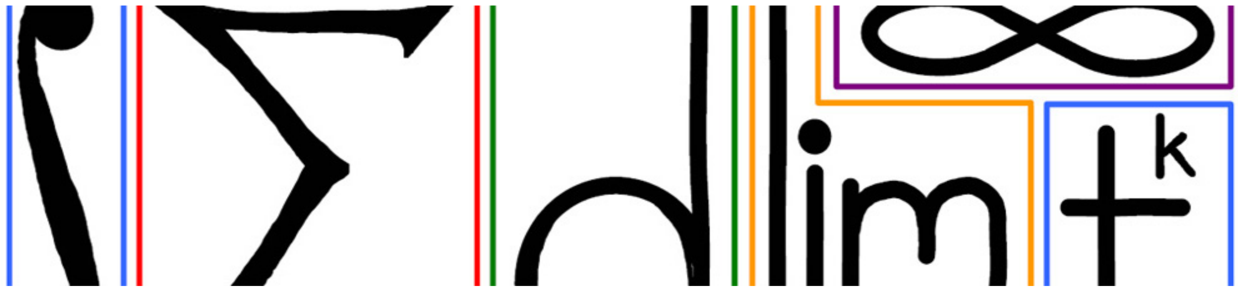
The Penn Calc Companion

Part III: Discrete Calculus

About this Document

This material is taken from the wiki-format Penn Calc Wiki, created to accompany Robert Ghrist's Single-Variable Calculus class, as presented on Coursera beginning January 2013. Past editors and contributors include: Prof. Robert Ghrist, Mr. David Lonoff, Dr. Subhrajit Bhattachayra, Dr. Alberto Garcia-Raboso, Dr. Vidit Nanda, Ms. Lee Ling Tan, Mr. Brett Bernstein, Prof. Antonella Grassi, and Prof. Dennis DeTurck.

Contents



45 Sequences

The remainder of the course is a look at discrete calculus, which is a study of all the previous sections (functions, derivatives, integrals) applied to a different kind of function: sequences. A sequence is a function, but instead of taking any real number as input, a sequence takes an integer as input.

45.1 Sequence

A *sequence* a is a function from the non-negative integers $0, 1, 2, 3, \ldots$ to the real numbers $\mathbb{R}$. The usual functional notation $a(n)$ is sometimes replaced with a_n. There are several ways to define a sequence. Here are three of the most common ways, demonstrated on the powers of 2.

1. An *explicit formula* gives a_n as a function of n, i.e. $a_n = f(n)$. This is usually the most convenient, since it typically gives the most information about the sequence. e.g. $a_n = 2^n$ for $n \geq 0$.
2. A *recursion relation* gives a_n as a function of previous terms in the sequence. Note that some initial conditions must be given as well. $a_n = f(a_{n-1}, a_{n-2}, \ldots, a_{n-k})$. e.g. $a_n = 2a_{n-1}$; $a_0 = 1$.
3. Finally, *listing terms* can be used if no explicit or recursive formula is available. This is sometimes used in experimental settings so that one can study the terms and look for a pattern. e.g. $a = (1, 2, 4, 8, 16, 32, \ldots)$.

Example

Write out the first six terms of the sequence defined by $a_n = 2a_{n-1} + 1$; $a_0 = 0$. Look for a pattern to try to find an explicit formula for a_n. (See Answer 1)

45.2 Limits of sequences

Recall that limits of functions came in two flavors. First, there were limits of the form $\lim_{x \to c} f(x)$. The second type of limits were of the form $\lim_{x \to \infty} f(x)$.

Only the second type of limit is sensible for a sequence. (To see why the first type of limit does not make sense for sequences, go back to the definition of a limit.) The definition of $\lim_{n \to \infty} a_n$ for sequences is the same as for continuous functions:

The Limit of a Sequence

We say that

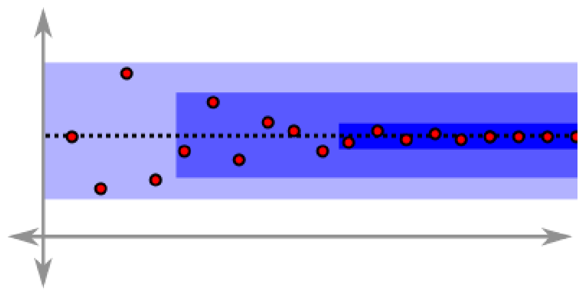
$$\lim_{n\to\infty} a_n = L$$

if for any $\epsilon > 0$ there exists M such that for all $n > M$,

$$|a_n - L| < \epsilon.$$

In other words, the sequence a_n has limit L if a_n gets arbitrarily close to L for sufficiently large n. If the limit L exists, then the sequence a_n is said to *converge* to L.

Intuitively, a sequence a_n has a limit if for any band around L, there is some point where all the terms of a_n are within the band around L:



Recall that Newton's method defined a sequence of numbers defined by the recursion relation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Here, the sequence *hopefully* converged to a root of the function $f(x)$. This gives an example of an application where it is useful to know about the convergence of a sequence.

Example

Let $a_n = 4 + \frac{(-1)^n}{n}$. Find $\lim_{n\to\infty} a_n$, if the limit exists. If the limit does not exist, explain why. (See Answer 2)

Example

Let $a_n = (-1)^n$. Find $\lim_{n\to\infty} a_n$, if the limit exists. If the limit does not exist, explain why. (See Answer 3)

45.3 Methods for computing limits

Many of the methods for computing limits of continuous functions carry over to computing limits of sequences. In particular, all of the big-O notation still applies.

Example

Compute the limit of the sequence $a_n = 3n - \sqrt{9n^2 + 6n}$. (See Answer 4)

45.4 Monotone, bounded sequences

In general, it can be difficult to find the limit of a sequence, but for certain sequences it is possible to prove that the limit exists.

A sequence is *monotone increasing* if it is non-decreasing, i.e., $a_0 \leq a_1 \leq a_2 \leq \ldots$. A sequence is *monotone decreasing* if it is non-increasing, i.e., $a_0 \geq a_1 \geq a_2 \geq \ldots$. A sequence that is either monotone increasing or monotone decreasing is *monotone*.

A sequence is *bounded above* if there exists some real number B such that $a_n \leq B$ for all $n \geq 0$. Similarly, a sequence is *bounded below* if there exists a real number C such that $a_n \geq C$ for all $n \geq 0$. A sequence is *bounded* if it is both bounded above and bounded below.

Monotone Convergence Theorem

If a sequence a_n is bounded and monotone, then the sequence converges.

Example

Let a_n be the sequence defined by $a_n = \frac{5+a_{n-1}}{2}$; $a_0 = 1$. Show that the sequence converges by using the Monotone Convergence Theorem. (See Answer 5)

45.5 Recursion relations and limits

When a sequence is defined by a recursion relation and the limit of the sequence exists, one can find the limit by simply taking the limit of both sides of the recursion relation and solving. This is best demonstrated by example.

Example

Find the limit $L = \lim_{n\to\infty} a_n$, where $a_n = \frac{5+a_{n-1}}{2}$; $a_0 = 1$, as in the previous example. (See Answer 6)

Example

Find

$$L = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1+\ldots}}}$$

by expressing L as a limit of a recursively defined sequence a_n which begins $(1, 1 + \frac{1}{1}, 1 + \frac{1}{2}, \ldots)$. Assume that the limit exists. (See Answer 7)

Example

Find $\sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}$. Assume the limit exists. (See Answer 8)

The golden ratio, often denoted

$$\varphi = \frac{1+\sqrt{5}}{2},$$

appears in many settings, both man made and natural. The golden ratio is also deeply connected with the Fibonacci numbers, as we will see in this example and in the future.

Example

The *Fibonacci sequence*, F_n is defined by the recursion relation and initial conditions

$$F_{n+2} = F_{n+1} + F_n; \quad F_0 = 0, F_1 = 1.$$

So the first few Fibonacci numbers are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

Show that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi.$$

You may assume that the limit of the sequence exists. Hint: Observe that

$$\frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}.$$

(See Answer 9)

45.6 EXERCISES

- Compute the limit of the sequence $a_n = 2n^2 - (8n^6 + 6n^4)^{1/3}$.

45.7 Answers to Selected Examples

1. $$\begin{aligned} a_0 &= 0 \\ a_1 &= 2a_0 + 1 = 1 \\ a_2 &= 2a_1 + 1 = 3 \\ a_3 &= 2a_2 + 1 = 7 \\ a_4 &= 2a_3 + 1 = 15 \\ a_5 &= 2a_4 + 1 = 31. \end{aligned}$$

   One might notice that adding 1 to each term in the sequence gives the sequence $(1, 2, 4, 8, 16, 32, \ldots)$, which look like the powers of 2. So it appears that $a_n = 2^n - 1$ for $n \geq 0$.

   (Return)

2. We claim the limit is 4. We know that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore, for any $\epsilon > 0$ we can choose M so that $\frac{1}{M} < \epsilon$. Then for any $n > M$, we have

$$\begin{aligned}|a_n - 4| &= \left|4 + \frac{(-1)^n}{n} - 4\right| \\ &= \left|\frac{(-1)^n}{n}\right| \\ &= \frac{1}{n} \\ &< \frac{1}{M} \\ &< \epsilon,\end{aligned}$$

as desired. For this course, we will not typically be this formal, but it is useful to see this type of argument at least a few times.

(Return)

3. From the intuitive understanding of a limit, it is clear that the terms of this sequence are not getting closer together, and so the limit does not exist.

   More formally, suppose the limit, say L, existed. Then from the definition of the limit of a sequence, we could find a number M such that

   $$|a_n - L| < \frac{1}{3}$$

   for all $n > M$. Since the terms of the sequence are -1 and 1, this would imply

   $$|-1 - L| < \frac{1}{3}$$

   This implies

   $$\begin{aligned}-\frac{1}{3} &< -1 - L < \frac{1}{3} \\ \frac{2}{3} &< -L < \frac{4}{3} \\ -\frac{2}{3} &> L > \frac{-4}{3}.\end{aligned}$$

   Similarly,

   $$|1 - L| < \frac{1}{3}$$

   which implies by similar algebra that

   $$\frac{4}{3} > L > \frac{2}{3}.$$

   Since L cannot be simultaneously positive and negative, we have reached a contradiction. And so we see that the limit L cannot exist.

   (Return)

4. A little factoring from the radical, and using the binomial series gives

$$\begin{aligned} a_n &= 3n - \sqrt{9n^2 + 6n} \\ &= 3n - 3n\sqrt{1 + \frac{2}{3n}} \\ &= 3n\left(1 - \sqrt{1 + \frac{2}{3n}}\right) \\ &= 3n\left(1 - (1 + \frac{1}{2}\frac{2}{3n} + O(1/n^2))\right) \\ &= -1 + O\left(\frac{1}{n}\right). \end{aligned}$$

   So the limit is -1.

   (Return)

5. It is not obvious that this sequence is either bounded or monotone. Writing out the first few terms, though, gives $a = (1, 3, 4, 4.5, 4.75, \ldots)$. It appears that the sequence is increasing, and that $a_n \leq 5$ for all $n \geq 0$.

   To show that the sequence is increasing, use induction. The first few terms are increasing, so assume that $a_{n-1} \leq a_n$ for some n. Then adding 5 and dividing by 2 throughout gives $\frac{5+a_{n-1}}{2} \leq \frac{5+a_n}{2}$, which implies $a_n \leq a_{n+1}$ by the recursive definition of a_n. Thus, the sequence is increasing.

   To see that $a_n \leq 5$, again use induction. Assume $a_n \leq 5$ for some n. Then adding 5 and dividing by 2 gives $\frac{5+a_n}{2} \leq \frac{5+5}{2} = 5$. This means $a_{n+1} \leq 5$ by the definition of a_n. Thus, $a_n \leq 5$ for all n. Finally, note that any increasing sequence is bounded below. So the sequence is bounded.

   Thus, by the Monotone Convergence Theorem, the sequence a_n converges.

   (Return)

6. The limit exists by the previous example, so taking limits of both sides of the recursion relation and simplifying gives

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{5 + a_{n-1}}{2} \\ L &= \frac{5 + \lim_{n\to\infty} a_{n-1}}{2} \\ L &= \frac{5 + L}{2}. \end{aligned}$$

   (Note that $\lim a_{n-1} = \lim a_n$ because the limit of a sequence does not depend on the indexing of the terms.) Now, solving for L gives $2L = 5 + L$, so $L = 5$.

   (Return)

7. The sequence a_n can be defined recursively by $a_n = 1 + \frac{1}{a_{n-1}}$; $a_0 = 1$. Then taking limits of both sides gives $L = 1 + \frac{1}{L}$. Multiplying through and collecting terms gives $L^2 - L - 1 = 0$, and solving for L gives $L = \frac{1\pm\sqrt{5}}{2}$.

   Note though that $a_n > 0$ for all n, so it must be that $L = \frac{1+\sqrt{5}}{2}$. This is the celebrated *golden ratio*.

   (Return)

8. One can express this as the limit of a recursively defined sequence a_n given by $a_0 = 1$ and $a_n = \sqrt{1 + a_{n-1}}$. Then letting $L = \lim a_n$, and taking limits of the recursion relation gives

   $$L = \sqrt{1 + L}.$$

   Squaring both sides and collecting terms gives $L^2 - L - 1 = 0$, which is the same equation from the previous example. Thus, the limit is again the golden ratio $L = \frac{1+\sqrt{5}}{2}$.

   (Return)

9. Let

   $$L = \lim_{n\to\infty} \frac{F_{n+1}}{F_n}.$$

   Then, beginning with the hint and taking the limit of both sides, we have

   $$\begin{aligned} \lim_{n\to\infty} \frac{F_{n+1}}{F_n} &= \lim_{n\to\infty} \left(1 + \frac{F_{n-1}}{F_n}\right) \\ &= 1 + \lim_{n\to\infty} \frac{F_{n-1}}{F_n} \\ &= 1 + \lim_{n\to\infty} \frac{1}{F_n/F_{n-1}} \\ &= 1 + \frac{1}{\lim\limits_{n\to\infty} F_n/F_{n-1}}. \end{aligned}$$

   But this means

   $$L = 1 + \frac{1}{L},$$

   which is an equation which we solved in an above example, where we found

   $$L = \frac{1 + \sqrt{5}}{2},$$

   as desired.

   (Return)

46 Differences

What is the derivative of a sequence? The original definition will not work because change in input is discrete and so one cannot take the limit as the change in input goes to 0. Instead, using the interpretation of the derivative as a rate of change leads to two different discrete derivatives. They are called *difference operators*, and are defined below.

46.1 Difference operators

The discrete analog of the derivative is the difference operator, defined as follows.

Difference operators

Given a sequence a_n, the *forward difference of a*, denoted $(\Delta a)_n$, is defined by

$$(\Delta a)_n = a_{n+1} - a_n.$$

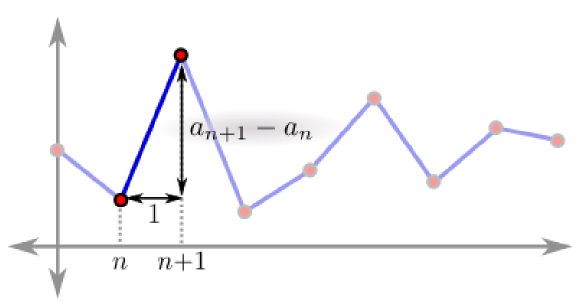
The *backward difference of a*, denoted by $(\nabla a)_n$, is defined by

$$(\nabla a)_n = a_n - a_{n-1}.$$

This can be interpreted as the change in output over the change in input:

$$(\Delta a)_n = \frac{a(n+1) - a(n)}{(n+1) - n} = \frac{a_{n+1} - a_n}{1} = a_{n+1} - a_n,$$

which resembles the definition for the derivative of a continuous function, but without the limit. If we plot the points of the sequence as if we were graphing the function, and then connect the dots with linear segments, then the forward difference can also be interpreted as the slope between adjacent points:



Example

The sequence $(4n) = 0, 4, 8, 12, 16, \ldots$ has forward difference sequence

$$\Delta(4n) = 4, 4, 4, 4, \ldots$$

Example

Find the forward difference sequence of the Fibonacci sequence $F = 0, 1, 1, 2, 3, 5, 8, 13, 21 \ldots$. (See Answer 1)

Example

Find the forward difference of the powers of two: $(2^n) = 1, 2, 4, 8, 16, 32, \ldots$. (See Answer 2)

Product rules

These difference operators have their own versions of the differentiation rules for continuous functions. For example, there is a product rule. For sequences $a = (a_n)$ and $b = (b_n)$, define the new sequence (ab) by $(ab)_n = (a_n b_n)$. Then

$$\begin{aligned}\Delta(ab)_n &= a_n \Delta b_n + b_n \Delta a_n + \Delta a_n \Delta b_n\\ \nabla(ab)_n &= a_n \nabla b_n + b_n \nabla a_n - \nabla a_n \nabla b_n.\end{aligned}$$

This should be reminiscent of the product rule $(fg)' = fg' + f'g$.

46.2 Higher order difference operators

Just as the derivative of a function gives another function, the difference operator of a sequence a gives another sequence Δa. Then one can take the difference operator of Δa which gives yet another sequence $\Delta^2 a$. And so on. Consider, for instance, the sequence $a_n = n^3$ for $n \geq 0$. Then

$$\begin{array}{rcccccccccc} a & = & 0 & 1 & 8 & 27 & 64 & 125 & 216 & 343 & \ldots \\ \Delta a & = & 1 & 7 & 19 & 37 & 61 & 91 & 127 & \ldots & \\ \Delta^2 a & = & 6 & 12 & 18 & 24 & 30 & 36 & \ldots & & \\ \Delta^3 a & = & 6 & 6 & 6 & 6 & 6 & \ldots & & & \\ \Delta^4 a & = & 0 & 0 & 0 & 0 & \ldots & & & & \end{array}$$

A little bit of pattern matching shows the following:

$$\begin{aligned} a_n &= n^3\\ (\Delta a)_n &= 3n^2 + 3n + 1\\ (\Delta^2 a)_n &= 6n + 6\\ (\Delta^3 a)_n &= 6\\ (\Delta^4 a)_n &= 0\end{aligned}$$

Just as taking higher derivatives of a polynomial eventually gives 0, taking higher order difference operators of a polynomial eventually gives 0. Moreover, if a is a polynomial of degree p, then $\Delta^{p+1} a = (0)$.

Note that the power rule is not quite the same as for regular derivatives (e.g. $n^3 \mapsto 3n^2 + 3n + 1$). This is an artifact of the binomial expansion. The next section shows a convenient way to avoid these problems.

46.3 Falling powers

The *falling power* $n^{\underline{k}}$ is defined to be

$$n^{\underline{k}} = n(n-1)(n-2)\cdots(n-k+1), \qquad n^{\underline{0}} = 1.$$

The falling power can be thought of as a discrete version of the monomial x^k. One nice feature of the falling power is that

$$\begin{aligned}
\Delta n^{\underline{k}} &= (n+1)^{\underline{k}} - n^{\underline{k}} \\
&= (n+1)n(n-1)\cdots(n-k+2) - n(n-1)\cdots(n-k+1) \\
&= n(n-1)\cdots(n-k+2)\,(n+1-(n-k+1)) \\
&= kn^{\underline{k-1}},
\end{aligned}$$

which is the familiar power rule.

Example

Express n^3 in terms of falling factorials and use the power rule above to find Δn^3. (See Answer 3)

46.4 Discrete e

With the falling power in hand, consider the discretized version of the exponential function, found by replacing the usual monomials x^k with $n^{\underline{k}}$ in the Taylor series for the exponential:

$$\sum_{k=0}^{\infty} \frac{n^{\underline{k}}}{k!} = 1 + n + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \cdots$$

If one evaluates this at $n = 1$ to find the "discrete e", note that all the terms after the first two disappear because of the $n-1$ factor in all the higher terms. Thus, the discrete version of e is 2. This is consistent with the earlier note that 2^n acted like the exponential function (it is its own discrete derivative). Indeed, the above series is 2^n, which can be seen by noting it is simply the binomial series $(1+x)^n$ evaluated at $x = 1$.

46.5 Sequence operators

Just as there were operators on functions (e.g. integration, differentiation, logarithm, exponentiation), there are operators on sequences:

Operator	Notation	What it does	Notes
Identity	I	$(Ia)_n = a_n$	$I^2 = I$
Left shift	E	$(Ea)_n = a_{n+1}$	E^2 shifts twice, etc.
Right shift	E^{-1}	$(E^{-1}a)_n = a_{n-1}$	$E^{-1}E = I$
Forward difference	Δ	$(\Delta a)_n = a_{n+1} - a_n$	$\Delta = E - I$
Backward difference	∇	$(\nabla a)_n = a_n - a_{n-1}$	$\nabla = I - E^{-1}$

Higher derivatives

The expressions for the forward and backward differences in terms of I and E can be used to compute the higher derivatives of sequences more easily than it would be to compute them by hand. For example, the third derivative can be found by expanding the expression as a binomial:

$$\begin{aligned}\Delta^3 a &= (E-I)^3 a \\ &= (E^3 - 3E^2 I + 3EI^2 - I^3)a \\ &= (a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n).\end{aligned}$$

More generally, the kth derivative can be written similarly:

$$\begin{aligned}\Delta^k &= (E-I)^k \\ &= \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} E^i\end{aligned}$$

Indefinite integral

As shown above, the forward difference can be expressed in terms of the operators E and I:

$$\Delta = E - I.$$

A logical question is to ask whether it is possible to take the anti-difference, just as there is the antiderivative for functions. The claim is that

$$\Delta^{-1} = -(I + E + E^2 + E^3 + E^4 + \cdots)$$

This looks reminiscent of the geometric series. By taking the inverse of Δ (and a few liberties with the algebra), we have

$$\begin{aligned}\Delta^{-1} &= (E-I)^{-1} \\ &= -(I-E)^{-1} \\ &= -(I + E + E^2 + E^3 + \cdots),\end{aligned}$$

by thinking of $(I-E)^{-1}$ as the reciprocal of $1-E$ in a sense (because I acts like 1), and applying the geometric series. This derivation is not rigorous, but the above formula does give the anti-difference (up to an additive constant), so long as the sequence a_n is eventually 0.

This is just a hint of the calculus of sequences, which we explore a little bit more in the modules to come.

46.6 Answers to Selected Examples

1. The difference sequence is

   $$\Delta F = 1, 0, 1, 1, 2, 3, 5, 8, \ldots$$

   Note that this is just the Fibonacci sequence again, but slightly shifted. This makes sense since the difference

   $$(\Delta F)_n = F_{n+1} - F_n = F_{n-1},$$

   by rearranging the Fibonacci recursion relation.

   (Return)

2. The difference sequence is

$$\Delta(2^n) = 1, 2, 4, 8, 16, 32, \ldots$$

   This shows that 2^n can be thought of as the discrete analog of the exponential e^x, in the sense that it is its own (discrete) derivative.

   (Return)

3. This requires a little bit of algebra. We know that we need a $n^{\underline{3}}$ to get a n^3. Note that

$$n^{\underline{3}} = n(n-1)(n-2) = n^3 - 3n^2 + 2n.$$

   So we have our n^3, but we also have some unwanted lower order terms. We must add $3n^{\underline{2}}$ to both sides to cancel the $-3n^2$. Note that

$$n^{\underline{2}} = n(n-1) = n^2 - n,$$

   and so

$$\begin{aligned} n^{\underline{3}} + 3n^{\underline{2}} &= (n^3 - 3n^2 + 2n) + 3(n^2 - n) \\ &= n^3 - n. \end{aligned}$$

   So to get rid of the final term of $-n$, we note that $n^{\underline{1}} = n$, and so we can add this to both sides to find

$$n^{\underline{3}} + 3n^{\underline{2}} + n^{\underline{1}} = n^3.$$

   This tells us that

$$\begin{aligned} \Delta n^3 &= \Delta\left(n^{\underline{3}} + 3n^{\underline{2}} + n^{\underline{1}}\right) \\ &= 3n^{\underline{2}} + 6n^{\underline{1}} + 1 \\ &= 3(n^2 - n) + 6n + 1 \\ &= 3n^2 + 3n + 1, \end{aligned}$$

   which matches the observation about $\Delta(n^3)$ in the above section.

   (Return)

47 Discrete Calculus

The previous two modules have laid the groundwork for the discretization, or digitization, of calculus:

1. The discrete version of a function is a sequence.
2. The discrete version of the derivative is the difference operator.
3. The discrete version of the integral is the sum.
4. The discrete version of a differential equation is a recurrence relation.

This module mostly deals with #3, the integral of a discrete function. For continuous functions, there was the Fundamental Theorem of Integral Calculus which made computing integrals easy under certain conditions. Essentially, it said that the integral of the derivative is the function itself, evaluated at the endpoints. The discrete version says the same thing:

The Discrete Fundamental Theorem of Integral Calculus (FTIC)

Given a sequence u,

$$\sum_{n=A}^{B} \Delta u = u\Big|_{n=A}^{B+1},$$

and

$$\sum_{n=A}^{B} \nabla u = u\Big|_{n=A-1}^{B}$$

(See Answer 1)

Example

Note that

$$\Delta\frac{1}{n} = \frac{1}{n+1} - \frac{1}{n} = \frac{-1}{n^2+n}.$$

Use this, along with FTIC, to find $\displaystyle\sum_{n=A}^{B} \frac{1}{n^2+n}$. (See Answer 2)

Example

Use the fact that

$$\Delta n! = (n+1)! - n! = (n+1)n! - n! = (n+1-1)n! = n \cdot n!,$$

along with the FTIC to find

$$\sum_{n=A}^{B} n!n$$

(See Answer 3)

Example

Let F denote the Fibonacci sequence defined by

$$F_{n+2} = F_{n+1} + F_n; \quad F_0 = 0, F_1 = 1.$$

Note that

$$\Delta(F_{n+1}) = F_{n+2} - F_{n+1} = F_n,$$

by rearranging the above recursion relation. Use this, along with the FTIC, to find

$$\sum_{n=1}^{k} F_n.$$

(See Answer 4)

Power rule for falling powers

Recall from the previous module that the falling power

$$n^{\underline{k}} = n(n-1)(n-2)\cdots(n-k+1)$$

has a nice power rule for the difference:

$$\Delta(n^{\underline{k}}) = kn^{\underline{k-1}}.$$

By running this in reverse, we can find the anti-difference (or antiderivative) of the falling power, which is very similar to the power rule for integration:

$$\Delta^{-1}(n^{\underline{k}}) = \frac{1}{k+1}n^{\underline{k+1}} + C,$$

where C is a constant. Using this, along with the FTIC, allows one to find closed formulas for the sum of polynomials.

Example

Using the power rule and the FTIC, find

$$\sum_{n=1}^{k} n.$$

(See Answer 5)

Example

Find
$$\sum_{n=1}^{k} n^2.$$
(See Answer 6)

Example

With a little algebra (shown in the previous module), one finds that
$$n^3 = n^{\underline{3}} + 3n^{\underline{2}} + n^{\underline{1}}.$$
Use this fact, along with FTIC, to find
$$\sum_{n=1}^{k} n^3$$
(See Answer 7)

47.1 Integration by parts

There is also a discrete version of integration by parts. First, the following product rule can be established for the forward difference:
$$\Delta(uv) = u\Delta v + Ev\Delta u.$$
(See Details 8) Then, just like for continuous functions, one can integrate both sides (i.e. sum both sides), rearrange, and apply FTIC to find
$$\sum_{n=A}^{B} u\Delta v = uv\Big|_{n=A}^{B+1} - \sum_{n=A}^{B} Ev\Delta u.$$

Example

Find $\sum_{n=0}^{k} n2^n$. (See Answer 9)

47.2 Differential equations

The discrete version of a differential equation is a recurrence relation. This is an equation relating one term of a sequence u with one or more previous terms in the sequence. In this context, the shift operator acts like the derivative.

Example

Consider the linear first-order recurrence relation
$$u_{n+1} = \lambda u_n,$$
which can be written as
$$Eu = \lambda u$$

or, with some rearrangement,

$$(E - \lambda I)u = 0.$$

By inspection, one can see the solution $u_n = C\lambda^n$, where $C = u_0$ is some constant (it can be thought of as an initial condition).

Example

Now, consider using the following linear first-order difference equation

$$\Delta u = \lambda u.$$

This is reminiscent of the differential equation $\frac{dx}{dt} = ax$, considered earlier in the course. Recall that $\Delta = E - I$, and so this difference relation can be written

$$(E - I)u = \lambda u.$$

With a little more rearranging one finds

$$(E - (\lambda + 1)I)u = 0,$$

which is almost identical to the equation from the previous example, but with λ replaced by $\lambda + 1$. Therefore, the solution is $u_n = C(\lambda + 1)^n$.

Compare this to the solution in the continuous case, which was

$$\frac{dx}{dt} = \lambda x \quad \Longrightarrow \quad x = Ce^{\lambda t}.$$

When $\lambda = 1$, we have the solution

$$x = Ce^t.$$

If we let $\lambda = 1$ in the discrete difference equation, we have

$$u = C(\lambda + 1)^n = C \cdot 2^n,$$

which again shows that 2^n is the discrete version of the exponential function e^t.

47.3 Fibonacci numbers revisited

We can take the recurrence relation for the Fibonacci numbers:

$$F_{n+2} = F_{n+1} + F_n$$

and rearrange it to be rewritten in terms of the shift operator:

$$\begin{aligned} F_{n+2} - F_{n+1} - F_n &= 0 \\ (E^2 - E - I)F &= 0. \end{aligned}$$

Note that the solutions to the equation

$$x^2 - x - 1 = 0$$

are

$$\varphi = \frac{1}{2}(1 + \sqrt{5}) \quad \text{and} \quad \psi = \frac{1}{2}(1 - \sqrt{5}).$$

So we can factor the operator $E^2 - E - I$ to see that

$$(E^2 - E - I)F = 0$$
$$(E - \varphi I)(E - \psi I)F = 0.$$

It is a fact that solutions to this equation are combinations (that is, a sum) of solutions to

$$(E - \varphi I)F1 = 0$$
$$(E - \psi I)F2 = 0.$$

The solutions to these equations are

$$F1_n = c_1 \varphi^n$$
$$F2_n = c_2 \psi^n.$$

So we have that

$$F_n = c_1 \varphi^n + c_2 \psi^n,$$

for some constants c_1 and c_2, which depend on the initial conditions of the sequence. To find those constants, we can plug in convenient values of n and solve.

Letting $n = 0$, we have

$$F_0 = 0 = c_1 + c_2 \quad \Longrightarrow \quad c_2 = -c_1.$$

Letting $n = 1$, we have

$$\begin{aligned} F_1 = 1 &= c_1 \varphi + c_2 \psi \\ &= c_1 \varphi - c_1 \psi \\ &= c_1(\varphi - \psi) \\ &= c_1 \cdot \sqrt{5}. \end{aligned}$$

So we find

$$c_1 = \frac{1}{\sqrt{5}}$$
$$c_2 = -\frac{1}{\sqrt{5}}.$$

This gives an explicit, closed form equation for the Fibonacci numbers:

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

47.4 EXERCISES

- (a) What is $\Delta(\frac{1}{n^2})$? (b) Using part (a), find $\sum_{n=A}^{B} \frac{-2n-1}{n^2(n+1)^2}$.

47.5 Answers to Selected Examples

1. To see why this holds, evaluate the sum and carefully note the cancellation. This type of sum is called a *telescoping* sum. First, for the forward difference operator, one finds

$$\begin{aligned}\sum_{n=A}^{B} \Delta u &= \sum_{n=A}^{B} u_{n+1} - u_n \\ &= (u_{A+1} - u_A) + (u_{A+2} - u_{A+1}) + (u_{A+3} - u_{A+2}) + \cdots + u_{B+1} - u_B \\ &= -u_A + u_{A+1} - u_{A+1} + u_{A+2} - u_{A+2} + u_{A+3} + \cdots - u_B + u_{B+1} \\ &= -u_A + u_{B+1},\end{aligned}$$

   as desired. Similarly, with the backwards difference operator, one finds

$$\begin{aligned}\sum_{n=A}^{B} \nabla u &= \sum_{n=A}^{B} u_n - u_{n-1} \\ &= u_A - u_{A-1} + u_{A+1} - u_A + u_{A+2} - u_{A+1} + \cdots + u_B - u_{B-1} \\ &= -u_{A-1} + u_B,\end{aligned}$$

   as claimed.

   (Return)

2. By FTIC,

$$\begin{aligned}\sum_{n=A}^{B} \frac{1}{n^2+n} &= -\sum_{n=A}^{B} \Delta \frac{1}{n} \\ &= -\left(\frac{1}{n}\Bigg|_{n=A}^{B+1} \right) \\ &= \frac{1}{A} - \frac{1}{B+1} \\ &= \frac{B-A+1}{A(B+1)}\end{aligned}$$

   (Return)

3. By the fact above, we have

$$\begin{aligned}\sum_{n=A}^{B} n!n &= \sum_{n=A}^{B} \Delta n! \\ &= n!\Big|_{n=A}^{B+1} \\ &= (B+1)! - A!.\end{aligned}$$

   (Return)

4. By the above fact, we have

$$\begin{aligned}\sum_{n=1}^{k} F_n &= \sum_{n=1}^{k} \Delta(F_{n+1}) \\ &= F_{n+1}\Big|_{n=1}^{k+1} \\ &= F_{k+2} - F_2 \\ &= F_{k+2} - 1.\end{aligned}$$

(Return)

5. Note that $n = n^{\underline{1}}$, and so

$$\begin{aligned}\sum_{n=1}^{k} n &= \sum_{n=1}^{k} n^{\underline{1}} \\ &= \frac{1}{2} n^{\underline{2}}\Big|_{n=1}^{k+1} \\ &= \frac{1}{2} n(n-1)\Big|_{n=1}^{k+1} \\ &= \frac{k(k+1)}{2}.\end{aligned}$$

(Return)

6. Note that

$$n^2 = n^{\underline{2}} + n^{\underline{1}}.$$

Therefore,

$$\begin{aligned}\sum_{n=1}^{k} n^2 &= \sum_{n=1}^{k} n^{\underline{2}} + n^{\underline{1}} \\ &= \frac{1}{3} n^{\underline{3}} + \frac{1}{2} n^{\underline{2}}\Big|_{n=1}^{k+1} \\ &= \frac{(k+1)\,k(k-1)}{3} + \frac{(k+1)k}{2} \\ &= (k+1)k\left(\frac{k-1}{3} + \frac{1}{2}\right) \\ &= (k+1)k\left(\frac{2k-2+3}{6}\right) \\ &= \frac{k(k+1)(2k+1)}{6}.\end{aligned}$$

(Return)

7. Using the above fact and FTIC, one finds

$$\begin{aligned}\sum_{n=1}^{k} n^3 &= \sum_{n=1}^{k} \left(n^{\underline{3}} + 3n^{\underline{2}} + n^{\underline{1}}\right) \\ &= \frac{1}{4} n^{\underline{4}} + n^{\underline{3}} + \frac{1}{2} n^{\underline{2}}\Big|_{n=1}^{k+1}.\end{aligned}$$

Now, note that $n^{\underline{m}} = 0$ when $n = 1$ for all $m > 1$, because of the factor of $n-1$. Therefore, the evaluation at the bottom limit is 0. Continuing with the algebra, we find

$$\begin{aligned}
\sum_{n=1}^{k} n^3 &= \frac{1}{4}(k+1)k(k-1)(k-2) + (k+1)k(k-1) + \frac{1}{2}(k+1)k \\
&= k(k+1)\left(\frac{1}{4}(k-1)(k-2) + k - 1 + \frac{1}{2}\right) \\
&= k(k+1)\left(\frac{(k-1)(k-2) + 4(k-1) + 2}{4}\right) \\
&= \frac{k(k+1)}{4}\left(k^2 - 3k + 2 + 4k - 4 + 2\right) \\
&= \frac{k(k+1)}{4}k(k+1) \\
&= \left(\frac{k(k+1)}{2}\right)^2.
\end{aligned}$$

Recalling that $1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$, this shows the remarkable fact that

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = (1 + 2 + 3 + \cdots + k)^2.$$

(Return)

8. By subtracting and adding a common term and rearranging, one finds

$$\begin{aligned}
(\Delta(uv))_n &= u_{n+1}v_{n+1} - u_n v_n \\
&= u_{n+1}v_{n+1} - u_n v_{n+1} + u_n v_{n+1} - u_n v_n \\
&= (u_{n+1} - u_n)v_{n+1} + u_n(v_{n+1} - v_n) \\
&= [\Delta u E v + u \Delta v]_n.
\end{aligned}$$

(Return)

9. Let $u = n$ and $\Delta v = 2^n$. It follows that $\Delta u = 1$, $v = 2^n$, and $Ev = 2^{n+1}$. Thus, by the integration by parts formula,

$$\begin{aligned}
\sum_{n=0}^{k} n2^n &= n2^n\Big|_{n=0}^{k+1} - \sum_{n=0}^{k} 2^{n+1} \\
&= (k+1)2^{k+1} - (2^{k+2} - 2) \\
&= (k+1)2^{k+1} - 2\cdot 2^{k+1} + 2 \\
&= (k-1)2^{k+1} + 2.
\end{aligned}$$

(Return)

48 Numerical ODEs

The previous few modules have discretized functions, derivatives, and integrals. This module shows how discrete methods can be used to approximate solutions to problems in the continuous realm. One such application was Newton's method for approximating roots of functions, seen back in Linear Approximations. This module deals with approximating solutions of continuous differential equations.

In certain situations, differential equations can be solved exactly. For example, a separable differential equation

$$\frac{dx}{dt} = f(x)g(t)$$

is solved by rearranging and integrating, as seen in Antidifferentiation. A linear first order differential equation

$$\frac{dx}{dt} = f(t)x + g(t)$$

is solved by an application of the product rule, as seen in More differential equations. However, there are many differential equations of the form

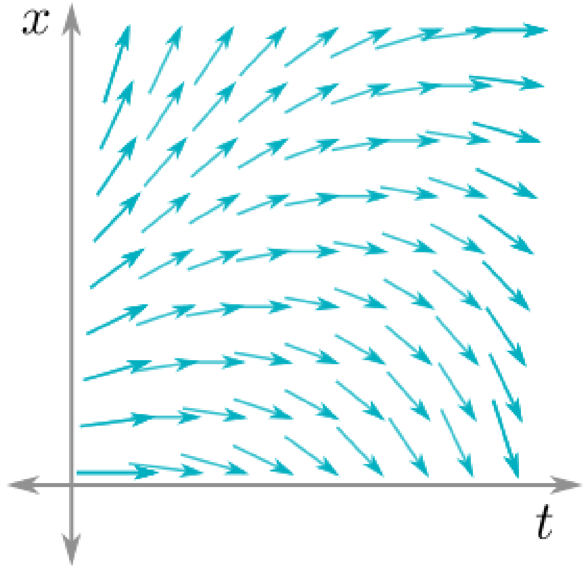
$$\frac{dx}{dt} = f(x, t)$$

which cannot be solved exactly by the above methods. This is the situation where techniques known as *numerical ordinary differential equations* can be used to approximate a solution. There are three methods covered in this module:

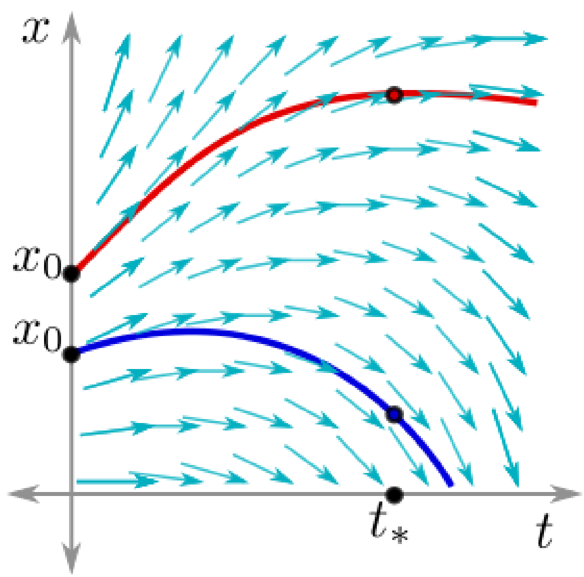
1. Euler's method
2. Midpoint method
3. Runge-Kutta method

48.1 Euler's method

Euler's method uses a difference equation to approximate the solution of an initial value problem. More specifically, given the differential equation $\frac{dx}{dt} = f(x, t)$, and initial value $x_0 = x(t_0)$, Euler's method approximates $x(t_*)$ for some $t_* > t_0$.



One way to visualize the problem is to imagine a river where the water is going in different directions at different locations. If one drops something that floats (perhaps a very small rock, or a duck) at different locations in the river, where would the object end up further down the river?



To compute the approximation, first a positive integer N is chosen, and the t axis is split into N intervals, giving the sequence of time points $t = (t_0, t_1, \ldots, t_N)$, where $t_N = t_*$. The time step is $h = \frac{t_* - t_0}{N}$. Then there is a corresponding sequence $x = (x_0, x_1, \ldots, x_N)$ where $x_0 = x(t_0)$ is the initial condition and each subsequent x_n is an approximation of $x(t_n)$, given by the update rule

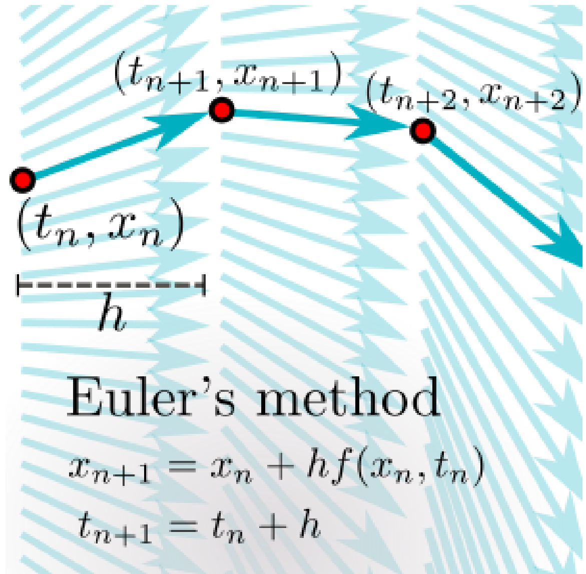
$$\begin{aligned} x_{n+1} &= x_n + hf(x_n, t_n) \\ t_{n+1} &= t_n + h. \end{aligned}$$

This recurrence is sensible by considering the discretization of the original differential equation:

$$\frac{dx}{dt} = f(x,t) \quad \Rightarrow \quad \frac{\Delta x}{\Delta t} = f(x,t)$$
$$\Rightarrow \quad \frac{x_{n+1} - x_n}{h} = f(x_n, t_n),$$

and then rearranging.

This process is using a linearization at each point (t_n, x_n) to get to the next point (t_{n+1}, x_{n+1}), then this is repeated to get the next point, and so on:



Remember that a linearization is only good when the change in input, h in this case, is small. Therefore, a bigger value of N gives a better approximation, but requires more computation.

Example

Let $\frac{dx}{dt} = x$ with $x_0 = x(0) = 1$. Use Euler's method with N left as a variable to approximate $x(1)$. What happens as $N \to \infty$? (See Answer 1)

Example

Consider the differential equation

$$\frac{dx}{dt} = t + x^2.$$

Use Euler's method to estimate (t_*, x_*), where $t_* = 1$, the initial conditions are $t_0 = 0$ and $x_0 = 1$, and the step size is $h = \frac{1}{2}$. (See Answer 2)

Taylor series perspective

If we think of $x = x(t)$ as a function of t, and expand the Taylor series of x about $t = t_0$, we have

$$\begin{aligned} x(t_0 + h) &= x(t_0) + h \cdot \left.\frac{dx}{dt}\right|_{t=t_0} + O(h^2) \\ &= x_0 + h \cdot f(x_0, t_0) + O(h^2), \end{aligned}$$

so Euler's method can be seen as taking the first order Taylor approximation and using it to form the recursion relation. The above equation shows that the error for a single step is in $O(h^2)$, so the error over all N steps is

$$\begin{aligned} \text{Error} &= N \cdot O(h^2) \\ &= \frac{t_* - t_0}{h} \cdot O(h^2) \\ &= O(h). \end{aligned}$$

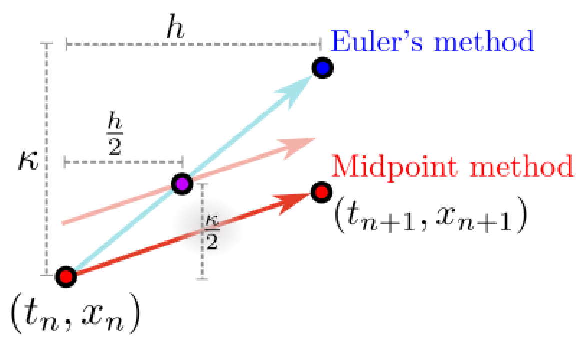
48.2 Midpoint method

Another method for solving the differential equation

$$\frac{dx}{dt} = f(x, y)$$

is known as the *midpoint method*. There is still an update rule which is similar to the one used in Euler's rule, but the function f is evaluated at a different point. The idea is to consider the point where Euler's rule would have taken us, and find the midpoint of that with our starting point. Use that midpoint as the point to evaluate f in the update rule.

As described, it is a little bit complicated, but with some extra notation and a diagram it becomes clearer. Let $\kappa = h \cdot f(x_n, t_n)$. So κ is the quantity which would be added to x_n in the update rule for Euler's rule:



Then the eponymous midpoint which is used in the update rule is

$$\left(t_n + \frac{h}{2}, x_n + \frac{\kappa}{2}\right).$$

So the update rule for the midpoint method is

$$\begin{aligned} x_{n+1} &= x_n + h \cdot f\left(x_n + \frac{\kappa}{2}, t_n + \frac{h}{2}\right) \\ \kappa &= h \cdot f(x_n, t_n). \end{aligned}$$

The midpoint method is a bit more complicated than Euler's method, but it has a benefit. The midpoint method is a second order approximation, and as a result the error turns out to be $O(h^3)$ for an individual step, and hence $O(h^2)$ for the full approximation process. Therefore, it is a more accurate method of approximation.

48.3 Runge-Kutta method

The final method for solving the above differential equation is called the *Runge-Kutta method.* It is a fourth order approximation. Its error for an individual step is $O(h^5)$ and for the whole process is $O(h^4)$. This makes it the most accurate model of the three in this module. It is difficult to describe in an intuitive manner other than to say it is a sort of average of Euler's method, the midpoint method, and other methods. The update rule is as follows:

$$\begin{aligned}
x_{n+1} &= x_n + \frac{1}{6}\left(\kappa_1 + 2\kappa_2 + 2\kappa_3 + \kappa_4\right) \\
\kappa_1 &= h \cdot f(x_n, t_n) \\
\kappa_2 &= h \cdot f\left(x_n + \frac{\kappa_1}{2}, t_n + \frac{h}{2}\right) \\
\kappa_3 &= h \cdot f\left(x_n + \frac{\kappa_2}{2}, t_n + \frac{h}{2}\right) \\
\kappa_4 &= h \cdot f\left(x_n + \kappa_3, t_n + h\right).
\end{aligned}$$

Note that κ_1 is the increase used in Euler's method, and κ_2 is the increase used in the Midpoint method.

48.4 Comparison of methods

We already know that Euler's method is the most basic, then the Midpoint method, and finally Runge-Kutta. We should expect, then, that Runge-Kutta should give the best approximation, followed by the Midpoint method, followed by Euler's method.

Example

Use each of the three methods to approximate the solution of

$$\frac{dx}{dt} = x$$

with initial conditions

$$\begin{aligned}
t_0 &= 0 & x_0 &= 1 \\
t_* &= 1 & x_* &= e
\end{aligned}$$

(we already know that $x_* = e$ because we have solved this differential equation several times already), using a step size of $h = 1$. (See Answer 3)

48.5 EXERCISES

- Given $\frac{dx}{dt} = x^3 + x^2 t$ with initial condition $t_0 = 0, x_0 = 1$, approximate $x(2)$ using the midpoint method with size step $h = 1$.

48.6 Answers to Selected Examples

1. Note that

$$\begin{aligned} x_{n+1} &= x_n + \frac{x_n}{N} \\ &= x_n(1 + \frac{1}{N}) \end{aligned}$$

(this is independent of t). Iterating this gives $x_N = \left(1 + \frac{1}{N}\right)^N \approx x(1)$. As N increases, the approximation gets better and better, and one finds that

$$\lim_{N\to\infty} x_N = \lim_{N\to\infty} \left(1 + \frac{1}{N}\right)^N .$$

Call this limit L and take the ln of both sides. Then we find that

$$\begin{aligned} \ln L &= \ln\left(\lim_{N\to\infty}\left(1+\frac{1}{N}\right)^N\right) \\ &= \lim_{N\to\infty} N \ln\left(1 + \frac{1}{N}\right) \\ &= \lim_{N\to\infty} N\left(\frac{1}{N} + O\left(\frac{1}{N^2}\right)\right) \\ &= \lim_{N\to\infty} 1 + O\left(\frac{1}{N}\right) \\ &= 1. \end{aligned}$$

Therefore, $L = e$. So $x(1) = e$, which matches the value from solving the problem by separation of variables.

(Return)

2. Here, we have

$$f(x,t) = t + x^2.$$

So, using the initial conditions, we have

$$f(x_0, t_0) = 0 + 1 = 1.$$

Then we have that

$$\begin{aligned} x_1 &= x_0 + h \cdot f(x_0, t_0) \\ &= 1 + \frac{1}{2}\cdot 1 \\ &= \frac{3}{2}. \end{aligned}$$

And $t_1 = \frac{1}{2}$. So

$$\begin{aligned} f(x_1, t_1) &= t_1 + x_1^2 \\ &= \frac{1}{2} + \left(\frac{3}{2}\right)^2 \\ &= \frac{11}{4}. \end{aligned}$$

Thus,

$$\begin{aligned} x_2 &= x_1 + h \cdot f(x_1, t_1) \\ &= \frac{3}{2} + \frac{1}{2} \cdot \frac{11}{4} \\ &= \frac{23}{8} \end{aligned}$$

And thus we have $x_* = \frac{23}{8}$.

(Return)

3. It may seem ill advised to use a step size equal to the distance from the start time to the end time, but this is just for the sake of comparison. Note that in this example,

$$f(x, t) = x.$$

Euler's method gives

$$\begin{aligned} x_1 &= x_0 + h \cdot f(x_0, t_0) \\ &= 1 + 1 \cdot 1 \\ &= 1 + 1, \end{aligned}$$

which is a pretty rough estimate of e. From this computation we have that $\kappa = 1$ for the midpoint method.

The **Midpoint method** gives

$$\begin{aligned} x_1 &= x_0 + h \cdot f\left(x_0 + \frac{\kappa}{2}, t_0 + \frac{h}{2}\right) \\ &= 1 + 1 \cdot f\left(1 + \frac{1}{2}, 0 + \frac{1}{2}\right) \\ &= 1 + 1 \cdot \left(1 + \frac{1}{2}\right) \\ &= 1 + 1 + \frac{1}{2}, \end{aligned}$$

so we see that we have gotten another term closer to e.

For **Runge-Kutta**, we need to compute several different κs. From the computations we did in Euler's method and the Midpoint method, we have that

$$\begin{aligned} \kappa_1 &= 1 \\ \kappa_2 &= \frac{3}{2}. \end{aligned}$$

Computing the other values, we have

$$\begin{aligned} \kappa_3 &= h \cdot f\left(x_0 + \frac{\kappa_2}{2}, t_0 + \frac{h}{2}\right) \\ &= 1 \cdot f\left(1 + \frac{3}{4}, 0 + \frac{1}{2}\right) \\ &= 1 + \frac{3}{4} \\ &= \frac{7}{4}. \end{aligned}$$

And

$$\begin{aligned}
\kappa_4 &= h \cdot f\,(x_0 + \kappa_3, t_0 + h) \\
&= 1 \cdot f\left(1 + \frac{7}{4}, 0 + 1\right) \\
&= 1 + \frac{7}{4} \\
&= \frac{11}{4}.
\end{aligned}$$

Putting these together (and rearranging the fractions a little bit), we have that

$$\begin{aligned}
x_1 &= x_0 + \frac{1}{6}(\kappa_1 + 2\kappa_2 + 2\kappa_3 + \kappa_4) \\
&= 1 + \frac{1}{6}\left(1 + 2 \cdot \frac{3}{2} + 2 \cdot \frac{7}{4} + \frac{11}{4}\right) \\
&= 1 + \frac{1}{6} + \frac{1}{2} + \frac{7}{12} + \frac{11}{24} \\
&= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24},
\end{aligned}$$

which has two more terms of e, making it a very good approximation, despite only having one step.
(Return)

49 Numerical Integration

The topic of this module is the discretization of the definite integral. How does one approximate the definite integral of a function which does not have an easily computable anti-derivative? The answer is with finite sums, which is the discrete analog of the definite integral.

Recall that one interpretation for the definite integral is area under the curve. The goal is to find a finite sum which approximates the area under the curve. There are three common techniques for making this approximation: Riemann sums, trapezoid rule, and Simpson's rule. Each gives an approximation of the integral $\int_a^b f(x)dx$.

Another way to think about this problem (and the practical applications) is to consider some sporadic sample points, thought of as a sequence

$$x = (x_n) = (x_0, x_1, \ldots, x_n).$$

Corresponding to these sample points is the sequence of function values

$$f = (f_n) = (f_0, f_1, \ldots, f_n),$$

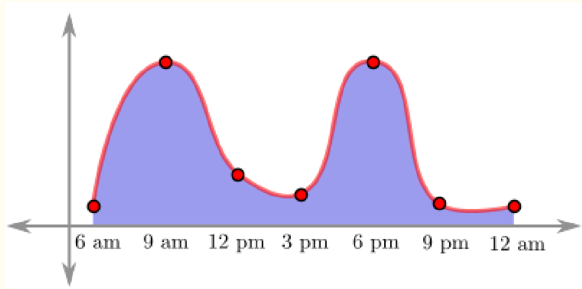
where $f_i = f(x_i)$. The goal is to approximate the definite integral of the underlying function f using these function values. Of course, in real world applications the function may not be continuous, let alone a familiar function with an easily computed anti-derivative. The method of numerical integration gives an approximation of the definite integral in this situation with imperfect information.

Example

Consider the problem of estimating the number of people who pass through a certain busy subway station each day. One could sit in the station for the entire day and count every person, or one could get an estimate by going in periodically and counting the number of people who come in over a short time.
This gives a sampling of the rate of passengers entering the station at different times:

Time	6 am	9 am	12 pm	3 pm	6 pm	9 pm	12 am
Passengers per hour	120	1200	360	240	1200	180	120

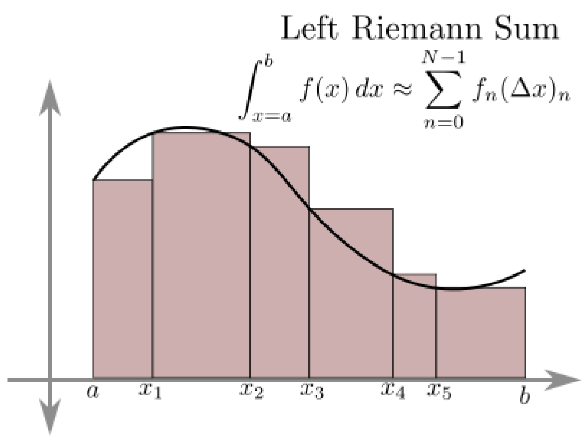
The definite integral of the rate of passenger arrival gives the total number of passengers using the station that day:



49.1 Riemann Sums

The most rudimentary approximation is given by *Riemann sums*, which should be familiar from the definition of the definite integral.

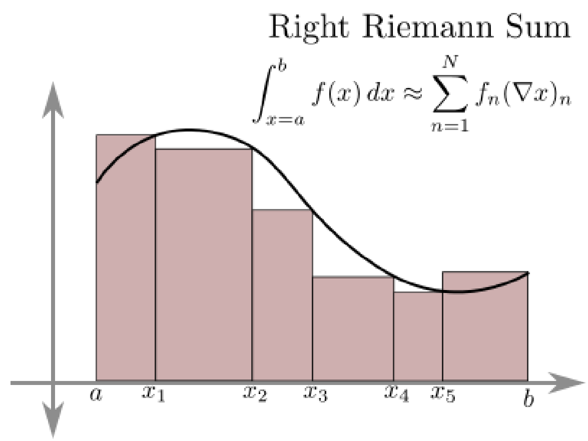
The *left Riemann sum* uses the left endpoint of the ith subinterval as the sample point to compute the height of the the ith rectangle:



Thus, the area of the nth rectangle is. So the *left Riemann sum* is given by

$$\int_a^b f(x)dx \approx \sum_{n=0}^{N-1} f_n \cdot (x_{n+1} - x_n)$$

The *right Riemann sum* uses the right endpoint of the ith subinterval to compute the height of the ith rectangle:



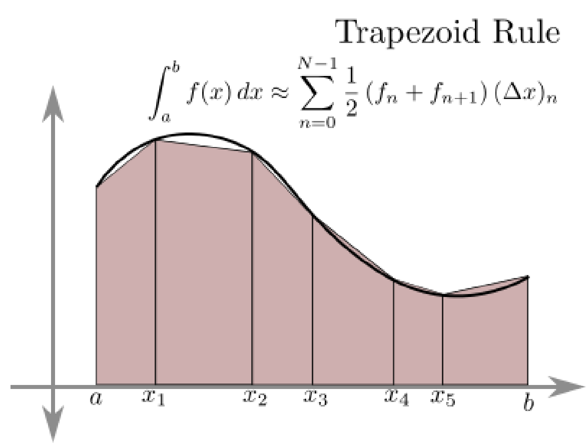
In this case, the area of the nth rectangle is $f_n \cdot (\nabla x)_n$. So the *right Riemann sum* is given by

$$\int_a^b f(x)dx \approx \sum_{n=1}^{N} f_n \cdot (x_n - x_{n-1})$$

An improvement on the left and right Riemann sums, called the trapezoid rule, is given in the next section.

49.2 Trapezoid rule

The *trapezoid rule* uses trapezoids instead of rectangles to approximate the area above each subinterval:



Recall that the area of a trapezoid with bases (parallel segments) of length p and q, with height h has area $\frac{1}{2}h(p+q)$. The area of the nth trapezoid, then, is $\frac{1}{2}(\Delta x)_n(f_n + f_{n+1})$. Thus, the trapezoid rule gives the approximation:

$$\int_a^b f(x)dx \approx \sum_{n=0}^{N-1} \frac{1}{2}(\Delta x)_n(f_n + f_{n+1})$$
$$\approx \sum_{n=0}^{N-1} \frac{1}{2}(f_n + f_{n+1})(x_{n+1} - x_n).$$

If the sample points are evenly spaced uniformly, then the formula for the trapezoid rule simply becomes

$$\int_a^b f(x)dx \approx h\left(\frac{1}{2}(f_0 + f_N) + \sum_{n=1}^{N-1} f_n\right),$$

where $h = \frac{b-a}{N}$ is the width of each trapezoid. Note that the trapezoid rule is the average of the left and right Riemann sums.

49.3 Simpson's rule

Think about how the previous approximations interpolate the function f. The Riemann sum is a piece-wise constant approximation (also called a step function). The trapezoid rule is a piece-wise linear approximation. The logical next step is to use piece-wise quadratic approximations. That is how Simpson's rule works. Another way to think of it is that Simpson's rule will compute the area under a parabola exactly (whereas Riemann sums and trapezoid rule will have errors in general).

One way that Simpson's rule differs from the above rules is that the sample points must be evenly spaced. Let $h = \frac{b-a}{N}$ be the distance between sample points. The *Simpson's rule* approximation is given by

$$\int_a^b f(x)dx \approx \frac{h}{3}\left(f_0 + 4f_1 + 2f_2 + 4f_3 + \ldots + 2f_{N-2} + 4f_{N-1} + f_N\right).$$

Here, N must be even. (See Derivation 1)

Example

Approximate the definite integral $\int_0^2 x^3 dx$ with $N = 4$ using (uniformly spaced) right and left Riemann sums; trapezoid rule; and Simpson's rule. Here is a table of pertinent values to make the computation easier:

n	x_n	f_n
0	0	0
1	.5	.125
2	1	1
3	1.5	3.375
4	2	8

(See Answer 2)

49.4 Errors bounds

With any approximation, it is good to get some idea of how far off the approximation is from the true value. Let E_T be the error using the trapezoidal rule, and E_S be the error from using Simpson's rule.

Using advanced calculus beyond the scope of this course, one can bound these errors as follows.

$$E_T \leq \frac{M(b-a)^3}{12N^2}$$

where M is the maximum value of $|f''(x)|$ on the interval $[a, b]$.

$$E_S \leq \frac{M(b-a)^5}{180N^4}$$

where M is the maximum value of $|f^{(4)}(x)|$ on the interval $[a, b]$.

49.5 EXERCISES

- Using the fact that $\sum_{n=0}^{j} n^2 = \frac{j(j+1)(2j+1)}{6}$, approximate $\int_0^2 x^2 dx$ using the right Riemann sum with N number of intervals.

49.6 Answers to Selected Examples

1. **Note** This derivation is a little bit different from the one in lecture, and perhaps more elementary.

   The idea of Simpson's rule is to fit a parabola to the first three points (x_0, f_0), (x_1, f_1), (x_2, f_2), and then find the area under that parabola. Then, fit a parabola to the next three points (overlapping the endpoints) (x_2, f_2), (x_3, f_3), (x_4, f_4), find the area under that parabola, and so on.

   Consider the parabola $g(x) = ax^2 + bx + c$ determined by the points $(-h, f_{n-1}), (0, f_n), (h, f_{n+1})$. Solving the resulting system of equations

$$\begin{aligned} f_{n-1} &= ah^2 - bh + c \\ f_n &= c \\ f_{n+1} &= ah^2 + bh + c, \end{aligned}$$

   gives the parabola which fits these points. One finds that

$$\begin{aligned} a &= \frac{1}{h^2}\left(\frac{f_{n-1}}{2} - f_n + \frac{f_{n+1}}{2}\right) \\ b &= \frac{f_{n+1} - f_{n-1}}{2h} \\ c &= f_n. \end{aligned}$$

   And so the area

$$\begin{aligned} \int_{x=-h}^{h} g(x)dx &= \int_{x=-h}^{h} ax^2 + bx + c dx \\ &= a\frac{x^3}{3} + b\frac{x^2}{2} + cx\Big|_{x=-h}^{h} \\ &= 2a\frac{h^3}{3} + 2ch, \end{aligned}$$

by using symmetry. Plugging in the earlier values for a and c and simplifying gives

$$\int_{x=-h}^{h} g(x)dx = \frac{h}{3}\left(f_{n-1} + 4f_n + f_{n+1}\right).$$

Carrying this out for all each triplet of points gives the approximation

$$\begin{aligned}\int_{x=a}^{b} f(x)dx &\approx \frac{h}{3}\left(f_0 + 4f_1 + f_2 + f_2 + 4f_3 + f_4 + \cdots + f_N\right)\\ &\approx \frac{h}{3}\left(f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + f_N\right),\end{aligned}$$

as desired.

(Return)

2. Let L, R, T, and S be the respective approximations. Then

$$\begin{aligned}L &= \frac{1}{2}\left(0 + .125 + 1 + 3.375\right) = 2.25\\ R &= \frac{1}{2}\left(.125 + 1 + 3.375 + 8\right) = 6.25\\ T &= \frac{1}{2}\left(0 + .125 + 1 + 3.375 + 4\right) = 4.25\\ S &= \frac{1}{6}\left(0 + .5 + 2 + 13.5 + 8\right) = 4.\end{aligned}$$

Note that the true value is

$$\int_{x=0}^{2} x^3\,dx = \frac{1}{4}x^4\Big|_{x=0}^{2} = 4,$$

so Simpson's rule gets the answer exactly and the trapezoid rule is the next closest.

(Return)

50 Series

The previous module discussed finite sums as the discrete analog of definite integrals with finite bounds. Then, logically, the discrete analog of improper integrals with infinite bounds should be infinite sums, referred to as *infinite series* or just *series* when there is no confusion.

50.1 Definition of infinite series

Recall that when computing definite integrals with bounds at infinity, one replaces the infinite bound with a variable and then takes the limit:

$$\int_{x=1}^{\infty} f(x)dx = \lim_{T\to\infty} \int_{x=1}^{T} f(x)dx.$$

For infinite series, the definition is analogous:

$$\sum_{n=1}^{\infty} a_n = \lim_{T\to\infty} \sum_{n=1}^{T} a_n.$$

The expression $\sum_{n=1}^{T} a_n$ is called the Tth *partial sum* of the series $\sum_{n=1}^{\infty} a_n$. Then a series *converges* if the sequence of partial sums converges. If the sequence of partial sums does not converge, we say the series *diverges*.

Example

Give the first few terms of the sequence of partial sums for the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}.$$

(See Answer 1)

50.2 Taylor series revisited

Recall some of the Taylor series from earlier modules:

$$\begin{aligned}
e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \dots && (|x| < 1) \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots && (|x| < 1)
\end{aligned}$$

These provide many examples of series which not only converge, but can be evaluated exactly.

Example

Compute $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots$. (See Answer 2)

Example

Compute $1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \dots$. (See Answer 3)

50.3 Classifying series

There are some series which cannot be evaluated exactly, though it is known that the series converges. For example, the series

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^3} &= 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \dots \\
&\approx 1.2
\end{aligned}$$

converges, but it is not known what the exact value is (though one can calculate as many digits as one likes). This is in contrast with an apparently similar series,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2} &= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots \\
&= \frac{\pi^2}{6},
\end{aligned}$$

for which an exact value is known (though the proof of this value is beyond the scope of this course).

Yet another similar series, called the *harmonic series*,

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots,$$

diverges, as will be shown in the next module.

There are two questions then. First, does a series converge or not? Second, if it does converge, to what does it converge? This course deals mostly with the first question, in this module and the next few modules. More advanced analysis classes can help answer the second question.

When intuition fails

Determining the convergence of a series using intuition can be dangerous. As one example of how intuition can fail, consider what happens when we plug $x = 1$ into the series for $\ln(1 + x)$:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots .$$

Although $x = 1$ is not in the radius of convergence for this series, it turns out that this series still converges. Now, consider what happens if we multiply both sides by $\frac{1}{2}$:

$$\frac{1}{2}\ln 2 = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots .$$

Adding this equation together with the above one, some of the terms cancel and some combine to show that

$$\frac{3}{2}\ln 2 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$

Now, notice that this series has all the terms of the series for $\ln 2$, but in slightly different order. And yet, the series evaluates to a different value. That is certainly counter intuitive. Perhaps even more alarming, the terms of the series for $\ln 2$ can be rearranged so that the resulting series evaluates to *any* real number. The takeaway is that we must tread carefully and not trust intuition but instead rely on logic.

50.4 The nth term test for divergence

The first, and usually simplest, test *for divergence* of a series is the nth term test.

The nth term test for divergence

If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=0}^{\infty} a_n$ diverges.

(See Proof 4) This test applies to all series, and it is easy to apply (just take the limit of the terms of the series). However, many of the series encountered in this course will have terms which go to 0 in the limit, in which case the test is inconclusive (see the caveat below).

Example

Show that the series

$$\sum_{n=0}^{\infty} \frac{4^n - 3^n}{4^n + 2^n}$$

diverges. (See Answer 5)

Example

Show that the series

$$\sum_{n=0}^{\infty}(-1)^n = 1 - 1 + 1 - 1 + 1 - \dots$$

diverges. (See Answer 6)

Example

Show that the series

$$\sum_{n=1}^{\infty}\cos\left(\frac{1}{n}\right) = \cos(1) + \cos(1/2) + \cos(1/3) + \dots$$

diverges. (See Answer 7)

Caveat

This is not a test for convergence! In particular, if $\lim_{n\to\infty} a_n = 0$, then the test is inconclusive (the series might converge or diverge). If the test is inconclusive, one of the other tests from the upcoming modules must be used.

In logical terms, this says that the *converse* of the nth term test does not hold. On the other hand, the *contrapositive* does hold:

$$\sum_{n=0}^{\infty} a_n \text{ converges} \quad \Rightarrow \quad \lim_{n\to\infty} a_n = 0$$

(The contrapositive of a true statement is always true, but the converse is not always true).

Example

What does the nth term test say about the series $\sum_{n=2}^{\infty}\frac{\ln(n)}{n}$? (See Answer 8)

Example

What does the nth term test say about the series

$$\sum_{n=1}^{\infty}\arctan n?$$

(See Answer 9)

Example

Suppose the series

$$\sum_{n=1}^{\infty} a_n = L,$$

converges, and that $a_n > 0$ for all n. What, if anything, can be said about the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{a_n}?$$

(See Answer 10)

50.5 EXERCISES

- Determine whether the following series converges or diverges

$$\sum_{n=2}^{\infty} \frac{n^3}{n \ln(n^{100})}$$

$$\sum_{n=1}^{\infty} \frac{2^n}{n^{100}}$$

50.6 Answers to Selected Examples

1. Adding the first term, then the first two terms, then the first three terms, and so on, gives

$$\frac{1}{2}, \frac{1}{2} + \frac{1}{4}, \frac{1}{2} + \frac{1}{4} + \frac{1}{8}, \ldots$$

which becomes

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots.$$

This sequence appears to be converging to 1, which is consistent with what we know this series to be (by the geometric series).

(Return)

2. Note that by the geometric series,

$$\frac{1}{1 - 1/3} = 1 + \frac{1}{3} + \frac{1}{9} + \ldots$$
$$= \frac{3}{2}.$$

So $\frac{1}{3} + \frac{1}{9} + \ldots = \frac{1}{2}$.

(Return)

3. Note that this is the Taylor series for $\cos(x)$ with $x = \pi$. Thus, the series evaluates to $\cos(\pi) = -1$.

(Return)

4. We give a proof by contrapositive. That is, we prove that if $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Assume $\sum_{n=0}^{\infty} a_n$ converges. Then by the definition of convergence, we know that the sequence of partial sums s_T defined by

$$s_T = \sum_{n=1}^{T} a_n$$

converges also. Therefore,

$$\lim_{T\to\infty} s_T = \lim_{T\to\infty} s_{T-1} = L$$

for some L. Then we have

$$\lim_{T\to\infty} (s_T - s_{T-1}) = 0,$$

by linearity of the limit. But notice that

$$\begin{aligned}
s_T - s_{T-1} &= \sum_{n=1}^{T} a_n - \sum_{n=1}^{T-1} a_n \\
&= (a_1 + a_2 + \cdots + a_{T-1} + a_T) - (a_1 + a_2 + \cdots + a_{T-1}) \\
&= a_T.
\end{aligned}$$

Putting this together with the above limit, we find

$$\begin{aligned}
\lim_{T\to\infty} (s_T - s_{T-1}) &= \lim_{T\to\infty} a_T \\
&= 0,
\end{aligned}$$

which is what we were trying to prove.

(Return)

5. Here,

$$a_n = \frac{4^n - 3^n}{4^n + 2^n}.$$

Applying the nth term test, we find

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{4^n - 3^n}{4^n + 2^n} \\
&= \lim_{n\to\infty} \frac{4^n(1-(3/4)^n)}{4^n(1+(1/2)^n)} \\
&= \lim_{n\to\infty} \frac{1-(3/4)^n}{1+(1/2)^n} \\
&= 1 \neq 0,
\end{aligned}$$

because both $\left(\frac{3}{4}\right)^n \to 0$ and $\left(\frac{1}{2}\right)^n \to 0$. Therefore, by the nth term test, the series diverges.

(Return)

6. Since $\lim_{n\to\infty}(-1)^n$ does not exist (the sequence oscillates), the series diverges by the nth term test.

(Return)

7. Note that $\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1 \neq 0$. Thus, by the nth term test, the series diverges.

(Return)

8. Note that $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$, either by using l'Hospital's rule or by recalling that $\ln(n)$ grows much more slowly than n. Thus, the nth term test is **inconclusive**.

   (Return)

9. Note that
$$\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0,$$
   and so by the nth term test, the series **diverges**.

   (Return)

10. By the contrapositive of the nth term test mentioned above, we know that since the series
$$\sum_{n=1}^{\infty} a_n$$
   converges, we know that
$$\lim_{n\to\infty} a_n = 0.$$
   But this implies that
$$\lim_{n\to\infty} \frac{1}{a_n} = \infty$$
   so by the nth term test, we know that the series
$$\sum_{n=1}^{\infty} \frac{1}{a_n}$$
   diverges.

   (Return)

51 Convergence Tests 1

Unlike the nth term test for divergence from the last module, this module gives several tests which, if successfully applied, give a definitive answer of whether a series converges or not. A common feature of the tests in this module is that they use a comparison.

51.1 Integral test for convergence and divergence

This is a test which can definitively tell whether a series converges or diverges. However, it can be harder to apply.

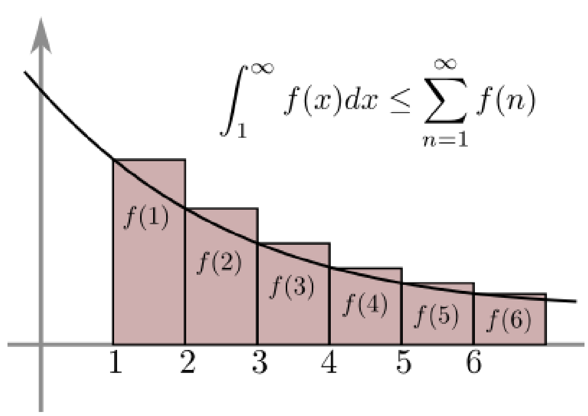
Integral test for convergence and divergence

If $f(x)$ is a positive, decreasing function, then for any integer m,

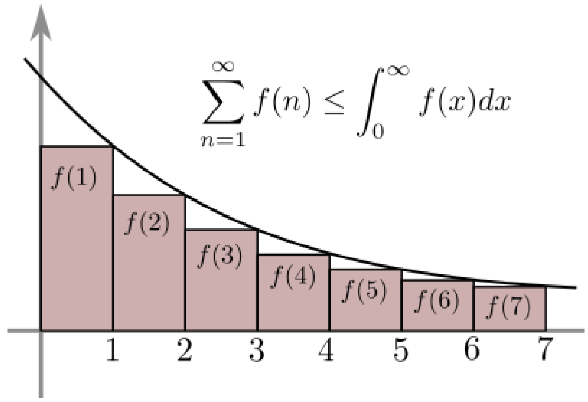
$$\sum_{n=m}^{\infty} f(n) \text{ converges} \iff \int_{m}^{\infty} f(x)dx \text{ converges.}$$

The double arrow $\iff$ means *if and only if*. So the series and integral either both converge, or they both diverge.

To see why, visualize the series as a sum of rectangles with base 1 and height $f(n)$. If these rectangles are drawn to the right of the curve $f(x)$, then the result is the following figure. Each rectangle is labeled with its area. The combined area of the rectangles completely contains the area under $f(x)$, which establishes the inequality shown:



On the other hand, if the rectangles are drawn to the left of the curve $f(x)$, then all the rectangles lie below the curve. Their combined area is less than the area under the curve $f(x)$, which establishes the inequality shown:



Combining these two inequalities gives

$$\int_1^{\infty} f(x)dx \leq \sum_{n=1}^{\infty} f(n) \leq \int_0^{\infty} f(x)dx.$$

If the integral diverges, then the series diverges (by the first inequality). And if the integral converges, then the series converges (by the second inequality). This establishes the integral test.

Example

Use the integral test to determine if

$$\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$$

converges or diverges. (See Answer 1)

Example

Use the integral test to determine if

$$\sum_{n=1}^{\infty} \frac{n}{e^n}$$

converges or diverges. (See Answer 2)

51.2 The p-series test

The next example is important enough that it gets its own name: *the p-series*. This makes use of the p-integrals that we computed earlier.

Example

Find the values of p for which the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges. (See Answer 3)

The p-series test

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Harmonic series

The series $\sum \frac{1}{n}$, known as the *harmonic series*, diverges by to the p-series test. This is a significant fact to keep in mind because the harmonic series diverges even though the terms of the series go to 0. Thus, the harmonic series is a demonstration that the nth term test is a test for *divergence* only and *cannot* be used to show a series converges.

Note that the harmonic series is a sort of boundary between convergence and divergence. The series $\sum \frac{1}{n^{.999}}$ diverges, but the series $\sum \frac{1}{n^{1.001}}$ converges.

Example

The p-series test proves the convergence of two examples from the last module: $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n^3}$.
Example
Determine the values of p for which the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

converges. (See Answer 4)

51.3 Comparison test

The integral test compared a series to its related integral. This test compares one series to another.

Comparison test

Let a_n and b_n be positive sequences such that $b_n > a_n$ for all n. It follows that

1. if $\sum b_n$ converges, then $\sum a_n$ converges.
2. if $\sum a_n$ diverges, then $\sum b_n$ diverges.

In other words, if a series is smaller than a convergent series, then it converges too. If a series is bigger than a divergent series, then it diverges too.

Caveat

It is critical that the inequality be in the correct direction. A series which is larger than a convergent series might converge or diverge. A series which is smaller than a divergent series might converge or diverge.

Example

Show that $\sum \frac{\ln(n)}{n}$ diverges. (See Answer 5)

Example

Show that

$$\sum_{n=4} \frac{n^3 - 2n^2 - 10}{n^5 + 7}$$

converges. (See Answer 6)

Example

Determine whether

$$\sum_{n=2}^{\infty} \frac{1}{\ln(n!)}$$

converges or diverges. Hint: try for a rough upper bound or lower bound on $n!$ and see which one gives the right comparison. (See Answer 7)

51.4 Limit test

The final test of this module is a slightly different type of comparison. Recall that when comparing two functions f and g to see which is "bigger" asymptotically, one computes the limit

$$\lim_{n\to\infty} \frac{f(n)}{g(n)}.$$

If this limit is infinite, then f is bigger. If the limit is 0, then g is bigger. If the limit is L where $0 < L < \infty$, then the two functions are roughly equal (up to a constant multiple). It is this third case that is used for this test (sometimes called the *Limit comparison test*):

Limit test

Let a_n and b_n be positive series. If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ and $0 < L < \infty$, then the series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

The key to the limit test is finding a suitable sequence b_n which is approximately equal to a_n in the limit. Often, a_n will be a ratio, in which case the lower order terms in the numerator and denominator can be dropped, and what is left will be b_n. Ideally, it will be easy to see if $\sum b_n$ converges or diverges. Finally, one must check that $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$.

Example

Show that
$$\sum \frac{n^2 - n}{n^3 + 7}$$
diverges. (See Answer 8)

Example

Determine whether
$$\sum \sin\left(\frac{1}{n}\right)$$
converges or diverges. (See Answer 9)

51.5 EXERCISES

- Determine whether the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{n+4}{n(2+n^4)^{1/3}}$$
$$\sum_{n=1}^{\infty} \frac{|\sin(n)^n|}{n^2}$$

51.6 Answers to Selected Examples

1. Computing, one finds (using the u-substitution $u = \ln(x)$) that
$$\int_2^{\infty} \frac{1}{x\ln(x)}\,dx = \int_{\ln(2)}^{\infty} \frac{1}{u}\,du$$
$$= \ln(u)\Big|_{\ln(2)}^{\infty},$$
which diverges since $\ln(u) \to \infty$ as $u \to \infty$. Since the integral diverges, the series also diverges by the integral test.
(Return)

2. The integral test says the series converges if and only if
$$\int_{x=1}^{\infty} \frac{x}{e^x}\,dx = \int_{x=1}^{\infty} xe^{-x}\,dx$$
converges. We know this integral converges (recall that xe^{-x} is the PDF for the exponential distribution). But we can also compute it again.
This integral is a good candidate for integration by parts, with
$$\begin{aligned} u &= x & du &= dx \\ dv &= e^{-x}\,dx & v &= -e^{-x}. \end{aligned}$$

Thus,

$$\begin{aligned}\int_{x=1}^{\infty} xe^{-x} &= -xe^{-x}\Big|_{x=1}^{\infty} - \int_{x=1}^{\infty} -e^{-x}\,dx \\ &= -xe^{-x} - e^{-x}\Big|_{x=1}^{\infty} \\ &= 0-(-e^{-1}-e^{-1}) \\ &= \frac{2}{e}.\end{aligned}$$

Since the integral converges, the series converges too, by the integral test.

(Return)

3. We know from the integral test that

$$\sum_{n=1}^{\infty} frac1n^p$$

converges if and only if

$$\int_{x=1}^{\infty} \frac{1}{x^p}\,dx$$

converges. But this integral converges if and only if $p > 1$, as we saw in the module on p-integrals. Therefore, the p-series converges if and only if $p > 1$.

(Return)

4. Making the substitution

$$\begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\,dx, \end{aligned}$$

we see that

$$\int_{x=2}^{\infty} \frac{1}{x(\ln x)^p}\,dx = \int_{u=\ln 2}^{\infty} \frac{1}{u^p}\,du.$$

This integral (again from our knowledge of the p-integral) converges if and only if $p > 1$. Therefore, by the integral test, the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

converges if and only if $p > 1$.

(Return)

5. Note that $\frac{\ln(n)}{n} > \frac{1}{n}$. Since $\sum \frac{1}{n}$ diverges, the series $\sum \frac{\ln(n)}{n}$ diverges too, by the comparison test.

(Return)

6. Note that

$$\frac{n^3-2n^2-10}{n^5+7} < \frac{n^3}{n^5} = \frac{1}{n^2},$$

since the numerator on the left is smaller, and the denominator on the left is bigger. So

$$\sum_{n=4}^{\infty} \frac{n^3-2n^2-10}{n^5+7} < \sum_{n=4}^{\infty} \frac{1}{n^2}.$$

$\sum \frac{1}{n^2}$ converges (p-series test from above), and so

$$\sum_{n=4}^{\infty} \frac{n^3 - 2n^2 - 10}{n^5 + 7}$$

converges as well, by the comparison test.

(Return)

7. A lower bound for $n!$ might be 2^n (or any exponential). This gives

$$\begin{aligned}
\sum \frac{1}{\ln(n!)} &< \sum \frac{1}{\ln(2^n)} \\
&= \sum \frac{1}{n \ln(2)} \\
&= \frac{1}{\ln 2} \sum \frac{1}{n},
\end{aligned}$$

which diverges, since it is a constant multiple of the harmonic series. This comparison does not go in the right direction, since our original series is *smaller* than a divergent series. Thus, we should try going in the other direction to find an upper bound for $n!$.

A rough upper bound for $n!$ is n^n. This gives

$$\begin{aligned}
\sum \frac{1}{\ln(n!)} &> \sum \frac{1}{\ln(n^n)} \\
&= \sum \frac{1}{n \ln(n)}.
\end{aligned}$$

This series diverges (see the example earlier in this module). Therefore, the original series, which is *bigger* than a divergent series, also diverges.

(Return)

8. By dropping the lower order terms in the numerator and denominator, one finds

$$\frac{n^2 - n}{n^3 + 7} \approx \frac{n^2}{n^3} = \frac{1}{n}.$$

So $b_n = \frac{1}{n}$ is a good choice. Assuming the above approximation is not too rough, the work is done since the harmonic series, $\sum \frac{1}{n}$, diverges.

To make sure the approximation is not too rough, compute the limit

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\left(n^2 - n\right) / \left(n^3 + 7\right)}{1/n} \\
&= \lim_{n\to\infty} \frac{n(n^2 - n)}{n^3 + 7} \\
&= \lim_{n\to\infty} \frac{n^3 - n^2}{n^3 + 7} \\
&= 1.
\end{aligned}$$

Thus, by the limit test, $\sum \frac{n^2-n}{n^3+7}$ and $\sum \frac{1}{n}$ either both converge or both diverge. Since $\sum \frac{1}{n}$ diverges, so too must $\sum \frac{n^2-n}{n^3+7}$.

(Return)

9. With an unusual series like this, the nth term test is a good first thing to try. But it is inconclusive since $\sin(1/n) \to 0$ as $n \to \infty$.

   To get a handle on how this function acts, note that when n is large, $1/n$ is small, and so we can use the Taylor series for $\sin x$ about 0:
$$\sin\left(\frac{1}{n}\right) = \frac{1}{n} + O\left(\frac{1}{n^3}\right).$$

   Thus, $\sin(1/n) \approx \frac{1}{n}$, which means $\frac{1}{n}$ is a good candidate for b_n in the limit test. This will work:
$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} \\ &= \lim_{n\to\infty} \frac{1/n + O(1/n^3)}{1/n} \\ &= 1.\end{aligned}$$

   Since $\sum b_n = \sum \frac{1}{n}$ diverges (harmonic series), it follows by the limit test that $\sum \sin(1/n)$ diverges also.

   (Return)

52 Convergence Tests 2

This module deals with the root test and the ratio test for convergence. Unlike the tests from the previous module, the tests of this module can be applied without having to find a good series to compare.

Recall that the geometric series

$$1 + x + x^2 + x^3 + \cdots$$

converges to $\frac{1}{1-x}$ provided that $|x| < 1$. This fact is at the heart of both the root test and the ratio test.

52.1 Root test

The Root test

Given a series $\sum a_n$, with all $a_n > 0$, let

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n}.$$

If $\rho < 1$, then $\sum a_n$ converges. If $\rho > 1$, then $\sum a_n$ diverges. Finally, if $\rho = 1$, then the test is inconclusive.

(See Justification 1) The root test works best when the term a_n involves an nth power, or can be expressed as an nth power, although it can be used in other situations as well.

Example

Determine if

$$\sum \left(\frac{n}{2n+1}\right)^n$$

converges or diverges. (See Answer 2)

Example

Determine if

$$\sum \left(\frac{n}{n+1}\right)^{n^2}$$

converges or diverges. (See Answer 3)

Example

Use the root test on the p-series
$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$
(See Answer 4)

52.2 Ratio test

The Ratio Test

Given a series $\sum a_n$, with all $a_n > 0$, let
$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$
If $\rho < 1$, then the series $\sum a_n$ converges. If $\rho > 1$, then the series $\sum a_n$ diverges. If $\rho = 1$, then the test is inconclusive (the series might converge or diverge).

(See Justification 5) The ratio test works best when a_n involves exponential functions and factorials, since in these situations there is a lot of cancellation. It does not work well with ratios of polynomials, because the test is inconclusive.

Example

Determine if the series
$$\sum \frac{n}{3^n}$$
converges or diverges. (See Answer 6)

Example

Determine if the series
$$\sum \frac{n!}{(2n)!}$$
converges or diverges. (See Answer 7)

Example

Show that the ratio test is inconclusive on the p-series $\sum \frac{1}{n^p}$. (See Answer 8)

Example

Determine if the series
$$\sum_{n=0}^{\infty} \frac{4^n}{(2n)!}$$
converges or diverges. (See Answer 9)

52.3 Summary of methods for a positive series

There is no foolproof method for determining the convergence or divergence of a series. However, here is a rough guide for tests to try. Given a series $\sum a_n$, where $a_n > 0$ for all n:

1. Do the terms go to 0? If $\lim_{n\to\infty} a_n \neq 0$, then by the nth term test, the series diverges. (If $\lim_{n\to\infty} a_n = 0$, then the test is inconclusive).
2. Does a_n involve exponential functions like c^n where c is constant? Does a_n involve factorial? Then the ratio test should be used.
3. Is a_n of the form $(b_n)^n$ for some sequence b_n? Then use the root test.
4. Does ignoring lower order terms make a_n look like a familiar series (e.g. p-series or geometric series)? Then use the comparison test or the limit test.
5. Does the sequence a_n look easy to integrate? Then use the integral test.

52.4 EXERCISES

- Determine whether the following series converges or diverges

$$\sum_{n=2}^{\infty} \frac{1}{\ln^n(n)}$$

$$\sum_{n=1}^{\infty} \frac{2^n 3^{4n}}{n!}$$

- What does the ratio test say about the convergence of

$$\sum_{n=1}^{\infty} \frac{2n(2n-2)(2n-4)\dots 2}{(2n-1)(2n-3)\dots 1}$$

52.5 Answers to Selected Examples

1. Recall what it means that $\lim \sqrt[n]{a_n} = \rho$. It means that for a given $\epsilon > 0$, there exists an M such that for all $n > M$,

$$|\sqrt[n]{a_n} - \rho| < \epsilon.$$

In other words, for n sufficiently big, $\sqrt[n]{a_n} \approx \rho$, and so $a_n \approx \rho^n$. This says that, roughly speaking, the series is eventually geometric with common ratio ρ. Hence, the series converges for $\rho < 1$ and diverges for $\rho > 1$.

(Return)

2. Computing, one finds that

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \sqrt[n]{a_n} \\
&= \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{2n+1}\right)^n} \\
&= \lim_{n\to\infty} \frac{n}{2n+1} \\
&= \frac{1}{2}.
\end{aligned}$$

Since $\rho < 1$, the series converges by the root test.

(Return)

3. Computing, one finds that

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} \\
&= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \\
&= \lim_{n\to\infty} \frac{1}{(1+1/n)^n} \\
&= \frac{1}{e}.
\end{aligned}$$

The last step above follows from the fact that $\lim_{n\to\infty}(1+1/n)^n = e$. Thus, $\rho < 1$ and so the series converges by the root test.

(Return)

4. Trying the root test gives

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^p}} \\
&= \lim_{n\to\infty} \left(\frac{1}{n^{1/n}}\right)^p \\
&= 1,
\end{aligned}$$

and so the root test is **inconclusive**.

The last step above follows from the fact that $\lim_{n\to\infty} n^{1/n} = 1$. Let $y = \lim_{n\to\infty} n^{1/n}$. Taking the natural log of both sides gives

$$\begin{aligned}
\ln(y) &= \lim_{n\to\infty} \frac{1}{n}\ln(n) \\
&= \lim_{n\to\infty} \frac{\ln(n)}{n} \\
&= 0.
\end{aligned}$$

Thus $y = e^0 = 1$, as desired.

(Return)

5. As in the justification for the root test, $\frac{a_{n+1}}{a_n}$ is eventually very close to ρ, and remains close to ρ ever after. Roughly speaking, then $a_{n+1} \approx \rho a_n$ for all sufficiently big n. But that means that after a while, the series becomes roughly geometric with common ratio ρ. Therefore, when $\rho < 1$ the series converges, and when $\rho > 1$ the series diverges.

   (Return)

6. Here, $a_n = \frac{n}{3^n}$. Computing, one finds

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n\to\infty} \frac{(n+1)/\left(3^{n+1}\right)}{n/3^n} \\
&= \lim_{n\to\infty} \frac{n+1}{3^{n+1}} \cdot \frac{3^n}{n} \\
&= \lim_{n\to\infty} \frac{1}{3}\frac{n+1}{n} \\
&= \frac{1}{3}.
\end{aligned}$$

   $\rho < 1$, and so by the ratio test, the series converges.

   (Return)

7. In this example, $a_n = \frac{n!}{(2n)!}$. Computing gives

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n\to\infty} \frac{(n+1)!}{[2(n+1)]!} \cdot \frac{(2n)!}{n!} \\
&= \lim_{n\to\infty} \frac{n+1}{(2n+2)(2n+1)} \\
&= \lim_{n\to\infty} \frac{n+1}{4n^2+6n+2} \\
&= 0.
\end{aligned}$$

   So $\rho < 1$, and the series converges by the ratio test.

   (Return)

8. Computing shows that

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{1/(n+1)^p}{1/n^p} \\
&= \lim_{n\to\infty} \frac{n^p}{(n+1)^p} \\
&= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p \\
&= 1,
\end{aligned}$$

   and so the ratio test is inconclusive.

   (Return)

9. There is a quick, tricky way to see that this converges, which is to note that

$$\sum_{n=0}^{\infty} \frac{4^n}{(2n)!} = \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!},$$

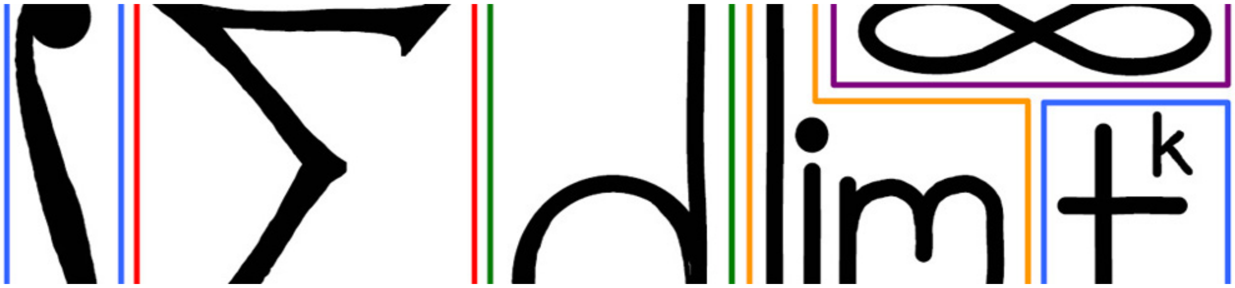
which is the Taylor series for $\cosh x$ with $x = 2$. We know cosh converges everywhere, so we know this series converges.

Alternatively, we can use the ratio test. We have $a_n = \frac{4^n}{(2n)!}$. Therefore

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n\to\infty} \frac{4^{n+1}}{(2(n+1))!} \cdot \frac{(2n)!}{4^n} \\
&= \lim_{n\to\infty} \frac{4 \cdot 4^n}{(2n+2)\cdot(2n+1)\cdot(2n)!} \cdot \frac{(2n)!}{4^n} \\
&= \lim_{n\to\infty} \frac{4}{(2n+2)\cdot(2n+1)} \\
&= 0.
\end{aligned}$$

Since $\rho < 1$, this series converges by the ratio test, confirming what we already knew.

(Return)



53 Absolute And Conditional

Up until now, the convergence tests covered by this course have only covered series with positive terms. What happens when a series has some positive and some negative terms? This module describes some tools for determining the convergence or divergence of such a series.

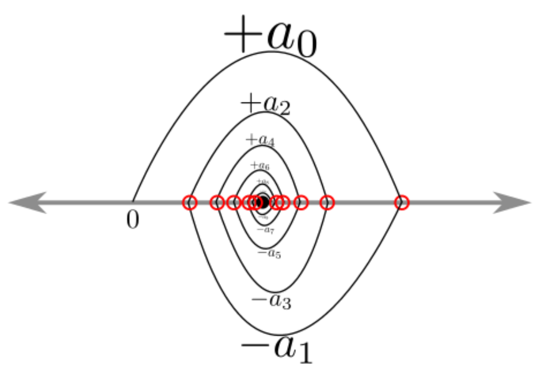
53.1 Alternating series test

One particular type of series is fairly simple to test for convergence. A series $\sum a_n$ is *alternating* if it is of the form $\sum(-1)^n b_n$, where b_n is a positive sequence. In other words, a series is alternating if its terms are alternately positive and negative.

Alternating Series Test

An alternating series $\sum(-1)^n b_n$ with decreasing terms converges if and only if $\lim_{n\to\infty} b_n = 0$.

The intuition behind this test is that if the terms are alternating, decreasing, and go to zero, then the partial sums of the series gradually zero in on the true value. The first partial sum is greater than the true value, the second partial sum is less than the true value, and so on:



Example

The series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

is alternating. Its terms are decreasing, and the terms go to zero, so by the alternating series test, the series converges.

Example

Determine if

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n}$$

converges. (See Answer 1)

Example

Determine if

$$\sum_{n=2}^{\infty}(-1)^n\frac{\ln(n)}{n}$$

converges. (See Answer 2)

53.2 Conditional and absolute convergence

Consider the two previous examples. Note that both $\sum(-1)^n\frac{1}{2^n}$ and $\sum(-1)^n\frac{\ln(n)}{n}$ converge. However, if these series were not alternating, then $\sum\frac{1}{2^n}$ still converges (it is geometric), whereas $\sum\frac{\ln(n)}{n}$ does not converge (comparison with the harmonic series $\sum\frac{1}{n}$).

These series demonstrate a distinction between two types of convergence.

Absolute and Conditional Convergence

The series $\sum a_n$ *converges absolutely* if $\sum|a_n|$ converges.
A series $\sum a_n$ *converges conditionally* if $\sum a_n$ converges, but $\sum|a_n|$ diverges.
In other words, a series is conditionally convergent if it is convergent but not absolutely convergent.

Example

Using this terminology, the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n}$$

converges absolutely, but

$$\sum_{n=2}^{\infty}(-1)^n\frac{\ln(n)}{n}$$

converges conditionally.

Example

Find the values of p for which the alternating p-series

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^p}$$

diverges, converges conditionally, and converges absolutely. (See Answer 3)

Example

Determine if

$$\sum_{n=1}^{\infty}(-1)^n\frac{e^n}{n^5}$$

converges, and if so, whether it is conditional or absolute convergence. (See Answer 4)

Example

Determine if

$$\sum_{n=1}^{\infty}(-1)^n\frac{n!}{n^n}$$

converges, and if so, whether it is conditional or absolute convergence. (See Answer 5)

53.3 Absolute convergence test

Some series are not strictly alternating, but have some positive and some negative terms, sporadically. In this situation, it can be difficult to determine whether the series converges directly, but the following test sometimes makes the determination easier.

Absolute Convergence Test

If the series $\sum |a_n|$ converges, then $\sum a_n$ converges. In other words, if a series converges absolutely, then the series converges.

(See Justification 6)

Example

Determine if

$$\sum_{n=1}^{\infty}\frac{\sin(n)}{n^2}$$

converges or diverges. (See Answer 7)

53.4 EXERCISES

- Determine whether the following series converges or not. If it converges, is it conditionally convergent or absolutely convergent?

$$\sum_{n=2}^{\infty}(-1)^n\frac{1}{n\ln(n)}$$

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^{1/2}}{((n+1)(n+2)(n+3))^{1/2}}$$

- Determine whether the following series is convergent or divergent

$$\sum_{n=1}^{\infty}\cos(n)\frac{\ln(n)}{n!}$$

53.5 Answers to Selected Exercises

1. The series is alternating, the terms are decreasing, and $\lim_{n\to\infty}\frac{1}{2^n}=0$. Thus, by the alternating series test, the series converges.

   Alternatively, we can observe that this series is geometric:

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n} &= \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n \\ &= \frac{1}{1-(-1/2)} \\ &= \frac{2}{3}.\end{aligned}$$

   (Return)

2. The series is alternating, terms are decreasing, and

$$\lim_{n\to\infty}\frac{\ln(n)}{n}=0.$$

   And so by the alternating series test, the series converges.

   (Return)

3. When $p\leq 0$, the terms of the series do not go to 0. That is,

$$\lim_{n\to\infty}\frac{1}{n^p}\neq 0.$$

   So by the nth term test for divergence, the series diverges for these values of p.

   When $0<p\leq 1$, the terms of the series are decreasing, alternating, and going to 0. So by the alternating series test, the series converges for this range of p. However, if absolute values are taken, then the resulting series, $\sum\frac{1}{n^p}$, does not converge by the p-series test. Thus, for $0When1$ (Return)

4. The series is alternating, but notice that the terms do not go to 0, since $e^n > n^5$. Therefore, by the nth term test for divergence, this series diverges.

   (Return)

5. The series is alternating, and note that $n^n > n!$, which leads us to believe that the series converges. In fact, the series converges absolutely, i.e.

$$\sum \frac{n!}{n^n}$$

   converges, as the ratio test shows, along with some careful algebra:

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \\
&= \lim_{n\to\infty} \frac{(n+1)n^n}{(n+1)^{n+1}} \\
&= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} \\
&= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \\
&= \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n} \\
&= \lim_{n\to\infty} \frac{1}{(1+1/n)^n} \\
&= \frac{1}{e} < 1.
\end{aligned}$$

   So $\rho < 1$, and so the series converges absolutely.

   (Return)

6. Consider the series $\sum(a_n + |a_n|)$. Note that

$$0 \le a_n + |a_n| \le 2|a_n|$$

   So $\sum(a_n + |a_n|) \le \sum 2|a_n|$, and by the comparison test we find $\sum(a_n + |a_n|)$ converges. Then

$$\begin{aligned}
\sum a_n &= \sum((a_n + |a_n|) - |a_n|) \\
&= \sum(a_n + |a_n|) - \sum |a_n|
\end{aligned}$$

   converges too, being the difference of two convergent, positive series.

   (Return)

7. $\sin(n)$ is a messy function because it is sometimes positive and sometimes negative, but not in a simple alternating pattern. However, one nice thing about $\sin(n)$ is that, in absolute value, it is bounded by 1. So

$$\sum_{n=1}^{\infty} \left|\frac{\sin(n)}{n^2}\right| \le \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

   Since $\sum \frac{1}{n^2}$ converges by p-series test, it follows by the comparison test that $\sum \left|\frac{\sin(n)}{n^2}\right|$ converges. Hence $\sum \frac{\sin(n)}{n^2}$ converges absolutely, and so it converges by the absolute convergence test.

   (Return)

54 Power Series

Sequences can be thought of as the discretization of a function. This module goes in the opposite direction: turning a sequence into a function called a power series.

54.1 Power series

Given a sequence a_n, the *power series* associated with a_n is the infinite sum $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

Example

The power series associated with the sequence $a_n = 1$ is the function $f(x) = 1 + x + x^2 + x^3 + \ldots$.

Example

All of the Taylor series encountered earlier in the course are power series. For instance, the Taylor series for the exponential is the power series associated with the sequence $a_n = \frac{1}{n!}$.

Example

The *Lucas numbers* L_n are like the Fibonacci numbers but with different initial conditions. $L_0 = 2$ and $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$. So the sequence begins $L = (2, 1, 3, 4, 7, 11, 18, \ldots)$.
Find the closed-form function $L(x)$ given by the power series

$$\begin{aligned} L(x) &= L_0 + L_1 x + L_2 x^2 + L_3 x^3 + \cdots \\ &= \sum_{n=0}^{\infty} L_n x^n. \end{aligned}$$

This is known in the field of combinatorics as the *generating function* for the sequence L_n. (See Answer 1)

These power series have lots of useful applications in enumeration and asymptotic analysis, among other things. But the rest of this module will deal with convergence. Given a sequence (a_n), for what values of x does its associated power series $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ converge?

54.2 Interval and radius of convergence

Recall that some Taylor series had restrictions on the values of x for which the series equaled the function (e.g. geometric series, $\ln(1+x)$, $\arctan(x)$). In other words, the series converges for some values of x and diverges for other values of x.

In general, given a power series $f(x) = \sum a_n x^n$, the goal is to find the values of x for which the series converges. The following theorem tells us that the set of such values is always an interval:

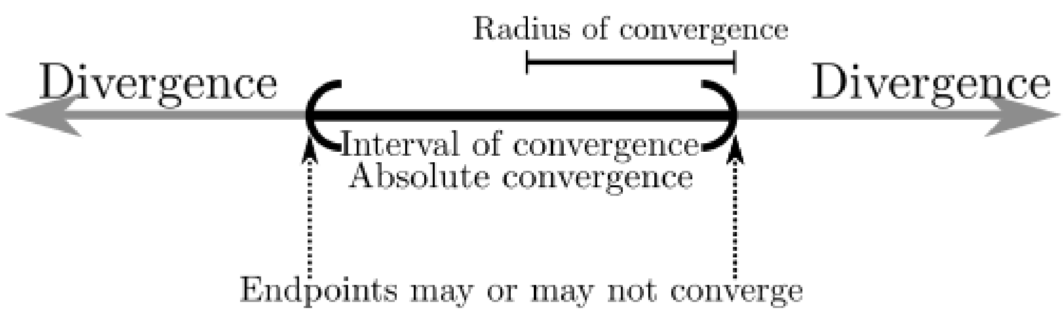
Given

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

there exists some $0 \leq R \leq \infty$ such that

- the series converges absolutely if $|x| < R$;
- the series diverges if $|x| > R$;
- might converge or diverge when $x = R$ or $x = -R$.

The *interval of convergence* is the interval $(-R, R)$, possibly with the endpoints included (they need to be individually checked in general). The *radius of convergence* is half the length of the interval of convergence.



The method for finding the interval of convergence is to use the ratio test to find the interval where the series converges absolutely and then check the endpoints of the interval using the various methods from the previous modules.

Previously, when using the ratio test, one computed ρ and then checked if $\rho < 1$, $\rho > 1$, or $\rho = 1$. Now, the goal is to find the values of x for which the series $\sum a_n x^n$ converges absolutely, i.e. for which $\rho < 1$. So ρ is computed, in terms of x, and is set to be less than 1. This gives an interval of values for x which the series converges absolutely. Once this interval has been determined, the endpoints must be checked for convergence as well.

Radius of convergence

For a general power series $f(x) = a_0 + a_1x + a_2x^2 + \cdots$, what is the radius of convergence R in terms of the sequence a_n? Note that

$$\begin{aligned}\rho &= \lim_{n\to\infty} \frac{|a_{n+1}x^{n+1}|}{|a_nx^n|} \\ &= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}|x|.\end{aligned}$$

And for absolute convergence $\rho < 1$. So in terms of the a_n, we get absolute convergence when

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}|x| < 1,$$

or equivalently,

$$|x| < \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}.$$

Thus, we have shown that the radius of convergence of the series $f(x) = a_0 + a_1x + a_2x^2 + \cdots$ is given by

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}.$$

Example

Find the interval and radius of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{n}{2^n}x^n$$

(See Answer 2)

Example

Find the interval of convergence for

$$\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n}$$

(See Answer 3)

54.3 Shifted power series

Recall that the Taylor series for a function can be computed at a point c other than 0. In this case the series took the form

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

Such a power series is said to be centered at c, since the interval of convergence for the series will be centered at c. To see why, carry out the calculation as above (replacing all the $|x|$'s with $|x-c|$'s) to find that the series converges absolutely when

$$|x-c| < \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = R.$$

So the radius of convergence is the same (it only depends on the sequence a_n), and it is only the center of the interval that has changed. Thus, the interval of convergence is $(c - R, c + R)$, and again one must individually check the endpoints.

Example

Find the interval of convergence for

$$\sum_{n=1}^{\infty} (-1)^n \frac{(x+3)^n}{n^2}$$

(See Answer 4)

Example

Find the interval of convergence for

$$\sum_{n=2}^{\infty} \frac{(2x+5)^n}{\ln n}.$$

(See Answer 5)

54.4 Answers to Selected Examples

1. Begin by writing out the series $L(x)$, $xL(x)$, and $x^2L(x)$:

$$\begin{aligned} L(x) &= L_0 + L_1x + L_2x^2 + L_3x^3 + L_4x^4 + \cdots \\ xL(x) &= \phantom{L_0 +{}} L_0x + L_1x^2 + L_2x^3 + L_3x^4 + \cdots \\ x^2L(x) &= \phantom{L_0 + L_1x +{}} L_0x^2 + L_1x^3 + L_2x^4 + \cdots \end{aligned}$$

Now note what happens when we take $L(x) - xL(x) - x^2L(x)$ and collect like terms. Because of the recurrence, all of the coefficients of the form $L_n - L_{n-1} - L_{n-2} = 0$, which leaves only two terms in the power series:

$$\begin{aligned} L(x) - xL(x) - x^2L(x) &= L_0 + (L_1 - L_0)x + (L_2 - L_1 - L_0)x^2 + (L_3 - L_2 - L_1)x^3 + \cdots \\ &= L_0 + (L_1 - L_0)x + 0x^2 + 0x^3 + 0x^4 + \cdots \\ &= 2 + (1-2)x \\ &= 2 - x. \end{aligned}$$

Now, solving for $L(x)$ by factoring and dividing gives

$$L(x) = \frac{2 - x}{1 - x - x^2}.$$

(Return)

2. Using the ratio test for absolute convergence, one computes

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}/2^{n+1}}{nx^n/2^n}\right| \\ &= \lim_{n\to\infty}\frac{n+1}{2^{n+1}}\cdot\frac{2^n}{n}|x| \\ &= \lim_{n\to\infty}\frac{1}{2}\frac{n+1}{n}|x| \\ &= \frac{|x|}{2}.\end{aligned}$$

Setting $\rho < 1$ means $\frac{|x|}{2} < 1$, hence $|x| < 2$. So for all $-2 < x < 2$, the series converges absolutely. Checking the endpoint $x = -2$ gives the series

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n}{2^n}(-2)^n &= \sum_{n=1}^{\infty}\frac{n}{2^n}(-1)^n 2^n \\ &= \sum_{n=1}^{\infty}(-1)^n n,\end{aligned}$$

which diverges by the nth term test for divergence. Similarly, the endpoint $x = 2$ gives the series

$$\sum_{n=1}^{\infty}\frac{n}{2^n}2^n = \sum_{n=1}^{\infty} n,$$

which diverges, also by the nth term test. Thus, both endpoints diverge and so the interval of convergence is $(-2, 2)$. The radius of convergence is 2.

(Return)

3. Using the ratio test gives

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\frac{n}{x^n}\right| \\ &= \lim_{n\to\infty}\frac{|x|n}{n+1} \\ &= |x|.\end{aligned}$$

Setting this less than 1 gives $|x| < 1$ so the interval is $(-1, 1)$. Checking the endpoint $x = 1$ gives the alternating harmonic series which converges. The other endpoint $x = -1$ gives the harmonic series which diverges. So the interval of convergence is $(-1, 1]$.

(Return)

4. The interval of convergence will be centered at -3. We can take a short cut and just compute the radius of convergence $R = \lim \frac{|a_n|}{|a_{n+1}|}$, where $a_n = \frac{(-1)^n}{n^2}$. This gives $R = 1$.

Then the interval of convergence is $(-3-1, -3+1) = (-4, -2)$, and it remains to check the endpoints. At $x = -4$, we get $\sum \frac{1}{n^2}$, which converges by p-series. At $x = -2$, we get the alternating p-series $\sum(-1)^n\frac{1}{n^2}$, which converges by alternating test. Thus, the interval of convergence is $[-4, -2]$.

(Return)

5. It takes a little rearranging before this takes the form of a shifted series as defined above (the problem is the coefficient of 2 on the x). The 2 can be factored out, and what results is

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{(2x+5)^n}{\ln n} &= \sum_{n=2}^{\infty} \frac{2^n (x+5/2)^n}{\ln n} \\ &= \sum_{n=2}^{\infty} \frac{2^n}{\ln n}(x+5/2)^n,\end{aligned}$$

which is now of the form given above. Note that this series is centered at $-5/2$. The radius of convergence is

$$\begin{aligned}R &= \lim_{n\to\infty} \frac{2^n}{\ln(n)} \cdot \frac{\ln(n+1)}{2^{n+1}} \\ &= \frac{1}{2}\end{aligned}$$

(since the ratio of logs goes to 1). Thus, the interval of convergence is $(-3, -2)$. Checking the endpoints, one finds convergence at $x = -3$ (by alternating series test) and divergence at $x = -2$ (by comparison of $\sum \frac{1}{\ln n}$ with the harmonic series, for example).

So the interval of convergence is $(-3, -2)$.

(Return)

55 Taylor Series Redux

The last module took a sequence (a_n) and turned it into a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$. This module turns this around and asks can we go from a function $f(x)$ to a sequence (a_n)? The answer is yes, and this is the familiar process of computing the Taylor series for the function. Recalling the Taylor series for f:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)x^n,$$

we see that $a_n = \frac{1}{n!} f^{(n)}(0)$ is the sequence corresponding to $f(x)$. We could also center the Taylor series at a different point, and get a different sequence, but for now let's keep things centered at 0.

Now that we have turned our function into its Taylor series, we come back to the questions deferred from earlier in the course:

1. For what values of x does a function's Taylor series converge?
2. Does the Taylor series converge to the function?

These questions are the topics of this module.

55.1 Taylor series convergence

We now have the tools to see when a power series converges, so the answer to the first question is that the series

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0} \frac{1}{n!} f^{(n)}(0)x^n$$

converges absolutely for $|x| < R$, where

$$\begin{aligned}
R &= \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} \\
&= \lim_{n\to\infty} \frac{f^{(n)}(0)}{n!} \cdot \frac{(n+1)!}{f^{(n+1)}(0)} \\
&= \lim_{n\to\infty} (n+1) \frac{f^{(n)}(0)}{f^{(n+1)}(0)}.
\end{aligned}$$

Within the interval of convergence, differentiation and integration of a power series are nice, in that they can be done term by term:

1. $\dfrac{df}{dx} = \sum_{n=0}^{\infty} n a_n x^{n-1}$

2. $\displaystyle\int f(x)\, dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} + C$

Why is this useful? Being able to differentiate and integrate term by term allows us to compute the Taylor series for various functions by differentiating or integrating the Taylor series for other functions.

Example

Compute the Taylor series for $\arctan x$ by noting that $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$. (See Answer 1)

Example

Show that the power series $f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$ can be written as $\frac{1}{(1-x)^2}$ for $|x| < 1$. Hint: try integrating both sides and see what familiar function you get. (See Answer 2)

55.2 Example

The *Fresnel Integral* $C(x)$ is defined by

$$C(x) = \int_{t=0}^{x} \cos(t^2) dt.$$

There is no elementary expression for this integral, but it can be expressed as a series by expanding the series for cos and then integrating term by term:

$$\begin{aligned} C(x) &= \int_{t=0}^{x} \cos(t^2) dt \\ &= \int_{t=0}^{x} \left(\sum_{n=0}^{\infty} (-1)^n \frac{(t^2)^{2n}}{(2n)!} \right) dt \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{t^{4n+1}}{(2n)!\,(4n+1)} \bigg|_{t=0}^{x} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n)!\,(4n+1)}. \end{aligned}$$

55.3 Taylor series convergence to a function

Now, we consider the second question above: when a Taylor series converges, does it always converge to the function? Unfortunately, not always. Even with a smooth f, and x within the interval of convergence, it is possible that the Taylor series does not converge to f. The following definition is used for functions whose Taylor series *do* converge to the functions themselves:

Definition: Real-analytic function

A function f is *real-analytic* at $x = a$ if for x sufficiently close to a,

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n.$$

That is, a function f is real-analytic at a if the Taylor series for f converges to f near a.

Almost all the functions we have encountered in this course are real-analytic. However, there are examples of smooth functions which are not real-analytic, as the next example shows.

Example

Consider the function

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

To show that this function is smooth, we must show that its derivative exists at $x = 0$ (everywhere else it is a composition of nice functions, so we need not worry). We use the definition of the derivative:

$$\begin{aligned} \left.\frac{df}{dx}\right|_{x=0} &= \lim_{h \to 0^+} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0^+} \frac{1}{h} e^{-1/h}. \end{aligned}$$

This can either be thought of as a $0/0$ case for l'Hospital's rule, or we can do a change of variables $t = \frac{1}{h}$, which gives the limit

$$\begin{aligned} \lim_{h \to 0^+} \frac{1}{h} e^{-1/h} &= \lim_{t \to \infty} t e^{-t} \\ &= \lim_{t \to \infty} \frac{t}{e^t} \\ &= 0, \end{aligned}$$

since the exponential beats a polynomial asymptotically. So $f'(0) = 0$. It turns out that all of the higher derivatives of f at 0 are 0 as well. So if we tried to expand this as a Taylor series, we would have

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{1}{2!} f''(0)x^2 + \cdots \\ &= 0 + 0x + 0x^2 + \cdots \\ &= 0. \end{aligned}$$

So the Taylor series for f converges to 0, despite the fact that f is non-zero for $x > 0$.

55.4 Answers to Selected Examples

1. We have that

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$
$$= 1 - x^2 + x^4 - x^6 + x^8 - \cdots,$$

for $|x| < 1$. Thus, for $|x| < 1$, we can safely integrate both sides of this equation to find

$$\arctan x = \int \left(1 - x^2 + x^4 - x^6 + x^8 - \cdots\right) dx$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + C.$$

By checking $\arctan(0) = 0$, we find that the integration constant $C = 0$. Thus for $|x| < 1$ we have

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

(Return)

2. Integrating both sides gives

$$\int f(x)dx = \int (1 + 2x + 3x^2 + \cdots)dx$$
$$= x + x^2 + x^3 + \cdots$$
$$= \frac{x}{1-x},$$

by recognizing that this is the geometric series. Now, differentiating both sides gives that

$$f(x) = \frac{1}{(1-x)^2},$$

as desired. Note that this only holds within the interval of convergence for the geometric series, $|x| < 1$.
(Return)

56 Approximation And Error

Given a series that is known to converge but for which an exact answer is not known, how does one find a good approximation to the true value? One way to get an approximation is to add up some number of terms and then stop. But how many terms are enough? How close will the result be to the true answer? That is the motivation for this module.

56.1 Error defined

Given a convergent series

$$s = \sum_{n=0}^{\infty} a_n.$$

Recall that the partial sum s_k is the sum of the terms up to and including a_k, i.e.,

$$\begin{aligned} s_k &= a_0 + a_1 + a_2 + \ldots + a_k \\ &= \sum_{n=0}^{k} a_n. \end{aligned}$$

Then the *error* E_k is the difference between s_k and the true value s, i.e.,

$$\begin{aligned} E_k &= s - s_k \\ &= \sum_{n=0}^{\infty} a_n - \sum_{n=0}^{k} a_n \\ &= a_{k+1} + a_{k+2} + a_{k+3} + \ldots \\ &= \sum_{n=k+1}^{\infty} a_n. \end{aligned}$$

In other words, the error is the sum of all the terms from the infinite series which were not included in the partial sum.

56.2 Alternating series error bound

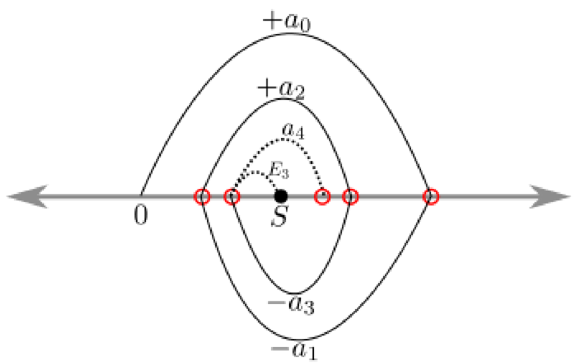
For a decreasing, alternating series, it is easy to get a bound on the error E_k:

$$|E_k| \leq |a_{k+1}|.$$

In other words, the error is bounded by the next term in the series.

Note

If the series is strictly decreasing (as is usually the case), then the above inequality is strict.



To see why the alternating bound holds, note that each successive term in the series overshoots the true value of the series. In other words, if S is the true value of the series,

$$\begin{aligned} a_0 &> S \\ a_0 - a_1 &< S \\ a_0 - a_1 + a_2 &> S. \end{aligned}$$

The above figure shows that if one stops at $a_0 - a_1 + a_2 - a_3$, then the error E_3 must be less than a_4.

Example

What is the minimum number of terms of the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$$

one needs to be sure to be within $\frac{1}{100}$ of the true sum?
The goal is to find k so that $|E_k| \leq \frac{1}{100}$. Since $|E_k| \leq |a_{k+1}|$, the question becomes for which value of k is $|a_{k+1}| \leq \frac{1}{100}$? If $k = 9$, then $|a_{k+1}| = \frac{1}{100}$, and so by the alternating series error bound, $|E_9| \leq \frac{1}{100}$. Thus 9 terms are required to be within $\frac{1}{100}$ of the true value of the series.

56.3 Integral test for error bounds

Another useful method to estimate the error of approximating a series is a corollary of the integral test. Recall that if a series $\sum f(n)$ has terms which are positive and decreasing, then

$$\int_{m+1}^{\infty} f(x)dx < \sum_{n=m+1}^{\infty} f(n) < \int_{m}^{\infty} f(x)dx.$$

But notice that the middle quantity is precisely E_m. So

$$\int_{m+1}^{\infty} f(x)dx < E_m < \int_{m}^{\infty} f(x)dx.$$

This bound is nice because it gives an upper bound *and* a lower bound for the error.

Example

How many terms of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

must one add up so that the Integral bound guarantees the approximation is within $\frac{1}{100}$ of the true answer? (See Answer 1)

56.4 Taylor approximations

Recall that the *Taylor series* for a function f about 0 is given by

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots. \end{aligned}$$

The *Taylor polynomial of degree N* is the approximating polynomial which results from truncating the above infinite series after the degree N term:

$$\begin{aligned} f(x) &\approx \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n \\ &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(N)}(0)}{N!}x^N. \end{aligned}$$

This is a good approximation for $f(x)$ when x is close to 0. How good an approximation is it? That is the purpose of the last error estimate for this module.

As in previous modules, let $E_N(x)$ be the error between the Taylor polynomial and the true value of the function, i.e.,

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n + E_N(x).$$

Notice that the error $E_N(x)$ is a function of x. In general, the further away x is from 0, the bigger the error will be.

A first, weak bound for the error is given by

$$E_N(x) \leq Cx^{N+1}$$

for some constant C and x sufficiently close to 0. In other words, $E_N(x)$ is $O(x^{N+1})$. A stronger bound is given in the next section.

Taylor remainder theorem

The following gives the precise error from truncating a Taylor series:

Taylor remainder theorem

The error $E_N(x)$ is given precisely by

$$E_N(x) = \frac{f^{(N+1)}(t)}{(N+1)!}x^{N+1},$$

for some t between 0 and x, inclusive. So if $x < 0$, then $x \le t \le 0$, and if $x > 0$, then $0 \le t \le x$.

Example

Consider the case when $N = 0$. The Taylor remainder theorem says that

$$f(x) = f(0) + f'(t)x,$$

for some t between 0 and x. Solving for $f'(t)$ gives

$$f'(t) = \frac{f(x) - f(0)}{x - 0},$$

for some $0 < t < x$ if $x > 0$ and $x < t < 0$ if $x < 0$, which is precisely the statement of the Mean value theorem. Therefore, one can think of the Taylor remainder theorem as a generalization of the Mean value theorem.

Taylor error bound

As it is stated above, the Taylor remainder theorem is not particularly useful for actually finding the error, because there is no way to actually find the t for which the equation holds. There is a slightly different form which gives a bound on the error:

Taylor error bound

$$|E_N(x)| \le \frac{C}{(N+1)!}|x|^{N+1},$$

where C is the maximum value of $|f^{(N+1)}(t)|$ over all t between 0 and x, inclusive.

Example

Estimate $\sqrt{e}$ using

$$e^{1/2} \approx 1 + \frac{1}{2} + \frac{(1/2)^2}{2!} + \frac{(1/2)^3}{3!} \approx 1.64,$$

and bound the error. (See Answer 2)

56.5 Answers to Selected Exercises

1. If one adds up the first m terms, then by the integral bound, the error E_m satisfies

$$\begin{aligned} E_m &< \int_m^\infty \frac{dx}{x^3} \\ &= \left.\frac{x^{-2}}{-2}\right|_m^\infty \\ &= \frac{1}{2m^2}. \end{aligned}$$

Setting $\frac{1}{2m^2} \le \frac{1}{100}$ gives that $m^2 \ge 50$, so $m \ge 8$. Thus, $m = 8$ is the minimum number of terms required so that the Integral bound guarantees we are within $\frac{1}{100}$ of the true answer.

Note

If you actually compute the partial sums using a calculator, you will find that 7 terms actually suffice. But remember, we want the *guarantee* of the integral test, which only ensures that $\frac{1}{128} < E_7 < \frac{1}{98}$, despite the fact that in reality, $E_7 \approx .009 < .01$.

(Return)

2. The function is $f(x) = e^x$, and the approximating polynomial used here is

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

Then according to the above bound,

$$|E_3(x)| \le \frac{C}{4!}|x|^4,$$

where C is the maximum of $f^{(4)}(t) = e^t$ for $0 \le t \le x$. Since e^t is an increasing function, $C = e^x$. Thus,

$$|E_3(x)| \le \frac{e^x}{4!}x^4.$$

Thus,

$$|E_3(1/2)| \le \frac{e^{1/2}}{4!}(1/2)^4 < \frac{1}{100}.$$

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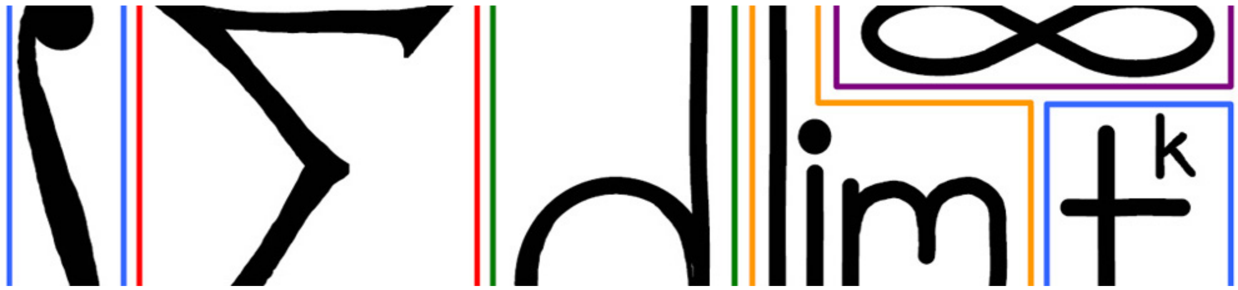

57 Calculus

In this course, we've learned skills in five key areas:

- Limits
- Differentiation
- Integration
- ODEs
- Series

Some of the things we can do are pretty impressive. However, there are many simple-seeming questions in single-variable calculus that show us just how much we have left to learn.

57.1 PROBLEM 1

Why is the standard Gaussian a probability density function? In other words, why is

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi} \tag{1}$$

Why is it that with all our methods, we cannot evaluate this integral easily? Let's try...

MORAL

This integral is *easy* with a little bit of multivariable calculus.

57.2 PROBLEM 2

Recall that we began this class with the definition of the exponential function e^x and then, to obtain series for sin and cos, we invoked **Euler's formula**:

$$e^{it} = \cos t + i \sin t$$

Why is this true? We certainly could substitute in our favorite Taylor series and verify that it is true, but wouldn't it be better to have a *principled* reason for why this is so? Let's try...

Let $z = e^{it}$. Then $z' = iz$ by that very familiar differential equation. Now, name the real and imaginary parts of z by x and y respectively. Then $z = x + iy$, and $z' = x' + iy'$. On the other hand, multiplying by i gives

$$\begin{aligned} z &= x + iy \\ iz &= ix + i^2 y = -y + ix. \end{aligned}$$

Therefore, $z' = iz$ becomes $x' + iy' = -y + ix$, and so by equating the real and imaginary parts in this equation we get the system

$$\begin{aligned} x' &= -y \\ y' &= x \end{aligned}$$

This is a system of differential equations which is easy to solve with some multivariable calculus, but for now we are stuck. We can observe that $x = \cos t$ and $y = \sin t$ provides a solution, but we cannot say it is the only solution without more tools.

MORAL

This system of ODEs is *easy* to solve with a little bit of multivariable calculus.

57.3 PROBLEM 3

On several occasions, we have referenced the famous series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Why is this true? Could we use discrete calculus to derive it? I don't think so...but here is proof using Taylor series and integration.

Let $u = \arcsin(x)$. Consider the integral

$$\int_{u=0}^{\pi/2} u\,du = \frac{u^2}{2}\bigg|_{u=0}^{\pi/2} = \frac{\pi^2}{8}.$$

On the other hand, the integral in terms of x is

$$\int_{x=0}^{1} \arcsin(x) \frac{1}{\sqrt{1-x^2}}\,dx$$

One can show that the Taylor series for $\arcsin(x)$ is given by

$$\arcsin(x) = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{x^{2n+1}}{2n+1}.$$

Plugging this in gives

$$\frac{\pi^2}{8} = \int_{x=0}^{1} \left(x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{x^{2n+1}}{2n+1} \right) \frac{1}{\sqrt{1-x^2}}\,dx.$$

One can find with careful integration by parts and induction that the inner integral evaluates to

$$\int_{x=0}^{1} \frac{x^{2n+1}}{\sqrt{1-x^2}}dx = \frac{2\cdot 4\cdot 6\cdots(2n)}{3\cdot 5\cdot 7\cdots(2n+1)},$$

(See Details 1)

Plugging this in cancels almost everything, leaving

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

This is the sum of the *odd* reciprocals squared. Giving names to these various sums:

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2}$$
$$S_o = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$
$$S_e = \sum_{n=1}^{\infty} \frac{1}{(2n)^2}.$$

Note that $S = S_o + S_e$ (S_o has the odd terms, and S_e has the even terms). Further,

$$\begin{aligned} S_e &= \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\ &= \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} \\ &= \frac{1}{4}S. \end{aligned}$$

Substituting, we find that $S = S_o + S/4$, and so $S = \frac{4}{3}S_o$. Since $S_o = \frac{\pi^2}{8}$ as shown above, it follows that $S = \frac{\pi^2}{6}$, as desired.

MORAL

This series is *easy* to evaluate with a little bit of multivariable calculus. Well, actually, no: it's not easy, but it is a bit simpler. In the end, some math problems are hard.

57.4 Answers to Selected Exercises

1. If we encountered this integral earlier in the course, we would hit it with a trigonometric substitution $x = \sin\theta$ (and $dx = \cos\theta\, d\theta$, which changes the integral to

$$\int_{\theta=0}^{\pi/2} \sin^{2n+1}\theta\, d\theta.$$

This integral can be found inductively using the following reduction formula:

$$\int \sin^n\theta\, d\theta = \frac{-\sin^{n-1}\theta\cos\theta}{n} + \frac{n-1}{n}\int \sin^{n-2}\theta\, d\theta.$$

(See Proof of Reduction Formula 2)

2. Use integration by parts. Let $u = \sin^{n-1}\theta$ and $dv = \sin\theta\, d\theta$. Then $du = (n-1)\sin^{n-2}\theta\cos\theta\, d\theta$ and $v = -\cos\theta$. It follows that

$$\begin{aligned}\int \sin^n\theta\, d\theta &= -\sin^{n-1}\theta\cos\theta + (n-1)\int \sin^{n-2}\theta\cos^2\theta\, d\theta \\ &= -\sin^{n-1}\theta\cos\theta + (n-1)\int \sin^{n-2}\theta(1-\sin^2\theta)\, d\theta \\ &= -\sin^{n-1}\theta\cos\theta + (n-1)\int \sin^{n-2}\theta\, d\theta - (n-1)\int \sin^n\theta\, d\theta.\end{aligned}$$

Now, solving for the integral $\int \sin^n\theta\, d\theta$ (by adding $(n-1)\int \sin^n\theta\, d\theta$ to both sides of the above equation and dividing by n) gives the desired result.

Applying the reduction formula in the situation at hand gives

$$\begin{aligned}\int_0^{\pi/2} \sin^{2n+1}\theta\, d\theta &= \left.\frac{-\sin^{2n}\theta\cos\theta}{2n+1}\right|_0^{\pi/2} + \frac{2n}{2n+1}\int_0^{\pi/2}\sin^{2n-1}\theta\, d\theta \\ &= \frac{2n}{2n+1}\int_0^{\pi/2}\sin^{2n-1}\theta\, d\theta,\end{aligned}$$

since the first quantity evaluates to 0. Now, induction gives the result.

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58 Foreshadowing

In this module we give a hint at what is to come in multivariable calculus.

58.1 Functions

As in single variable calculus, multivariable calculus is primarily a study of functions. But instead of functions with one input and one output, multivariable calculus looks at functions with multiple inputs and outputs. The notation for a function f with n real inputs and m real outputs is

$$f : \mathbb{R}^n \to \mathbb{R}^m.$$

58.2 Matrices

When dealing with multiple inputs and multiple outputs, it becomes necessary to keep track of several pieces of information when dealing with, say, the derivative. The data structure which makes this possible is a *matrix*, which is an array of numbers arranged in rows and columns. For example, a 4×3 matrix has 4 rows and 3 columns, and might look like

$$\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \\ 2 & 6 & 5 \\ 3 & 5 & 8 \end{bmatrix}.$$

A *square matrix* has the same number of rows and columns. A particular square matrix with a special name is the *identity matrix*, which has 1's on the main diagonal and 0's everywhere else. For example, the 3×3 identity matrix is given by

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Matrix algebra

One feature of matrices is that they can be multiplied, by a slightly unintuitive process. Consider the product of a 2×3 matrix with a 3×3 matrix:

$$\begin{bmatrix} 1 & -1 & 4 \\ 2 & 5 & 7 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 0 \\ 4 & 4 & 3 \\ -2 & 7 & 1 \end{bmatrix} = \begin{bmatrix} -10 & 25 & 1 \\ 10 & 71 & 22 \end{bmatrix}$$

To see where these numbers come from, arrange the matrices a little differently:

(Matrix Multiplication Animated GIF)

To get an entry in the resulting matrix (bottom right), take the corresponding row from the matrix to the left and the corresponding column from the matrix above, multiply their corresponding entries together, and add. The above example shows the calculation for two entries in the result.

Note that for this multiplication to be defined, the number of columns in the first matrix must match the number of rows in the second matrix (otherwise there would not be the correct number of entries to multiply together and add).

There are some nice features of matrix multiplication, and some features which are a little bit different than regular multiplication:

- The identity matrix can be thought of as the matrix equivalent of 1, since multiplying by the identity (of the appropriate size) gives back the same matrix with which we began.
- Matrix multiplication is associative (i.e. $(AB)C = A(BC)$ for appropriately sized matrices A, B, C), but it is *not* commutative (i.e. $AB \neq BA$) in general. Indeed, both orders of multiplication is only defined if A and B are square matrices of the same size.
- There is a matrix version of $\sqrt{-1}$ (again thinking of I as 1). Note that

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I.$$

- It is possible that the product of two non-zero matrices to give the zero matrix:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

58.3 Vectors

Another data structure of importance is a *vector*, which can be thought of as a single row or single column matrix (depending on context). A useful way to visualize a vector is as a difference between two points, or an arrow from one point to another. So a vector can be thought of as a line segment with both a magnitude (the distance between the two points) and a direction (which way the arrow points).

Vectors can be added by visualizing placing the tail of one vector at the head of the other. Vectors can also be multiplied by a matrix to give another vector (the multiplication is just matrix multiplication, again by thinking of the vector as a matrix with just a single column).