Lecture 2

Generating functions in probability and combinatorics

For this chapter, a more complete discussion may be found in Chapters 2 and 3 of my lecture notes on Analytic Combinatorics in Several Variables. These chapters are excerpted on the Cornell Summer School website. Throughout this chapter, the notation $[n]$ is used for the set $\{1, \ldots, n\}$.

2.1 Formal power series and combinatorial enumeration

Let $\mathbb{C}[[z_1, \ldots, z_d]]$ denote the ring of **formal power series** in the variables $z_1, \ldots, z_d$. Formally, elements of $\mathbb{C}[[z_1, \ldots, z_d]]$ are collections $\{a_\mathbf{r} : \mathbf{r} \in (\mathbb{Z}^+)^d\}$ of complex numbers, via the correspondence $\{a_\mathbf{r}\} \mapsto \sum_\mathbf{r} z_\mathbf{r} \mathbf{z}^\mathbf{r}$. Addition is defined by $(f + g)_\mathbf{r} = f_\mathbf{r} + g_\mathbf{r}$ and multiplication is defined by convolution: $(f \cdot g)_\mathbf{r} = \sum_\mathbf{s} f_\mathbf{s} g_{\mathbf{r}-\mathbf{s}}$. The sum in this convolution is always finite, so there is no question of convergence. If $q_\mathbf{0} \neq 0$ then q is a unit. Therefore, the rational functions p/q with $q_\mathbf{0} \neq 0$ are identified with a subring of $\mathbb{C}[[z_1, \ldots, z_d]]$.

Rational operations

The operations of addition, multiplication and $f \mapsto 1/(1 - f)$ have useful combinatorial interpretations. Addition corresponds to a disjoint union. Multiplication corresponds to convolution. The operation $f \mapsto 1/(1 - f)$ corresponds to building a new combinatorial class whose elements are all finite sequences of members of the old class, counted by weight defined to equal the sum of the weights of the elements of the sequence.

Example 2.1. Let E be a finite subset of $(\mathbb{Z}^+)^d$ not containing $\mathbf{0}$ and let $\mathcal{A}$ be the class of finite sequences $(\mathbf{0} = x_0, x_1, \ldots, x_k)$ of elements of $(\mathbb{Z}^+)^d$ with $x_j - x_{j-1} \in E$ for $1 \leq j \leq k$. We call these **paths with steps in** E.

Let $B(\mathbf{z}) = \sum_{\mathbf{r} \in E} \mathbf{z}^{\mathbf{r}}$ generate E by step size. Then $1/(1 - B(\mathbf{z}))$ counts paths with steps in E by ending location. This includes examples we have already seen. Multinomial coefficients count paths ending at $\mathbf{r}$ with steps in the standard basis directions $e_1, \ldots, e_d$; the generating function $1/(1 - \sum_{j=1}^d z_j)$ follows from the generating function $\sum_{j=1}^d z_j$ for E.

Suppose we wish to allow only some transitions and to give each path a multiplicative weight: Let M be any matrix and define the weight of a paths of n steps to be the product over the n transitions $i \to j$ of M_{ij}. We can forbid some transitions by taking some of the M_{ij} to be zero. The **transfer matrix method** uses matrix algebra to give the following elementary but very useful generalization of the previous example.

Theorem 2.2 (transfer matrix method)**.** *Let V be a finite alphabet and let M be a matrix whose rows and columns are indexed by V. For each path $v_0, \ldots, v_n$, define its weight $\omega := \prod_{j=1}^n M(v_j, v_{j-1})$. Let $W(i, j; n)$ be the sum of all weights of paths of length n from $v_0 = i$ to $v_n = j$. Let $f_{ij}(z)$ be the generating function $\sum_n W(i, j; n) z^n$. Then the matrix F whose (i, j) element is the generating function $f_{ij}(z)$ is given by*

$$F(z) = (\mathbf{I} - z\mathbf{M})^{-1} .$$

□

Exercise 2.1 (counting domino tilings)**.** Let a_{nk} be the number of ways of placing k non-overlapping dominoes on a $2 \times n$ grid. Find the generating function for these numbers.

The exponential formula

The exponential e^f of a formal power series in any number of variables may be defined as $1 + \sum_{n=1}^{\infty} f^n/n!$, which is well defined as long as $f_{\mathbf{0}} = 0$. Exponentiation turns out to have a very useful combinatorial interpretation.

Let $\mathcal{B}$ be a combinatorial class with $b_n := |\mathcal{B}_n|$. Define a **$\mathcal{B}$-partition** of $[n]$ to be a set of pairs $\{(S_\alpha, G_\alpha) : \alpha \in I\}$, where the collection $\{S_\alpha : \alpha \in I\}$ is a partition of the set $[n]$

and each G_α is an element of $\mathcal{B}_{|S_\alpha|}$. Define the class $\exp(\mathcal{B})$ to be the class of $\mathcal{B}$-partitions enumerated by n, that is, $\exp(\mathcal{B})_n$ is the class of $\mathcal{B}$-partitions of $[n]$.

Example 2.3. Take $\mathcal{B}$ to be the class of connected graphs with labelled vertices, enumerated by number of vertices. Given $S \subseteq [n]$ and a graph G with $|S|$ vertices, labelled $1, \ldots, |S|$, let $\langle S, G \rangle$ denote the graph G with each label j replaced by s_j, where $S = \{s_1 < \cdots < s_{|S|}\}$. Replacing each pair (S_α, G_α) by $\langle S, G \rangle$, we have an interpretation of $\exp(\mathcal{B})_n$ as a collection of connected graphs whose labels are $1, \ldots, n$, each used exactly once. In other words, the exponential of the class of labelled connected graphs is the class of all labelled graphs.

The use of the word "exponential" for the combinatorial operation described above is justified by the following theorem; a proof may be found in [Pem09, Theorem 2.31] or see [Wil94, theorem 3.4.1].

Theorem 2.4 (exponential formula)**.** *Let $g(z)$ be the exponential generating function for the class $\mathcal{B}$, that is, $g(z) = \sum_{n=0}^{\infty} \frac{b_n}{n!} z^n$. Then $\exp(g(z))$ is the exponential generating function for $\exp(\mathcal{B})$ and $\exp(y\, g(z))$ is the* ***semi-exponential generating function*** *whose $y^k z^n$-coefficient is $1/n!$ times the number of elements of $\exp(\mathcal{B})_n$ with $|I| = k$.* □

To use the exponential formula, one needs exponential generating functions to input.

Example 2.5 (egf for permutations)**.** The number of permutations of $[n]$ is $n!$, so the exponential generating function for permutations is

$$f(z) = \sum_{n=0}^{\infty} \frac{n!}{n!} z^n = \frac{1}{1-z} \,. \tag{2.1}$$

Subtracting 1, the generating function for non-empty permutations is $z/(1-z)$.

Example 2.6 (egf for cycles)**.** A proportion of $1/n$ of permutations of size n are a single n-cycle. This follows, for instance, by computing recursively the probability that $\pi^k(1) = 1$ given $\pi^j(1) \neq 1 \,\forall\, j \leq k-1$. The exponential generating function for non-empty n-cycles is therefore

$$g(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n = \log\left(\frac{1}{1-z}\right) \,. \tag{2.2}$$

Applying the exponential formula to this example gives another way of looking at the previous example.

Example 2.7 (permutations by number of cycles)**.** A permutation is the commuting product of its cycles. Thus the class $\mathcal{A}$ of permutations is the exponential of the class $\mathcal{B}$ of non-empty cycles. By the exponential formula, the relation $f = \exp(g(z))$ holds between them, which agrees with (2.1) and (2.2): $1/(1-z) = \exp(\log(1/(1-z)))$. Enumerating permutations by cycle, we get the exponential generating function

$$F(y,z) = \exp\left(y \log \frac{1}{1-z}\right).$$

We may write this compactly as $\frac{1}{(1-z)^y}$, though this is not contentful since we have no definition of the y-power of a series other than exponentiation of y times the logarithm.

Example 2.8 (2-regular graphs)**.** A 2-regular graph is a graph with no self-edges in which every vertex has degree 2. A labelled 2-regular graph is the union of labelled, undirected cycles, whence the class of labelled 2-regular graphs is the exponential of the class of labelled undirected cycles. Let $\mathcal{A}$ denote this class. We do not allow parallel edges, so the cycles must have length at least 3. What is the number $a_n := |\mathcal{A}_n|$ of labelled 2-regular graphs on n vertices?

Every undirected cycle of length $n \geq 3$ corresponds to two directed cycles. Counting a permutation π as having weight $w = 2^{-N(\pi)}$ where $N(\pi)$ is the number of cycles, and letting p be the proportion of permutations having no short cycles (cycles of length less than 3), we see that $a_n = n!\, p\overline{w}$, where $\overline{w}$ is the average of w over permutations having no short cycles. It is known that $p = \Theta(1)$ and $N(\pi) \sim \log n$ for all but a vanishing proportion of permutations so it would seem likely that $a_n/n! = \Theta(2^{-\log n}) = \Theta(n^{-\log 2})$. This gives a rigorous lower bound: by convexity of 2^{-x}, the average of 2^{-N} over permutations with no short cycles is at least $2^{-\overline{N}}$. It takes a generating function, however, to correct this to a sharp estimate.

Let $u(z)$ is the exponential generating function for undirected cycles of length at least 3. By (2.2)

$$u(z) = \frac{1}{2}\left(\log \frac{1}{1-z} - z - \frac{z^2}{2}\right).$$

Applying the exponential formula shows that the exponential generating function for labelled 2-regular graphs is

$$\frac{e^{-\frac{1}{2}z - \frac{1}{4}z^2}}{\sqrt{1-z}}.$$

Methods in Section 2.3 convert this quickly into a good estimate.

2.2 Applications in probability

Branching processes

Rather than a rather abstract discussion of cominatorial operations associated with composition of generating functions, let us focus on a single example where probabilisitic analysis is always carried out via generating functions. Let $\phi(z) = \sum_{n=0}^{\infty} p_k z^k$ be the generating function for a probability distribution on $\mathbb{Z}^+$. Let Z_n be the size of the n^{th} generation of a Galton-Watson branching process beginning with a single individual and let $f_n(z) := \sum_{k=0}^{\infty} \mathbb{P}(Z_n = k) z^k$ be the generating function for Z_n. Thus $f_0(z) = z$ and $f_1(z) = \phi(z)$. Conditioning on Z_1 and observing that the conditional probability generating function for Z_n is $f_{n-1}(z)^{Z_1}$ shows that $f_n(z) = \phi(f_{n-1}(z))$. Induction recovers the well known fact that f_n is the n-fold iteration

$$f_n(z) = \phi(\cdots(\phi(z)))\,. \tag{2.3}$$

Proofs of all the basic results in the theory of branching processes are based on (2.3). Because ϕ and its derivatives are positive on $[0,1]$ and $\phi(1) = 1$, there is a $p \in (0,1)$ for which $\phi(p) = p$ if and only if $\phi'(1) > 1$. The quantity $\phi'(1) = \sum kp_k$ is the mean number of offspring. The extinction probability after n generations is $f_n(0)$ so the probability of eventual extinction is the increasing limit of iterations of ϕ. This is the least fixed point of ϕ in $[0,1]$, so it is equal to 1 if and only if the mean number of offspring is at most 1.

If $\phi'(1) = \mu < 1$, taking expectations shows the survival probability in generation n to be at most $[\phi'(1)]^n$, thus exponentially small. Suppose now that $\phi'(1) = 1$. Let $\psi(x) = 1 - \phi(1-x)$ so the n^{th} iterate $\psi^{(n)}(1)$ gives the n^{th} survival probability. Manipulating the Taylor series for ϕ we see that if the offspring distribution has a second moment then

$$\psi(x) = x - (c + o(1))x^2 \tag{2.4}$$

where $c = \phi''(1)/2$. Letting X be a sample from the offspring distribution, and letting σ^2 denote the offspring variance, we see because $\mathbb{E}X = 1$, that $\phi''(1) = \sum k(k-1)p_k = \mathbb{E}X^2 - X = \sigma^2$. Iterating (2.4), we see inductively that $n\psi^{(n)}(1)$ converges to $1/c$. Thus we have recovered a result from [AN72, 1.9.1]:

Theorem 2.9 (Yaglom's law)**.** *If $\phi'(1) = 1$ and $0 < \phi''(1) < \infty$ then the survival probability to the n^{th} generation is estimated by*

$$\mathbb{P}(Z_n > 0) \sim \frac{2}{n\phi''(1)}\,.$$

□

Lattice random walks

Let $\{p_{\mathbf{r}} : \mathbf{r} \in \mathbb{Z}^d\}$ be nonnegative numbers summing to 1 and let $\phi(\mathbf{z}) := \sum_{\mathbf{r}} p_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ be the associated generating function. The generating function for the n-step transition probabilities of the random walk with these increments is the n^{th} power, $\phi^n(\mathbf{z})$. The spacetime generating function generating function is

$$F(\mathbf{z}, y) := \sum_{\mathbf{r},n} p_n(\mathbf{r}) \mathbf{z}^{\mathbf{r}} y^n = \frac{1}{1 - y\phi(\mathbf{z})} .$$

Multiplication of Laurent series is not defined in $\mathbb{C}[[z_1, \ldots, z_d]]$, but because $\phi \in L^1$, the coefficients of ϕ^n are defined by convergent series. This allows $1/(1 - y\phi(\mathbf{z}))$ to be defined as well. The function ϕ is analytic in some poly-annulus only if the coefficients decay at least exponentially. Nevertheless, ϕ is always defined on the unit torus, which is sufficient for the application of classical Fourier techniques.

The great majority of the results in [Spi64] are proved using generating functions. One sees this also in [Fel68] and in the more recent [Law91]. In the next lecture, we will use generating function techniques to derive local large deviation and central limit results for lattice random walks with small tails. In the last lecture, we will study a quantum analogue of a lattice random walk (QRW). The QRW is defined on $\mathbb{Z}^d \times [k]$ where the second coordinate is an auxilliary state variable called **chirality** that can take one of k different values. In the quantum world, the transition matrix is unitary rather than stochastic. To specify a QRW requires a $k \times k$ unitary matrix U, along with k integer vectors $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(k)}$ specifying the possible increments of the QRW. Define a diagonal matrix M by

$$M_{ij} = \delta_{ij} \mathbf{x}^{\mathbf{v}^{(i)}} .$$

We will see that the spacetime generating function matrix F satisfies

$$F = (I - yMU)^{-1} .$$

This innocent looking generating function produces an asymptotic profile that is visually quite unexpected.

Random trees and graphs

The recursive definition of a binary tree is that it is either empty, or consists of a root together with an ordered pair of binary trees, respectively known as the left and right subtrees. Let $N(k)$ be the number of binary trees with k vertices; the numbers $N(k)$ are known as the **Catalan numbers** and count many other interesting classes. The recursion implies $N(k+1) = \sum_{j=0}^{k} N(j)N(k-j)$. This convolution identity, together with the boundary condition $N(0) = 1$, immediately implies the generating function identity

$$f(z) = 1 + zf(z)^2 \, . \tag{2.5}$$

Solving this explicitly using the quadratic formula yields

$$f = \frac{1 \pm \sqrt{1-4z}}{2z} \, .$$

It is obvious that the recursion has exactly one solution. This leads one to ponder whether the quadratic equation for f has two solutions, as it appears, or just one or none at all. In the ring $\mathbb{C}[[z_1, \ldots, z_d]]$ of formal power series, the square root operation is well defined (as long as $f_0 \neq 0$, which is true here). Division by $2z$, however, requires the leading term to be zero. We easily check that precisely one of the two series $1 \pm \sqrt{1-4z}$ has a vanishing leading term, whence (2.5) has the unique solution

$$f(z) = \frac{1 - \sqrt{1-4z}}{2z} = \frac{2}{1 + \sqrt{1-4z}} \, . \tag{2.6}$$

While the specification of the correct solution to (2.5) may appear somewhat *ad hoc*, the **kernel method** accomplishes this methodically and in any number of variables. Expositions of the kernel method may be found in [FS08, BMP00] as well as in [Pem09].

Families of trees and graphs defined by recursions often have nice generating functions. Many of these can be found in [FS08]. The complexity of the recursion and the resulting graphs can far outstrip what we saw in the binary tree example. For instance, in [BBMD$^+$02], the family $\mathcal{F}$ of **maps** are considered, where a map is a planar graph, rooted at a distinguished directed edge, and two are equivalent if their planar embeddings are topologically equivalent. For $G \in \mathcal{F}$, denote the number of vertices by $N(G)$ and let $C(G)$ denote the size of the **core**, namely the largest 2-connected rooted subgraph. The generating function $M(z) := \sum_G z^{N(G)}$ couting maps by size is algebraic, as is the generating function counting all non-equivalent cores by size. Furthermore, the number of maps of size n having core size k is shown in [BBMD$^+$02] to be given by Lagrange's inversion formula, leading to a limit law for the core size of a uniformly chosen map of n vertices.

2.3 Coefficient extraction and transfer theorems

In Example 1.5, we saw how knowledge of an explicit generating function may lead to an asymptotic formula for the coefficients a_n. The ability to transfer knowledge about a generating function to estimates on its coefficients is very handy because we often have a nice description of a generating function and no independent means of producing asymptotic formulae. In one variable, this is relatively well understood. Example 1.5 is a special case of an **admissible function**. This class, defined in the original work of Hayman [Hay56], satisfies a general theorem on coefficient asymptotics which may be proved along the lines of the analysis in Example 1.5.

Hayman's method works well for entire functions or functions with iosolated singularities, such as the exponential generating function $\exp(z/(1-z))$ for order set partitions. Asymptotics are given in terms of a sequence of numbers a_n that must be computed when checking the definition of admissibility. The first combinatorial application of this method appears to have been by Hardy and Ramanujan [HR17], though Hayman's was the first general theorem. A great number of results have since been proved that extend the scope of the method, my favorite among which are the **transfer theorems** of Flajolet and Odlyzko [FO90]. The contribution of Flajolet and Odlyzko was not only to extend the scope of the method to functions with **algebraic- logarithmic** singularities, but more importantly, to "commute" asymptotic information from the hypotheses to the conclusions. The usual regularity assumptions are therefore not required: any estimate, sharp or crude, on f near its dominant singularity leads to a correspondingly sharp or crude estimate on the coefficients of f. The rest of this section details Flajolet and Odlyzko's transfer theorem and discusses a few applications.

To state the main theorem of [FO90], we define the class alg-log to be the class of functions that are a product of a power of $R-z$, a power of $\log(1/(R-z))$ and a power of $\log\log(1/(R-z))$. The first step is to give asymptotics for all functions in the class alg-log. The reader is referred to [FO90] for the proof of the following lemma.

Lemma 2.10 ([FO90, Theorem 3B]). *Let α, γ and δ be any complex numbers other than nonnegative integers and let*

$$f(z) = (1-z)^\alpha \left(\frac{1}{z}\log\frac{1}{1-z}\right)^\gamma \left(\frac{1}{z}\log\left(\frac{1}{z}\log\frac{1}{1-z}\right)\right)^\delta .$$

Then the Taylor coefficients $\{a_n\}$ of f satisfy

$$a_n \sim \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}(\log n)^\gamma (\log\log n)^\delta .$$

□

Remark 2.11. When α, γ or δ is a nonnegative integer, different formulæ hold. For example, for the case $\alpha \in \mathbb{Z}^+, \gamma \notin \mathbb{Z}^+, \delta = 0$, the estimate

$$a_n \sim Cn^{-\alpha-1}(\log n)^{\gamma-1} \tag{2.7}$$

is known: the coincidence of α with a nonnegative integer causes an extra log in the denominator.

Given a positive real R and an $\epsilon \in (0, \pi/2)$, the so-called **Camembert-shaped region**,

$$\{z : |z| < R + \epsilon, z \neq R, |\arg(z - R)| \geq \pi/2 - \epsilon\},$$

denoted $\Delta(R, \epsilon)$, is shown in figure 2.1.

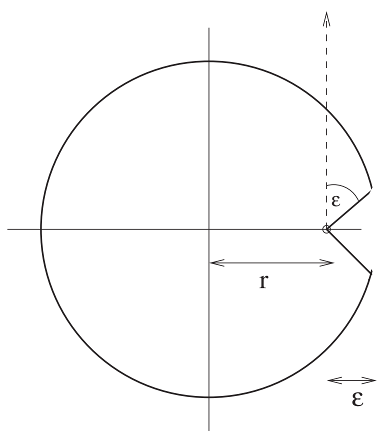

Figure 2.1: a Camembert-shaped region

Theorem 2.12 (Transfer Theorem)**.** *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be analytic in a Camembert-shaped region* $\Delta(R, \epsilon)$*. If* $g(z) = \sum_{n=0}^{\infty} b_n z^n \in$ alg-log*, then the following hold.*

(i)

$$f(z) = O(g(z)) \Rightarrow a_n = O(b_n);$$

(ii)

$$f(z) = o(g(z)) \Rightarrow a_n = o(b_n);$$

(iii)

$$f(z) \sim g(z) \Rightarrow a_n \sim b_n$$

Discussion: One may refer to [FO90] for the full proof, or to [Pem09] for a proof in the case of algebraic singularities, which illustrates the argument in [FO90] relatively painlessly. Here, I will give the ideas but cut out all computational details.

The main work is in proving the big-O part. The hypothesis $f = O(g)$ as $z \to R$ implies $|f| \le Kg$ on the whole region Δ. The engine of the proof is a four-piece contour $\gamma_n \subseteq \Delta$, such that the Cauchy integral over γ_n may be bounded above when f is replaced by $|f|$. This leads to the upper bound

$$a_n \le \theta(K)\, b_n$$

which finishes the proof of the big-O part. The little-o part follows as well, once one observes that $\theta(K) \to 0$ as $K \to 0$. Finally, the asymptotic statement follows from the lemma and little-o statement: $f = g + o(g)$ implies that $a_n = b_n + o(b_n)$. □

Example 2.13 (Catalan numbers)**.** Let $a_n := \frac{1}{n+1}\binom{2n}{n}$ be the n^{th} Catalan number. Recall from (2.6) that the generating function for the Catalan numbers is given by $f(z) = (1-\sqrt{1-4z})/2z$. The function f has an algebraic singularity at $r = 1/4$, near which the asymptotic expansion for f begins

$$f(z) = 2 - 4\sqrt{\frac{1}{4} - z} + 8(\frac{1}{4} - z) - 16(\frac{1}{4} - z)^{3/2} + O(\frac{1}{4} - z)^2 .$$

Observing that $(1-z)$ do not contribute to the asymptotics, Theorem 2.12 gives

$$\begin{aligned} a_n &\sim \left(\frac{1}{4}\right)^{1/2-n} n^{-3/2}\frac{-4}{\Gamma(-1/2)} + \left(\frac{1}{4}\right)^{3/2-n} n^{-5/2}\frac{-16}{\Gamma(-3/2)} + O(n^{-7/2}) \\ &= 4^n n^{-3/2}\frac{(-4)(\frac{1}{4})^{1/2}}{\Gamma(-1/2)} + 4^n n^{-5/2}\frac{(-16)(\frac{1}{4})^{3/2}}{\Gamma(-3/2)} + O(n^{-7/2}) \\ &= 4^n\left(\frac{n^{-3/2}}{\sqrt{\pi}} - n^{-5/2}\frac{3}{2\sqrt{\pi}} + O(n^{-7/2})\right) . \end{aligned}$$

Example 2.14 (cycle-weighted permutations)**.** Let $N(\pi)$ denote the number of cycles in the permutation π. For $\sigma > 0$, define the weight of π by $w(\pi) := \sigma^{N(\pi)}$ and let $W_n := \sum_{\pi\in S_n} w(\pi)$ be the total weight of permutations in S_n. Let

$$f_\sigma(z) := 1 + \sum_{n=1}^{\infty}\frac{W_n}{n!}z^n$$

be the generating function for total weight. Recalling from Example 2.7 the bivariate generating function

$$F(y, z) = \sum_n \frac{1}{n!} \sum_{\pi \in S_n} y^{N(\pi)} z^n$$

for cycles by size of ground set and number of cycles, we see trivially that

$$f_\sigma(z) = F(\sigma, z) = \exp\left(\sigma \log \frac{1}{1-z}\right) .$$

When σ is a positive real number, we can interpret this as the branch of $(1-z)^{-\sigma}$ that is analytic in a neighborhood of the origin. This leads to the exact formula

$$\frac{W_n}{n!} = (-1)^n \binom{-\sigma}{n} \sim \Gamma(\sigma) n^{\sigma - 1} .$$

Example 2.15 (permutations with restricted cycles)**.** The preprint [Lug09], counts unweighted permutations in which all cycle lengths be in a fixed infinite set S. This is equivalent to weighting cycles of length n by $\mathbf{1}_{n \in S}$. If S has asymptotic density σ then the exponential generating function g for cycles with length in S satisfies

$$g(z) \sim \sigma \log\left(\frac{1}{1-z}\right)$$

as $z \to 1$. The same argument as in the previous example then gives

$$f_S(z) = (1-z)^{-\sigma + o(1)}$$

as $z \to R := 1$ within a camembert-shaped region, where f_S counts permutations with cycles in the allowed set S, by size of ground set. Applying the Flajolet-Odlyzko transfer theorem to $(1-z)^{\sigma+\epsilon}$, we find that

$$\frac{\log W_n}{\log n} \to \sigma - 1 .$$

Lugo is interested in computing a limit law for the length L of the cycle containing the element 1 in a uniformly chosen permutation with cycle lengths in S. The logarithmic estimate is not quite sharp enough, but under the regularity assumption $g(z) = \sigma \log(1/(1-z)) + K + o(1)$, Lugo is able to prove that L/n converges to the disribution whose cdf is $1 - (1-x)^\sigma$.