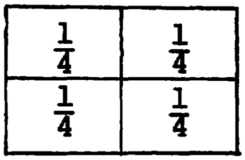


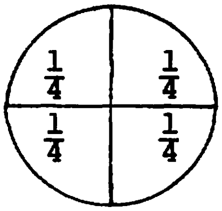
If we divide the square and circle in half, what fraction could we use to label each part?

Who can come to the board and label the square?

Who can come to the board and label the circle?

Board work:





If we wanted to divide the square and circle in fourths, where should we draw the line?

How many agree?

Who can come to the board and label each part?

Who can come to the board and shade in $\frac{1}{2}$ of the square?

To shade in half of the square, how many portions were shaded?

To shade in half the square, how many $\frac{1}{4}$'s were shaded?

Who can come to the board and shade in $\frac{1}{2}$ of the circle?

How many fourths had to be shaded?

Is $\frac{1}{2}$ the same as $\frac{2}{4}$?

Is one-half the same as two-fourths?

Is one over two the same as two over four?

If I erase this $\frac{1}{4}$ and that $\frac{1}{4}$, what is one new name I could give to this shaded portion?

How many agree?

Who can come to the board and shade in another $\frac{1}{4}$ of the square?

How many agree with the shading?

How much more did she/he shade?

How much is shaded in now?

What is the name of the shaded portion?

What is the name of the old shaded part?

The new part? All the shaded parts?

Who can make up a sentence using $\frac{1}{2}$ and $\frac{1}{4}$?

How many of you know what $\frac{1}{2} + \frac{1}{4} = ?$

Which answer does the shaded portion of the square give?

Do we have a '$\frac{1}{2}$' shaded in the square?

Do we have a '$\frac{1}{4}$' shaded in the square?

So altogether what do we have?

Then who can close $\frac{1}{2} + \frac{1}{4} = \square$?

How many agree with $\frac{3}{4}$?

What's another name for $\frac{1}{2}$?

If I erase $\frac{1}{2}$, what's the other name I can put in its place?

Who can read the sentence now?

Is it true that two fourths plus one fourth equals three fourths?

If I erase two fourths, what other fraction can I write in its place?

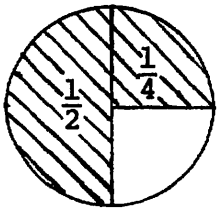
How many agree that $\frac{2}{4} = \frac{1}{2}$?

Who can read the sentence now?

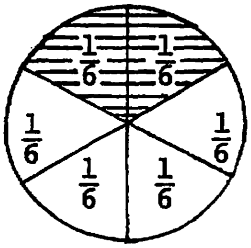
A modified repetition of the above questions about the square should now follow for the case of the circle. The questioning pace should be quicker.

This geometric approach helped the class arrive at $\frac{3}{4}$ as the solution to the open sentence $\frac{1}{2} + \frac{1}{4} = \square$. Many students still will have an inclination to close that sentence with $\frac{2}{6}$. To dispel this misconception further, the following approach can be used.

If the class is not comfortable with reducing, let them compare the following:



$\frac{1}{2} + \frac{1}{4}$



$\frac{2}{6}$

Is the shaded portion on the right side the same as the shaded portion on the left side?

Is the shaded portion on the right side the same size as the portion shaded on the left?

Which looks larger?

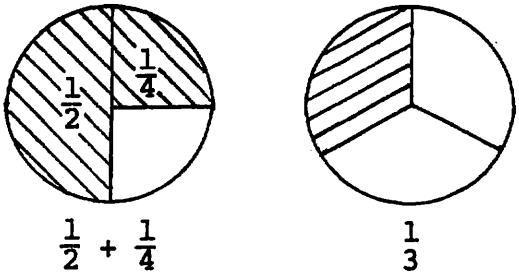
Which looks smaller?

Do you think it makes sense for $\frac{1}{2} + \frac{1}{4}$ to be the same as $\frac{2}{6}$?

If the class can reduce, then try

$$\frac{1}{2} + \frac{1}{4} = \frac{2}{6} = \frac{1}{3}$$

and compare



with similar questions.

The interpretation of fractions as shaded parts of a whole helps the student to visualize one fractional number as larger or smaller than another. Fractions should also be ordered on the number line. Here is another approach:

Who can tell me what kind of numbers we have talked about so far?

Who can think of a number less than one?

Who can think of a number greater than zero?

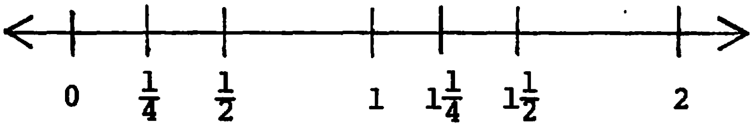
Who can think of a number that's less than one but greater than zero?

Who knows what kind of number that is?

Can anyone think of another fraction?

How many numbers are there between 0 and 1? (An example of an open-ended question, which may be quickly discussed or pursued at length.)

Draw a number line on the board. (Do not include the numbers. Students will supply that information during the questioning sequence. Get the whole numbers first.)



Who can tell us what number goes here? And here?

Raise your hand if you know what number is halfway between 0 and 1.

What number is midway between 1 and $\frac{1}{2}$?

What number is halfway between 1 and 2?

Who knows where $1\frac{1}{4}$ would be on the number line?

Who can locate $2\frac{1}{2}$ on the number line?

Look at the number line and tell me which is greater, $\frac{1}{2}$ or $\frac{1}{4}$?

Why do you think $\frac{1}{2}$ is greater?

Which is greater, 1 or $\frac{1}{2}$?

Which is greater, $1\frac{1}{2}$ or 1?

Which is greater, $1\frac{1}{2}$ or $1\frac{1}{4}$?

Which is less, $1\frac{1}{2}$ or $\frac{1}{2}$?

Which is less, 2 or $1\frac{1}{4}$?

Which is greater, $\frac{1}{3}$ or $\frac{1}{4}$?

Who remembers the symbol for greater than?

Write

$$>$$

Who remembers the symbol for less than?

Write

$$\frac{1}{2} \qquad \frac{1}{4}$$

Raise your hand if you think you know which symbol I should use to make the sentence true.

Suppose the $\frac{1}{4}$ was written to the left of the $\frac{1}{2}$; what symbol would I use then?

(Use several examples of comparing <u>unit</u> fractions.)

Raise your hand if you can tell me what symbol should be used in comparing these two fractions.

$$\frac{1}{2} \qquad \frac{2}{4}$$

Is $\frac{1}{2}$ greater than $\frac{2}{4}$?

Is $\frac{1}{2}$ less than $\frac{2}{4}$?

How much is $\frac{1}{2}$ of a dollar?

Suppose I had $\frac{2}{4}$ of a dollar; how much would I have?

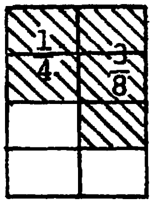
Who thinks they now know what symbol to use to compare $\frac{1}{2}$ and $\frac{2}{4}$?

If $\frac{2}{4}$ is another name for $\frac{1}{2}$, who has another name for $\frac{1}{2}$?

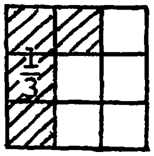
Who can think of 10 names for $\frac{1}{2}$? 12 names? 15 names? 50 names?

Repeat this process with several other addition problems that can easily be looked at geometrically, such as

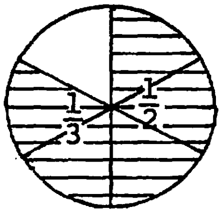
$\frac{3}{8} + \frac{1}{4} = \frac{5}{8}$



$\frac{1}{3} + \frac{1}{9} = \frac{4}{9}$



$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$



If you have two half-dollars, how much money do you have?

How much is $\frac{2}{2}$?

$\frac{4}{4}$ is another name for what number?

Who can think of another name for 1?

How many names for 1 can you think of?

1 times 5 equals what?

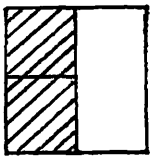
What happens when you multiply a number by 1?

What's 1 times $\frac{1}{2}$?

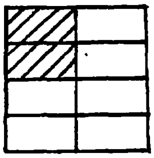
What's $\frac{2}{2}$ times $\frac{1}{2}$?

What's true about 1 times $\frac{1}{2}$ and $\frac{2}{2}$ times $\frac{1}{2}$?

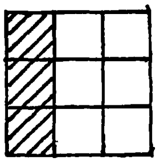
Now look more formally at equivalent fractions algebraically and geometrically, asking questions similar to the ones above.



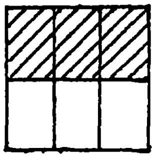
$$\begin{array}{ccccc} \frac{1}{2} & \times & 1 & = & \frac{1}{2} \\ \downarrow & & \downarrow & & \downarrow \\ \frac{1}{2} & \times & \frac{2}{2} & = & \frac{2}{4} \end{array}$$



$$\begin{array}{ccccc} \frac{1}{4} & \times & 1 & = & \frac{1}{4} \\ \downarrow & & \downarrow & & \downarrow \\ \frac{1}{4} & \times & \frac{2}{2} & = & \frac{2}{8} \end{array}$$



$$\begin{array}{ccccc} \frac{1}{3} & \times & 1 & = & \frac{1}{3} \\ \downarrow & & \downarrow & & \downarrow \\ \frac{1}{3} & \times & \frac{3}{3} & = & \frac{3}{9} \end{array}$$



$$\begin{array}{ccccc} \frac{1}{2} & \times & 1 & = & \frac{1}{2} \\ \downarrow & & \downarrow & & \downarrow \\ \frac{1}{2} & \times & \frac{3}{3} & = & \frac{3}{6} \end{array}$$

What is a general rule for finding a fraction which is equivalent to (names the same number as) $\frac{1}{2}$? $\frac{1}{3}$? $\frac{1}{n}$? $\frac{a}{b}$?

Return to the addition problems that have been solved geometrically and look at their formal solution. Be sure to reinforce the computational results with the geometric interpretation.

$$\begin{array}{ccccc} \frac{1}{2} & + & \frac{1}{4} & = & ? \\ \downarrow & & \downarrow & & \uparrow \\ \frac{2}{4} & + & \frac{1}{4} & = & \frac{3}{4} \end{array} \qquad \begin{array}{ccccc} \frac{3}{8} & + & \frac{1}{4} & = & ? \\ \downarrow & & \downarrow & & \uparrow \\ \frac{3}{8} & + & \frac{2}{8} & = & \frac{5}{8} \end{array} \qquad \begin{array}{ccccc} \frac{1}{3} & + & \frac{1}{9} & = & ? \\ \downarrow & & \downarrow & & \uparrow \\ \frac{3}{9} & + & \frac{1}{9} & = & \frac{4}{9} \end{array} \qquad \begin{array}{ccccc} \frac{1}{2} & + & \frac{1}{3} & = & ? \\ \downarrow & & \downarrow & & \uparrow \\ \frac{3}{6} & + & \frac{2}{6} & = & \frac{5}{6} \end{array}$$

The class should now be able to extract from these examples a method for adding any two fractions. Until the students have had a considerable amount of practice and are very secure, it is wise to keep the numbers small. The arithmetic difficulty of a problem like $\frac{3}{17} + \frac{2}{19} = ?$ can frustrate a student who can solve $\frac{3}{7} + \frac{2}{5} = ?$

2.4 Division of Fractions

In mathematics, we speak of subtraction and division as being the 'inverse operations' to multiplication and division. For example, knowing that 8x7=56 is equivalent to 56÷7=8 and knowing that 3+4=7 is equivalent to 7-4=3. This inverse property for multiplication and addition can be very useful when we begin to deal with fractions, negatives, and other types of numbers other than whole numbers, for it provides us with a strategy for understanding how to divide and subtract. Actually, subtraction of fractions is very straightforward and should be woven in with practice problems on addition. Students will not have trouble seeing that since $\frac{1}{3}+\frac{1}{3}=\frac{2}{3}$ then $\frac{2}{3}-\frac{1}{3}=\frac{1}{3}$ and that it is necessary to look for a common denominator before attempting to compute $\frac{1}{7}-\frac{1}{8}$.

Division, however, is a bit more complex. Most adults remember that to divide fractions they should "invert and multiply," although they may forget which term to invert and few ever stop to wonder why this is so. This is exactly what a discovery teacher must do, however, for she/he is dealing with concepts and not just rote algorithms.

When we looked at addition of fractions, we began with a thorough investigation of a single problem that could be approached intuitively as well as formally. A similar approach is recommended for division.

To motivate the need for special rules for working with division, we begin by creating a false sense of security. Review multiplication of fractions, then put on the board several problems like:

$$\frac{8}{15} \div \frac{2}{3} = ?$$

$$\frac{9}{20} \div \frac{3}{4} = ?$$

$$\frac{12}{25} \div \frac{3}{5} = ?$$

The students' natural inclination to divide numerators and divide denominators works perfectly. Ask for the multiplication problem that checks each division problem.

Now put up

$$\frac{1}{2} \div \frac{1}{3} = ?$$

Our intuitive method leads us to

$$\frac{1}{2} \div \frac{1}{3} = \frac{1 \div 1}{2 \div 3} = \frac{1}{\frac{2}{3}} = ?$$

Unless the class has developed a procedure for simplifying complex fractions, this is a conceptual dead end.*

Rather than work with complex fractions at this time, we take a more intuitive route. Ask a student to record $\frac{1}{2} \div \frac{1}{3} = ?$ as an unsolved problem and turn to a more intuitive set of problems.

$$30 \div 5 = ?$$

How do you know the answer is 6?

How many 5's are there in 30?

How can you check it?

---

*Note that if the class can simplify complex fractions, $\frac{a}{b} \div \frac{c}{d}$ becomes $\frac{\frac{a}{b}}{\frac{c}{d}} \cdot \frac{\frac{d}{c}}{\frac{d}{c}} = \frac{a}{b} \cdot \frac{d}{c}$ or the familiar rule and the material which follows can be used simply to reinforce a rule which they can derive algebraically.

$$(5+5+5+5+5+5) \div 5 = ?$$

$$(4+4+4+4+4) \div 4 = ?$$

$$(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}) = ?$$

$$3 \div \frac{1}{2} = ?$$

What number times $\frac{1}{2}$ equals 6?

How many $\frac{1}{2}$'s are there in 3?

Can someone come to the board and draw a picture to illustrate it?

$$4 \div \frac{1}{2} = ?$$

$$5 \div \frac{1}{2} = ?$$

$$3 \div \frac{1}{3} = ?$$

$$12 \div \frac{1}{3} = ?$$

etc.

Do a sequence of similar problems until the class conjectures the rule

$$k \div \frac{1}{a} = k \times a$$

This can be verified by using the equivalence between addition and multiplication

$$k \div \frac{1}{a} = \square \Longleftrightarrow \square \times \frac{1}{a} = k$$

$$\begin{array}{cccc} k \times & \underbrace{a \times \frac{1}{a}} & = & ? \\ \downarrow & \downarrow & & \uparrow \\ k \times & 1 & = & k \end{array}$$

What is the multiplication sentence that checks this division sentence?

Who sees two numbers whose product is one?

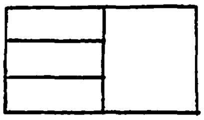
What do I bring down for the k?

What's kxl?

Next look at problems of the type $\frac{1}{k} \div a$. These can be looked at both geometrically and algebraically.

Example: $\frac{1}{2} \div 3 = ?$

Geometric interpretation:



Algebraic interpretation:

$$\frac{1}{2} \div 3 = \square \Longleftrightarrow \square \times 3 = \frac{1}{2}$$

Note that solving $\square \times 3 = \frac{1}{2}$ is easy if we think of it as a two-step process.

What would I have to multiply 3 by to get 1?

What would I multiply 1 by to get $\frac{1}{2}$?

What is $\frac{1}{2} \times \frac{1}{3} \times 3$ equal to?

$\frac{1}{2} \times \frac{1}{3}$ equals what?

$\frac{1}{6} \times 3$ equals what?

What is $\frac{1}{2} \div 3$?

This process will be useful in solving more general problems.

Before turning to a more general problem, quickly review reciprocals.

$\square \times 2 = 1$

$\frac{1}{2} \times \square = 1$

$\frac{2}{3} \times \square = 1$

$a \times \square = 1$

$\frac{a}{b} \times \square = 1$

At this point, you may wish to return to examples like $3 \div \frac{1}{2} = 6$ and $\frac{1}{2} \div 3 = \frac{1}{6}$ and see if students can explain a rule for finding the answer that uses the word 'reciprocal'. An alternative approach is to establish the general rule and then verify that the examples that were solved before do fit it.

Now look at the unsolved problem $\frac{1}{2} \div \frac{1}{3} = \square$.

Who can write a multiplication sentence equivalent to this division sentence?

$$\frac{1}{2} \div \frac{1}{3} = \square \Longleftrightarrow \square \times \frac{1}{3} = \frac{1}{2}$$

Focus on $\square \times \frac{1}{3} = \frac{1}{2}$

What is the reciprocal of $\frac{1}{3}$?

What if I put a 3 in the square variable?

What is $3 \times \frac{1}{3}$?

What do I have to multiply 1 by to get $\frac{1}{2}$?

Who can multiply $\frac{1}{3}$ by something to get 1?

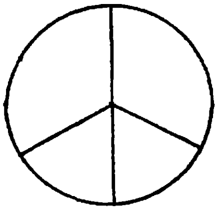
Now that we've got 1, what do we have to multiply by to get $\frac{1}{2}$?

Who can give me two numbers to multiply $\frac{1}{3}$ by to get $\frac{1}{2}$?

What is $\frac{1}{2}$x3 equal to?

What number is $\frac{1}{2}\div\frac{1}{3}$ equal to?

This particular result can be reinforced by a geometric interpretation.



Who can divide this circle into $\frac{1}{2}$'s?

Who can divide the circle into $\frac{1}{3}$'s?

How many $\frac{1}{3}$'s are there in $\frac{1}{2}$? (Answer: $1\frac{1}{2}$ or $\frac{3}{2}$.)

Continue with another problem. (Since the questions parallel the above development, we have included only the board work.)

$$\frac{3}{4} \div \frac{2}{5} = \boxed{} \Longleftrightarrow \boxed{\frac{3}{4}\times\frac{5}{2}} \times \frac{2}{5} = \frac{3}{4}$$

$$\downarrow \qquad \downarrow$$

$$\frac{3}{4} \times 1 = \frac{3}{4}$$

$$\text{So } \frac{3}{4} \div \frac{2}{5} = \frac{3}{4} \times \frac{5}{2}$$

After several more problems, the class should develop the usual 'invert and multiply' rule. To reinforce the rule and maintain conceptual understanding, review problems should include examples that can be solved without using the rule.

Examples:

$5 \div \frac{1}{2} = \square$, $10 \div \frac{1}{2} = \triangle$, ..., $k \div \frac{1}{a} = \bigtriangledown$

$\frac{1}{2} \div 3 = \square$, $\frac{1}{2} \div 5 = \triangle$, ..., $\frac{1}{a} \div k = \bigtriangledown$

$1 \div \frac{1}{2} = \square$, $1 \div \frac{1}{3} = \triangle$, ..., $1 \div \frac{1}{a} = \bigtriangledown$

How many $\frac{1}{2}$'s in 1?

How many $\frac{3}{4}$'s in 1?

$1 \div \frac{2}{3} = \square$, $1 \div \frac{3}{4} = \triangle$, ..., $1 \div \frac{a}{b} = \bigtriangledown$

How many $\frac{2}{3}$'s in 1?

How many $\frac{1}{3}$'s in 1?

Who can draw a picture illustrating this?

Note that the last sequence leads to an alternate approach to the general rule.

If apples are 5 for one dollar, how many apples can you buy for \$2? \$3? \$5? \$10? \$17? \$5$\frac{1}{2}$? \$k?

How many $\frac{2}{3}$'s are there in 1?

So how many $\frac{2}{3}$'s are there in 2? In 3? In 5? In 10? In k? In $\frac{a}{b}$?

In general, how many $\frac{c}{d}$'s are there in 1? (Answer: $\frac{d}{c}$.)

So how many $\frac{c}{d}$'s are there in $\frac{a}{b}$? (Answer: $\frac{a}{b} \times \frac{d}{c}$.)

2.5 Least Common Multiples

'Reducing to lowest terms' and finding 'least common denominators' normally occupy a great deal of time in the study of fractions. We have omitted them from our discussion so far because neither is mathematically necessary to understanding the basic operations with fractions. Because they can be useful for simplifying computations and because convention generally requires them, we outline briefly an approach to each in this section.

In Section 2.3, we established a method of finding equivalent fractions using the multiplication property of one:

$$\frac{a}{b} = \frac{a}{b} \times 1 = \frac{a}{b} \times \frac{k}{k} = \frac{a \times k}{a \times k}$$

Reducing fractions to lowest terms (finding an equivalent fraction with the smallest possible numerator and denominator) is simply the reverse of this process and we will not dwell on it here. Students can derive a method for reducing fractions from examples such as

$$\frac{9}{12} = \frac{3 \times 3}{4 \times 3} = \frac{3}{4} \times \frac{3}{3} = \frac{3}{4} \times 1 = \frac{3}{4}$$

The question of 'least common multiple' might arise in the course of doing a problem like $\frac{1}{6}+\frac{1}{8}$. What is the smallest number both 6 and 8 will divide? Mathematicians call this number the least common multiple of 6 and 8. One method of arriving at the least common multiple is to simply list out the multiples of 6 and the multiples of 8 and to see where the two lists first coincide. Now we will look at this another way.

First, factor 6 and 8 to get:

6 = 2 x 3, and

8 = 2 x 4 = 2 x 2 x 2

Now, using the old method, we find that the least common multiple of 6 and 8 is 24, which factors into:

24 = 2 x 12 = 2 x 2 x 6 = 2 x 2 x 2 x 3

Now ask the students whether they can find the prime factors of 8 within the list, 2x2x2x3. They will say, "Yes." Similarly, get them to agree that they can find the factors of 6, that is, 2x3. Now challenge them to come to the board and erase any of the factors in the list, 24=2x2x2x3, so that they still see the factors of 8 and the factors of 6. They will find that they cannot.

This example suggests our new technique for finding the least common multiple of two numbers, say 12 and 18. First, we can factor the numbers into products of primes:

12 = 2 x 2 x 3

18 = 2 x 3 x 3

Now we simply 'build' our least common multiple, using the minimum amount of material. For 12, we will need two 2's and one 3:

2 x 2 x 3,

and for the 18 we will need one 2 and two 3's. Notice, however, that we already have enough 2's to make 18 and that we are short only one 3, so all we have to do is add in another 3 to get:

$$2 \times 2 \times 3 \times 3$$

This, then, is the least common multiple, since if we were to erase any of the numbers, we would no longer be able to make either the 12 or the 18. So the least common multiple (LCM) of 12 and 18, written LCM(12,18), equals 2x2x3x3 equals 36.

Sometimes a teacher will have some trouble getting the students to see the logic here. One way to clarify the concept, after obtaining:

$$12 = 2 \times 2 \times 3$$

$$18 = 2 \times 2 \times 3 \times 3,$$

is to write down all the factors,

$$2 \times 2 \times 3 \times 2 \times 2 \times 3 \times 3$$

and invite the students to come to the board and erase as much as they can while still being able to see the factors of 12 and the factors of 18. (It's OK if they share factors.) Students like this exercise, and it tends to drive the point home.

In subsequent examples the teacher can put up the factors of 12 first:

$$2 \times 2 \times 3,$$

and then ask the class what else must go up so that the number contains 18. Thus the teacher can convey information by the very form in which the question is asked.

2.6 Exercises[*]

1\. a. $\frac{1}{2} \times 2 = \underline{\hspace{2em}}$

b. $3 \times \frac{1}{3} = \underline{\hspace{2em}}$

c. $\boxed{} \times \frac{1}{4} = 1$

d. $100 \times \frac{1}{100} = \underline{\hspace{2em}}$

e. $\boxed{} \times n = 1$

f. $\frac{3}{3} \times 3 \times 5 \times \frac{1}{5} \times 4 = \boxed{}$

g. $\frac{2}{3} \times \frac{3}{2} = \boxed{}$

h. $\frac{a}{b} \times \frac{b}{a} = \boxed{}$

2\.
$$\underbrace{\frac{1}{4} \times \frac{1}{5}}_{\downarrow} \times \underbrace{5 \times 4}_{\downarrow} = \underset{\downarrow}{\boxed{}}$$
$$\triangle \times \bigtriangledown = \boxed{}$$
$$\frac{1}{4} \times \frac{1}{5} = \underline{\hspace{2em}}$$

3\.
$$\frac{2}{3} \times 3 = \boxed{}$$
$$\frac{2}{3} \times 3 \times \frac{5}{8} \times 8 = \boxed{}$$
$$\underbrace{\frac{2}{3} \times \frac{5}{8}}_{\downarrow} \times \underbrace{3 \times 8}_{\downarrow} = \underset{\downarrow}{\boxed{}}$$
$$\triangle \times \bigtriangledown = \boxed{}$$

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[*] Because there are numerous sources of exercises on fractions, we include here only examples which illustrate the conceptual foundations covered in this chapter or which gain interest by leading to a general result.

4. a. $\frac{1}{2} + \frac{1}{3} = \square$

b. $\frac{1}{3} + \frac{1}{4} = \square$

c. $\frac{1}{4} + \frac{1}{5} = \square$

d. $\frac{1}{5} + \frac{1}{6} = \square$

e. $\frac{1}{99} + \frac{1}{100} = \square$

5. a. $\frac{1}{5} + \frac{1}{5} = \square$

b. $\frac{1}{5} + \frac{1}{10} = \square$

c. $\frac{1}{5} + \frac{1}{15} = \square$

d. $\frac{1}{5} + \frac{1}{20} = \square$

e. $\frac{1}{5} + \frac{1}{25} = \square$

6. a. $\frac{1}{2} + \frac{1}{4} = \square$

b. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \square$

c. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \square$

d. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \square$

7. a. $\frac{3}{4} - \frac{1}{2} = \square$

b. $\frac{4}{9} - \frac{1}{3} = \square$

c. $\frac{5}{16} - \frac{1}{4} = \square$

d. $\frac{6}{25} - \frac{1}{5} = \square$

What is the next problem in this sequence?

8. a. $\frac{1}{3} - \frac{1}{4} = \square$

b. $\frac{1}{4} - \frac{1}{5} = \square$

c. $\frac{1}{5} - \frac{1}{6} = \square$

d. $\frac{1}{6} - \frac{1}{7} = \square$

9. a. $\frac{3}{4} + \frac{5}{6} = \square$

b. $\frac{3}{8} + \frac{5}{6} = \square$

c. $\frac{5}{6} - \frac{3}{4} = \square$

d. $\frac{5}{6} - \frac{3}{8} = \square$

10. $\frac{4}{7} \div \frac{2}{3} = \square$

Write and solve the equivalent multiplication sentence.

11. a. $\frac{3}{4} \div \frac{1}{4} = \square$

b. $\frac{3}{8} \div \frac{1}{8} = \square$

c. $\frac{3}{12} \div \frac{1}{12} = \square$

12. a. $\frac{3}{4} \div \frac{3}{4} = \square$

b. $\frac{7}{8} \div \frac{7}{8} = \square$

c. $\frac{9}{10} \div \frac{9}{10} = \square$

d. $\frac{a}{b} \div \frac{a}{b} = \square$

2.7 Chapter Summary

Multiplicative Inverses

$$k \times \frac{1}{k} = 1 \text{ for } k \neq 0$$

Multiplication of Fractions

$$\frac{1}{a} \times \frac{1}{b} = \frac{1}{a \times b}$$

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

Addition of Fractions

$$\frac{a}{b} + \frac{c}{d} = \frac{(a \times d) + (b \times c)}{b \times d}$$

Division of Fractions

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \times d}{b \times c}$$

Least Common Multiples

Find the LCM of a and b using the prime factors of a and b.

Vocabulary

Multiplicative inverse

Reciprocal

Numerator

Denominator

Least common multiples

Equivalent fractions

Equivalent sentences

2.8 Chapter Test

1. $\frac{1}{4} \times \square = 1$

   $k \times \square = 1$

   $\frac{p}{q} \times \frac{q}{p} = \square$

2. What is the reciprocal of p?

   What is the reciprocal of $\frac{3}{4}$?

3. $$\underbrace{\frac{1}{5} \times \frac{1}{3}}_{\downarrow} \times \underbrace{3\times5}_{\downarrow} = \underset{\downarrow}{\square}$$

   $$\text{⬡} \times \triangle = \square$$

   So $\frac{1}{5} \times \frac{1}{3} =$ ____

4. $$\underbrace{\frac{3}{4} \times \frac{5}{7}}_{\downarrow} \times \underbrace{4\times7}_{\downarrow} = \underset{\downarrow}{\square}$$

   $$\text{⬡} \times \triangle = \square$$

   So $\frac{3}{4} \times \frac{5}{7} =$ ____

5. $\frac{1}{8} \times \frac{1}{9} = \square$

   $\frac{2}{3} \times \frac{4}{5} = \square$

6. Find $\frac{1}{3} + \frac{1}{6}$ using a picture and using equivalent fractions.

7. $\frac{3}{8} + \frac{1}{4} = \square$

   $\frac{5}{6} + \frac{1}{4} = \square$

   $\frac{5}{6} - \frac{1}{4} = \square$

8. $\frac{5}{6} \div \frac{3}{7} = \square \Longleftrightarrow \square \times \triangle = \triangledown$

9. $6 \div \frac{1}{2} = \square$

$\frac{6}{7} \div 2 = \square$

10. Find the least common multiple of 16 and 12 using prime factors.

MODULE 3: NEGATIVE NUMBERS

3.0 Introduction

The study of the integers provides an excellent opportunity to extend the algebraic type of thinking of the chapter on fractions. Students discover rules for working with a new type of number. The parallels between addititive and multiplicative inverses create a context for reviewing the basic concepts involving fractions. Review of addition and subtraction of whole numbers also is embedded in the study of the integers.

The basic concept of additive inverses (Section 3.2) is all that is necessary for the applications in Module 4. The section on addition of positive and negative numbers (3.3) uses the same theoretical approach as the multiplication of reciprocals and provides an alternative approach to some of the concepts in the next chapter. The section on subtraction of positive and negative numbers (3.4) may be omitted, although since it is analogous to the procedure for division of fractions, it provides a natural context for reviewing that process. The section on the numberline (3.5) may be used for review or as an alternate approach to the study of negative numbers.

3.1 Identity Elements

'Zero' and 'one' play a unique role in our mathematical system, and students usually encounter the special behavior of these numbers early in their mathematical careers. In our investigation of fractions in the preceding chapter, we frequently used the unique property of 'one' with respect to multiplication to simplify the solution of complex problems. The concept of identity elements is an important one in higher mathematics, and we pause briefly to investigate it here. We adopt an abstract notation in order to emphasize the underlying concept and to avoid the appearance of belaboring the obvious.

Begin by writing on the board

$$5 + I_+ = \square$$

Ask for a student to read the sentence. If no one volunteers because of the unfamiliar symbol, ask if they can read any part of it. When the only part left is the 'I_+', tell the class it is read 'I sub plus' or 'the additive identity'. Ask again for individual students and the whole class to read it.

Now ask if anyone knows how to close the sentence to make it true. Since the class has never seen the symbol I_+ before, they should be encouraged to make educated guesses. Your comments after each response can be used to narrow the field of choices and keep the guesses from being random. This is also an opportunity to embed review concepts in the lesson. Some examples follow:

> The number which makes this a true sentence is an odd number.
>
> The answer is a prime number.
>
> The answer is less than 2E3.
>
> The answer is greater than 5E0.
>
> Seven is close but it doesn't preserve the identity of the 5.

Note that if, at any time, a student makes an error on one of the review concepts, you may choose to stop and conduct a brief lesson on that concept. You might, for example, ask a short, fast-paced sequence of questions on odd and even numbers culminating in a student's explanation of a rule for determining whether a number is odd or even. There are two purposes for such a digression. The first, and obvious, one is that it gives you a chance to review and reinforce a part of the curriculum you want students to master. The second is that by leaving the original problem unsolved for longer, you actually increase the students' interest in it. On the other hand, exploring every possible sidetrack that comes up can bring progress on the main topic to a standstill and you will want to treat lightly many incorrect answers involving a secondary concept. For example, if a student gives an even rather than an odd number, you might ask him or her to call on someone who is disagreeing and then move on to the next question. In this case, it is important that you plan a way to spend some time on the concept later.

Once you have arrived at 5 as the number which makes the sentence true, continue with a rapid sequence of similar questions leading to a general result.

$$\begin{array}{rcl} 5 + I_+ & = & \boxed{5} \\ 7 + I_+ & = & \boxed{} \\ I_+ + \frac{3}{4} & = & \boxed{} \\ 1.982 + I_+ & = & \boxed{} \\ I_+ + Y & = & \boxed{} \end{array}$$

What happens when you add the additive identity to a number?

When you add I_+ to a number, do you get the identical number you started with?

What number is the additive identity acting like?

What number to you think I_+ equals?

Do you think we can find an identity element for any other operation?

What symbol do you think we'd use for the identity element for multiplication?

If I_+ is called the 'additive identity', what do you think mathematicians call I_x?

Who can make up a true mathematical statement using I_x?

How many true mathematical statements can we make up using I_x?

Who can give me a generalization using I_x?

What number is the multiplicative identity acting like?

What number do you think I_x equals?

This is a good point to employ the technique of erasing everything on the board but the most important results or erasing the entire board and reproducing the most important results. You should end up with something like:

$$\gamma + I_+ = \gamma \qquad \gamma \times I_x = \gamma$$
$$I_+ = 0 \qquad I_x = 1$$

Students should record these in their notebooks.

There are several activities which you can do at this point to demonstrate that identity elements are not trivial. The first is to look at whether identity elements exist for other operations.

If * were the symbol for some unknown operation, what symbol would you use for the identity element for *?

$$k * I_{*} = ? \qquad I_{*} * k = ?$$

Now look at whether the familiar operations subtraction, division, and exponentiation have identity elements. In other words, is there a number I_- so that for all numbers k, $k-I_-=k$ and $I_- - k=k$? What if we change the '-' to '÷' or 'E'. It is not hard for the students to figure out that each of these operations has a 'right' identity but not a left. That is, for all numbers k, $k-0=k$, $k \div 1=k$, and kE1=k but none of these sentences is true when you reverse the order of the terms.

What is true about the identity elements for addition and multiplication that is not true about these other identities? (Answer: They are both right and left identities.)

Another activity to reinforce the concept of identity elements involves the investigation of nonstandard operations. This is a concept which can be brought up from time to time during the term. For example, when you are reviewing the four basic operations and exponentiation, you might introduce the class to a new operation. Determining how a nonstandard operation works is an exercise in discovery which provides a change of pace and a challenge to the students. You might keep it as an ongoing mystery, giving the students a few more examples or hints each day until they crack the code.

Consider, for example, the operation '$\otimes$' defined by

$$a \otimes b = (a-1) \times (b-1) + 1$$

This can be presented to the class as a mystery operation. Give them problems such as the ones below and use the techniques for focusing guesses described previously to have the class build a collection of examples to use in formulating a general rule for this operation.

Examples:
$3 \otimes 4 = 7$
$3 \otimes 5 = 9$
$3 \otimes 6 = 11$
$3 \otimes 7 = ?$

$4 \otimes 4 = 10$
$4 \otimes 5 = 13$
$4 \otimes 6 = 16$
$4 \otimes 7 = ?$

$5 \otimes 4 = 13$
$5 \otimes 5 = 17$
etc.

If the class doesn't seem to be able to figure out the role of the 3 and the 4 in $3 \otimes 4$ after a large number of examples, you can help them focus by putting a similar set of ordinary multiplication problems on another board.

$2 \times 3 = ?$ $\quad 3 \times 3 = ?$ $\quad$ etc.
$2 \times 4 = ?$ $\quad 3 \times 4 = ?$ $\quad$ etc.
etc.

Comparing the problems and their solutions usually leads to an understanding of $\otimes$.

Now that the class understands this new operation, return to the investigation of identity elements.

Is there a number $I_{\otimes}$ so that $I_{\otimes} \otimes k = k$ and $k \otimes I_{\otimes} = k$ for all numbers k? Students will typically respond "1," which turns out to be incorrect although 'circle multiplication' by 1 turns out to have an interesting result. Eventually, the class discovers that the identity for this operation is 2.

Before leaving this operation, you may wish to look at the corresponding operation $\oplus$ defined by

$$a \oplus b = (a-1) + (b-1) + 1$$

and find its identity.*

If the class has looked at properties of operations like commutativity, you might want to investigate whether $\oplus$ and $\otimes$ also have these properties.

$\oplus$ and $\otimes$ can also be used to motivate computational practice with whole numbers. Students enjoy being able to mystify their friends and visitors and will perform calculations using $\oplus$ and $\otimes$ that they would consider boring in 'ordinary arithmetic'.

You might also encourage students to make up their own operations such as $a * b = (a+1) \times (b+1)$ and $a\#b = ab + 1$. They can give the class examples until the rule has been discovered. The class can also look at questions of identity elements, commutativity, etc.

3.2 Additive Inverses

Begin by reviewing open sentences and the additive identity. Build up the students' confidence with several examples like:

$$\begin{aligned} \square + 17 &= 20 \\ 4 + \triangle &= 72 \\ 17 + I_+ &= \text{⬡} \\ I_+ + 5 &= \bigtriangledown \end{aligned}$$

---

*The map $f: R \to R$ defined by $f: k \to k+1$ represents an isomorphism of R onto R where addition and multiplication in the image space are defined by $\oplus$ and $\otimes$ as above. In particular, this explains why $1=0+1$ is the identity for $\oplus$ and $2=1+1$ is the identity for $\otimes$. It also means that $\oplus$ and $\otimes$ will be commutative and associative and satisfy the distributive law.

Leaving these problems and their solutions on the board, write

$$5 + \square = I_+$$

Ask someone to read it. Ask what number will make it true.

Because this problem looks similar to the previous ones, students will be confident of their ability to solve it and will begin to volunteer answers. Accept each answer and explore its consequences on another part of the board. Students usually try 0, 5, and I_+ and quickly realize that they are not solutions. Then they suggest you change the problem so that it can be solved, e.g.:

Change the + to −.	$5 - \square = I_+$
Change the I_+ to 5.	$5 + \square = 5$
Change the + to x.	$5 \times \square = I_+$

In each case, solve the new problem. You might also use this opportunity to involve more students by solving a sequence of problems similar to the changed one, which leads to a generalization, e.g.:

$7-\square=I_+$	$19-\square=I_+$	$27.3-\square=I_+$	$\gamma-\square=I$
$7+\square=7$	$19+\square=19$	$27.3+\square=27.3$	$\gamma+\square=\gamma$
$7x\square=I_+$	$19x\square=I_+$	$27.3x\square=I_+$	$\gamma x\square=I_+$

Keep returning to the original problem. Occasionally, students will offer a solution containing two numbers such as 0-5 or 2-7. In this case you might look at the set of all possible ways of naming the solution and choose one representative to name the number which you add to 5 to get I_+. The easiest one to choose, of course, is 0-5, which can then be shortened to $^-5$. Some instructors like to guide the students onto this path by asking for two numbers that together will solve the problem. Another way to introduce this concept is to look at different ways to name the solution to an equation like $5+\square=10$. 5-0, 7-2, 9-4, etc., all belong to the equivalence class for 5.

Some of the students may be aware of negative numbers from the weather reports or science classes. You might get "5 below 0" or "minus 5" as an answer. If the majority of the class seems ready to accept those answers, say that those are both correct names for the answer but that mathematicians usually call it 'negative 5', written $^-5$. Sometimes mathematicians use $^+5$, 'positive 5', instead of 5, but they usually omit it since $^+5$ and 5 operate the same way.

If no one has seen negative numbers, the students may suggest that the problem can't be done. If it is early in the discussion, you might encourage the class to think about it further: "Are you sure we can't solve it if we try a little harder?" If the entire class agrees that there is no solution, point out that mathematicians worried for centuries about that problem and eventually decided to invent a solution, $^-5$, read 'negative five'.

Now pose several more problems of this type:

$$8 + \square = I_+$$

$$\square + 17 = I_+$$

$$123 + \triangle = 0$$

$$\bigtriangledown + 86 = 0$$

$$^-1276 + 1276 = \text{⬡}$$

$$\square + {}^-29 = 0$$

Generalize to $\alpha+{}^-\alpha=I_+$ or $\alpha+{}^-\alpha=0$.

While doing these problems, introduce the terminology 'additive inverse' by asking who can tell you the additive inverse of 8 when you write $8+\square=I_+$. Similarly write $\square+{}^-29=0$ and ask for the additive inverse of $^-29$.

The question of $^-0$ sometimes arises. The following parallel sequence of problems can be used to show that $0={}^-0$:

$$\begin{array}{lll}
6+\square=0 & \qquad & 6+\square=6 \\
4+\square=0 & \qquad & 4+\square=4 \\
2+\square=0 & \qquad & 2+\square=2 \\
0+\square=0 & \qquad & 0+\square=0
\end{array}$$

Now compare $0+\boxed{^-0}=0$ and $0+\boxed{0}=0$ by drawing arrows between parts of the sentences that are the same.

A similar argument can be used to demonstrate that $k={}^{--}k$. When you have a result like $^-5+5=0$ on the board, look at $\square+{}^-\square=0$. After the class has substituted 5, 17, 92, etc., and determined that any number will make the sentence true, ask what happens when you put $^-5$ in the square variable. Now compare $^-5+5=0$ and $\boxed{^-5}+{}^-\boxed{^-5}=0$.

To reinforce the concept of additive inverses, classwork and homework should include several problems like the following:

$$^-2+2+{}^-3+3+5+{}^-5=\square$$

$$^-2+5+{}^-3+3+2+{}^-5=\square$$

$$^-2+2+5=\square$$

$$^-4+5+{}^-5+4+{}^-3+3+17=\square$$

$$^-13+13+{}^-5+2+3=\square$$

Students should be asked to find all the zeros in solving these problems.

An important review activity is to look at the parallel with multiplication:

$$5 + \square = I_+ \qquad 5 \times \triangle = I_x$$

3.3 Addition of Positive and Negative Numbers

The addition of positive and negative numbers breaks up into three cases:

Case 1: $a,b>0$ $\ ^-a+b = ?$ where $|a|<|b|$

Case 2: $a,b>0$ $\ ^-a+{}^-b = ?$

Case 3: $a,b>0$ $\ ^-a+b = ?$ where $|a|>|b|$.

Case 1

Begin the investigation of problems like $^-5+7=?$ by reviewing the basic properties of additive inverses. Your review problems might include:

$$3 + \square = 0$$
$$\bigtriangledown + {}^-5 = 0$$
$$\alpha + {}^-\alpha = \triangle$$
$$^-5 + 5 + 2 = \square$$

Now look at $^-5+7=\square$.

Who has an idea about what $^-5+7$ equals?

What number do we have to add to $^-5$ to get 0?

Is there a 5 hidden in that sentence?

Who can give me another name for 7 that has 5 in it?

Is there an 0 hidden in this sentence?

Who can rewrite the 7 so that the hidden zero is more obvious?

What do I bring down for the $^-5$?

What do I put for + and = in the equivalent sentence?

Who knows what $^-5+7$ equals?

Who can prove it?

On the board you should end up with

$$\begin{tikzcd}
{}^-5 \arrow[d] & + & 7 \arrow[d] \arrow[drr] & & & = & \boxed{2} \\
{}^-5 \arrow[dr] & + & 5 \arrow[dl] & + & 2 \arrow[d] & = & \boxed{2} \arrow[u] \\
& 0 & & + & 2 & = & \boxed{2} \arrow[u]
\end{tikzcd}$$

Repeat this process with several more examples. Allow the students to develop a rule for solving this type of problem, although it is important to have them review the conceptual process of 'splitting' occasionally.

Once they have established a procedure for problems of this type, weave in review of basic arithmetic.

Examples: $^-1982 + 4976 = \square$

$17.3 + {}^-14.2 = \square$

$\frac{3}{4} + \frac{-1}{2} = \triangle$

Note that we could have used a similar process for dealing with problems like $\frac{1}{2}$x10. Instead of writing out ten addends of $\frac{1}{2}$, we could have split the 10 so that we found a 1 in the sentence, i.e.:

$$\frac{1}{2}x10 = \frac{1}{2}x2x5 = 1x5 = 5.$$

You may wish to investigate the comparison between $\frac{1}{2}x10$ and $^-2+10$, etc., as a review lesson.

Case 2

The answer to problems like $^-2+^-3=\square$ seems obvious, although if we simply accept this there is no reason not to accept $^-2x^-3=^-6$.

The procedure for solving this problem is exactly parallel to the procedure we used for determining that $\frac{1}{2}x\frac{1}{3}=\frac{1}{6}$,

We're not certain we know what $^-2+^-3$ equals, so let's put it in a true mathematical sentence and see what it acts like.

When we were trying to discover the value of $\frac{1}{2}x\frac{1}{3}$ we used multiplicative inverses.

Who can make up a sentence that shows that $\frac{1}{2}x\frac{1}{3}=\frac{1}{6}$?

Review

$$\begin{tikzcd}[column sep=small] \bigcirc\!\!\!\!\!\!\frac{1}{2}x\frac{1}{3} \arrow[d] & x & \underbrace{3x2} \arrow[d] & = 1 & \text{or} & \frac{1}{2}x\frac{1}{3} \; x \; 3x2 = I_x \\ \bigcirc\!\!\!\!\!\!\frac{1}{6} & x & 6 & = 1 & & \end{tikzcd}$$

What kind of an inverse is $\frac{1}{2}$?

What kind of identity element did we use to find out what $\frac{1}{2}x\frac{1}{3}$ was?

What kind of an inverse is ${}^-2$?

What kind of an identity element do you think we should use to find out what ${}^-2+{}^-3$ equals?

What if I add 3 to it?

What if I add 2 to it?

Who can give me a true mathematical sentence containing ${}^-2+{}^-3$?

Let's look at

$$\underset{\downarrow}{\boxed{{}^-2+{}^-3}} + \underset{\downarrow}{\boxed{3+2}} = 0$$

Who can come to the board and prove that ${}^-2+{}^-3 + 3+2 = 0$?

What does 3+2 equal?

What do I bring down for '+'? '='? 0?

What is the additive inverse of 5?

What is ${}^-2+{}^-3$ acting like?

What do you think ${}^-4+{}^-5$ equals?

Can we use it in a sentence that equals 0?

What two numbers would we need in the sentence?

Who can give me the sentence?

Who can prove that it's true?

What's ${}^-4+{}^-5$ acting like?

Who can prove it?

Now do several similar problems.

In the answer to ${}^-9+{}^-6$, where does the 15 come from?

Where does the 'negative' come from?

Who can describe in words what you do when you add two negative numbers?

How can we write the 15 to show where it comes from?

Have on the board:

$${}^-9+{}^-6 = {}^-(9+6)$$

Use the erase-and-change technique to arrive at a generalization:

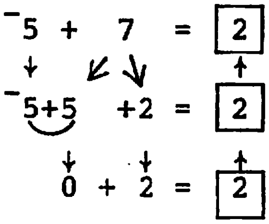
$${}^-\alpha+{}^-\beta = {}^-(\alpha+\beta)^{*}$$

An alternative is to prove this result directly using the sentence ${}^-\alpha+{}^-\beta + \alpha+\beta = 0$. A third approach is to ask the class to write in symbols what they have said in words.

<u>Case 3</u>

Now that the class can add two negative numbers, the solution of problems like $5+{}^-7$ is entirely analogous to the 'split system' used with ${}^-5+7$.

Begin by reviewing the solution to ${}^-5+7$:



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* Note that what we have done here and with reciprocals is simply a special case of the proof that for any associative operation *, if a' denotes the * inverse of a, then $(a*b)' = b'*a'$.

You can arrive at a solution to $5+{}^{-}7$ by simply changing the ${}^{-}5$ to 5 and the 7 to ${}^{-}7$. Ask the class what additional changes have to be made so that all the statements on the board are true. The students will make the correct changes somewhat mechanically, so it is important to go back through each step and verify that the substitutions do, in fact, give equivalent expressions and true statements.

An alternate approach is to simply put up the problem $5+{}^{-}7=\square$. If the class is secure with the split system, they will be able to give you the answer and prove it with very little guidance from you. If they are not confident, guide them with a sequence of questions analogous to those used originally to solve ${}^{-}5+7$.

Now do several more examples including a circulation problem. As long as the students can explain their answer if asked, it is not necessary to prove their result each time.

To review and consolidate the results on addition of integers, put on the board several sets of problems like:

$$\begin{array}{rclcrcl}
4 + 9 &=& \square & \qquad & 17 + 26 &=& \square \\
{}^{-}4 + 9 &=& \triangle & & {}^{-}17 + 26 &=& \triangle \\
{}^{-}4 + {}^{-}9 &=& \bigtriangledown & & {}^{-}17 + {}^{-}26 &=& \bigtriangledown \\
4 + {}^{-}9 &=& \text{⬡} & & 17 + {}^{-}26 &=& \text{⬡}
\end{array}$$

Ask the class to tell you in words all the possible cases when you are adding two positive and negative numbers. Have them formulate a rule for computing each case.[*]

---

[*] It is preferable for the students to use colloquial terminology like the 'number part' than to have an incorrect impression of absolute value. Examples like $|{}^{-}7|=7$ lead to the

Note that if the class has studied ordering of fractions, problems like $\frac{1}{2}+{}^{-}(\frac{1}{4})$, $\frac{1}{3}+{}^{-}(\frac{1}{2})$, $\frac{2}{3}+{}^{-}(\frac{3}{4})$, etc., can be used to motivate review.

3.4 Subtraction of Positive and Negative Numbers

The conceptual development of the process for subtracting positive and negative numbers is exactly parallel to the development of division of fractions. The parallel is illustrated below.

Write the equivalent:

Multiplication sentence	Addition sentence
$7\div\frac{1}{3} = \square \Longleftrightarrow \square\ x\frac{1}{3} = 7$	$7-{}^{-}3 = \square \Longleftrightarrow \square + {}^{-}3 = 7$
What number do I multiply by $\frac{1}{3}$ to get to 1?	What number do I add to ${}^{-}3$ to get to 0?
Now what number do I have to multiply by 1 to get to 7?	Now what number do I have to add to 0 to get to 7?
(Answer: first 3, the multiplicative inverse of $\frac{1}{3}$, then 7.	(Answer: first 3, the additive inverse of ${}^{-}3$, then 7.
So $7\div\frac{1}{3} = 7x3$.	So $7-{}^{-}3 = 7+3$.

In other words, the 'invert and multiply' rule and the 'change the sign and add' rule are both saying "change the

---

impression that $|{}^{-}k|=k$ for all k. Absolute value should be introduced only when students can interpret it as the distance from 0 on the number line or when they can handle the definition $|k|=k$ for $k\geq 0$, $|k|={}^{-}k$ for $k<0$.

operation to its inverse operation and change the second term to its inverse for that operation." (No one is expected to remember this mouthful although the students should be able to tell you what's the same or similar about the two problems above.)

The following set of examples illustrates the method in all possible combinations involving a 7 and a 3. In each case, the process of solution is the same:

(1) Write the equivalent addition sentence.

(2) Add the inverse of the subtrahend. (What do I have to add to the second term to get to zero?)

(3) Add the minuend. (Now what do I have to add to get back to the first term?)

(4) To find the answer, perform the computation in steps (2) and (3).

Examples:

$$7 - 3 = \boxed{} \Leftrightarrow \boxed{7 + {}^{-}3} + 3 = 7$$

$${}^{-}7 - 3 = \boxed{} \Leftrightarrow \boxed{{}^{-}7 + {}^{-}3} + 3 = {}^{-}7$$

$$7 - {}^{-}3 = \boxed{} \Leftrightarrow \boxed{7 + 3} + {}^{-}3 = 7$$

$${}^{-}7 - {}^{-}3 = \boxed{} \Leftrightarrow \boxed{{}^{-}7 + 3} + {}^{-}3 = {}^{-}7$$

$$3 - 7 = \boxed{} \Leftrightarrow \boxed{3 + {}^{-}7} + 7 = 3$$

$${}^{-}3 - 7 = \boxed{} \Leftrightarrow \boxed{{}^{-}3 + {}^{-}7} + 7 = {}^{-}3$$

$$3 - {}^{-}7 = \boxed{} \Leftrightarrow \boxed{3 + 7} + {}^{-}7 = 3$$

$${}^{-}3 - {}^{-}7 = \boxed{} \Leftrightarrow \boxed{{}^{-}3 + 7} + {}^{-}7 = {}^{-}3$$

In general, for any a and b

$$a - b = \boxed{} \Leftrightarrow \boxed{} + b = a$$

and since $a + {}^{-}b + b = a$, $a - b = a + {}^{-}b$.

If your class has had a lot of practice with the development of the invert and multiply rule, you might introduce the subtraction of positive and negative numbers somewhat formally following the outline above. Some questions which elicit the intermediate steps are given below.

> Who knows what 7-3 equals?
>
> How do you check that?
>
> What's an addition sentence equivalent to 7-3=4?
>
> What's an addition sentence equivalent to $7-3=\square$?
>
> What number makes $\square+3=7$ true?
>
> What number makes $7-3=\square$ true?
>
> Who has an idea about $3-7=\square$?
>
> Who can write an addition sentence equivalent to that subtraction sentence?
>
> Who knows what number I have to add to 7 to get 3?
>
> What is 3-7 equal to?

Note that in both these cases, once you have written the equivalent addition sentence, the solution is readily apparent. Leave both sentences on the board and write:

$$7 \div 3 = \square \text{ and } 3 \div 7 = \triangle$$

Have the class write the equivalent multiplicaiton sentences and solve each by the two-step process of getting to 1 and then getting to the dividend.

Now ask questions like:

> Who can tell me what the multiplicative identity equals?

What happens when you multiply a number by 1?

What number acts the same way for addition?

What is the additive identity?

What do I multiply 3 by to get 1?

What is the reciprocal of 3?

What is another name for reciprocal?

What is the additive inverse of 3?

What do I add to 3 to get 0?

What if I change the '÷' to '-'? What else would I have to change?

If I change the 'x' to '+' and $\frac{1}{3}$ to ${}^{-}3$, is this still a true sentence?

How do we check?

Now repeat the process with a problem like $7-{}^{-}3=\square$ and the rest of the examples above.

If the class is not confident about the division of fractions, you will want to develop the subtraction of positive and negative numbers independently and then go back and draw the analogy with division to reinforce that process.

One way to do this is to review the relationship between subtraction and addition, e.g.,

$$8 - 3 = 5 \Leftrightarrow 5 + 3 = 8$$

$$9 - 4 = \square \Leftrightarrow \square + 4 = 9$$

Now put up several problems like

$$0 - 4 = \square \text{ and } 0 - {}^{-}4 = \triangle$$

Ask for the equivalent addition sentences:

$$\square + 4 = 0 \text{ and } \square + {}^{-}4 = 0$$

In each case, the solution is apparent from our work with additive inverses. Repeat the process, changing the 4 to other numbers.

You have now established a special case of the rule for subtraction although it has not been articulated, and you have laid the groundwork for the general case.

Next write

$$7 - 4 = \square \text{ and } 7 - {}^{-}4 = \bigcirc$$

Ask for the equivalent addition sentences:

$$\square + 4 = 7 \text{ and } \bigcirc + {}^{-}4 = 7$$

If the 7 were a 0, what would the solution be?

If we put that number in the variable, how far are we off?

What do I have to add to that number to get to 7?

Who can give me two numbers and an operation that will make this a true sentence?

What do I have to add to ${}^{-}4$ to get 0?

Now what do I have to add to get 7?

What does $7-{}^{-}4$ equal?

What does 4+7 equal?

What do I have to add to 4 to get 0?

Now what do I have to add to that number to get to 7?

What does ${}^{-}4+7$ equal?

Does this process give the correct answer for $7-4$?

Repeat this process with a variety of problems until the class develops a general rule.

Patterns can be used effectively as reinforcement.

$5 - 3 = 2$

$4 - 3 = 1$

$3 - 3 = 0$

$2 - 3 = ?$

$1 - 3 = ?$

$0 - {}^{-}3 = 3$

$1 - {}^{-}3 = 4$ If you start with one more, what do you have left?

$2 - {}^{-}3 = 5$ If you start with one more than that, what do you have left?

${}^{-}1 - {}^{-}3 = ?$ If you start with one less, what do you you have left?

$5 - 5 = \square$

$17 - 17 = \square$

$Y - Y = \square$

${}^{-}3 - {}^{-}3 = \square$

${}^{-}17 - {}^{-}17 = \square$

Another review and reinforcement activity that is appropriate at this time is to introduce some 'real-world' interpretations of negative numbers.

Examples:

It's 17 degrees and the temperature drops 4 degrees. What's the temperature now? What's a subtraction sentence that illustrates this problem?

It's 0 degrees and the temperature drops 4 degrees. What's the new temperature? What's a subtraction sentence that illustrates this problem?

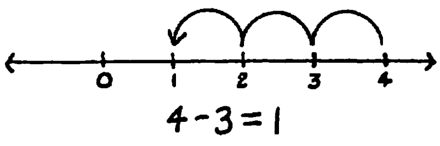
You have $17 in the bank. The bank discovers it has incorrectly charged you a $4 service charge. It credits your account $4 (takes away the $4 charge). How much money is in your account? Write a subtraction sentence that illustrates this problem.

If you've run out of money and you owe $25 on your credit card, how much money do you have? (Students may respond zero. You can help them see that they're actually below zero or ¯25 by asking, "If you have $0 and you get $25, how much money do you have?" "If you owe $25 and you get $25, how much money do you have?") You return something that cost $10 (subtract a charge of $10). Now how much do you owe? What subtraction sentence illustrates this problem?

Have the class make up their own word problems using positive and negative numbers.

3.5 The Numberline

The numberline can be used as an alternative introduction to negative numbers. It also is an excellent vehicle for review and reinforcement when negative numbers have been approached theoretically as above.



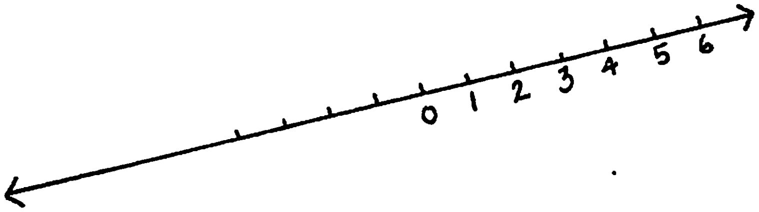
One way to lead into negative numbers is to ask the class to tell you what problem the picture represents and work several similar examples. Then ask them to draw the picture for some problems you give: 3-1= , 3-2= , 3-3= , and then 3-4= .

Where do we start? Which way do we go? How many steps do we go?

Mathematicians have a name for this spot just one to the other side of zero: ¯1.

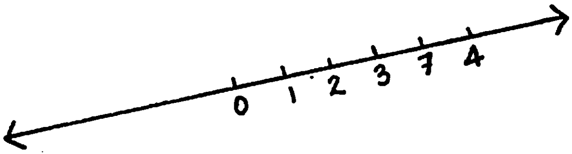
Fill in other spots to the left of 0.

If the class has already seen negative numbers, put up a partial numberline.



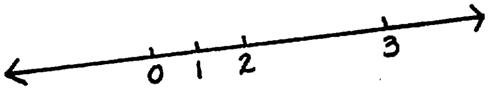
Putting up the numberline provides a good place to make deliberate errors which force the students to articulate the properties of the numberline. For example:

(a)



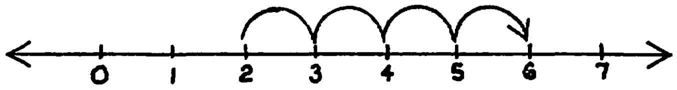
The numbers have to go in order.

(b)



The distance between consecutive integers has to be the same. (More formally, the distance between any two numbers on the numberline is the same length as the segment from 0 to the number which is the absolute value of their difference.)

Now ask what number goes with the point on the other side of 0 from 1. After several false starts, you should get $^{-}1$. If the class is uncertain about it and to reinforce the addition process, look at the illustration of addition problems using the numberline.



How would you represent 2+4 on the numberline?

Where do you start?

What direction do you go?

How many steps do you go?

Where do you end up?

Now look at problems like $\square+2=7$, $\triangle+1=0$, $\nabla+2=0$, etc. No matter how the numberline has been introduced, once the class has become familiar with the pictorial procedure for adding a positive number, use it to solve problems like $^{-}3+5=\square$ and $^{-}7+5=\nabla$. If the class has learned the algebraic process for solving these problems, it should be reviewed to verify that it leads to the same results.

Now put up $^{-}3+3=\square$. Review its solution on the numberline with the standard questions:

Where do I start?

What direction do I go?

How many steps?

Where do I end up?

What does $^{-}3+3$ equal?

Now look at $3+{}^-3=\square$. Because addition is commutative, we want the sum to be 0.

If we start at the 3, which direction do I go?

How many steps?

Where do I end up?

What if I had $5+{}^-5$?

When you add a negative number, which direction do you go?

If I have $5+{}^-3$, where do I start?

What direction do I go?

How many steps?

Where do I end up?

What does $5+{}^-3$ equal?

Who can find $5+{}^-7$ using the numberline?

Who can find the sum of ${}^-3$ and ${}^-2$ on the numberline?

There are several activities that can establish the procedure for subtraction using the numberline. First compare addition and subtraction using positive numbers and then extend the reasoning to negative numbers. Put on the board a set of examples like

(a) $5 + 3 = \square$ (b) $5 + {}^-3 = \triangle$

(c) $5 - 3 = \nabla$ (d) $5 - {}^-3 =$ ⬡

Solve (a)-(c) using the numberline. Ask questions which bring out the equivalence of (b) and (c) and the contrast between (a) and (c).

Where do I start?

What direction do I go?

What operation is the opposite of addition?

When you add a positive number, what direction do you go?

When you subtract a positive number, what direction do you go?

When you add a negative number, what direction do you go?

Adding a negative number is the same as subtracting what kind of number?

What operation is the opposite of subtraction?

When you add a negative number, what direction do you go?

When you subtract a negative number, what direction do you think you'd go?

Look at problems like ${}^-3-{}^-3=\square$ to verify this conjecture.

An alternative approach to operations on the numberline is to first study the algebraic analysis of positive and negative numbers using the methods of the previous sections. Choose a representative set of problems:

(a) $8 + 2 = \square$ (b) $8 + {}^-2 = \triangle$

(c) $8 - 2 = \triangledown$ (d) $8 - {}^-2 =$ ⬡

Solve each and then ask the class how each problem could be represented on the numberline. Repeat this process until the class has discovered the rules. Many students find that the pictorial representation showing subtracting a negative is the opposite of adding it is a helpful guide to remembering how to perform the operations.

Students often have difficulty accepting the concept of ordering the negative numbers such as ${}^-5 < {}^-3$. The numberline can be used to introduce this concept. Beginning with positive numbers, ask questions like, "As I move to the right, do the numbers get larger or smaller?" "What about the left?" Then extend this reasoning to the left of zero.

An algebraic approach to order begins with an example like:

$$4 + {}^-1 \bigcirc 4$$

Students will quickly determine that '<' goes in the circle. Continue with additional examples and arrive at a generalization.

$$5 + {}^-1 < 5$$

$$10 + {}^-1 < 10$$

$$\alpha + {}^-1 < \alpha$$

Now substitute numbers for α including ${}^-3$. If $\alpha + {}^-1 < \alpha$ for all α, then we must conclude that ${}^-4 < {}^-3$.

Patterns also help support this result:

$3 - 1 < 3$

$3 - 2 < 3$

$3 - 3 < 3$

$3 - 4 < 3$

$3 - 5 < 3$

$3 - 6 < 3$

$3 - 7 < 3$

Real-world examples are perhaps the most convincing. Is $^{-}25^{\circ}$ colder or warmer than $^{-}15^{\circ}$? Would you rather owe \$25 or \$15?

3.6 Exercises

1. a. $5 + I_+ = \square$

   b. $I_+ + \frac{3}{4} = \square$

   c. $17 \times I_x = \square$

   d. $I_x \times 43.7 = \square$

   e. $I_+ + I_+ + I_+ + 16 = \square$

   f. $I_x + I_x + I_x + 16 = \square$

2. a. $5 + {}^-5 = \square$

   b. ${}^-27 + 27 = \square$

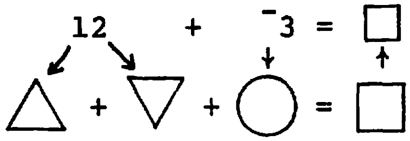
   c. $\frac{1}{2} + \square = 0$

   d. $\frac{-3}{4} + \square = I_+$

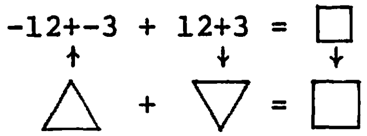
   e. ${}^-4 + 5 + 4 + 3 + {}^-5 + 17 = \square$

   f. $\alpha + {}^-\alpha = \square$

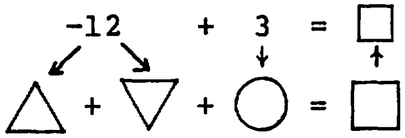
3. a.



   b.



   c.



4. Use additive inverses to show that

   a. $15 + {}^-4 = 11$

   b. ${}^-15 + {}^-4 = {}^-19$

   c. ${}^-15 + 4 = {}^-11$

5. a. $^{-}3 + 17 =$

b. $^{-}15 + {}^{-}27 =$

c. $^{-}14 + 2 =$

d. $^{-}(\frac{1}{2}) + {}^{-}(\frac{1}{4}) =$

e. $^{-}(\frac{1}{2}) + (\frac{3}{4}) =$

f. $^{-}(\frac{7}{8}) + \frac{1}{4} =$

g. $^{-}37.2 + 43.8 =$

h. $45.92 + {}^{-}91.84 =$

6. Write the equivalent addition sentence and solve each subtraction problem.

a. $9 - 5 = \square \Longleftrightarrow \square + 5 = 9$

b. $9 - {}^{-}5 = \square$

c. $^{-}9 - 5 = \square$

d. $^{-}9 - {}^{-}5 = \square$

e. $5 - 9 = \square$

f. $5 - {}^{-}9 = \square$

g. $^{-}5 - 9 = \square$

h. $^{-}5 - {}^{-}9 = \square$

7. a. $5 - 5 =$

b. $^{-}5 - {}^{-}5 =$

c. $^{-}7 - {}^{-}2 =$

d. $0 - 5 =$

e. $14.5 - 13.2 =$

f. $14.5 - {}^{-}13.2 =$

g. $\frac{3}{4} - \frac{1}{2} =$

h. $\frac{3}{4} - {}^{-}(\frac{1}{2}) =$

8. a. Draw a numberline that contains $^{-}5$, 3, 1, $^{-}4$, 2, 0, $^{-}1$, 4, 5, $^{-}3$, $^{-}2$.

Use the numberline to show how to solve $^{-}3+^{-}2$.

b. Draw another numberline and show the solution of $^{-}3-^{-}2$.

9. At 9 A.M. the temperature was 17 below zero. By noon it had warmed up 10 degrees. Between noon and 5 o'clock it fell 5 degrees. What was the temperature at 5 o'clock? (Write a sentence using positive and negative numbers to show how you got your answer.)

10. At the beginning of October, Mrs. Smith owes $243.17 on her credit card. During the month she charges a sweater for $27.95 and socks for $4.50 and returns a dress which cost $35.75. On October 10, she makes a payment of $50. How much does she owe at the end of the month? (Write a sentence using positive and negative numbers to show how you get your answer.)

3.7 Chapter Summary

Identity Elements

$\forall k,\ k + I_+ = k$ and $k \times I_x = k$

Zero is the additive identity ($I_+ = 0$).

One is the multiplicative identity ($I_x = 1$).

Additive Inverses

$\alpha + {}^{-}\alpha = 0$

Addition of Positive and Negative Numbers

Addition of two positive or negative numbers using the 'split system' or 'use it in a sentence method'.

E.g.:

$$\begin{tikzcd}
{}^{-}5 \arrow[d] & + & 9 \arrow[d, shift left] \arrow[d, shift right] & = & \boxed{4} \\
{}^{-}5 & + & 5+4 & = & 4 \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\underbrace{{}^{-}5+{}^{-}4} \arrow[d] & + & \underbrace{5+4} \arrow[d] & = & 0 \\
\boxed{{}^{-}9} & + & 9 & = & 0
\end{tikzcd}$$

Subtraction of Positive and Negative Numbers

Subtraction of two positive and negative numbers using the equivalent addition sentence and the 'go to zero' process:

E.g.: $a - {}^{-}b = \boxed{} \Leftrightarrow \boxed{} + {}^{-}b = a$

Numberline

Location of negative numbers

Solution of addition and subtraction problems using the numberline

Optional Topics

Nonstandard operations

Ordering of negative numbers

Vocabulary

- Identity element
- Additive identity
- Multiplicative identity
- Additive inverse
- Positive
- Negative
- Equivalent sentence
- Numberline

3.8 Chapter Test

1. a. $16 + I_+ =$ □

   b. $I_x \times \frac{3}{8} =$ △

   c. $I_+ =$ ▽

   d. $I_x =$ ⬡

2. a. $9 +$ □ $= 0$

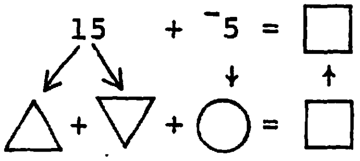
   b. $^-5 +$ △ $= 0$

   c. $k + {}^-k =$ ▽

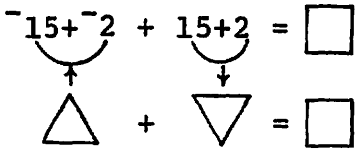
3. What is the additive inverse of 7?

   What is the additive inverse of $^-47$?

4.



5.



   So $^-15 + {}^-2 =$ ______

6. a. $47 + 3 =$

   b. $47 + {}^-3 =$

   c. $^-47 + 3 =$

   d. $^-47 + {}^-3 =$

7. a. $^-7 - {}^-3 =$ □ $\Leftrightarrow$ □ + △ = ⬡

   b. $3 - 7 =$ □ $\Leftrightarrow$ □ + △ = ⬡

8. a. $5 - {}^{-}2 = \square$

b. ${}^{-}3 - {}^{-}9 = \square$

9. a. Draw a numberline and use it to show the solution to ${}^{-}4-2$.

b. Draw a numberline and use it to show the solution to ${}^{-}4-{}^{-}2$.

10. At the beginning of the month, Mr. Jones owes $500 on his credit card. During the month he makes a payment of $75. He charges gas for his car costing $21 and a pair of shoes which cost $45. He returns a shirt which cost $15. Write a sentence using positive and negative numbers which shows how much money he owes at the end of the month.

MODULE 4: NEGATIVE AND FRACTIONAL EXPONENTS

4.0 Introduction

The material in this module extends the thinking of Module 1 and combines it with the concepts of Modules 2 and 3. First and foremost, it embeds review in a new conceptual context. This is a hallmark of good discovery teaching, providing an opportunity for reinforcement and practice without the onus of repetitious drill. Both the conceptual results and the problem-solving strategies of the previous modules are reinforced by this chapter.

In addition, Chapter 4 provides some alternate conceptual paths to previously established results. These are particularly useful for students who are unsure of themselves or who may find an alternate approach conceptually easier to understand.

In this chapter we begin to present question sequences in a conceptually comprehensive but less meticulously detailed manner than in previous units. Many of the details in the previous sections are examples of recurring types of question sequences. They are included to illustrate the process of asking many small questions in the development of a larger concept. Having worked through the previous chapters, you will probably find that you have internalized the process and developed the knack of asking the variety of questions which maintains student focus and involvement and a smooth mathematical flow. You should also find the students becoming more confident and more adept at mathematical problem solving with less guidance from you. If, as you teach from this chapter, you feel the need for more details, review some of the expanded question sequences in the previous units and try to formulate a similar sequence for the current concepts.

It is also essential to keep in mind the various techniques from the "Guidelines for Discovery Teaching," particularly review and feedback. For example, whereas we have included one or two examples here of a type of problem that leads to the conclusion 2E¯1=$\frac{1}{2}$, in the classroom you will want to review and repeat the process with additional examples until the class has firm grip on the concept. Section 4.4 contains several applications which review arithmetic skills. You may wish to look ahead to this section and weave in these topics as you go along rather than waiting.

4.1 Negative Exponents

Begin with a rapid oral review of additive inverses (5+?=0, 3+¯3=? , ¯14+?=0, etc.), then review exponents and the additive property of exponentiation, focusing on powers of two. Asking the class for the numerical values in each case, put on the board:

2E4 = 16

2E3 = 8

2E2 = 4

2E1 = 2

(Leave room for several more entries below 2E1.)

Now ask for the next problem. If the class has resolved 2E0=1, put it on the board and review the various arguments, including putting it in a true sentence using the additive propery of exponentiation. If the class is very confident about 2E0, this is a good opportunity to add some excitement to the lesson by arguing the opposite position using their original argument that 2E0=0 "because you don't put any two's down." If the class has not yet resolved, 2E0, put a '?' and move on to the next question. (In the development which follows, we first present an approach to negative exponents that depends on 2E0=1 and then two which can be investigated independently of that result.)

> Do you think we can take the table any lower than 2E0?
>
> What do you think $2E^{-}1$ is equal to? (Record all conjectures.)
>
> Can we put it in a sentence to find out what it should be?

Guide the class to:

$$\begin{array}{ccccc} (2E^{-}1) & x & (2E1) & = & 2E0 \\ \downarrow & & \downarrow & & \downarrow \\ \square & x & 2 & = & 1 \end{array}$$

> What does $2E^{-}1$ act like in this sentence?
>
> What do you think mathematicians have defined $2E^{-}1$ to be?

Note that at this point, it is necessary for the class to recall that $\frac{1}{2}x2=1$. You might wish to insert this in the review at the beginning of the lesson. We chose not to in this development because the review already contained several concepts and it seemed more motivating to bring up $\frac{1}{2}$ in context. If the class stumbles on $\frac{1}{2}$, a quick repetition of the conceptual development in Chapter 2 should reestablish multiplicative inverses. Even if they have no problem, you might use this opportunity to review more extensively.

There is no hard and fast rule about when to introduce review material. It is a matter of personal preference based as much on the personality of the class and the need for variety as on any logical mathematical structure. This particular lesson has been taught successfully with the review of reciprocals both at the beginning and in context. In fact, in planning the course, some instructors omit the work on fractions, introducing first the concept of additive inverses and then use negative exponents to motivate the study of fractions. It doesn't matter as long as the students have a sense of mathematical growth and accomplishment and whatever computational skills they need to acquire are interwoven with the algebra.

Once the class has decided that $2E^{-}1=\frac{1}{2}$, the students can vary the process to establish $2E^{-}2=\frac{1}{4}$, $2E^{-}3=\frac{1}{8}$, etc.

Some students might discover an approach to $2E^{-}2$ that uses what they have learned about $2E^{-}1$.

$$\begin{tikzcd}
(2E^{-}1) \arrow[d] & \times & (2E^{-}1) \arrow[d] & = & 2E^{-}2 \\
\frac{1}{2} & \times & \frac{1}{2} & = & \frac{1}{4} \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
(2E^{-}1) \arrow[d] & \times & (2E^{-}2) \arrow[d] & = & 2E^{-}3 \\
\frac{1}{2} & \times & \frac{1}{4} & = & \frac{1}{8} \arrow[u]
\end{tikzcd}$$

As you do more problems and in subsequent review lessons, the class should arrive at the generalizations $2E^{-}\alpha = \frac{1}{2E\alpha}$ and in general for $\alpha\neq 0$, $\alpha E^{-}\beta = \frac{1}{\alpha E\beta}$ since

$$\begin{tikzcd}
(\alpha E\beta) \arrow[d] & \times & (\alpha E^{-}\beta) & = & \alpha E0 \arrow[d] \\
(\alpha E\beta) & \times & \boxed{\frac{1}{\alpha E\beta}} \arrow[u] & = & 1
\end{tikzcd}$$

An alternate approach to negative exponents which does not depend on 2E0 uses mathematical sentences like

$$\begin{tikzcd}
(2E^{-}1) \arrow[d] & \times & (2E3) \arrow[d] & = & 2E(^{-}1+3) & = & 2E2 \arrow[dll] \\
\boxed{} & \times & 8 & = & 4 & &
\end{tikzcd}$$

Often students who are certain that 2E0 is 0 are able to see that 2E^{-}1 must be $\frac{1}{2}$. After you have looked at several more examples, you may want to present them with (2E^{-}1)x(2E1)=? This leads to one more piece of evidence that 2E0 should be 1:

$$\begin{tikzcd}
(2E^{-}1) \arrow[d] & \times & (2E1) \arrow[d] & = & 2E(^{-}1+1) & = & 2E0 \arrow[dll] \\
\frac{1}{2} & \times & 2 & = & \boxed{} & &
\end{tikzcd}$$

Negative exponents can also be approached through the 'subtractive property'. The argument also is an excellent review activity for classes which have arrived at negative exponents using one of the paths above.

Begin with several review problems like $\frac{2E4}{2E3}=2E1$ and review the generalization $\frac{\alpha E\beta}{\alpha E\gamma} = \alpha E(\beta-\gamma)$.

Now look at
$$\begin{tikzcd}
\frac{2E3}{2E4} \arrow[d] & = & 2E(3-4) & = & 2E^{-}1 \arrow[dll] \\
\frac{8}{16} & = & \boxed{} & &
\end{tikzcd}$$

Patterns can be used to reinforce the results on negative exponents:

$$\begin{array}{lll} 2E3 & = 8_{\downarrow} & \\ 2E2 & = 4_{\downarrow} & \div 2 \\ 2E1 & = 2_{\downarrow} & \div 2 \\ 2E0 & = 1_{\downarrow} & \div 2 \\ 2E{}^{-}1 & = \frac{1}{2_{\downarrow}} & \div 2 \\ 2E{}^{-}2 & = \frac{1}{4} & \div 2 \end{array}$$

4.2 Multiplicative Property of Exponentiation

The general formula $(\alpha E\beta)E\gamma = \alpha E(\beta x\gamma)$ is an immediate outgrowth of the additive property of exponentiation. Many instructors teach it and fractional exponents along with the material in Module 1, introducing and reviewing the necessary concepts with fractions only as they arise in context. In these notes, we included a more comprehensive look at fractions first because these topics form a major portion of the general mathematics curriculum. We also wanted to illustrate early in this volume how a common arithmetic topic can be taught from a conceptual, problem-solving point of view. We include the multiplicative property now because it can be used to reinforce the concepts in the next section and the next unit. It would not destroy the logical continuity of the presentation to omit it entirely along with the subsequent applications or to wait until the class is ready for the applications before introducing it.

After a brief review of exponents and the additive property, write on the board

$$(2E3)E4$$

Ask if anyone can write an equivalent expression using only one E. Record conjectures and investigate the most promising. If the class concludes that (2E3)E4=2E12, try several more examples, then move on to the questions below to establish the conceptual foundation for the generalization. Usually the class will not arrive at the correct conclusion quickly, and you should leave the question open and move on to the conceptual questions which follow.

First review the basic concepts of whole number exponents and base emphasizing:

(a) What does the base tell us?

(b) What does the exponent tell us?

(c) What is the base in this expression?

(d) What is the exponent in this expression?

Gradually introduce more and more complicated bases. For example, you might ask for the factor form of each of the following exponential expressions:

2E3 =

4.982E5 =

$(\frac{3}{4})$E4 =

☆E3 =

□ E4 =

(3+2)E4 =

(3x2)E4 =

(3E2)E4 =

If the students have difficulty with the last three, try substituting in the open variable in the preceding problem. Asking "What is the base for the exponent 4?" is usually clarifying. It is also helpful to cover the base with a sheet of paper and get the factor form of 'paper' E4. Then ask what happens if you write 3E2 on the paper.

Next review and extend the additive property with examples like:

$$(2E3) \text{ x } (2E3) = 2E(3+3)$$

$$(2E3) \text{ x } (2E3) \text{ x } (2E3) = 2E(3+3+3)$$

$$(2E3) \text{ x } (2E3) \text{ x } (2E3) \text{ x } (2E3) = 2E(3+3+3+3)$$

Pointing to each occurrence of (2E3) and asking how many times 2 is used as a factor in each helps verify the results that students may arrive at by following a pattern.

Returning to (2E3)E4, it is easy to establish:

$$(2E3)E4 = (2E3) \text{ x } (2E3) \text{ x } (2E3) \text{ x } (2E3) = 2E(3+3+3+3) = 2E12$$

At some point, you will want to rewrite the 12 as 3x4. This can be accomplished by looking at the parallel problems

$$3x3x3x3 = 3E4 \text{ and } 3+3+3+3 = 3 \bigcirc 4$$

or by simply asking for a way to write 12 that shows where it comes from in the problem.

To reach the conclusion (2E3)E4 = 2E(3x4), it is helpful to rearrange the problem using vertical arrows to indicate equivalent expressions:

$$\begin{tikzcd}
\underline{(2E3)E4} \arrow[d] & {} & 2E(3x4) \\
(2E3)x(2E3)x(2E3)x(2E3) & = & \overline{2E(3+3+3+3)} \arrow[u] \\
 & & \text{What symbol goes here?} \arrow[uul, dashed, bend right]
\end{tikzcd}$$

If you have not already done so, this is a good time to investigate the transitive property of equality. In particular, if ▭ = ☁ and ☁ = ⬭ and ⬭ = 〰, what is true about ▭ and 〰? Students can come to the board and put the appropriate shapes around the different parts of this problem. (The reader is also referred to the section on equality and transitivity in the "Guidelines for Discovery Teaching.")

Another way to refinforce this result is to compare the tables of powers of 2 and 8 to find solutions to

8E□ = 2E△

E.g.:

8E1 = 2E3

8E2 = 2E6

8E3 = 2E9

8E4 = 2E12

Replacing 8 by the equivalent expression 2E3, we get:

(2E3)E1 = 2E3

(2E3)E2 = 2E6

(2E3)E3 = 2E9

(2E4)E4 = 2E12

Now repeat the analysis with additional examples such as (3E2)E4=3E(2x4), (2E4)E3=2E(4x3), etc., and then use the 'erase and change' method to arrive at the generalization $(\alpha E\beta)E\gamma=\alpha E(\beta x\gamma)$.

4.3 Fractional Exponents

In this section, we reinforce the conceptual type of thinking used to explore zero and negative exponents by applying it to fractional exponents. There also are ample opportunities to weave in review of operations with fractions and decimals.

Begin by having the class make a table of powers of 9.

9E3 = 729

9E2 = 81

9E1 = 9

9E0 = 1

Insert $9E\frac{1}{2}$ and ask for conjectures. Use the 'put it in a sentence' technique to define $9E\frac{1}{2}$. It is instructive for students to make up their own sentences like $(9E\frac{1}{2})x(9E1)=9E\frac{3}{2}$ so they will see that not every example gives useful information. Using questions like, "What can I add to $\frac{1}{2}$ to get a whole number?" eventually arrive at:

$$\begin{array}{ccccc} (9E\frac{1}{2}) & x & (9E\frac{1}{2}) & = & \underline{9E(\frac{1}{2}+\frac{1}{2})} \\ \downarrow & & \downarrow & & \downarrow \\ \square & x & \square & = & 9 \end{array}$$

Be certain to review the rule of substitution. Whatever value $9E\frac{1}{2}$ takes on for the first variable must be its value for the second. After concluding that $9E\frac{1}{2}$ must be 3, move to problems like $16E\frac{1}{2}$, $25E\frac{1}{2}$, $\frac{4}{9}E\frac{1}{2}$, $\frac{1}{100}E\frac{1}{2}$, $.01E\frac{1}{2}$, etc.

The question of $2E\frac{1}{2}$ usually arises. If not, you may wish to bring it up. Putting it in a sentence leads to

$$\begin{tikzcd} (2E\frac{1}{2}) \arrow[d] & \times & (2E\frac{1}{2}) \arrow[d] & = & \underline{2E(\frac{1}{2}+\frac{1}{2})} \arrow[d] \\ \square & \times & \square & = & 2 \end{tikzcd}$$

There will be a variety of responses. Two common ones are "Put 1 in the first square, 2 in the second" and "Change the x to a +." When these have been ruled out, ask if the class can tell you anything about $2E\frac{1}{2}$?

What does 2E0 equal?

What does 2E1 equal?

Who can give me a number greater than $2E\frac{1}{2}$?

Who can give me a number less than $2E\frac{1}{2}$?

Write $2E0<2E\frac{1}{2}<2E1$ or $1<2E\frac{1}{2}<2$. Have the students read it and explain why it it's true.

Now ask for conjectures again. The usual guess is $2E\frac{1}{2}=1\frac{1}{2}$. Check by substituting

$$\begin{tikzcd} \frac{3}{2} \times \frac{3}{2} \arrow[d] & \overset{?}{=} & 2 \arrow[d] & \text{or} & 1.5\times1.5 \arrow[d] & \overset{?}{=} & 2 \arrow[d] \\ \frac{9}{4} & \neq & 2 & & 2.25 & \neq & 2 \end{tikzcd}$$

Is $2E\frac{1}{2}$ greater or less than $1\frac{1}{2}$?

Change the board to read $1<2E\frac{1}{2}<1.5$.

Continuing to refine and check the guesses provides additional practice with fractional or decimal multiplication. It is also interesting for the students (or a class representative) to use a calculator so that the work proceeds more quickly. The class will soon arrive at a number which when squared gives you an answer extremely close to 2. They will also begin to develop a sense that no matter how many decimal places they use, they are not going to find it exactly.

Have the students find approximations for $3E\frac{1}{2}$ and $5E\frac{1}{2}$.

Many instructors introduce the notation for square roots at this point and work the same and some additional problems using that notation.*

We next extend the work with the exponent $\frac{1}{2}$ to other unit fractions. Begin with $8E\frac{1}{3}=\square$.

---

*If you want the class to have extra practice dividing decimals, you can introduce Newton's method for approximating square roots. This method is based on the fact that if a is an approximation for $\sqrt{b}$, then either $a<\sqrt{b}$ and $\frac{b}{a}>\sqrt{b}$ or vice versa. The average, $(a+\frac{b}{a})\div 2$, is a much closer approximation. The steps of Newton's method are (1) make a guess at the square root of b, (2) divide b by your guess, (3) average your guess and the quotient in (2), (4) repeat steps (2) and (3) using the average as a new approximation to $\sqrt{b}$. Newton's method converges very rapidly. For instance, using Newton's method in the case of $\sqrt{2}$, our original approximation of 1.5 leads to the sequence of approximations 1.5, 1.42, 1.414.

The students will first try $(8E\frac{1}{3})$ x $(8E\frac{1}{3}) = 8E\frac{2}{3}$. Look at what they do know about $\frac{1}{3}$:

Use $\frac{1}{3}$ in a true mathematical sentence.

What does $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}$ equal?

Who knows what $3x\frac{1}{3}$ equals?

What do I have to do with $\frac{1}{3}$ to get a whole number?

What do I have to do with $\frac{2}{3}$ to get a whole number?

Who can think of something else to multiply the left side by to make the exponent on the right-hand side a whole number?

What if we put another factor of $(8E\frac{1}{3})$?

You should now have:

$$\begin{array}{ccccccc} (8E\frac{1}{3}) & x & (8E\frac{1}{3}) & x & (8E\frac{1}{3}) & = & 8E1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \square & x & \square & x & \square & = & 8 \end{array}$$

The students will quickly conclude that $8E\frac{1}{3}=2$. This method can be easily extended to other results such as $64E\frac{1}{3}$, $1000E\frac{1}{3}$, $16E\frac{1}{4}$, $32E\frac{1}{5}$, etc. $8E\frac{2}{3}$ can be used as a challenge question. Leave it open and allow students to work on it on their own. They will probably arrive at the correct conclusion using either $(8E\frac{1}{3})x(8E\frac{1}{3})=8E\frac{2}{3}$ or $(8E\frac{1}{3})x(8E\frac{2}{3})=8E1$.

If the class has developed the multiplicatiave property, it provides another, less cumbersome approach to fractional exponents.

For example, to find $8E\frac{1}{3}$, look at either

$$\begin{tikzcd}
(8E\frac{1}{3}) \arrow[d] & E3 \arrow[d] & = & \underline{8E(\frac{1}{3}x3)} \arrow[d] & \text{or} & 8E\frac{1}{3} \arrow[d] & = & \boxed{} \\
\boxed{} & E3 & = & 8 & & \overline{(2E3)E\frac{1}{3}} & = & \overline{2E(3x\frac{1}{3})} \arrow[u]
\end{tikzcd}$$

To find $8E\frac{2}{3}$, look at either

$$\begin{tikzcd}
(8E\frac{2}{3}) \arrow[d] & E3 \arrow[d] & = & \underline{8E(\frac{2}{3}x3)} \arrow[d] & \text{or} & 8E\frac{2}{3} \arrow[d] & = & \boxed{} \\
\boxed{} & E3 & = & 64 & & \overline{(2E3)E\frac{2}{3}} & = & \overline{2E(3x\frac{2}{3})} \arrow[u]
\end{tikzcd}$$

Other problems which the class can solve include $16E\frac{3}{4}$, $32E\frac{2}{5}$, $32E\frac{3}{5}$, $64E\frac{5}{6}$, etc.

If you have introduced the square root notation, you may want to review the above problems using the notation $aE\frac{1}{b}=\sqrt[b]{a}$.

4.4 Review and Related Activities

The examples below show how negative and fractional exponents can be used to review and reinforce some topics in the general mathematics curriculum. The emphasis you place on them both in class and on assignments will depend on the needs of your class for additional practice in these skills.

(1) Derive the negative powers of 10. Use them to review and reinforce decimal place value.

$10^{-1} = \frac{1}{10}$	$= .1$	one-tenth
$10^{-2} = \frac{1}{100}$	$= .01$	one-hundredth
$10^{-3} = \frac{1}{1000}$	$= .001$	one-thousandth
$10^{-4} = \frac{1}{10000}$	$= .0001$	one-ten-thousandth

etc.

3,496.725 =

(3x1,000)+(4x100)+(9x10) +(6x1) +(7x.1) +2x(.01) +(5x.001)

↓ ↓ ↓

$(3x10^3)$ $+(4x10^2)+(9x10^1)+(6x10^0)+(7x10^{-1})+(2x10^{-2})+(5x10^{-3})$

Ask students why our number system is called 'base 10'. You may also want to take a brief look at number systems in other bases.

(2) Use multiplication of exponential terms to review and reinforce the rules for multiplying decimals.

Example:

$10^{-3} \text{ x } 10^{-2} = 10^{-5}$ .001x.01=.00001

$$\begin{array}{r} 25.6 \\ \underline{x.32} \\ 512 \\ \underline{768} \\ 8.192 \end{array}$$

$(256x10^{-1})x(32x10^{-2})=(256x32)x(10^{-1}x10^{-2})$

$=8192x10^{-3}=8192x\frac{1}{1000}=8.192$

(3) Introduce scientific notation, using powers of 10 to express any number as the product of a number between 1 and 10 and a power of 10.

Examples:

$375,000 = 3.75x10^5$

$.0047 = 4.7x10^{-3}$

(4) Use negative exponents as an alternate method of computing certain multiplication and division problems with fractions.

Examples:

(a)

$$\begin{tikzcd} \frac{1}{4} \arrow[d] & \times & \frac{1}{8} \arrow[d] & = & \boxed{} \\ (2E{}^{-}2) & \times & (2E{}^{-}3) & = & 2E{}^{-}5 \arrow[u] \end{tikzcd}$$

(b)

$$\begin{tikzcd} 32 \arrow[d] & \times & \frac{1}{8} \arrow[d] & = & \boxed{} \\ (2E5) & \times & (2E{}^{-}3) & = & 2E2 \arrow[u] \end{tikzcd}$$

(c)

$$\begin{tikzcd} 8 \arrow[d] & \div & \frac{1}{16} \arrow[d] & = & \boxed{} & & \\ (2E3) & \div & (2E{}^{-}4) & = & 2E(3-{}^{-}4) & = & 2E7 \arrow[llu] \end{tikzcd}$$

(d)

$$\begin{tikzcd} \frac{1}{32} \arrow[d] & \div & \frac{1}{8} \arrow[d] & = & \boxed{} & & \\ (2E{}^{-}5) & \div & (2E{}^{-}3) & = & 2E({}^{-}5-{}^{-}3) & = & 2E{}^{-}2 \arrow[llu] \end{tikzcd}$$

(5) Establish the general rule (aEc)x(bEc)=(axb)Ec by looking at examples such as

(2E4)x(3E4)=(2x2x2x2)x(3x3x3x3)=(2x3)x(2x3)x(2x3)x(2x3)=(2x3)E4

This rule can be used to simplify problems like

$$(2E\tfrac{1}{2})\text{x}(8E\tfrac{1}{2})=(2\text{x}8)E\tfrac{1}{2}=16E\tfrac{1}{2}=4$$

i.e.,

2 x 8 = 2x8 = 16 = 4

4.5 Exercises

1. $(3E^{-}1) \times (3E1) = 3E\triangle$

$$\begin{array}{ccccc} \uparrow & & \downarrow & & \downarrow \\ \square & \times & \bigtriangledown & = & ⬡ \end{array}$$

$3E^{-}1 =$ _____

2. Make up a sentence to show what 3^{-2} is equal to.

3. $(\frac{1}{3}E^{-}1) \times (\frac{1}{3}E1) =$ ___E___

$$\begin{array}{ccccc} \uparrow & & \downarrow & & \downarrow \\ \square & \times & \bigtriangledown & = & ⬡ \end{array}$$

$\frac{1}{3}E^{-}1 =$ _____

4. Write each expression using only positive exponents. Then find the numerical values of each.

a. $5E^{-}2 =$ __________ = __________

b. $10E^{-}1 =$ __________ = __________

c. $10E^{-}2 =$ __________ = __________

d. $10E^{-}3 =$ __________ = __________

e. $10E^{-}4 =$ __________ = __________

f. $\frac{2}{3}E^{-}1 =$ __________ = __________

g. $kE^{-}Q =$ __________ =

5. Compute each problem two ways. First use the rules for fractions. Then use exponential forms.

a. $$\begin{array}{ccccc} \frac{1}{4} & \times & \frac{1}{8} & = \\ \downarrow & & \downarrow & \end{array}$$

b. $$\begin{array}{ccccc} 8 & \times & \frac{1}{4} & = \\ \downarrow & & \downarrow & \end{array}$$

c. $\frac{1}{4} \div 8 =$

$\quad\downarrow \quad \downarrow$

d. $\frac{1}{4} \div \frac{1}{8} =$

6. What number do these equal?

7.98 x (10E1) = _____

7.98 x (10E2) = _____

7.98 x (10E3) = _____

7.98 x (10E^{-}1) = _____

7.98 x (10E^{-}2) = _____

When you multiply by a positive power of 10, what happens to the decimal point?

When you multiply by a negative power of 10, what happens to the decimal point?

7. Fill in the missing exponents:

456 = 4.56 x 10E ☐

4.56 = 4.56 x 10E ☐

45,600 = 4.56 x 10E ☐

.456 = 4.56 x 10E ☐

.0456 = 4.56 x 10E ☐

8. Fill in the missing numerals:

a. (16E2)E3 = ( ) x ( ) x ( ) = ___E___

b. (16E3)E4 = ___E___

c. (16E)E2 = 16E8

d. (16E$\frac{3}{4}$)E4 = ___E___

e. (2E4)E$\frac{3}{4}$ = ___E___

f. (16E   )E   = 16E12

Is there more than one solution to this problem?

How many whole number solutions are there?

List as many as you can find.

9. $(49E\frac{1}{2})$ x $(49E\frac{1}{2})$ = ___E___

↓ ↓ ↓

□ x □ = △

What is the numerical value of $49E\frac{1}{2}$?

10. Find the numerical value of each of the following:

a. $64E\frac{1}{2}$ =

b. $4E\frac{1}{2}$ =

c. $9E\frac{1}{2}$ =

d. $\frac{4}{9}E\frac{1}{2}$ =

e. $.01E\frac{1}{2}$ =

f. $\frac{1}{4}E\frac{1}{2}$ =

g. $\frac{1}{4}E-\frac{1}{2}$ =

h. $\frac{4}{9}E-\frac{1}{2}$ =

11. 2E4 = □

$(2E4)E\frac{3}{4}$ = □ E △ = _____

What is the numerical value of $16E\frac{3}{4}$?

12. Find the numerical value of each of the following:

a. $8E\frac{1}{3}$ =

b. $27E\frac{1}{3}$ =

c. $27E\frac{2}{3}$ =

d. $16E\frac{1}{4}$ =

e. $8E-\frac{1}{3}$ =

f. $\frac{1}{8}E\frac{1}{3}$ =

g. $\frac{1}{8}E-\frac{1}{3}$ =

h. $\frac{8}{27}E\frac{2}{3}$ =

4.6 Chapter Summary

Negative Exponents

$$2E^{-}\alpha = \frac{1}{2E\alpha}$$

For $\beta \neq 0$, $\beta E^{-}\alpha = \frac{1}{\beta E\alpha}$

Multiplicative Property of Exponentiation

$$(\alpha E\beta)E\gamma = \alpha E(\beta x\gamma)$$

Fractional Exponents

Numerical value of $kE\frac{1}{2}$ where k is a perfect square

Evaluation of $kE\frac{1}{b}$ where k is the form qEb

Optional Topics

Scientific notation

Roots

Approximation of square roots

Vocabulary

Square root

Approximation

4.7 Chapter Test

Fill in the missing numerals.

1. $(5E^{-}2) \times (5E2) = 5E\triangle$

   ↑ ↓ ↓

   □ x ▽ = ⬡ $\qquad 5E^{-}2 =$ _____

2. $(\frac{1}{5}E^{-}2) \times (\frac{1}{5})E2 =$ ___E___

   ↑ ↓ ↓

   □ x ▽ = ⬡

3. Find the numerical value of each expression:

   a. $2E^{-}3 =$ _____

   b. $\frac{1}{4}E^{-}2 =$ _____

   c. $10E^{-}3 =$ _____

4. a. What number does 2.34 x (10E2) equal?

   b. What number does $2.34 \times (10E^{-}2)$ equal?

   c. Fill in the missing exponent:

   .00234 = 2.34 x 10E □

5. Find the following product using exponential forms:

   $16 \times \frac{1}{8} =$ □

   ↓ ↓ ↑

   ( ) x ( ) = ___E___

6. a. (5E2)E3 = □ E△

   b. $(8E\frac{2}{3})E3 =$ □ E△

   c. If (2Em)E7 = 2E21, m = _____

7. $$\begin{array}{ccccc} (25E\frac{1}{2}) & \times & (25E\frac{1}{2}) & = & \_\_E\_\_ \\ \downarrow & & \downarrow & & \downarrow \\ \square & \times & \square & = & \triangle \end{array}$$

What is the numerical value of $25E\frac{1}{2}$?

8. $$\begin{array}{ccccc} (25E{-\frac{1}{2}}) & \times & (25E\frac{1}{2}) & = & \_\_E\_\_ \\ \downarrow & & \downarrow & & \downarrow \\ \square & \times & \triangle & = & \triangle \end{array}$$

What is the numerical value of $25E{-\frac{1}{2}}$?

9. $(2E3)E\frac{2}{3} = \square\ E\triangle = \_\_\_\_$

Use the above to find the numerical value of $8E\frac{2}{3}$.

10. Find the numerical value of each of the following:

a. $8E\frac{1}{3}$

b. $16E\frac{3}{4}$

c. $(32E\frac{3}{5}) \times (32E\frac{2}{5})$

MODULE 5: THE DISTRIBUTIVE PROPERTY AND ITS APPLICATIONS

5.0 Introduction

The distributive property is the basis for our multiplication algorithms and is heavily used in algebra. In this module we look at the distributive property and some of its applications, including its use in deriving the rules for multiplication of negative numbers.

There is a striking comparison between the additive property of exponentiation and the distributive property. The same reasoning which led to $(\alpha E\beta)x(\alpha E\gamma)=\alpha E(\beta+\gamma)$ leads to $(\alpha x\beta)+(\alpha x\gamma)=\alpha x(\beta+\gamma)$ when multiplication is viewed as repeated addition in the same way that exponentiation was viewed as repeated multiplication. Section 5.4 contains, in outline form, a parallel analysis of exponentiation and multiplication including the additive property and the distributive property along with corresponding applications of both. This section is designed to be used at the end of the unit to review and consolidate previous results. Simply divide the board into two sections and present the problems as they appear, alternating between sides and asking questions which bring out the comparisons between the two sets of examples.

Section 5.4 can also be used as an alternate introduction to concepts in the first three sections of this module. In this case, use Section 5.4 as an outline. Review each exponential example and then make the appropriate changes to bring in the distributive property and negative numbers. If you follow this approach, go back over the development in Sections 5.1 and 5.3 as reinforcement. Also look at the applications in Section 5.2.

5.1 The Distributive Property

One way to introduce and motivate the study of the distributive property is to present a sequence of problems like the following. The problems should be presented one at a time and the students should be asked to perform the computation on paper.

$$(5\times6) + (5\times4) = ?$$
$$(17\times6) + (17\times4) = ?$$
$$(17\times7) + (17\times3) = ?$$
$$(9\times32) + (9\times68) = ?$$
$$(12\times32) + (12\times68) = ?$$
$$(12\times44) + (12\times56) = ?$$

etc.

While their classmates are continuing to compute each multiplication separately and then add, some students will begin to solve the problems in their heads, usually with a greater degree of accuracy. When a number of students have caught on to the 'trick', have them explain how they are getting their answer. You will end up with something like

$$(5\times6) + (5\times4) = 5\times10$$

Ask:

Where did the 10 come from?

Who can write 10 using a 6 and a 4?

How many 5's do you have to add to make 5x6?

How many times is 5 used as an addend in the second quantity?

How mny times is 5 used as an addend altogether?

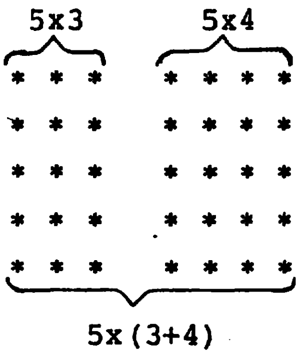
Have on the board:

$$\begin{array}{ccccc} (5x6) & + & (5x4) & = & 5x(6+4) \\ \downarrow & & \downarrow & & \uparrow \\ 5+5+5+5+5+5 & + & 5+5+5+5 & = & 5x10 \end{array}$$

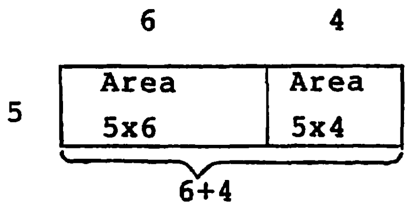
Since the students have used the same logic to derive the additive property, a few changes or additional examples will lead quickly to the generalization

$$(\alpha x\beta) + (\alpha x\gamma) = \alpha x(\beta+\gamma)$$

You might want to reinforce this result using a pictorial representation:



If the class knows how to find the area of a rectangle, there is a geometric interpretation:



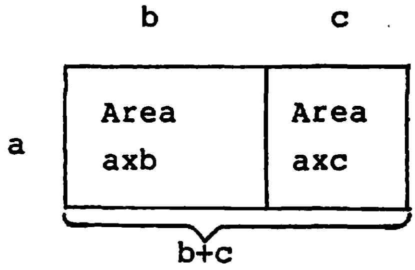
(5x6) + (5x4) = area of large rectangle = 5x(6+4)

The usual format for stating the distributive property $\alpha x(\beta+\gamma)=(\alpha x\beta)+(\alpha x\gamma)$ is, of course, a simple consequence of the symmetry of equality; however, it should be reinforced by some direct development.

Review the fact that 4x5=4+4+4+4+4=5+5+5+5. Then ask how many times (2x3) is used as an addend in 4x(2x3). Get

$$\begin{aligned} 4x(2+3) &= (2+3) + (2+3) + (2+3) + (2+3) \\ &= (2+2+2+2) + (3+3+3+3) \\ &= (4x2) + (4x3) \end{aligned}$$

The pictorial and area interpretations shown above can also be looked at from the other direction to reinforce this concept.



Area of large rectangle = ax(b+c) = (axb) + (axc)

Practice problems should be given which illustrate all forms of the distributive property.

Examples:

(13x5) + (7x5) = (13+7)x5

(26x3) + (26x7) = 26x(3+7)

4x(10+7) = (4x10) + (4x7)

(10+3)x5 = (10x5) + (3x5)

5.2 Applications of the Distributive Property

We have already seen how the distributive property can be used to simplify computations such as $(5x17)+(5x3)=\boxed{}$.

Another application should allow most students to multiply many two-digit numbers by a one-digit number in their heads. It also provides a conceptual foundation for the multiplication algorithm most students have learned by rote.

Give the class a set of problems to do using the distributive property such as

3x(10+6) = (3x10) + (3x6) = 30+18 = 48
5x(10+7) =
2x(40+3) =
5x(40+7) =
17x(20+1) =

In the later problems, ask if anyone can give the final numerical value without writing down the intermediate steps.

Now have students come to the board to write

$$\begin{array}{r} 16 \\ \underline{x3} \end{array} \qquad \begin{array}{r} 17 \\ \underline{x5} \end{array} \qquad \begin{array}{r} 43 \\ \underline{x2} \end{array} \qquad \begin{array}{r} 47 \\ \underline{x5} \end{array} \qquad \begin{array}{r} 21 \\ \underline{x17} \end{array}$$

Go over what they've done, asking questions which elucidate the role of the distributive property.

For example, in 47x5 ask

> 5x7 equals what?
>
> What do I write down?

What happens to the 3?

Is it really a 3?

What does the 3 stand for?

4x5 equals what?

Why don't you write 20 under the 7?

What does the 4 in 47 stand for?

What's 5x40?

Get on the board:

$$\begin{array}{r} 3 \\ 47 \\ \underline{\mathrm{x}5} \\ 235 \end{array} \qquad \begin{array}{r} (40+7) \\ \underline{\mathrm{x}\ \ 5} \\ 35 \\ \underline{200} \\ 235 \end{array} \qquad 5\mathrm{x}(40+7) = (5\mathrm{x}40) + (5\mathrm{x}7)$$

Ask the students to compare the different examples and to explain why the familiar algorithm leads to the correct result.

Repeat the procedure with additional examples. Note that the common algorithm for computing 21x17 actually is the condensation of

$$\begin{array}{rl} 20+1 & \\ \underline{\mathrm{x}10+7} & \\ 7 & \leftarrow (7\mathrm{x}1) \\ 140 & \leftarrow (7\mathrm{x}20) \\ 10 & \leftarrow (10\mathrm{x}1) \\ \underline{200} & \leftarrow (10\mathrm{x}20) \\ 357 & \end{array}$$

This leads us to the question of multiplying two binomials, a double application of the distributive property.

One way to develop this is to do more examples of two-digit multipliation problems such as

$$\begin{array}{rl} 37 & \\ \underline{x25} & \\ 185 & \leftarrow(5x37) \\ 74 & \leftarrow(20x37) \\ \hline 925 & \end{array} \qquad \begin{array}{rl} 30+7 & \\ \underline{x20+5} & \\ 35 & \leftarrow(5x7) \\ 150 & \leftarrow(5x30) \\ 140 & \leftarrow(20x7) \\ \underline{+600} & \leftarrow(20x30) \\ 925 & \end{array}$$

Then (a+b)x(c+d) becomes

$$\begin{array}{r} a+b \\ \underline{xc+d} \\ bxd \\ axd \\ cxb \\ \underline{+axd} \\ (bxd)+(axd)+(cxb)+(axd) \end{array}$$

A more algebraic approach is to look at a sequence of examples like

$$\begin{array}{rlrl} 5x(3+4) & = \qquad & (3+4)x5 & = \\ 5x(c+d) & = & (a+b)x5 & = \\ 10x(c+d) & = & (a+b)x10 & = \\ kx(c+d) & = & (a+b)xc & = \\ \square\, x(c+d) & = & (a+b)xd & = \end{array}$$

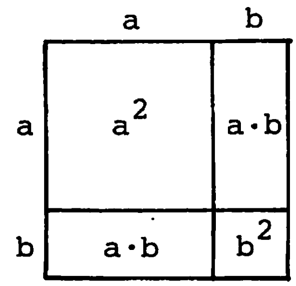
Now substitute a+b in the open variable and apply the distributive property to each term

$$(a+b)x(c+d)=(a+b)xc+(a+b)xd=(axc)+(bxc)+(axd)+(bxd)$$

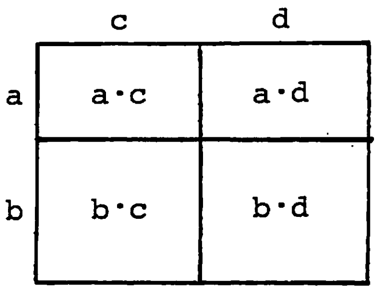
Verify this result with numerical examples. If the students are about to take algebra, you will want to introduce the less cumbersome notion 47x36=47·36 and axb=ab. You will also want them to practice with many additional examples. If they are not going to be taking algebra immediately, you will want to touch lightly on the abstract examples and make certain that they get sufficient computational practice.

The area argument used earlier can also be used to illustrate the multiplication of binomials:

Examples:



$(a+b)^2=a^2+2ab+b^2$



$(a+b)x(c+d)=ac+bc+ad+bd$

If the class needs practice multiplying and dividing decimals this is a good time to include it, bringing in the work with negative exponents from the previous unit. The students can also solve word problems which use the distributive property. An example is included below.

> Baseball cards come 5 to a pack. On Monday Joe bought 2 packs; on Tuesday he bought 3 packs; and on Wednesday he bought 4 packs. How many cards did he buy altogether?

5.3 Multiplication of Negative Numbers

We begin this section with a theoretical digression. In working with the multiplication of negative numbers, we need the result $\forall\alpha$, $0x\alpha=\alpha x0=0$. Students will not have any difficulty applying this well-known fact. $0x\alpha$ can thought of as "add 0 α times" while $\alpha x0$ also seems to say, "Don't put any α's down." This is the same type of thinking that led to

2E0=0 (you don't put any 2's down so you have zero). $\alpha x0=0$ can, however, be derived as a consequence of the identity properties of 0 and 1 and the distributive law in exactly the same way as we derived the zero exponent:

$$\begin{array}{ccccc} (\alpha x0) & + & (\alpha x1) & = & \alpha x(0+1) \\ \downarrow & & \downarrow & & \downarrow \\ \boxed{} & + & \alpha & & \alpha \end{array}$$

In our experience, for most classes, proving that $\alpha x0=0$ runs the risk of belaboring the obvious, and we recommend omitting it or touching on it only lightly when you reach the comparison review in Section 5.4. For some classes which particularly like to rise to a mathematical challenge, you may want to motivate the need for the proof by reminding them that everyone thought 2E0 was 0 and it turned out to be 1 so you think 2x0=1. Look at the derivation of 2E0=1 and let them discover the changes necessary to convince you that 2x0=0.

Begin the lesson on multiplying negative numbers with a review of additive inverses and the distributive law.

Write on the board

$$4x{}^{-}3 = ?$$

Ask for answers. (The usual ones are ${}^{-}12$ and 1.) Some students may argue that $4x{}^{-}3={}^{-}3+{}^{-}3+{}^{-}3+{}^{-}3$. In fact, this argument can be used with any positive whole number times a negative number. But what happens when you have $13.17x{}^{-}3$? (To keep the parallel with exponentiation, some instructors adopt the convention that the right-hand factor tells you how many times to use the left as an addend. This does force students to use the distributive law in this case but flies in the face of commutativity and the symmetry of the geometric interpretation of multiplication.)

Ask if anyone can put $4x{}^{-}3$ in a true sentence using the distributive law. The easiest one is

$$\begin{tikzcd} (4x{}^{-}3) \arrow[d] & + & (4x3) \arrow[d] & = & 4x({}^{-}3+3) \arrow[d] \\ \square & + & 12 & = & 0 \end{tikzcd}$$

After the students have simplified the parts they're certain about, they will conclude that for the distributive law to hold, $4x{}^{-}3$ has to be ${}^{-}12$.

Repeat this process with several other examples.

Patterns can be used to help reinforce multiplication by a negative. For example,

$$\begin{aligned} 4x3 &= 12 \\ 4x2 &= 8 \\ 4x1 &= 4 \\ 4x0 &= 0 \\ 4x{}^{-}1 &= ? \\ 4x{}^{-}2 &= ? \end{aligned}$$

A frequently used consequence is $\beta x{}^{-}1={}^{-}\beta$. It can be generalized from examples or can be derived directly.

$$\begin{tikzcd} (\beta x{}^{-}1) \arrow[d] & + & (\beta x1) \arrow[d] & = & \beta x({}^{-}1+1) & = & \beta x0 \arrow[dll] \\ \boxed{{}^{-}\beta} & + & \beta & = & 0 \end{tikzcd}$$

We now come to the concept that has been doubted and memorized by generations of algebra students. Put on the board

$$^{-}4x^{-}3 = \square$$

Students invariably answer $^{-}12$. Use $^{-}4x^{-}3$ in the distributive law to get the following analysis:

$$\begin{tikzcd}
(^{-}4x^{-}3) \arrow[d] & + & (^{-}4x3) \arrow[d] & = & ^{-}4x(^{-}3+3) & = & ^{-}4x0 \arrow[dll] \\
\square & + & ^{-}12 & = & 0 & &
\end{tikzcd}$$

What does $^{-}4x^{-}3$ act like in this sentence?

What do mathematicians define $^{-}4x^{-}3$ to be?

Repeat this analysis with several other examples.

Patterns can also be used to help make this result more plausible. E.g.:

$$\begin{aligned}
^{-}4x3 &= {}^{-}12 \\
^{-}4x2 &= {}^{-}8 \\
^{-}4x1 &= {}^{-}4 \\
^{-}4x0 &= 0 \\
^{-}4x^{-}1 &= ? \\
^{-}4x^{-}2 &= ?
\end{aligned}$$

An extension of this pattern can be used to complete a four-quadrant multiplication table whose symmetry again points to the product of two negatives being positive.

¯9	¯6	¯3	3	3	6	9
¯6	¯4	¯2	2	2	4	6
¯3	¯2	¯1	1	1	2	3
¯3	¯2	¯1		1	2	3
			¯1	¯1	¯2	¯3
			¯2	¯2	¯4	¯6
			¯3	¯3	¯6	¯8

This leads naturally to the graphing of ordered pairs in the plane. Once the class has discovered that the ordered pair solutions to y=2x, y=3x, y=4x, etc., lie on a straight line, look at y=¯2x, graphing first positive values of x and then looking at what y has to be for negative values of x to maintain the 'linear' quality of the equation.

If the students have established ¯¯k=k (through comparing $^{-}\boxed{^{-}k} + \boxed{^{-}k} = 0$ and k+¯k=0), you can look at γx¯1=¯γ and substitute negative numbers.

E.g.,

$$^{-}6 \text{x} ^{-}1 = {}^{--}6 = 6$$

Using these results, you get

$$\begin{aligned} ^{-}4\text{x}^{-}3 &= {}^{-}4\text{x}(3\text{x}^{-}1) \\ &= (^{-}4\text{x}3)\text{x}^{-}1 \\ &= {}^{-}12\text{x}^{-}1 \\ &= {}^{--}12 \\ &= 12 \end{aligned}$$

Using the relationship between division and multiplication, students should quickly derive rules for division with negative numbers. They should also recognize that $^{-}(\frac{a}{b}) = \frac{a}{b} = \frac{a}{^{-}b}$.

This result can also be established directly using the basic definition of inverses since

$$\frac{{}^-a}{b} + \frac{a}{b} = \frac{{}^-a+a}{b} = \frac{0}{b} = 0 \quad \text{and}$$

$$\frac{a}{{}^-b} + \frac{a}{b} = \frac{(axb)+(ax{}^-b)}{{}^-bxb} = \frac{ax(b+{}^-b)}{{}^-bxb} = \frac{ax0}{{}^-bxb} = \frac{0}{{}^-bxb} = 0$$

We point out, as we have throughout these notes, that in practicing with negative numbers, students can be given problems which require them to practice any or all of the basic compuations with fractions and decimals. In fact, for many students your emphasis will continue to be on basic arithmetic skills. You simply are providing them with a fresh context in which to practice.

5.4 A Parallel Analysis of Exponentiation and Multiplication

The pages which follow outline a sequence of questions which show the similarities and differences between exponentiation and multiplication.

Factor Form

What is being used as a factor in

2 x 2 x 2 x 2

How many times?

Now $\left\{\begin{array}{l} 3 \text{ x } 3 \text{ x } 3 \text{ x } 3 \\ 7 \text{ x } 7 \text{ x } 7 \text{ x } 7 \text{ x } 7 \\ 29 \text{ x } 29 \text{ x } 29 \end{array}\right.$

E Form

Put 2 x 2 x 2 x 2 in E form (exponential form)

Answer: 2 x 2 x 2 x 2 = 2E4

Now $\left\{\begin{array}{l} 3 \text{ x } 3 \text{ x } 3 \text{ x } 3 \\ 7 \text{ x } 7 \text{ x } 7 \text{ x } 7 \text{ x } 7 \\ 29 \text{ x } 29 \text{ x } 29 \end{array}\right.$

Write (2E3) x (2E4) on board. Have it read:

How many times is 2 being used as a factor in:

(2E3) x (2E4)

↓ ↓

2x2x2 x 2x2x2x2

Now (7E5) x (7E3)
(5E1) x (5E2)

Addend Form

What is being used as a factor in

2 + 2 + 2 + 2

How many times?

Now $\left\{\begin{array}{l} 3 + 3 + 3 + 3 \\ 7 + 7 + 7 + 7 + 7 \\ 29 + 29 + 29 \end{array}\right.$

X Form

Put 2 + 2 + 2 + 2 in X form (multiplicative form)

Answer: 2 + 2 + 2 + 2 = 2X4

Now $\left\{\begin{array}{l} 3 + 3 + 3 + 3 \\ 7 + 7 + 7 + 7 + 7 \\ 29 + 29 + 29 \end{array}\right.$

Write (2X3) + (2X4) on board. Have it read:

How many times is 2 being used as a factor in:

(2X3) + (2X4)

↓ ↓

2+2+2 + 2+2+2+2

Now (7X5) + (7X3)
(5X1) + (5X2)

$$(\underline{2E3})\ \text{x}\ (\underline{2E4})\ = \triangle\ E\ \square$$

$$\underbrace{\downarrow \quad \downarrow \quad \downarrow} \quad \downarrow$$

Put ↗ in E form

Now get

$$(7E5)\ \text{x}\ (7E3) = 7E8 = 7E\ (5+3)$$

$$(5E1)\ \text{x}\ (5E2) = 5E3 = 5E\ (1+2)$$

$$(\lambda E\gamma)\ \text{x}\ (\lambda E\beta) =$$

Answer: $\lambda E(\gamma+\beta)$

$$(\underline{2X3}) + (\underline{2X4})\ = \triangle\ X\ \square$$

$$\underbrace{\downarrow \quad \downarrow \quad \downarrow} \quad \downarrow$$

Put ↗ in X form

Now get

$$(7X5) + (7X3) = 7X8 = 7X\ (5+3)$$

$$(5X1) + (5X2) = 5X3 = 5X\ (1+2)$$

$$(\lambda X\gamma) + (\lambda X\beta) =$$

Answer: $\lambda X(\gamma+\beta)$

$5 \times \square = 1$ $\qquad$ $7 \times \frac{1}{7} = \square$

$2E0 = ?$

$(2E0) \times (2E1) = ?$

Get this

$$\begin{array}{cccccc} (\underline{2E0}) & \times & (\underline{2E1}) & = 2E\ (0+1) & = & \underline{2E1} \\ \downarrow & & \downarrow & & & \downarrow \\ (2E0) & \times & 2 & & = & 2 \end{array}$$

So what's 2E0 acting like?

Now do 7E0

$(2E^{-}1) = ?$

$(2E^{-}1) \times (2E1) = ?$

Get this

$$\begin{array}{cccccc} (\underline{2E^{-}1}) & \times & (\underline{2E1}) & = 2E\ (^{-}1+1) & = & \underline{2E0} \\ \downarrow & & \downarrow & & & \downarrow \\ (2E^{-}1) & \times & 2 & & = & 1 \end{array}$$

So what's $2E^{-}1$ acting like?

Now do $7E^{-}1$

$5 + \square = 0$ $\qquad$ $7 + {}^{-}7 = \square$

$2X0 = ?$

$(2X0) + (2X1) = ?$

Get this

$$\begin{array}{cccccc} (\underline{2X0}) & + & (\underline{2X1}) & = 2X\ (0+1) & = & \underline{2X1} \\ \downarrow & & \downarrow & & & \downarrow \\ (2X0) & + & 2 & & = & 2 \end{array}$$

So what's 2X0 acting like?

Now do 7X0

$(2X^{-}1) = ?$

$(2X^{-}1) + (2X1) = ?$

Get this

$$\begin{array}{cccccc} (\underline{2X^{-}1}) & + & (\underline{2X1}) & = 2X\ (^{-}1+1) & = & \underline{2X0} \\ \downarrow & & \downarrow & & & \downarrow \\ (2X^{-}1) & + & 2 & & = & 0 \end{array}$$

So what's $2X^{-}1$ acting like?

Now do $7X^{-}1$

$$9E\frac{1}{2} = ?$$

$$\begin{array}{llllll}
\text{Ask:} & (9E\frac{1}{2}) & \text{x} & (9E\frac{1}{2}) & = 9E\ (\frac{1}{2}+\frac{1}{2}) \ = & \underline{9E1} \\
 & \downarrow & & \downarrow & & \downarrow \\
 & (9E\frac{1}{2}) & \text{x} & (9E\frac{1}{2}) & \equiv & 9
\end{array}$$

So what's $9E\frac{1}{2}$ acting like?

Do $25E\frac{1}{2}$

$$6X\frac{1}{2} = ?$$

$$\begin{array}{llllll}
\text{Ask:} & (6X\frac{1}{2}) & + & (6X\frac{1}{2}) & = 6X\ (\frac{1}{2}+\frac{1}{2}) \ = & \underline{6X1} \\
 & \downarrow & & \downarrow & & \downarrow \\
 & & + & & \equiv & 6
\end{array}$$

So what's $6X\frac{1}{2}$ acting like?

Do $10X\frac{1}{2}$

AN IMPORTANT DIFFERENCE

$$2E3 \overset{?}{=} 3E2$$

$$5E1 \overset{?}{=} 1E5$$

When does $\lambda E\gamma = \gamma E\lambda$?

$$2X3 \overset{?}{=} 3X2$$

$$5X1 \overset{?}{=} 1X5$$

When does $\lambda X\gamma = \gamma X\lambda$?

$$\begin{array}{llllll}
(^{-}5X^{-}1) & + & (^{-}5X1) & = & ^{-}5X\ (^{-}1+1) \ = & ^{-}5X0 \\
\downarrow & & \downarrow & & & \downarrow \\
(^{-}5X^{-}1) & + & ^{-}5 & & \equiv & 0
\end{array}$$

So what's $^{-}5X^{-}1$ acting like?

5.5 Exercises

1. a. Prove that $(2\underset{\downarrow}{x}3) + (2\underset{\downarrow}{x}4) = 2x(3+4)$ using addends.

   b. Show that 3x(4+2)=(3x4)+(3x2) using areas of rectangles.

   c. Prove by finding the numerical value of each term that

$$(\frac{1}{2}x6) + (\frac{1}{2}x4) = \frac{1}{2}x10$$

2. Fill in the missing numerals:

   a. $(3x5) + (3x7) = \square \text{ x } \triangle$

   b. $(3x\square) + (3x9) = 3x493$

   c. $(3x14.7) + (3x23.6) = \square \text{ x } \triangle$

   d. $(3x\frac{3}{4}) + (3x\frac{1}{8}) = \square \text{ x } \triangle$

   e. $(3x5) + (3x^{-}2) = \square \text{ x } \triangle$

   f. $(3x^{-}14.7) + (3x23.6) = \square \text{ x } \triangle$

   g. $(7x2) + (5x2) = \square \text{ x } \triangle$

   h. $(9x\frac{1}{2}) + (1x\frac{1}{2}) = \square \text{ x } \triangle$

3. Use the distributive property to compute each of the following. Show the step where you use it.

   a. (17x3) + (17x7) =

   b. (17x30) + (17x70) =

   c. (6x37) + (4x37) =

   d. (66x37) + (34x37) =

   e. 5x(40+9) =

   f. 17x(10+2) =

   g. 5x37 =

   h. 17x14 =

4. Fill in the parentheses to show where each partial product comes from.

$$\begin{array}{r l} 30+7 & \\ \underline{\text{x}20+5} & \\ 35 & \leftarrow(\qquad\qquad) \\ 150 & \leftarrow(\qquad\qquad) \\ 140 & \leftarrow(\qquad\qquad) \\ \underline{+600} & \leftarrow(\qquad\qquad) \\ 925 & \end{array}$$

5. Use the distributive property to fill in the missing quantities.

a. (mx4) + (mx3) = □ x △

b. (4xm) + (3xm) = □ x △

c. 3x(10+5) = (□ x △) + (□ x ▽)

d. 3x(a+b) = (□ x △) + (□ x ▽)

e. Qx(a+b) = __x__ + __x__

6. Write the mathematical statement using binomials that is illustrated by the drawing below.

	12	3
8	8x12	8x3
2	8x2	2x3

7. Fill in the missing numerals:

a. (3x$^-$7) + (3x7) = ___x___

↓ ↓ ↓

□ + △ = ▽

b. ($^-$3x$^-$7) + ($^-$3x7) = ___x___

↓ ↓ ↓

□ + △ = ▽

8. a. $^{-}3 \times 9 =$

b. $4 \times -\frac{1}{2} =$

c. $1.5 \times {}^{-}3.2 =$

d. $\frac{3}{4} \times -\frac{1}{2} =$

e. $^{-}3 \times {}^{-}9 =$

f. $^{-}4 \times -\frac{1}{2} =$

g. $^{-}1.5 \times {}^{-}3.2 =$

h. $-\frac{3}{4} \times -\frac{1}{2} =$

9. a. $8 + 4 =$

b. $8 + {}^{-}4 =$

c. $^{-}8 + 4 =$

d. $^{-}8 + {}^{-}4 =$

e. $8 - 4 -$

f. $8 - {}^{-}4 =$

g. $^{-}8 - 4 =$

h. $^{-}8 - {}^{-}4 =$

i. $8 \times 4 =$

j. $8 \times {}^{-}4 =$

k. $^{-}8 \times 4 =$

l. $^{-}8 \times {}^{-}4 =$

m. $8 \div 4 =$

n. $8 \div {}^{-}4 =$

o. $^{-}8 \div 4 =$

p. $^{-}8 \div {}^{-}4 =$

10. a. $^{-}1\text{E}1 =$

b. $^{-}1\text{E}2 =$

c. $^{-}1\text{E}3 =$

d. $^{-}1\text{E}4 =$

e. $^{-}1\text{E}5 =$

f. $^{-}1\text{E}6 =$

g. $^{-}1\text{E}10 =$

h. $^{-}1\text{E}15 =$

i. $^{-}1\text{E}100 =$

j. $^{-}1\text{E}101 =$

k. $^{-}1\text{E}526 =$

l. $^{-}1\text{E}427 =$

11. a. $5 \times 5 = \square$

b. $^{-}5 \times {}^{-}5 = \square$

c. Find two solutions for $\triangle$ E2 = 25.

d. Find two solutions for $\triangle$ E2 = 49.

e. Find two solutions for $\triangle$ E2 = $\frac{4}{9}$.

12. Marbles come 12 to a package. Sue buys 3 packages. She gets 2 for her birthday. How many marbles does

she have altogether? Write a distributive property statement that simplifies the solution to this problem.

5.6 Chapter Summary

Distributive Property

$$(\alpha x \beta) + (\alpha x \gamma) = \alpha x (\beta + \gamma)$$

$$\alpha x (\beta + \gamma) = (\alpha x \beta) + (\alpha x \gamma)$$

Applications of the Distributive Property

Use the distributive law to simplify computations such as $(6x17)+(6x13)=?$ and $5x(30+7)=?$

Demonstration of how the distributive property is used in the common algorithm for multiplying 47x35

Multiplication of two binomials

Multiplication of Negative Numbers

Multiplication in all cases involving positive and negative numbers, e.g.,

$$5 \text{ x } 3 = 15$$

$$5 \text{ x } {}^-3 = {}^-15$$

$${}^-5 \text{ x } 3 = {}^-15$$

$${}^-5 \text{ x } {}^-3 = 15$$

$$\gamma \text{ x } {}^-1 = {}^-\gamma$$

Vocabulary

Addend

Distributive property

Area

Rectangle

Geometric

Algebraic

Algorithm

Binomial

Product

5.7 Chapter Test

1. a. Prove that (3x2) + (3x4) = 3x6 using addends.

   b. Show that 5x(2+4) = (5x2) + (5x4) using areas of rectangles.

   c. Prove that $(\frac{1}{2}x10) + (\frac{1}{2}x12) = \frac{1}{2}$ x 22 by finding the numerical value of each term.

2. Use the distributive property to compute each of the following. Show the step where you use it.

   a. (23x7) + (23x3) =

   b. 6 x (30+4) =

   c. 7 x 45 =

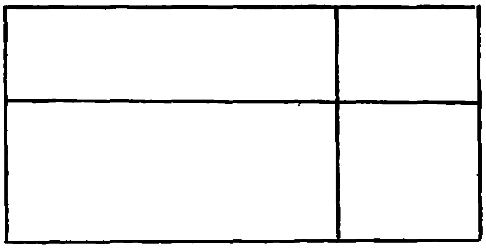
3. Use the distributive property to fill in the missing quantities.

   a. (kx3) + (kx4) = ___x___

   b. $(\gamma x \beta) + (\gamma x \alpha)$ = ___x(   )

   c. 5x(a+b) = (   ) + (   )

4. Label the diagram below so you can use areas to find (k+7) x (q+5).



Find the product (k+7)x(q+5).

5. Fill in the missing numerals:

$$\begin{array}{ccccc} (4\times 9) & + & (4\times {}^{-}9) & = & \_\_\times\_\_ \\ \downarrow & & \downarrow & & \downarrow \\ \triangle & + & \square & = & \bigtriangledown \end{array}$$

$4\times {}^{-}9 = \square$

6. Fill in the missing numerals:

$$\begin{array}{ccccc} ({}^{-}4\times {}^{-}9) & + & ({}^{-}4\times 9) & = & \_\_\times\_\_ \\ \downarrow & & \downarrow & & \downarrow \\ \square & + & \triangle & = & \bigtriangledown \end{array} \qquad {}^{-}4 \times {}^{-}9 = \square$$

7. a. $12 \times {}^{-}3 =$

b. ${}^{-}12 \times {}^{-}3 =$

c. $12 \div {}^{-}3 =$

d. ${}^{-}12 \div {}^{-}3 =$

8. a. ${}^{-}1\text{E}3 =$

b. ${}^{-}1\text{E}4 =$

c. ${}^{-}1\text{E}5 =$

d. ${}^{-}1\text{E}6 =$

e. If n is an odd number, ${}^{-}1\text{E}n =$ _____

9. a. ${}^{-}3 \times {}^{-}3 = \square$

b. Find two solutions to $\triangle\text{E}2 = 9$.

c. Find two solutions to $\triangle\text{E}2 = \frac{9}{16}$.

10. There are 17 pieces of candy in a package. Mrs. Smith buys 7 for a party on Wednesday. On Thursday

her daughter invites more friends and buys 3 more packages. How many pieces of candy did they buy altogether?

Write a distributive property statement that simplifies finding the answer to this problem.

MODULE 6: SUMMATION AND RELATED CONCEPTS

6.0 Introduction

The final module provides another example of the use of an advanced topic as a vehicle both to teach critical thinking and problem solving and to review and reinforce basic computational skills. The concepts of summation and infinite series are generally not taught until precalculus or calculus courses and yet, as will be shown on the following pages, the underlying concepts are readily accessible to general mathematics students. In fact, Project SEED instructors have successfully taught many of these concepts to fifth- and sixth-grade students. The motivational benefit of studying a twelfth-grade or college topic is considerable.

We begin this module with a problem. We have chosen this problem for a variety of reasons. It presents the students with an easily understood challenge whose solution is within their grasp. Its solution requires the students to systematize their thinking and organize their work, an important tool in mathematics and other subjects. It also illustrates a process for determining a general pattern from a sequence of examples. It demonstrates that there can be different conceptual ways of looking at a problem which lead to the same conclusion. It also incorporates the introduction of a fundamental geometric principle. Apart from its own intrinsic value, this problem can serve as a logical springboard for the investigation of several other topics.

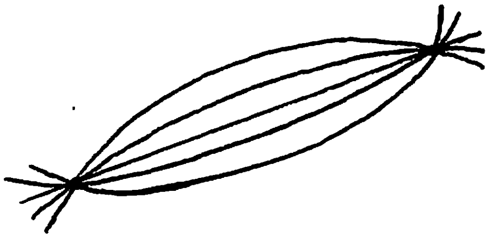
Section 6.1 mentions briefly several directions the class might take after solving the problem. The remainder of the chapter consists of a more thorough investigation of one direction, summations and infinite series, chosen because it contains more reinforcement of basic skills with fractions and decimals than the others. This material, while it arises out of Section 6.1, is not dependent on it.

6.1 An Interesting Problem and the Discovery Paths It Suggests

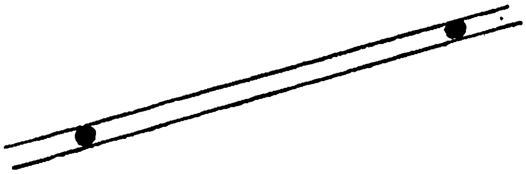
The main problem of this section can be stated formally as follows: how many lines are determined by n points, no three of which are collinear?

For most classes, you will want to do some preliminary work before posing the question.

Begin by putting two dots on the board and asking how many lines you can draw which pass through both of them. If you have students come to the board, you will end up with something like



Next point out that when mathematicians say 'line' they mean what is referred to in the vernacular as 'straight line'. (The reader is reminded that occasionally it is important to use nonmathematical words not familiar to the students. They should be asked to determine their meaning from the context or to look them up in the dictionary as an assignment.) Many students will try something like



Point out that you can't really draw it accurately but mathematicians think of a point as a location in space and a straight line as the shortest string of points between two of them. You might ask what would happen if you tried to stretch the thinnest possible thread between two of the thinnest possible pins. Ask the students to think of other models of points and lines. (This may end up in a discussion of subatomic physics.)

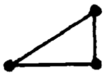
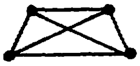
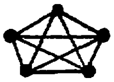
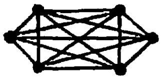
Eventually the students will gain an intuitive understanding of the axiom, 'there is one and only one line between two points.' You may also want to introduce 'line segment' here.

Before proceeding with the analysis of the original question, we would like to point out that how much you structure the presentation will depend on your students and how you are planning to use this problem. For example, you might simply throw out the question early in the term as a long-term bonus project, answering individual questions as they arise and giving occasional hints. Even if you work on the problem as an in-class project, you have choices about how much assistance you give the students. Do you, for example, begin labeling the points in each case you analyze, or do you let the students discover that naming the points helps them keep track of their work more accurately? Do you explore the alternate conceptual ways of achieving the final result, or do you guide the students to the one which leads to the rest of this chapter? As we proceed with the development, we will point out several alternative approaches. Feel free to choose one and omit the others or to use the others as a fresh approach to the same result. You may also wish to try your hand at developing lessons to explore more fully one or more of the topics which this problem suggests. Having led students through the first five modules successfully, you should have little difficulty structuring similar sequences of questions to lead them into the study of combinations, Pascal's Triangle, or the Binomial Theorem, which are mentioned here (or functions, graphing, linear equations, etc., which are not).

Now we return to the question, "How many lines are determined by n points, no three of which are collinear?" In some classes you might put this question on the board as a distant challenge and proceed with the analysis by saying, "Let's see what happens for some values of n." For most classes, you will simply want to give the class several specific cases to do. At some point when they gain confidence or start to recognize a pattern, ask how many lines there will be with 20 points or 100 and finally with n points. This point will vary with your students--you want to challenge and motivate them, not intimidate them.

Having established that there is one and only one line through two points, answer the question with a third point and then a fourth and perhaps a fifth. By the time you get to five or six points, it becomes apparent that you need to be somewhat systematic to be accurate. If you are asking students to tell you lines to draw from their seats, they will quickly see the need for labeling the points. They may also suggest that you begin to build a chart. (Here is a good place to throw out the challenge of 20 or 100 or n points, making sure there is some in- or out-of class time for interested students to work on it.)

On the board you have:

	Number of Points	**Number of Lines**
	2	1
	3	3
	4	6
	5	10
	6	15

You now have several options:

(1) <u>Summation Argument</u>

This argument leads to the work with summations in the rest of the chapter. It also offers several opportunities for recognizing interesting mathematical patterns.

Extend the table by looking for patterns. Students will quickly notice that each right-hand entry is the sum of the two numbers above. Others will notice that the differences between successive numbers of lines increase by 1 each time.

1 ↓2

3 ↓3

6 ↓4

10 ↓5

15 ↓?

Others might point out that each time you add a point, you add one less than the number of points to the previous result.

Exploring why these methods give the correct answer leads to the discovery that the only new lines you get from adding a point are the ones determined by the new point and each of the previous ones. Another interpretation is that the first point can be connected to n-1 other points, the next one only n-2 to avoid duplication, the next to n-3, etc. This leads to:

Number of Points	**Number of Lines**
2	1
3	3=1+2
4	6=1+2+3
5	10=1+2+3+4
6	15=1+2+3+4+5
.	
.	
.	
n	?=1+2+3+4+5+...+(n-2)+(n-1)*

This will lead very quickly into summations:

$$1+2+3+4+5 = \sum_{i=1}^{5} i$$

$$1+2+3+4+5+\ldots+(n-2)+(n-1) = \sum_{i=1}^{n-1} i$$

But at this point it doesn't give us an efficient way to compute the answer for large values of n. (You may introduce

---

*Questions like "How do you say the number before 8 without saying 8 (in any language)?" lead the students to the general statement n-1 is the number one less than n; n-2 is the number one before that, etc.

the summation notation now or wait until the next section.)

Looking again at the chart, ask if the students can see any relationship between the number of points and the number of lines. For odd numbers there is an obvious connection which leads to a pattern:

Number of Points	**Number of Lines**
2	1
3	3 = 1x3
4	6 $\leftarrow 1\frac{1}{2}x4$ or $\frac{3}{2}x4$
5	10 = 2x5
6	15 $\leftarrow 2\frac{1}{2}x6$ or $\frac{5}{2}x6$
7	21 = 3x7
8	28 $\leftarrow 3\frac{1}{2}x8$ or $\frac{7}{2}x8$
9	36 = 4x9

Students should now be able to conjecture the number of lines for any odd number of points, "Half of one less than the number times the number or $\frac{(n-1)}{2}xn$." To extend to even numbers either check this result with some examples or look again at the pattern.

When we have the 3 the right-hand factor is what?

What about 5? 7?

What do you think it should be for 4?

What number is halfway between 3 and 5?

What number is halfway between 1 and 2?

What's $1\frac{1}{2}x4$?

What number do I have to multiply 4 by to get 6?

What's the improper fraction name for $1\frac{1}{2}$?

What's $\frac{3}{2}$x4?

Who can tell me how to write 15 to follow the pattern?

We now see that the formula $\frac{(n-1)}{2}xn$ seems to work for any n.

The simplest way for students to see why this is the correct formula is to look at the chart yet another way.

Number of Points	**Number of Lines**	
2	1	$= \frac{1}{2}x2$
3	3 = 1+2	= 1x3
4	6 = 1+2+3	$= 1\frac{1}{2}x4$
5	10 = 1+2+3+4	= 2x5
6	15 = 1+2+3+4+5	$= 2\frac{1}{2}x6$
7	21 = 1+2+3+4+5+6	= 3x7

Ask who can find three 7's in 1+2+3+4+5+6.

Have a student come to the board and connect numbers which add up to 7. Repeat with four 9's in 1+2+3+4+5+6+7+8 and two and one-half 6's in 1+2+3+4+5.

Students will soon discover that in each case, pairing numbers from opposite ends of the sum 1+2+3+...+(n-1) gives you $\frac{n-1}{2}$ pairs that add up to n.

(2) <u>Combinatorics Argument</u>

Suppose you have six points labeled A-F.

First list all the lines that pass through A.

How many are there?

Repeat this process with each of the other points.

How many lines have you named altogether?

Where did the 30 come from?

Are there any duplications?

Is every line repeated?

Is any line repeated more than twice?

How many times is each line repeated?

If you eliminate one of each pair of duplicates, how many lines are left?

What do you have to do to 30 to get that number?

Repeat with several more specific cases until the students can generalize that for n points you can connect each point to n-1 other points but you divide by 2 to eliminate the duplication.

You can motivate or reinforce the result from the chart as we did previously by looking at the patterns: 1x3, 2x5, 3x7, 4x9, etc., and inserting the values for even numbers.

(3) Extensions

Without going into detail we suggest several topics that might be developed at this point.

The combinatorics argument leads naturally into an investigation of combinations and permutations. The result which we have derived is simply the formula for the number of combinations of n items taken two at a time.

Another way to look at combinations is in terms of subsets of a set of n elements. Recalling that the empty set is a subset of every set, students can develop the following chart:

# of Elements	# Empty Subsets	# of 1-Element Subsets	# of 2-Element Subsets	# of 3-Element Subsets	# of 4-Element Subsets	# of 5-Element Subsets
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1
	etc.					

which can be rewritten as the familiar Pascal's Triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
etc.

You can also ask students to sum the rows of the chart (give you the total number of subsets of a set of n elements). The sums 1, 2, 4, 8, 16, ... lead back into exponents and result from the fact that the entries in the chart represent the coefficients in the binomial expansion of $(1+1)^n$. You can extend the distributive law work of the previous chapter to look at, for example, $(x+1)^2$, $(x+1)^3$, $(x-1)^4$, etc., and recreate the triangle from another viewpoint.

6.2 Summations

On the simplest level, summations can be introduced for motivation and practice because students like working with complicated, out-of-the-ordinary mathematical symbols. The fact that certain summations help develop pattern recognition skills and that others have important applications in higher mathematics make them an even more rich and fruitful topic.

Put on the board $\sum_{1=1}^{4} (2xi)$. Read it for the students, "The summation from i=1 up to 4 of 2 times i." Have several students and the whole class read it.

> What mathematical word is hidden in summation?
>
> What does sum mean?
>
> What operation do we use when finding sums?
>
> What operation do you think we use with summation?

Use the technique of giving clues and refining guesses to have the students develop:

$$\sum_{i=1}^{4} (2xi) = (2x1) + (2x2) + (2x3) x (2x4)$$

Now introduce the vocabulary of summation and the meaning of the terms through examples:

> If I change the upper limit from 4 to 6, what has to change?
>
> If I change the lower limit from 1 to 3, what do I have to change?
>
> What if I change the argument from 2xi to 3xi? to 2i-1?

Do additional examples until the students can use the terms accurately.

The following summations lead fairly easily to a general formula for the sum of n terms. You will probably want to present these for the challenge of figuring out the pattern and not spend much time justifying the result. All are examples of arithmetic series and you could lead the students to the standard proofs if you wish or use the pairing arguments of the preceding section.

Have the students evaluate each summation for n=1, 2, 3, 4, etc., until they see the general pattern and can formulate an algebraic expression for the sum of n terms. The answers are listed in the footnote below.[*]

(a) $\sum_{k=1}^{n} (2xk) =$

(b) $\sum_{i=1}^{n} i =$

(c) $\sum_{i=1}^{n} axj =$

(d) $\sum_{i=1}^{n} [(2xi)-1] =$

The last problem has an interesting geometric interpretation:

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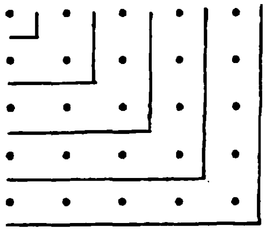
[*] (a) $nx(n+1)$ (b) $\frac{nx(n+1)}{2}$ (c) $\frac{axnx(n+1)}{2}$ (d) n^2



You might also want to look at some examples of geometric series.*

(e) $\sum_{1=0}^{n} 2^{i} = ?$

(f) $\sum_{i=0}^{n} 2x3^{i} = ?$

(g) $\sum_{i=0}^{n} 3^{i} = ?$

(h) $\sum_{i=0}^{n} a^{i} = ?$

6.3 Infinite Series

In this section we introduce the concept of the limit of an infinite series through a special example. We extend this thinking to several more instances including one which takes us into the study of repeating decimals. This material provides another illustration of an advanced conceptual topic which provides substantial arithmetic practice.

Begin by reviewing summations and negative exponents.

---

*(e) $2^{n+1}-1$ (f) $3^{n+1}-1$ (g) $\frac{3^{n+1}-1}{2}$ (h) $\frac{a^{n+1}-1}{a-1}$

Put on the board:

$$S_1 = \sum_{i=1}^{1} 2^{-i}$$

$$S_2 = \sum_{i=1}^{2} 2^{-i}$$

Introduce the word subscript and have the students read and evaluate each expression. Ask what they think S_3 is. Repeat with S_4.

On the board you should have:

$$S_1 = \sum_{i=1}^{1} 2^{-i} = \frac{1}{2}$$

$$S_2 = \sum_{i=1}^{2} 2^{-i} = \frac{1}{2}+\frac{1}{4} = \frac{3}{4}$$

$$S_3 = \sum_{i=1}^{3} 2^{-i} = \frac{1}{2}+\frac{1}{4}+\frac{1}{8} = \frac{7}{8}$$

$$S_4 = \sum_{i=1}^{4} 2^{-i} = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} = \frac{15}{16}$$

Ask if anyone can give you the sum for S_5 without writing the expanded form or doing the computation. At this point the pattern should be apparent. If not, have the class work it out. If the class has spent a lot of time on finding general sums in the previous section, they should be able to tell you

$$S_n = \sum_{i=1}^{n} 2^{-i} = \frac{2^n-1}{2^n}$$

Now focus on the terms of the sequence $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots$.

How long can we go on adding terms to the sequence?

Is $\frac{1}{2}+\frac{1}{4}$ greater or less than $\frac{1}{2}$?

Which is larger, $\frac{1}{2}$ or $\frac{3}{4}$?

Which term is the smallest term of the sequence?

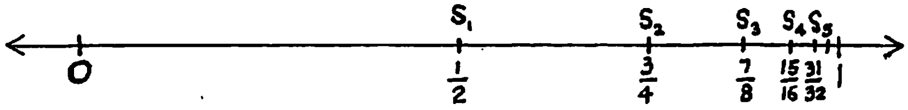
Do successive terms get larger or smaller or stay the same?

Can anyone give me a number that is larger than all the terms of the sequence?

What's the smallest number that is larger than all of the terms of the sequence?

What number do the terms of the sequence get closer and closer to?

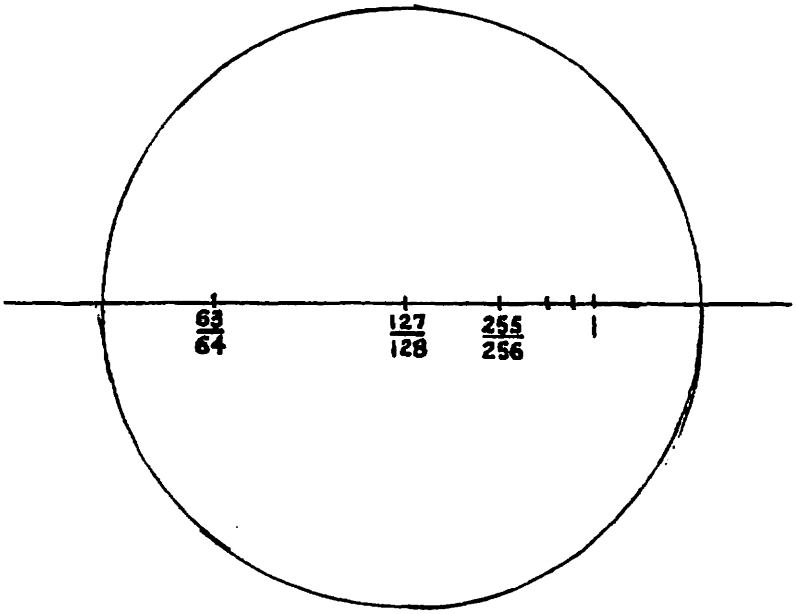
Before asking the last few questions, you may want to graph the terms of S_n on a numberline.



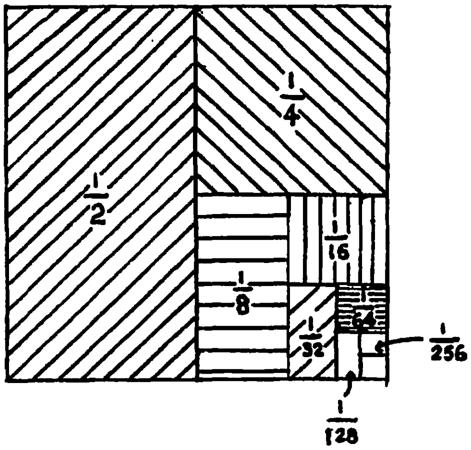
If the students have trouble locating these points, make several parallel numberlines marked off in units of $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$, respectively. Look for equivalent fractions. It is also helpful to go back to the summation $\frac{3}{4}=\frac{1}{2}+\frac{1}{4}$, $\frac{7}{8}=\frac{3}{4}+\frac{1}{8}$, etc.

It is readily apparent that the terms of the sequence are approaching 1. To emphasize this point, suggest that you made the numberline too short so you ran out of room. Draw a numberline with the distance between 0 and 1 the length of the board.

Have students come to the board and plot the points of S_n. You might also take a small section of the line and "put it under a microscope."



Another way to illustrate this process is to shade in parts of a square as you add each term of the summation.



As you go higher in the sequence, what happens to the part of the square that's shaded?

What happens to the part that's not shaded?

If you had shaded $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{1024}$ of the square and a visitor walked into the room, what would she think?

How much of the square is not shaded for S_5? S_6? S_{10}?

On the numberline, how much does S_5 miss 1 by? S_6? S_{10}?

How far is S_5 from 1? S_6? S_{10}?

How large would n have to be for all the S_n's to be closer to 1 than .01?

Could you go out far enough in the sequence so that all the terms would be closer to 1 than .001? How about .000001?

When n is very large, what number does S_n act like?

We are not looking for a precise definition here but for a solid conceptual understanding that the terms of the sequence become artitrarily close to 1. Intuitive terminoloy is helpful like 'acts like' and 'gets as close as you want'. You can also introduce examples that illustrate that 'as close as you want' is meaningful. If you were measuring a piece of window glass and you were off by $\frac{1}{64}$ of an inch, would it be a problem? What if you were building a computer chip or doing brain surgery? The important concept is that for this sequence no matter how close anyone wants you to get to 1, if n is large enough, you can do it. This is an intuitive interpretation of the standard epsilon and N definition.

Mathematicians say that the limit of S_n as n approaches infinity (becomes arbitrarily large) is 1 and write it

$$\lim_{n\to\infty} S_n = 1$$

As n gets larger and larger, the numerical value of $\sum_{i=1}^{n} 2^{-i} = S_n$ gets arbitrarily close to 1 and we write

$$\sum_{i=1}^{\infty} 2^{-i} = 1$$

You can also look at a general expression for the nth partial sum,

$$\sum_{i=1}^{n} 2^{-i} = \frac{2^n - 1}{2^n} = \frac{2^n}{2^n} - \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

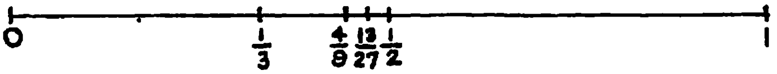
What happens to $\frac{1}{2^n}$ as n increases without bound?

Now look at $\sum_{i=1}^{\infty} 3^{-i}$. Get conjectures. It is helpful for the students to make a chart.

n	$\sum_{i=1}^{n} 3^{-i}$
1	$\frac{1}{3}$
2	$\frac{4}{9}$
3	$\frac{13}{27}$
4	$\frac{40}{81}$
5	$\frac{121}{243}$

Note how rich this material is for reinforcing skills with fractions. For instance, which is larger, $\frac{4}{9}$ or $\frac{13}{27}$? Students might recall how the table was generated and point out that $\frac{4}{9}+\frac{1}{27}=\frac{13}{27}$ so $\frac{4}{9}<\frac{13}{27}$. Without using that information, they might decide to find a common denominator and thus determine $\frac{12}{27}<\frac{13}{27}$.

Graphing the sequence $\frac{1}{3}, \frac{4}{9}, \frac{13}{27}, \frac{40}{81}$ on a numberline will sharpen the students' estimation skills with fractions and will give some students the idea that the limit might be $\frac{1}{2}$ although it is far from obvious.



It is useful at this point to look at various equivalent fractions for $\frac{1}{2}$.

> Who can give me a fraction equivalent to $\frac{1}{2}$?
>
> Who can give me an equivalent fraction to $\frac{1}{2}$ that has 3 as a numerator?
>
> Who can give me an equivalent fraction to $\frac{1}{2}$ that has 10 as a denominator?
>
> Who can give me an equivalent fraction to $\frac{1}{2}$ that has 5 as a denominator? 3?

Students will balk at writing $\frac{2\frac{1}{2}}{5}$ but they will eventually agree that it does equal $\frac{1}{2}$. Continue this exercise choosing 27, 81, 243, etc., as denominators. Now compare

S_n	$\frac{1}{2}$	difference between S_n and $\frac{1}{2}$
$\frac{1}{3}$	$\frac{1\frac{1}{2}}{3}$	$\frac{\frac{1}{2}}{3}$
$\frac{4}{9}$	$\frac{4\frac{1}{2}}{9}$	$\frac{\frac{1}{2}}{9}$
$\frac{13}{27}$	$\frac{13\frac{1}{2}}{27}$	$\frac{\frac{1}{2}}{27}$
$\frac{40}{81}$	$\frac{40\frac{1}{2}}{81}$	$\frac{\frac{1}{2}}{81}$

As n gets larger and larger, what happens to the difference between $\frac{1}{2}$ and the nth partial sum? Through similar questions to the ones used for the first example, conclude that

$$\sum_{i=1}^{\infty} 3^{-i} = \frac{1}{2}$$

Another way of seeing that the partial sums get close to $\frac{1}{2}$ is to compare fractions with the same numerators:

$$\frac{4}{9} \text{ and } \frac{4}{8},\ \frac{13}{27} \text{ and } \frac{13}{26},\ \frac{40}{81} \text{ and } \frac{40}{80}, \text{ etc.}$$

As k becomes large, the difference between $\frac{a}{k}$ and $\frac{a}{k+1}$ would seem to get very small. Computing exactly what the difference is provides an excellent opportunity to review subtraction of fractions.

Note that another way of achieving this result is to look at $\sum_{i=1}^{\infty} 2x3^{-i}$. The sequence of partial sums is $\frac{2}{3}$, $\frac{8}{9}$, $\frac{26}{27}$, $\frac{80}{81}$, $\frac{242}{243}$, ... From the work with $\sum_{i=1}^{\infty} 2^{-i}$, it is clear that this sequence also has 1 as its limit. Using a generalized distributive property, we conclude

$$2x \sum_{i=1}^{\infty} 3^{-i} = 1 \text{ and hence } \sum_{i=1}^{\infty} 3^{-i} = \frac{1}{2}$$

You can also reach the same conclusion by noticing that each term of the second series is one-half the corresponding term of the first so it should approach one-half the limit of the first. This approach can be used as reinforcement, but we prefer the first approach for developing the limit because of the practice with fractions it affords.

A similar analysis can be applied to additional problems to give more practice with fractions and generate a pattern. Some results are summarized in the chart below.

Summation	Sequence of Partial Sums	nth Partial Sum	Limit
$\sum_{i=1}^{\infty} 2^{-i}$	$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots$	$\frac{2^n-1}{2^n}$	1
$\sum_{i=1}^{\infty} 3^{-i}$	$\frac{1}{3}, \frac{4}{9}, \frac{13}{27}, \frac{40}{81}, \ldots$	$\frac{3^n-1}{3^n} \times \frac{1}{2}$	$\frac{1}{2}$
$\sum_{i=1}^{\infty} 4^{-i}$	$\frac{1}{4}, \frac{5}{16}, \frac{21}{64}, \frac{85}{256}, \ldots$	$\frac{4^n-1}{4^n} \times \frac{1}{3}$	$\frac{1}{3}$
$\sum_{i=1}^{\infty} 5^{-i}$	$\frac{1}{5}, \frac{6}{25}, \frac{31}{125}, \frac{156}{625}, \ldots$	$\frac{5^n-1}{5^n} \times \frac{1}{4}$	$\frac{1}{4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\sum_{i=1}^{\infty} 10^{-i}$	$\frac{1}{10}, \frac{11}{100}, \frac{111}{1000}, \frac{1111}{10000}, \ldots$	$\frac{10^n-1}{10^n} \times \frac{1}{9}$	$\frac{1}{9}$
$\sum_{i=1}^{\infty} 11^{-i}$	$\frac{1}{11}, \frac{12}{121}, \frac{133}{1331}, \frac{1464}{14641}, \ldots$	$\frac{11^n-1}{11^n} \times \frac{1}{10}$	$\frac{1}{10}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\sum_{i=1}^{\infty} a^{-i}$	$\frac{1}{a}, \frac{a+1}{a^2}, \frac{a^2+a+1}{a^3}, \frac{a^3+a^2+a+1}{a^4}, \ldots$	$\frac{a^n-1}{a^n} \times \frac{1}{a-1}$	$\frac{1}{a-1}$

For most of these examples, the results are fairly intuitive. It is easy to see, for instance, that $\frac{111}{1000}$ is close to $\frac{111}{999}$ or $\frac{1}{9}$ and that $\frac{133}{1331}$ is close to $\frac{133}{1330}$ or $\frac{1}{10}$. It also becomes easy to use a pattern to conjecture the result and then to verify it for specific examples.

The column of nth partial sums is included because many classes will want to find these formulas. Having the nth partial sum makes it easier to calculate the limit and to see how the distance between the nth sum and the limit is decreasing. Once you have found a few of them, the pattern becomes apparent.

You can also look at a variation of the standard proof:

$$\text{If } S = \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \ldots + \frac{1}{a^{n-1}} + \frac{1}{a^n}$$

$$\text{then } a \times S = 1 + \frac{1}{a} + \frac{1}{a^2} + \ldots + \frac{1}{a^{n-1}}$$

$$\text{and } (a-1)S = 1 - \frac{1}{a^n}$$

$$\text{So } S = \frac{a^n-1}{a^n} \times \frac{1}{a-1}$$

The students' grasp of the algebraic argument will be somewhat fragile at this point in their mathematical careers. Continuing to work with and check conjectures with the actual partial sums will reinforce the students' conceptual understanding of the limit concept as well as their intuitive grasp of very close or 'nearly equivalent' fractions. It is highly motivating for them to learn that the concept of limit is central to the study of calculus.

6.4 Repeating Decimals

Everyone knows that 873 means '8 hundreds plus 7 tens plus 3 ones'. It is much less apparent that .873 is the same as '8 tenths plus 7 hundredths plus 3 thousandths'. Students will want to write $.873=\frac{873}{1000}$. This can be transformed as follows:

$$
\begin{aligned}
.873 &= \frac{873}{1000} = \frac{800}{1000} + \frac{70}{1000} + \frac{3}{1000} \\
&= \frac{8}{10} + \frac{7}{100} + \frac{3}{1000} \\
&= (8x10^{-1}) + (7x10^{-2}) + (3x10^{-3})
\end{aligned}
$$

In other words, our notion of place value extends to the right of the decimal point as well as the left.

Note that this offers an interpretation of the familiar rule that when 'you add zeros after the last number to the right of the decimal point, you don't change the value of the number'. Rather than write $.5=\frac{5}{10}$ and $.500=\frac{500}{100}$ and reduce the fractions, we simply compare $\frac{5}{10}$ to $\frac{5}{10}+\frac{0}{100}+\frac{0}{1000}$.

To mathematicians, all real numbers are represented by infinite series. Some numbers (rational numbers) have series expansions which repeat, e.g., $.5=\frac{5}{10}+\frac{0}{100}+\frac{0}{1000}+\ldots$ and $\frac{1}{3}=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}$ .... Many more others (irrational numbers) are represented by infinite series where no block of terms ever repeats.

The infinite series .333... is usually written $.\overline{3}$ and usually arises from the division 1÷3.

Repeated applications of the division algorithm give us:

$$\frac{1}{3} = \frac{3}{10} + \frac{1}{30}$$
$$= (\frac{3}{10} + \frac{3}{100}) + \frac{1}{300}$$
$$= (\frac{3}{10} + \frac{3}{100} + \frac{3}{1000}) + \frac{1}{3000}$$
$$=(\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10,000}) + \frac{1}{30,000}$$

etc.

If we look at the sequence of remainders, $\frac{1}{30}, \frac{1}{300}, \frac{1}{3000}$, etc., we see that the difference between $\frac{1}{3}$ and each successive quotient ($\frac{3}{10}, \frac{3}{10}+\frac{3}{100}, \frac{3}{10}+\frac{3}{100}+\frac{3}{1000}$, etc.) is getting closer and closer to zero. But these quotients are precisely the partial sums of the infinite series $\sum_{i=1}^{\infty} 3x10^{-i}$. In other words, saying that $\frac{1}{3}$=.333... is the same as saying that the infinite series $\sum_{i=1}^{\infty} 3x10^{-i}$ converges to (has the limit) $\frac{1}{3}$.

We now have a way of converting a repeating decimal to a rational number. Simply write it as an infinite series and use the results of the previous section, $\sum_{i=1}^{\infty} a^{-i} = \frac{1}{a-1}$ and the generalized distributive law to find its limit.

Examples:

$$.\overline{3} = \sum_{i=1}^{\infty} 3x10^{-i} = 3x \sum_{i=1}^{\infty} 10^{-i} = 3x\frac{1}{9} = \frac{1}{3}$$

$$.\overline{7} = \sum_{i=1}^{\infty} 3x10^{-i} = 7x \sum_{i=1}^{\infty} 10^{-i} = 7x\frac{1}{9} = \frac{7}{9}$$

$$.\overline{10} = \sum_{i=1}^{\infty} 10x100^{-i} = 10x \sum_{i=1}^{\infty} 100^{-i} = 10x\frac{1}{99} = \frac{10}{99}$$

$$.\overline{9} = \sum_{i=1}^{\infty} 9x10^{-i} = 9x\frac{1}{9} = 1$$

$$.\overline{142857} = \sum_{i=1}^{\infty} 142857x1{,}000{,}000^{-i} = 142857 \text{ x } \frac{1}{999999} = \frac{1}{7}$$

Another way to look at this type of problem is the following:

$$\begin{aligned} \text{If } n &= .333... \\ \text{then } \underline{10n} &\underline{= 3.333...} \\ \text{so } 10n-n &= 3.000 \\ \text{or } 9n &= 3 \\ n &= \frac{3}{9} = \frac{1}{3} \end{aligned}$$

Similarly,

$$\begin{aligned} \text{If } m &= .101010... \\ \text{then } \underline{100m} &\underline{= 10.101010...} \\ \text{so } 100m-m &= 10 \\ 99m &= 10 \\ m &= \frac{10}{99} \end{aligned}$$

Two warnings are in order. First, for many students this argument involves symbolic manipulation and is not conceptually enlightening. Students can learn the process without understanding the underlying concepts. For this reason, we recommend that it be used only as reinforcement. Second, this reasoning depends on the existence of a well-defined sum for the infinite series in question. Without this fact, analogous reasoning can be used to derive the erroneous theorem that the sum of the positive powers of 2 is $^{-}1$. The 'proof' is as follows.[*]

$$\begin{aligned} \text{If } S &= 1+2+4+8+16+\ldots \\ \text{then } S &= 1+2(1+2+4+8+\ldots) \\ \text{So } S &= 1+2S \\ \text{Therefore } S &= {}^{-}1, \text{ i.e., } 1+2+4+8+16+\ldots = {}^{-}1 \end{aligned}$$

As a final activity to reinforce and tie together all the work of this volume, you might want to look at the representation of numbers less than one in other bases. In the same way that we can write $.873 = \frac{873}{1000}$ as $(8\text{x}10^{-1}) + (7\text{x}10^{-2}) + (3\text{x}10^{-3})$ we can write $\frac{7}{8} = \frac{1}{2}+\frac{1}{4}+\frac{1}{8} = (1\text{x}2^{-1}) + (1\text{x}2^{-2}) + (1\text{x}2^{-3}) = ._{\text{two}}111$. (In base 10, we refer to '.' as the 'decimal' point. Similarly, in base 2, '$._{\text{two}}$' is the 'bimal' or 'twomal' point.) Going the other direction, $._{\text{three}}221$ becomes $\frac{2}{3} + \frac{2}{9} + \frac{1}{27} = \frac{25}{27}$.

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[*] This 'proof' can be found in a letter by James Metz in <u>The Mathematics Teacher</u> 75, no. 9 (December 1982): 730.

Another fruitful exercise is to have students perform computations in different number bases. For example, in bimals, what is $\cdot_{two}101$ divided by $\cdot_{two}01$? First, if we do the problem directly using the rules for 'decimal points', we get $10\cdot_{two}1$. Next, if we check by converting $\cdot_{two}101$ into $\frac{1}{2}+\frac{0}{4}+\frac{1}{8}$ equals $\frac{5}{8}$ and $\cdot_{two}\cdot 01$ into $\frac{1}{4}$, we get $\frac{5}{8}$ divided by $\frac{1}{4}$ equals $\frac{5}{8}x\frac{4}{1}= 2\frac{1}{2}$. Recalling that 10.1 in bimals represents $2\frac{1}{2}$, we see that both methods lead to the correct answer. If anyone doubts the relevence of the above, other than it strengthens concepts, it should be remembered that modern computers operate in base 2, base 8, and base 16.

An interesting question arises when we want to represent a number like $\frac{1}{3}$ as a 'bimal'. If we happen to recall that $\frac{1}{3} = \sum_{i=1}^{\infty} 4^{-i}$, we can see that $\frac{1}{3}=\frac{0}{2}+\frac{1}{4}+\frac{0}{8}+\frac{1}{16}+\frac{0}{32}+\frac{1}{64}+\ldots$ $= \cdot_{two}010101$.... If we don't happen to recall this fact, using the standard division method works:

First change 3_{ten} to 11_{two} and 1_{ten} to 1_{two}.

Now divide:

$$\begin{array}{r}
\cdot_{two}010101 \\
11_{two}\overline{)1\cdot_{two}000000} \\
\underline{11} \\
100 \\
\underline{11} \\
100 \\
\underline{11} \\
1
\end{array}$$

In other words, $\frac{1}{3}_{\text{ten}} = \frac{1}{11}_{\text{two}} = ._{\text{two}}010101...$

Checking, we have $\frac{0}{2}+\frac{1}{4}+\frac{0}{8}+\frac{1}{16}+\frac{0}{32}+\frac{1}{64}+\ldots$, which we know converges to $\frac{1}{3}$.

6.5 Exercises

1. Fill in the argument, lower limit, and upper limit:

$$\sum_{1=\triangle}^{\square} (\qquad) = (3x7) + (4x7) + (5x7) + (6x7)$$

2. Write the expansion of this summation:

$$\sum_{k=3}^{5} (2xk) =$$

3. Expand each summation and find the sum:

a. $\sum_{j=3}^{5} 2xj =$ b. $2x \sum_{j=3}^{5} j =$

c. $\sum_{i=1}^{2} 2x3^{-i} =$ d. $2x \sum_{i=1}^{2} 3^{-i} =$

e. $\sum_{k=4}^{7} 3xk =$ f. $3x \sum_{k=4}^{7} k =$

g. What can you conclude from these problems?

4. Expand each summation and find the sum:

a. $\sum_{i=1}^{2} (2xi)$

b. $\sum_{i=1}^{3} (2xi)$

c. $\sum_{i=1}^{4} (2xi)$

d. $\sum_{i=1}^{5} (2xi)$

e. $\sum_{i=1}^{6} (2xi)$

5. Write each sum in (4) as a product that includes the upper limit.

a.

b.

c.

d.

e.

Can you guess what number $\sum_{i=1}^{20} (2xi)$ equals?

6. Write the expansion of each summation and find the sum:

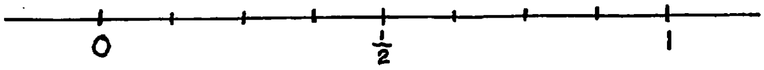
$\sum_{i=1}^{1} 3x4^{-i} =$

$\sum_{i=1}^{2} 3x4^{-i} =$

$\sum_{i=1}^{3} 3x4^{-i} =$

What do you guess the next sum will be?

7. Plot $\frac{3}{4}$, $\frac{15}{16}$, $\frac{63}{64}$ on this numberline:



What number is the limit of the sequence $\frac{3}{4}, \frac{15}{16}, \frac{63}{64}, \ldots$?

8. Fill in the missing partial sums of $\sum_{i=1}^{\infty} 10^{-i}$

$$\frac{1}{10}, \frac{11}{100}, \square, \triangle, \ldots$$

What number are these sums getting closer and closer to?

9. Write each fraction as a decimal:

a. $\frac{1}{2}$

b. $\frac{1}{8}$

c. $\frac{1}{4}$

d. $\frac{1}{10}$

e. $\frac{1}{100}$

f. $\frac{1}{3}$

g. $\frac{1}{9}$

h. $\frac{1}{11}$

i. $\frac{1}{7}$

j. $\frac{1}{12}$

10. Write each decimal as a sum of fractions whose bases are powers of 10.

a. .375

b. .5

c. .55

d. .25

e. .625

f. .333...

g. .555...

h. .666...

i. .1818...

j. .999...

11. Write each decimal as a fraction in lowest terms:

a. .5

b. .25

c. .375

d. .55

e. .625

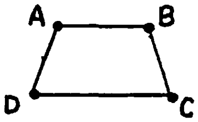
f. .333...

g. .555...

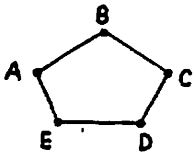
h. .666...

i. .1818...

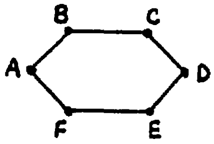
j. .999...

12. a. 

Draw the diagonals of ABCD. How many are there?

b. 

Draw the diagonals of ABCDE. How many are there?

c. 

Draw the diagonals of ABCDEF. How many are there?

d. How many diagonals would a 10-sided polygon have?

e. How many diagonals would an n-sided polygon have?

13. Write the base 10 numeral for each:

a. $\cdot_{\text{two}}{}^{101}$

b. $\cdot_{\text{two}}{}^{1111}$

c. $10\cdot_{\text{two}}{}^{11}$

d. $\cdot_{\text{three}}{}^{1}$

e. $\cdot_{\text{three}}{}^{01}$

f. $\cdot_{\text{three}}{}^{11}$

14. Write a base 2 numeral for each:

a. $\frac{1}{2}$

b. $\frac{1}{4}$

c. $\frac{3}{4}$

d. $\frac{5}{8}$

e. $\frac{7}{8}$

f. $\frac{13}{16}$

15. You have a whole cake. You give half to John and keep the rest. Then Sue comes and you give half of what's left to her. Carl comes and you give half of what you have to him. Then Kate arrives and you give her half the rest. How much cake do you have left? What is the total amount of cake you've given away?

6.6 Chapter Summary

Summations

Expansion of $\sum_{i=\alpha}^{\beta} f(i)$ for any whole numbers α and β and any function f.

Infinite Series

Find $\lim_{n\to\infty} S_n$ for selected sequences of partial sums.

Repeating Decimals

Change from fractional to decimal numerals

Change repeating decimals to fractions

Representing numbers less than 1 in bases other than 10

Vocabulary

Point
Line
Vernacular
Axiom
(Combination)
(Subset)
(Pascal's Triangle)
(Binomial Theorem)

Summation
Upper limit
Lower limit
Expansion
Partial sum
Argument
Infinity

Subscript
Sequence
Limit
Infinite

6.7 Chapter Test

1. Fill in the argument, lower limit, and upper limit:

$$\sum_{i=\triangle}^{\square}(\qquad) = (\frac{1}{2}x3) + (\frac{1}{2}x4) + (\frac{1}{2}x5)$$

2. Write the expansion of this summation:

$$\sum_{k=5}^{9}(3xk) =$$

3. Write the expansion and find each sum:

a. $\sum_{i=1}^{3}(2^{-i})$

b. $\sum_{i=1}^{4} 2^{-i} =$

4. a. Write the next two terms of the sequence:

$\frac{1}{2}, \frac{3}{4}, \frac{7}{8},$ _____, _____

b. Locate each of the numbers in (4.a.) on this numberline:

c. What number is this sequence approaching as a limit?

5. a. Circle the number which is closest to $\frac{1}{2}$:

$\frac{1}{3}, \frac{13}{27}, \frac{40}{81}, \frac{4}{9}$

b. Circle the number which is closest to $\frac{1}{9}$:

$\frac{1111}{10000}, \frac{11}{100}, \frac{1}{10}, \frac{111}{1000}$

6. If $S_1 = 2$

$S_2 = 4$

$S_3 = 6$

find $S_4 =$

$S_5 =$

$S_{10} =$

$S_n =$

7. Write each fraction as a decimal:

a. $\frac{1}{2}$ c. $\frac{2}{3}$

b. $\frac{1}{10}$ d. $\frac{5}{9}$

8. Write each decimal as a fraction in lowest terms:

a. .75 c. .333...

b. .11 d. .777...

9. Write a base 10 numeral for:

a. $._{\text{two}}111$

b. $._{\text{three}}101$

10. Write a base 2 numeral for

a. $\frac{1}{2}$

b. $\frac{3}{4}$