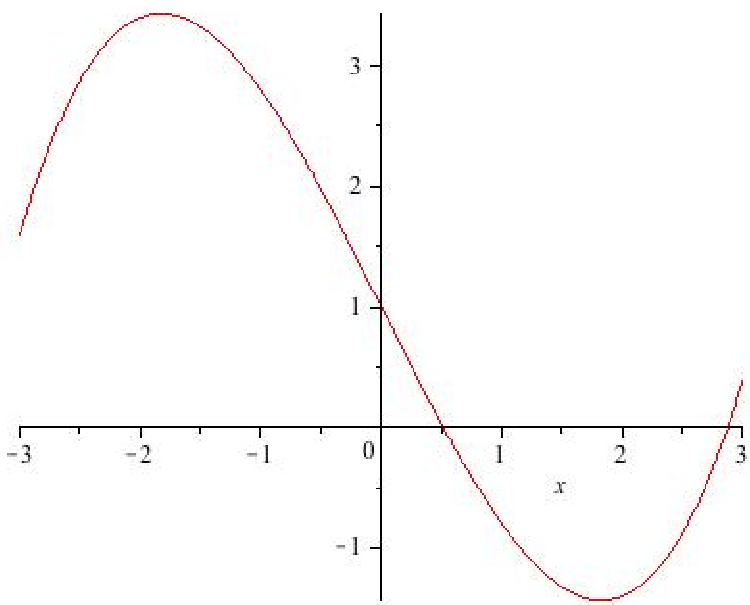
Math 110, Fall 2014

Study packet

1.1 Graphing

1. A graph of the function f is shown. On the same graph, sketch the functions
   (a) $f(x+1)$;
   (b) $f(x/2)$;
   (c) $f(x)-2$;
   (d) $(2/3)f(x)$.



2. Graph the function $x^2 e^{-x}$ on the positive half line $0 \le x < \infty$. Be sure to indicate the way the curve bends near zero, where the maximum occurs, and what is the behavior as $x \to \infty$.

3. Graph the function $x/\sqrt{1+x^2}$ from $-\infty$ to ∞. Indicate any maxima and minima.

4. Graph the function $\dfrac{1}{x^3 - 4x}$. Note all the good features: maxima, minima, asymptotes, a few points where you know the values, etc.

5. Sketch a graph of a function f with these properties:

   (i) $f(0) = 1$;

   (ii) $f(2) = 0$;

   (iii) $f'(x)$ is positive except when $x \in (0, 1)$.

6. Sketch a graph of a function g with these properties:

   (i) $g'(0) = 1$;

   (ii) $g'(2) = 0$;

   (iii) $g''(x)$ is positive except when $x \in (0, 1)$.

1.2 Units, proportionality and word problems

7. Let $P(x)$ be the pressure, in PSI (pounds per square inch) at a depth of x inches below the surface of a body of water. What are the units of $P'(x)$, and what is its interpretation?

8. In the statement, "The faster the acceleration changes, the more a human feels off balance," you feel off balance when what derivative of position with respect to time is large? Also: what are the units of this quantity?

   (i) $f'(t)$

   (ii) $f''(t)$

   (iii) $f'''(t)$

   (iv) $f''''(t)$

9. Suppose $f(T)$ is the number of dollars you have to earn before you owe T dollars in taxes. Write an expression in terms of f for the *marginal tax rate* (how many more dollars you pay in taxes per extra dollar earned) when you earn \$50,000. Be sure to state the units.

10. (a) Write a formula for the number of ball bearings that fit inside a cubical box.

(b) What happens to the number of bearings if you double the edge length of the box (it stays a cube but the length of each edge doubles)?

11. Suppose the amount of a drug remaining in your system after t days follows an exponential law: $A = Ce^{-kt}$. If you have 150 mg. in your system after 2 days and 96 mg. in your system after 4 days, how much do you have in your system after 6 days?

12. A study shows that the number of responses to a charity campaign is proportional to the 0.72 power of the advertising budget. (Do not worry about units because this was a result of empirical curve fitting.)

    (a) Write an equation for this.

    (b) By what factor does the response increase if the advertising budget is increased by a factor of 12? Don't compute it as a decimal, just write the exact expression in terms of powers of numbers and so forth.

    (c) Give an approximate numeric value for this by using your log cheatsheet.

13. The growth of a fund over time is modeled by an exponential $Ce^{\alpha t}$.

    (a) What are the units of C and α?

    (b) If the value is listed as $\$300,000,000$ on January 1, 2014, and as $\$330,750,000$ on July 1, 2014, what value would you project for the announced valuation at the start of the fourth quarter, on October 1, 2014?

1.3 Linear approximations and bounds

14. (a) What is the linear approximation to $\log_{10}(x)$ at $x = a$?

    (b) Choose a so that you can approximate $\log_{10} 111$.

15. If ε is small then what is a good approximation to $\sqrt{1+\varepsilon}$?

16. What is a good approximation to $\sqrt{A+\varepsilon}$ if you know $\sqrt{A}$?

17. Is the linear approximation to $1/(1+x^2)$ near $x = 5$ a lower bound or an upper bound for the function at $x = 6$?

18. If $u = 5^x$ then increasing x from 2 to 2.1 will increase u from 25 to approximately what?

19. Is the approximation $\log_{10} 13 \approx 1 + \dfrac{3}{10 \ln 10}$ an upper or lower bound for $\log_{10} 13$?

20. Because $\tan x$ is incerasing on $(0, \pi/2)$, easy bounds on $\tan(\pi/5)$ are given by $\tan(\pi/6) < \tan(\pi/5) < \tan(\pi/4)$, resulting in the lower bound of $1/\sqrt{3}$ and the upper bound of 1. Improve on either one of these with a bound you are able evaluate.

21. Draw the graph of $y = \sqrt{x}$ with the points $(49, 7)$ and $(64, 8)$ marked. Draw the chord between these points and use it to estimate $\sqrt{52}$. Is the estimate less than or greater than the true value, and why?

1.4 Limits, continuity and inverse functions

22. Does $\lim_{x\to 0^+} \lfloor \sqrt{x} \rfloor$ exist? If so, what is the value? If not, why not?

23. Evaluate $\displaystyle\lim_{x\to\infty} \frac{x^2+1}{3x^2+5}$ and give a reason why you know this is true.

24. If $\lim_{x\to 0^+} f(x) = 3$ and $\lim_{g\to 0^+} g(x) = 4$ then does it follow that $10f - g$ has a limit at 0^+? Please justify your answer.

25. Let $f(x)$ be the function $\sin(x)/x$. Is there a value we can assign $f(0)$ so that f is continuous at 0? If so, what value? If not, why not?

26. Evaluate $\lim_{x\to\infty} x^2 e^{-x}$.

27. Evaluate $\lim_{x\to 0^+} \sqrt{1+x}/(1-x)$.

28. Evaluate $\lim_{x\to 0} \sin(ax)/x$;

29. Does the function x^x have a limit as $x \to 0^+$? If so what? If not, why not?

30. Compute the inverse function for $f(x) = 2 + e^{3x}$ and give the domain and range of the inverse function.

31. Let $f(x) = \sqrt{13 - 4x}$ and let g be the inverse function of f. Evaluate $g(3)$. Be sure to justify any sign choices.

32. Let $h(x)$ be the number of millions of Pentium chips you can sell if you price them at \$$x$ per chip. Write an expression telling you how much to decrease the price per million chips you need to sell beyond the current demand of M million chips.

2.1 Computing exponents and logs

33. Simplify $\ln\sqrt{x}$.

34. Simplify $4^{\log_2 11}$.

35. Simplify $81^{-3/4}$.

36. Simplify $x^{1/\ln x}$.

37. Simplify $10^{\log_{10} 3 + 2\log_{10} 5}$.

38. Simplify $\ln\left(\frac{3 \times 10^4}{2^{10}}\right)$

39. Using the log cheatsheet but no calculator, estimate the following quantities well enough that you can say how many digits and have a guess as to the first digit.

    (a) 2^{40}

    (b) e^{45}

    (c) $(4/3)^{25}$

40. Using the log cheatsheet but no calculator, estimate the following quantities.

    (a) $\log_{10} 300$

    (b) $\ln 300$

    (c) $\ln 305$

2.2 Logarithmic relationships

41. Suppose M is a 30-digit number. Give good lower and upper bounds for $\log_{10} M$.

42. Suppose M is a 30-digit number. Give an estimate for $\log_2 M$.

43. Adding 1 to $\ln x$ does what to the value of x?

44. Doubling x does approximately what to $\ln x$?

45. What number z has its natural logarithm ($\ln z$) exactly halfway between $\ln x$ and $\ln y$?

46. Doubling $\log_{10} x$ does what to the value of x?

2.3 Orders of magnitude

47. In each case, find a function $g(x) = cx^p$ such that $f(x) \sim g(x)$ as $x \to \infty$. In part (a) prove your answer. In each remaining part, just write down what you think it is.

(a) $f(x) = \dfrac{x^2 + 2x + 3}{4x + 5}$

(b) $f(x) = \ln\left(2 + e^{3x}\right)$

(c) $f(x) = \sqrt[3]{x^4 + \cos x}$

(d) $f(x) = \sqrt{x^2 + 1} - x$ [Hint: estimate f by using the linear approximation of the square root function at the point x^2.]

48. True or false?

(a) $x^{10} = o(e^x)$ as $x \to \infty$

(b) $e^x \ll e^{2x}$ as $x \to \infty$

(c) $\sqrt{x} \sim \sqrt[3]{x}$ as $x \to \infty$

(d) $\sqrt{1+x^2} \asymp x$ as $x \to 0$ (note: x is going to zero, not infinity)

49. Find c and p such that $e^x - 1 - x \sim cx^p$ as $x \to 0$. Use L'Hôpital's rule to justify your answer. [Hint: see Problem 6 on Homework 01.]

3.1 Sums

50. These questions relate to the sum $\sum_{j=-3}^{4} A \cdot (3/2)^j$.

    (a) How many terms are there in the sum?

    (b) What is the first term of the sum?

    (c) What is the last term of the sum?

    (d) When the sum is evaluated, what variables will appear in the final expression?

51. Evaluate the sum $\sum_{i=1}^{19} 12i$.

52. Evaluate the sum $\sum_{q=13}^{105} (H + 14q)$.

53. Evaluate the difference between sums: $\sum_{q=1}^{19} \frac{e^q}{1+q} - \sum_{q=1}^{18} \frac{e^q}{1+q}$. [Hint: you can find the difference between these without evaluating either one individually.]

54. Evaluate the sum $\sum_{i=0}^{11} (1/3)^i$.

55. Write the sum $5 + 15 + 45 + 135 + \cdots + 32805$ in Sigma notation and evaluate it.

56. Evaluate the sum $\sum_{n=0}^{\infty} \frac{13}{3^n}$.

57. Write the sum $\frac{1}{6} - \frac{1}{12} + \frac{1}{24} - \frac{1}{48} + \cdots$ (don't overlook the negative signs) in Sigma notation and evaluate it.

58. At the University of California, employees are paid by their level (which increases with seniority and merit). Suppose the list of salary by level for levels 1 thorugh 14 increases in a regular pattern like this: \$36,000, \$41,500, \$47,000, ..., \$107,500.

    (a) What is a formula for a_n if a_n is the salary of an employee at level n?

    (b) Write an expression in Sigma notation for the total salary in a department with fourteen employees, one at each level.

    (c) Evaluate this sum.

59. Mari's credit card charges 1.8% interest per month. There is a minimum payment of \$100 per month to avoid a further finance charge. Suppose Mari charges \$300 per month and never pays more than the minimum.

    (a) How much does Mari owe after three months of this?

    (b) Write an expression in Sigma notation for what Mari owes after five years.

    (c) Evaluate this expression analytically – that means you will have expressions with powers in them.

    (d) Find a numerical approximation to this expression by using ln and its linearization near 1.

3.2 Integrals and Riemann sums

60. Write a Riemann sum in Sigma notation, that has 20 terms and uses the value at the left endpoint, for the integral $\displaystyle\int_3^9 \sin(2\pi x)\,dx$

61. Write an upper Riemann sum with 5 terms for $\int_0^1 x^2\,dx$ and evaluate it.

62. If $f(x)$ is the density of a rod three meters long at a position x meters from the left endpoint of the rod, give an interpretation of $\int_0^3 f(x)\,dx$ and state the units of f and of the integral.

63. If $f(t)$ is the vertical acceleration of an object at time t, then what is the interpretation of $\int_0^x f(t)\,dt$ and what are the units of this integral?

3.3 Comparing sums and integrals

64. Approximate the integral $\displaystyle\int_1^2 \frac{1}{x}\,dx$ by a Riemann sum with three terms, using the value at the left endpoint. Is this an upper or lower bound for $\ln 2$?

65. Approximate $\int_1^7 (1/x)\,dx$ by a right Riemann sum. Is this an upper bound or a lower bound for $\ln 7$?

66. Obtain upper and lower bounds for $\displaystyle\int_0^1 \frac{1}{2+\cos(\pi x)}\,dx$ by using upper and lower Riemann sums with six terms.

4.1 Integrals via substitution

67. Compute the indefinite integral $\int 3x^2\sqrt{x^3+10}\,dx$.

68. Compute the indefinite integral $\int \frac{(\ln x)^p}{x}\,dx$. Be sure to cover all cases: certain values of p may behave differently.

69. Compute the definite integral $\int_0^{\pi/4} e^{\tan\theta}\sec^2\theta\,d\theta$ by using the subsitition $u=\tan\theta$ and writing it as an integral in the u-variable.

4.2 Integration by parts

70. Compute the indefinite integral $\int x^2 e^{3x}\,dx$.

71. Compute the indefinite integral $\int \ln(x+1)\,dx$.

72. Compute the indefinite integral $\int e^{-x}\cos x\,dx$.

73. Compute the indefinite integral $\displaystyle\int x \sec^2 x \, dx$.

74. Compute the definite integral $\displaystyle\int_0^1 e^{\sqrt{x}} \, dx$.

75. Compute the definte integral $\displaystyle\int_0^{10} \frac{x}{\sqrt{1+x^2}} \, dx$.

5.1 Type I improper integrals

76. In each case write the integral as a limit, then say whether the integral converges. If it does not, but goes to ∞ or $-\infty$, please specify that.

(a) $\displaystyle\int_3^\infty \frac{dx}{\sqrt{x^3+1}}$

(b) $\displaystyle\int_{-\infty}^\infty e^{-5x}\,dx$

(c) $\displaystyle\int_{e^2}^\infty \frac{(\ln x)^3}{x^3}\,dx$

(d) $\displaystyle\int_1^\infty \frac{dx}{1+\ln x}$

77. In each case evaluate the improper integral: either a real number, ∞, $-\infty$ or DNE (do not just say DNE when the integral goes to $\pm\infty$). Please state how you know (for example, you can do the integral exaclty, or the integrand is $\sim cx^p$, etc.)

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

(b) $\displaystyle\int_{3}^{\infty} \frac{4}{\sqrt{1+x}}\,dx$

(c) $\displaystyle\int_{1}^{\infty} \frac{x\,dx}{(1+x^2)^2}$

(d) $\displaystyle\int_{0}^{\infty} xe^{-x}\,dx$

5.2 Probability densities

78. (a) For what value of C is Cx^{-2} a probability density on $[2, \infty)$?

    (b) Compute the mean of this probability distribution.

79. (a) For what value of C is Cx^3e^{-x} a probability density on $[0, \infty)$?

    (b) Compute the mean of this probability distribution.

80. For each of these, choose a probability distribution you feel is the best fit from among these choices:

    (i) Uniform (please specify interval $[a, b]$);
    (ii) Exponential (please specify parameter C);
    (iii) Normal (please specify mean and standard deviation).

    (a) The speed of cars passing a monitoring device on the PA turnpike

    (b) The Stanford-Binet IQ scores of children enrolling in New York City's Public School #53.

    (c) Data whose first ten samples look like this when ordered least to greatest::

    $$0.12, 0.17, 0.31, 0.45, 0.72, 0.80, 1.13, 1.46, 1.91, 3.02$$

5.3 Type II improper integrals

81. In each case, write the integral as one or more limits, say whether the integral converges, goes to $\pm\infty$ or DNE. If the entire integral converges to a finite value, find the value.

    (a) $\displaystyle\int_0^1 x^{-1/2}\,dx$

    (b) $\displaystyle\int_{-1}^1 x^{-2}\,dx$

    (c) $\displaystyle\int_{-\infty}^0 \frac{dx}{1+x^3}$

6.1 Taylor polynomials

82. Compute the quartic Taylor polynomial $P_4(x)$ for $f(x) = \dfrac{1}{1-x}$ at $x = 0$.

83. Compute the cubic Taylor polynomial $P_3(x)$ for $\ln(1 + x)$ at $x = 0$ and use it to approximate $\ln 1.2$.

84. For each of these functions compute the quadratic Taylor polynomial $P_2(x)$ centered at 0.

    (a) $f(x) = e^{3x}$

    (b) $f(x) = e^{-x^2}$

    (c) $f(x) = \dfrac{1}{1-x^2}$

85. Compute the cubic Taylor polynomial at $x = 1$ for the function $f(x) = x^{1/3}$.

86. Compute the quadratic Taylor polynomial P_2 for $f(x) = e^{2x}\sqrt{1 + x^2}$ by computing P_2 separately for e^{2x} and $\sqrt{1 + x^2}$ and multiplying.

87. Compute the fourth degree Maclaurin polynomial P_4 for $f(x) = e^{x^2+x^3}$.

6.2 Taylor's remainder theorem

88. (a) What is the linear Taylor polynomial P_1 for the function $f(x) = \tan x$ near $x = \pi/4$?

    (b) Compute $P_1(0.8)$ and state whether it is an over- or under-estimate of $\tan(0.8)$.

    (c) Give a bound using Taylor's theorem on the difference $\tan(0.8) - P_1(0.8)$.

89. (a) Compute the cubic Taylor polynomial at $x = 10$ for the function $1/x$.

    (b) Use this to approximate $1/11$.

    (c) What bound can you give on the remainder R_3?

90. (a) Compute the quadratic Taylor polynomial $P_2(x)$ for the function $\cos(x)$ centered at $x = \pi/4$.

(b) Use this to evaluate $\cos(\pi/5)$, leaving your results in terms of π, $\sqrt{2}$, etc.

(c) Give a bound on the remainder R_2.

7.1 Convergence tests for series and remainder estimates

91. Use the integral test, alternating test or the fact that the terms do not go to zero to determine if each of the following series converges or diverges. Give reasons.

(a) $\sum_{n=1}^{\infty} e^{\frac{1}{n^2}}$.

(b) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$.

(c) $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$.

92. Use the integral or going to zero test along with an asymptotic estimate to determine if each of the following series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{n^3+n+1}{2n^5-n^2-3}$.

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{e^n+n}}$

93. In each case, say whether the series $\sum_{n=1}^{\infty} a_n$ converges. If so, give upper and lower bounds on the difference $L - s_{100}$ where L is the infinite sum $\sum_{n=1}^{\infty} a_n$.

(a) $a_n = 4n^{-3}$

(b) $a_n = \dfrac{\ln n}{n}$

(c) $a_n = \dfrac{(-1)^n}{\ln n}$, where the sum starts at $n = 2$ rather than $n = 1$.

7.2 Ratio and root tests

94. Use the ratio or root test to determine if each of these series converges or diverges. Give reasons.

    (a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{2^n}$.

    (b) $\displaystyle\sum_{n=1}^{\infty} \frac{3^k k!}{(k+2)!}$.

    (c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$.

    (d) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n} + 3}{5^n}$.

95. By any means you can, determine whether each of the following series converges or diverges. Give a reason. You may find these problems more difficult than the previous ones because you are not told which method to use, however in two cases, it is just a matter of figuring out that the terms do not go to zero.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n^2 + 2^n}$.

(b) $\displaystyle\sum_{n=1}^{\infty} (\ln 2n - \ln n)$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1 + 4^n}{6^n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2 + 3n}$

7.3 Power series

96. For which values of x does the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n x^n$ converge?

97. For which values of x does the series $\displaystyle\sum_{n=1}^{\infty} 2^{n/2} x^n$ converge?

98. For which values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{14x^n}{n!}$ converge?

99. Find the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} n^{-3} x^n$.

100. (a) Compute the Taylor series for the function $\ln(x)$ centered at $x = 1$

(b) Determine the interval on which this series converges, including whether the series converges at each endpoint.

101. (a) Compute the Taylor series for the function e^{-x^2} centered at $x = 0$.

(b) Determine the interval on which this series converges. including whether the series converges at each endpoint.

102. Determine the value of $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n!}$ by recognizing the function whose power series is $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and evaluating it at $x = 3$.

8.1 Word problems

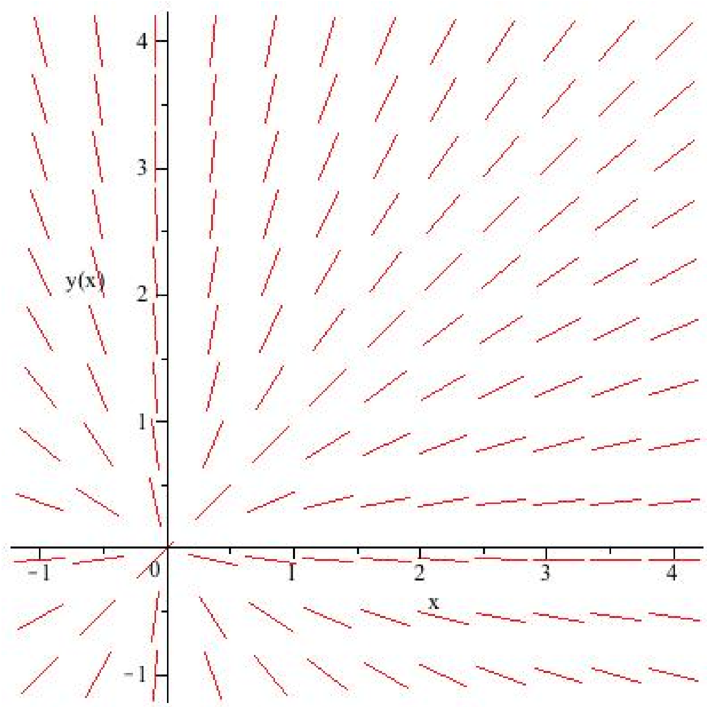
103. The rate of growth of a population of bacteria is proportional to the present population and to the difference $T - T_1$ between the temperature, T, and the equilibrium temperature, T_1. Write a differential equation for this. Be sure to say what each variable or constant stands for and to give units.

104. A disease affecting elm trees spreads so that the rate of new infections is proportional to to the square root of the number of infected trees. Write a differential equation for this. Be sure to say what each variable or constant stands for and to give units. [Bonus: Why would such a thing spread at a rate proportional to the square root of the size of the infected population?]

105. A certain chemical reaction depletes oxygen at a rate proportional to the square of the concentration of oxygen. Write a differential equation for this. Be sure to say what each variable or constant stands for and to give units.

8.2 Slope fields

106. This is the slope field for which one of the following differential equations? State briefly how you know.



(i) $y' = x + y$

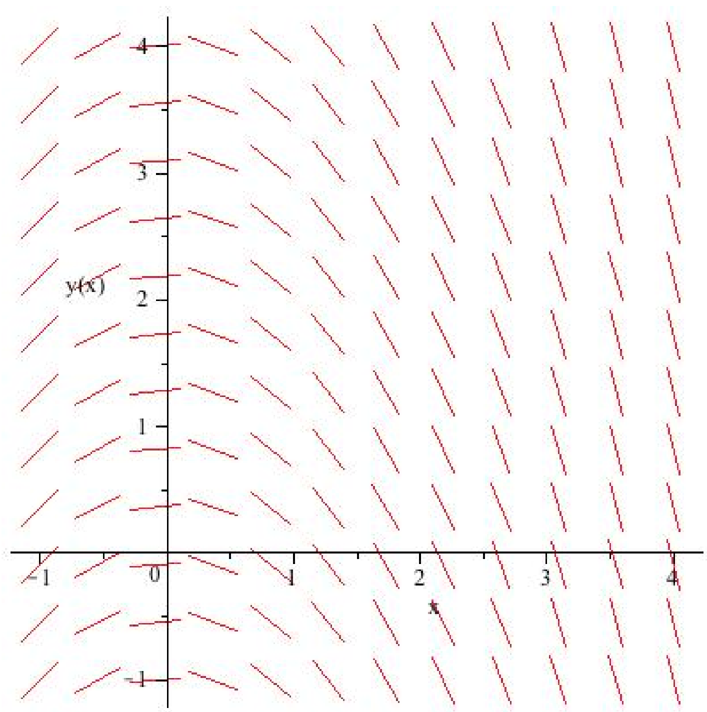
(ii) $y' = x$

(iii) $y' = \dfrac{y}{x}$

(iv) $y' = \dfrac{x}{x+y}$

(v) $y' = \tan\theta$

107. This is the slope field for which one of the following differential equations? State briefly how you know.



(i) $y' = x$

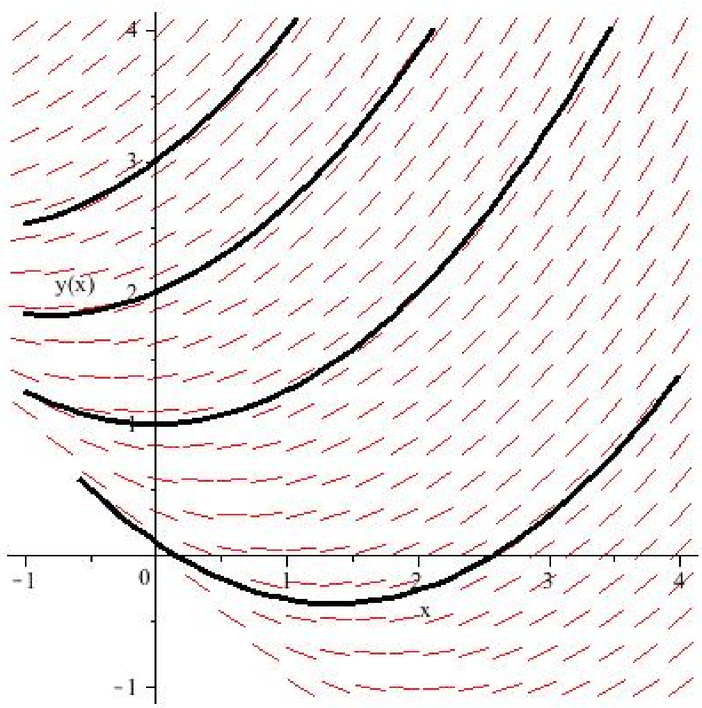
(ii) $y' = y$

(iii) $y' = -x$

(iv) $y' = y - x$

(v) $y' = x - y$

108. A slope field is shown along with several solution curves. Which one of the following could be the differential equation for which these were drawn? State briefly how you know.



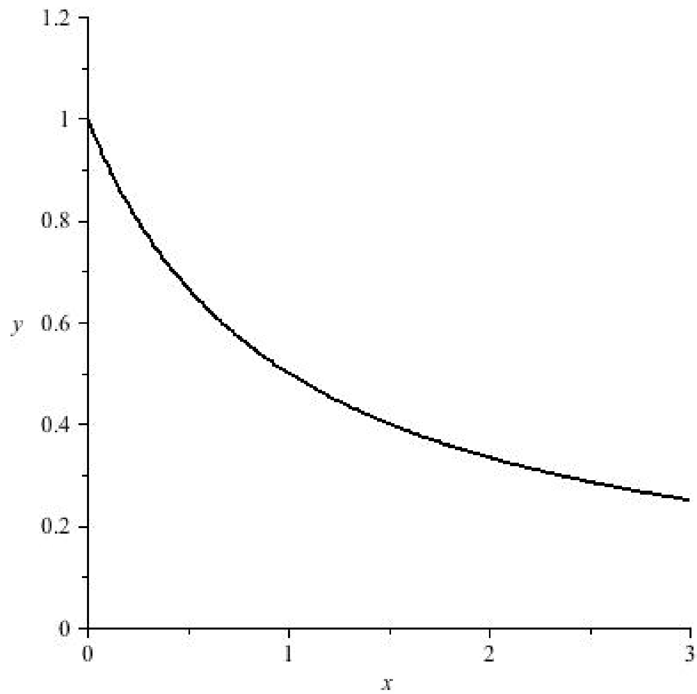
(i) $y' = y - 1$

(ii) $y' = x - 1$

(iii) $y' = 1 - \dfrac{1}{y}$

(iv) $y' = \sqrt{x + y} - 1$

109. The following function could be a solution to which one of these differential equations? State briefly how you know.



(i) $y' = -\dfrac{1}{x}$

(ii) $y' = -xy$

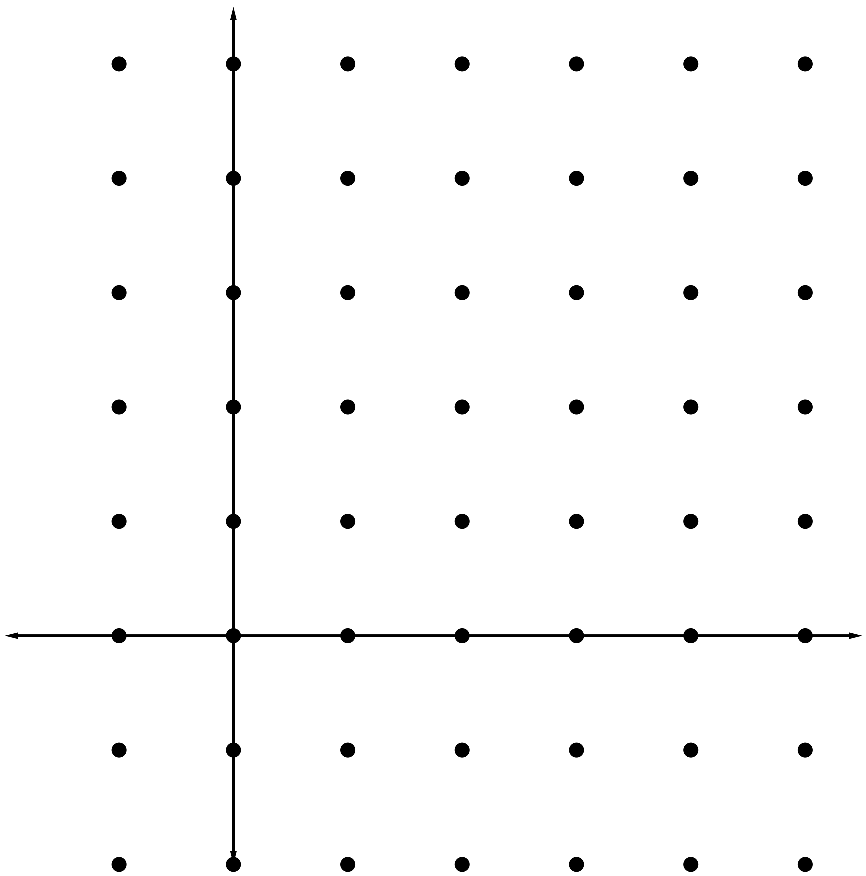
(iii) $y' = \dfrac{-y}{1+x}$

(iv) $y' = \dfrac{1}{y} - 1$

(v) $y' = e^{-x}$

110. Draw the slope field for $y' = (x-2)^2 - y$ for integer values of x and y between 0 and 4. Then use this to sketch the solution to the IVP

$$y' = (x-2)^2 - y\,; \qquad y(0) = 1\,.$$



111. The function $y(x) = \dfrac{x^2}{4} - x$ solves which of these differential equations (possibly more than one)?

(a) $y' = \dfrac{y}{x}$

(b) $y' = \dfrac{y + x^2/4}{x}$

(c) $y' = \sqrt{x+y} - 1$

8.3 Euler iteration

112. Suppose that $y' = x - y$ and $y(0) = 1$.

    (a) Approximate $y(2)$ using Euler iteration with step size 1.

    (b) Approximate $y(2)$ using Euler iteration with step size 1/2.

    (c) **This part will be done in Section 9.2:** Solve the differential equation and evaluate $y(2)$ precisely.

113. Suppose that $y' = 0.7y$ with $y(0) = 1$.

    (a) Find the formula for the approximation A_n to $y(1)$ when you use Euler's method with n steps.

    (b) What is $\lim_{n\to\infty} A_n$?

9.1 Exponential behavior

114. (a) Find the general solution for the differential equation $\dfrac{dN}{dx} = -\dfrac{N-29}{3}$.

(b) Find the particular solution if $N(1) = 47$.

(c) For this particular solution, what is $N(0)$?

115. (a) Solve the differential equation $\dfrac{dy}{dt} = k(500 - y)$.

(b) If $y(1) = 200$ and $y(3) = 488$, what is $y(0)$?

116. Suppose $P'(t) = CP(t)$ with $C = 1/45$ inverse years and $P(0) = 8 \times 10^9$ (the present world population). Approximately when will the world population exceed 1.6×10^{10} (sixteen billion)?

117. A balloon loses air pressure at a rate proportional to the difference between its pressure and the atmostpheric pressure of 14 PSI.

(a) Write a differential equation for this scenario assuming an initial pressure of size A. Be sure to say what each variable or constant stands for and to give units.

(b) Solve this initial value problem.

(c) If the balloon goes from 38 PSI to 26 PSI in one day, how many days will it take to get down to 17 PSI?

118. Describe the long term behavior of the solution $y(t)$ to the differential equation

$$\frac{dy}{dt} = k(B - y)\,.$$

You may assume that k and B are positive constants. If the long term behavior depends on the initial conditions, indicate how.

9.2 Separable equations

119. (Problem 7.2.12 from the book) Give the general solution of this equation.

$$\frac{dy}{dx} = 3x^2 e^{-y} .$$

120. (Problem 7.2.14 from the book) Give the general solution of this equation.

$$\sqrt{2xy}\frac{dy}{dx} = 1 .$$

121. A disease affecting elm trees spreads so that the rate of new infections is proportional to to the square root of the number of infected trees. In other words,

$$\frac{dS}{dt} = kS(t)^{1/2}$$

where $S(t)$ is the number of sick trees at time t.

(a) Find the general solution to this equation.

(b) At present there are 10,000 infected trees. A year ago there were only 8,100 infected trees. Compute the constant(s) to find the particular solution consistent with these data.

(c) How long ago should the first infected elm have been observed?

122. A certain chemical reaction depletes oxygen at a rate proportional to the square of the concentration of oxygen. In other words the amount of oxygen $y(t)$ satisfies the equation $y' = -ky^2$.

(a) Find the general solution to this equation.

(b) The concentrations of oxygen at times $t = 0$ and $t = 1$ minute are observed to be 0.018 moles per liter and 0.012 moles per liter respectively. At what time will the concentration drop below 0.01 moles per liter?

123. An investment account guarantees to grow at least as fast as the *prime rate*, which is an interest rate $r(t)$ that varies with time.

(a) Write a differential equation for this scenario assuming an initial deposit of size A. Be sure to say what each variable or constant stands for and to give units.

(b) Solve this initial value problem (the solution will be in terms of the unknown function, r).

(c) What will be the guaranteed size of the account after five years, again, in terms of the unknown function $r(t)$?

124. A snowball melts so that its volume changes at a rate proportional to its surface area. Initially the snowball has a radius of 10 cm. Write and solve the inital value problem giving the volume in terms of the elapsed time. There will be an undetermined constant. (Note: the new part of this problem is that you are supposed to solve the equation.)

9.3 Blowups

125. Does the equation in Problem 119 have a vertical asymptote (y goes to infinity at a finite x-value), no asymptote (y goes to infinity as x goes to infinity), or a horizontal asymptote (y remains bounded)?

126. Does the equation in Problem 120 have a vertical asymptote, horizontal asymptote or no asymptote?

9.4 First order linear differential equations

127. (Problem 9.2.6 from the book) Find the general solution to this equation.

$$(1+t)y' + y = \sqrt{t}$$

128. (Problem 9.2.20 from the book) Solve this initial value problem.

$$\frac{dy}{dx} + xy = x, \quad y(0) = 6$$

129. Write down the most explicit solution you can to the initial value problem

$$(1+t^2)\, y' - ty = t\,; \quad y(0) = 1\,.$$

130. Solve the initial value problem $y' = x - y$ with $y(0) = 1$. Then evaluate $y(2)$.

131. Money is deposited in a bank account with a nominal annual interest rate of 3% compounded continuously. Also, money is being added to the account continuously at a rate of \$500 per year, and no withdrawals are made. Write the differential equation for the amount in the account at time t, and solve the initial value problem corresponding to an initial deposit of size \$1,000.

132. A 20-quart juice dispenser in a cafeteria is filled with a juice mixture that is 10% mango and 90% orange juice. A pineapple-mango blend (40% pineapple and 60% mango) is entering the dispenser at a rate of 4 quarts an hour and the well-stirred mixture leaves at a rate of 5 quarts an hour. Write and solve the equation to say how much mango juice is in the container at time t.

10.1 Functions of two variables

133. Let R be the rectangle $[0,1] \times [1,3]$ and let $f(x,y) = \sqrt{y-x}$. Approximate f by dividing the rectangle in half in each direction (four pieces in total) and computing the Riemann sum obtained by evaluating f at the midpoint of each small rectangle. Use a calculuator.

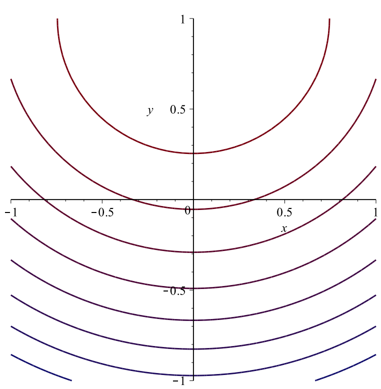
134. Which function has this contour plot?

(i) $|y-1|$

(ii) $|y-1|+|x|$

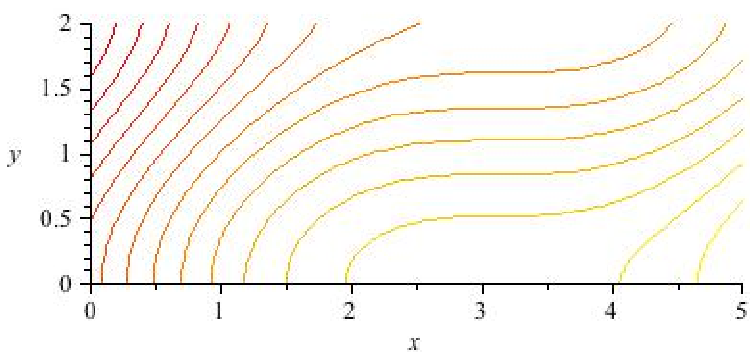
(iii) $x^2+(y-1)^2$

(iv) $x/(1+y^2)$



135. A hiker is at position $(0.5, 0.5)$ on the topographical map shown in the picture. Which of these directions will be closest to a level path? NW, N NW, N, N NE, NE, E NE, E

At which of these points is the terrain flattest? $(0.5, 0.5)$, $(3.5, 0.2)$, $(0.2, 1.6)$



10.2 Integrals over rectangles

136. Compute the following integrals.

(a) $\displaystyle\int_3^5 \int_{-2}^0 (1 + x + y + xy)\; dx, dy$

(b) $\displaystyle\int_1^2 \int_0^\infty e^{-yx}\, dx\, dy$

137. Write down the integral that computes the volume under the surface $z = y^2(1+x^2)$ over the square of side 2 centered at the origin. Then compute the integral.

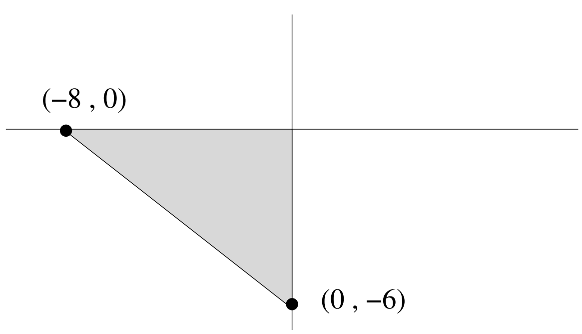
10.3 Integrals over general regions

138. In each case, draw the region (if not already drawn), then give limits of integration for the region in two ways: horizontal srips then vertical strips. Your answers should look like $\int_a^b \int_c^d (\cdots)\, dx\, dy$ or $\int_a^b \int_c^d (\cdots)\, dy\, dx$ where z, b, c and d are replaced by numbers or expressions involving x or y.

    (a) The region between the graph $y = e^x$ and the lines $x = 0$ and $y = e^2$.

    (b) The region inside the disk of radius 5 where $x + y \geq 7$.

    (c) The region in this picture.



139. In each case sktech the region of integration and evaluate the double integral.

(a) (Problem 24 in Section 15.2 of the textbook) $\displaystyle\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}}\, dy\, dx$

(b) (Problem 67 in Section 15.2 of the textbook) $\displaystyle\int_0^3 \int_0^{2-2x/3} \left(1 - \frac{1}{3}x - \frac{1}{2}y\right) dy\, dx$

(c) $\displaystyle\int_{-3}^3 \int_0^{3-|y|} x^2\, dx\, dy$

140. In each case compute the integral by reversing the order of integration. You will probably need to sketch the region.

(a) (Problem 50 in Section 15.2 of the textbook) $\displaystyle\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dy\, dx$

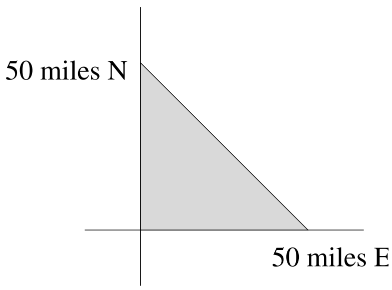
(b) $\displaystyle\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$

10.4 Applications of double integrals

141. For what value of C is the function $C(x+y)$ a probability density on the unit square $0 \leq x, y \leq 1$? What is the mean of the X variable for this probability density?

142. A point (X, Y) is chosen uniformly at random in the quarter of the unit disk that lies in the first quadrant. What is the corresponding probability density and what is the mean of the Y variable?

143. The rainfall in an equatorial region decreases as you go away from the equator. At a point x miles east and y miles north from the center the average annual rainfall is given by $f(x, y) = 100/(1 + y^2)$ centimeters. Suppose the region is triangular as in the picture. What is the average annual rainfall over the region?



144. Due to signal strength issues, the density of businesses per square mile at a distance r miles from the region's only cell tower is $100e^{-r^2}$. Write an integral that gives the total number of businesses in the region.

145. What is the average of the quantity $1 + xy^2$ over the square $0 \leq x, y \leq 2$?

11.1 Partial derivatives and the Increment Theorem

146. (Problems # 11, 16 and 19 from Section 14.3 of the book) Compute $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ for each of the following functions.

(a) $\dfrac{x+y}{xy-1}$

(b) $e^{xy}\ln y$

(c) x^y

147. A function f has the following values. Estimate the value of $\dfrac{\partial f}{\partial x}(15,23)$.

x	y	$f(x,y)$
15	23	100
23	15	102
15	24	98
17	23	99
22	15	103

148. Let $f(x,y) = \sqrt{x+\ln y}$. Use the increment theorem and the fact that $f(9,1) = 3$ to estimate the quantity $\sqrt{9.1+\ln(0.8)}$.

11.2 Chain rule

149. (Problems 13 and 15 from Section 14.4 of the book) In each case, draw a branch diagram. Then write a chain rule for the derivative of the dependent variable with respect to each of the independent variables.

   (a) $z = f(x, y)$, $x = g(t)$, $y = h(t)$.

   (b) $w = f(x, y, z)$, $z = f(x, y)$, x and y are independent variables.

150. Use the chain rule to evaluate $\dfrac{\partial f}{\partial t}$ at $(s, t) = (1, 2)$.

$$\begin{aligned} x &= \sqrt{s + 4t} \\ y &= \ln(t - s^2) \\ f(x, y) &= x^2 e^y \end{aligned}$$

151. Use the chain rule to evaluate $\dfrac{\partial z}{\partial t}$ at $(s, t) = (5, 1)$ if

$$\begin{aligned} z &= \sqrt{x^2 + y^2} \\ x &= s e^{s-5t} \\ y &= \frac{s^2 - t^2}{2} \end{aligned}$$

152. In each of these scenarios, f is a function of the two variables x and y, and you are asked to write the given rate using the notation of partial derivatives.

[Hint: two of these problems do not require the chain rule; one requires the chain rule in the case of a branch diagram that is a diamond; one requires the chain rule with the triangular branch diagram shown on the second page of the course notes for Unit 11.2.]

(a) If the y-coordinate remains constant, what is the rate of change of $f(x, y)$ with respect to the x-coordinate?

(b) If the x-coordinate remains constant and the y-coordinate increases by a little, the value of $f(x, y)$ increases by roughly what multiple of the increase in y?

(c) If the x- and y-coordinates vary so that $y = g(x)$, what is the rate of change of $f(x, y)$ with respect to x?

(d) If both x and y are functions of time, how rapidly is $f(x, y)$ changing?

11.3 Implicit differentiation

153. (Problem 3.7.7 from the book) Use implicit differentiation to find dy/dx if $y^2 = \dfrac{x+1}{x-1}$.

154. (Problem 3.7.41 from the book) Find the two points where the curve $x^2+xy+y^2 = 7$ crosses the x-axis. Use implicit differentiation to find the slopes of the tangents at these two points. Do the tangents intersect, coincide, or are they parallel without coinciding?

155. Suppose that $xy+e^x z+y\ln z$ is held constant. Differentiate implicitly with respect to x to find the rate of change of z with respect to x when y is held constant.

156. A point moves in three dimensions, always satisfying the equation

$$xy + xz + yz + xyz = xyz^2 .$$

   (a) If z is held constant, what is the rate of change of y with respect to x?

   (b) If x and y are both functions of time, how rapidly is z changing?

11.4 Word problems

157. Joe the Trader plans to buy shares of an undervalued stock. Suppose that a bid for y shares can be completed at a share price of u dollars per share and that Joe believes the value of the stock to be x dollars per share.

   (a) Write a formula for the amount W that Joe gains in net worth, as a function of x, y and u.

   (b) What partial derivative of W expresss how much Joe's net gain goes up or down as he changes the number of shares he buys, but not his valuation of the stock or the bid price?

   (c) The problem with a huge buy is that the more shares he bids for, the higher the share price will be. To model this, assume $u = f(y)$. Use the chain rule to say how Joe's net gain changes with the number of shares he bids for, assuming his private valuation of the stock remains constant.

   (d) If Joe wants to maximize his net gain, what equation must be satisfied? You may assume that all quantities x, y, f, f' are positive.

158. Housing market data indicates that the value of a house in a given neighborhood is proportional to the square footage and to the 2/3 power of the lot size. A buyer is considering a 2400 square foot house on a lot of 1.00 acres. If the buyer desires just a little more land but is unable to pay any more, how many square feet will the buyer have to forgo per acre of additional land?

12.1 Vectors

159. Let $\mathbf{v}$ be the vector $-3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$. Compute a unit vector parallel to $\mathbf{v}$.

160. Let $\mathbf{v} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ and let $\mathbf{w}$ be a vector of length 3 in the 45° northeast direction.

    (a) Compute $\mathbf{v} \cdot \mathbf{v}$.

    (b) Compute $\mathbf{v} \cdot \mathbf{w}$.

    (c) Compute a unit vector $\mathbf{u}$ in the direction of $\mathbf{w}$.

    (d) Compute $\mathbf{v} \cdot \mathbf{u}$.

161. The vector $\mathbf{v} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$ has magnitude 10. Its direction is 30° counterclockwise from pointing directly east. What are a and b?

162. The vector $5\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$ is parallel to which of these vectors?

    (i) $6\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$
    (ii) $5\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$
    (iii) $34\hat{\mathbf{i}} - 34\hat{\mathbf{j}}$
    (iv) $\hat{\mathbf{i}} - (3/5)\hat{\mathbf{j}}$
    (v) $(1/5)\hat{\mathbf{i}} - (1/3)\hat{\mathbf{j}}$
    (vi) $(\sqrt{34}/5)\hat{\mathbf{i}} - (\sqrt{34}/3)\hat{\mathbf{j}}$

163. What is the dot product of a vector of length 4 in the 45° northwest direction and a unit vector in the 45° southwest direction?

12.2 Gradients

164. Let $f(x, y) = \dfrac{x}{\sqrt{x + \sin y}}$.

   (a) Compute the gradient of f and evaluate it at the point $(2, \pi)$.

   (b) If the x-coordinate increases at rate one and the y-coordinate decreases at rate $1/2$, starting at the point $(2, \pi)$, approximately what value does f take after 0.1 time units?

   (c) At what rate is f changing if (x, y) is currently $(2, \pi)$ but is moving at velocity given in vector form by $3\hat{\mathbf{j}}$?

   (d) In what direction should you move from $(2, \pi)$ in order for f to increase the fastest?

   (e) What is the rate of change in f per unit move in that direction?

   (f) What direction should (x, y) move in order to stay roughly constant?

165. Suppose f is a function of two variables taking the value 3 at $(5, 10)$ and having gradient $\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$ there. Suppose g takes value 4 at $(5, 10)$ and has gradient $3\hat{\mathbf{j}}$ there.

   (a) Use the properties of the gradient on page 836 to compute the gradient of $f + g$ at $(5, 10)$.

   (b) Use the properties of the gradient on page 836 to compute the gradient of fg at $(5, 10)$.

   (c) In which direction should you move if you want $f + g$ to increase the fastest?

   (d) In which direction should you move if you want the function g to neither increase nor decrease?

166. Survey data is used to obtain consumer preference information. The data indicates a good model for utility among purchasers of dishwashers is $u = \sqrt{s - 2000}/n$ where s is the space in cubic inches and n is the noise in decibels.

   (a) Compute ∇u.

   (b) Evaluate ∇u for the current model of dishwasher, which has 4500 cubic inches of space and a super-quiet noise level of 20 decibels.

   (c) Each year, the new improved model is 100 cubic inches larger and 2 decibels quieter. What is the present rate of increase of the consumer utility function (in utility per year) for this line of dishwashers?

   (d) The level curve of $u(x, y)$ through the point $(4500, 20)$ has what slope?

   (e) What is the marginal rate of substitution: how many more cubic inches of space would be required per decibel increase in noise level in order to maintain the same customer satisfaction as with the 4500 cubic inch 20 decibel dishwasher?

12.3 Optimization on a curve

167. Find the maximum of $x + y$ on the curve $x^2 + 3xy + y^2 = 14$.

168. The maximum of $3x + 2y$ over the curve $f(x, y) = 10$ occurs

   (i) At one of the endpoints

   (ii) Where ∇f is equal to zero

   (iii) Where ∇f is parallel to $3\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$

   (iv) Where ∇f is perpendicular to $3\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$

169. Find the maximum and minimum of the function $f(x, y) = x + 2y$ on the astroid $\sqrt{x} + \sqrt{y} = 1$.

170. The cost to produce a dishwasher is given by the formula $c = (s + 500)/n$, where s is the space in cubic inches and n is the noise produced in decibels.

(a) What is ∇c?

(b) Recall from problem #166 that we model consumer utility function for dishwashers as $\sqrt{s - 2000}/n$. To find the minimum of c on the indifference curve $u = 5/2$, what equation must be satisfied in addition to $u = 5/2$?

(c) Are these equations satisfied at $s = 4500, n = 20$?

12.4 Optimization over a region

171. Find the maximum of $x^2 + 2y^2$ over the triangle $x \geq 0, y \geq 0, x + y \leq 10$.

172. Find the maximum of $x - x^2 - y^2$ over the square $-1 \leq x, y \leq 1$.

173. Cell towers sit at three points: $(-9, -11), (15, 0)$ and $(0, 8)$. The cost to build a cable network centered at (x, y) that coordinates the three towers is the sum of the squares of the distances to the towers from (x, y).

   (a) Write a formula for the cost of a hub at (x, y).

   (b) Find the location for which the cost is minimized.