7 Infinite series

This topic is addressed in Chapter 10 of the textbook, which has ten sections and gets about four weeks in Math 104. That's because it is an important foundation for differential equations, Bessel functions, and Fourier analysis. In Math 110, however, we need this topic for only two reasons, which can be covered in one week. (1) There are some useful infinite sums. In Unit 3 you already saw the most important example of this, the infinite geometric series, which is used for modeling the total lifetime value of an asset or debt. (2) Understanding the Taylor series will help to make sense of the material you just learned concerning Taylor polynomials. For example, is e^x really equal to the infinite sum $1 + x + x^2/2! + x^3/3! + \cdots$?

7.1 Convergence of series: integral test and alternating series

As mentioned before, we are not going to cover the more than a dozen variants of theorems about when infinite series converge. You can get by with just a few methods: comparing to an integral, comparing to a geometric series, and using sign alternation.

The integral test

Here's an example: Does $\sum_{n=1}^{\infty} n^{-2}$ converge? It's hard to tell from summing the first few terms $1 + 1/4 + 1/9 + \cdots$. This sum should be very similar to $\int_1^{\infty} x^{-2}\, dx$. Does this improper integral converge? Yes. How can we be sure the sum behaves like the integral? We have to somehow compare the sum and the integral. This will be your first in-class problem.

We can generalize this into a theorem, which may be found in Section 10.3 of the textbook on page 594. We will discuss the nuances of the theorem in class.

> **Integral test:** Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$ where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\, dx$ either both converge or both diverge.

Alternating series

If the terms $\{a_n\}$ alternate in sign and decrease in magnitude with a limit of zero then $\sum_{n=1}^{\infty} a_n$ must converge. This is more intuitive than it looks. For example, the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

fails to converge (integral test) but its alternating version

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

converges. To see why, let's write out the series in long form so it is not obscured by notation.

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}} + \cdots$$

In decimal approximations, that's

$$1 - 0.71 + 0.58 - 0.50 + 0.45 - 0.41 + \cdots \ .$$

Do you see why the partial sums converge? The partial sums are $1, 0.29, 0.87, 0.37, 0.82, 0.41, \ldots$. These alternate down, up, down up,... but notice that after each up the partial sum is not as it was before, and after each down is it not as low as it was before. That means that

> Each partial sum ending in a positive term is an upper bound for the infinite sum;
>
> Each partial sum ending in a negative term is a lower bound for the infinite sum.

This it useful: bounds for alternating series are easy! But also it should make it clear why the theorem is true: you have a bunch of upper and lower bounds getting closer to each other, so the squeeze the series to a limit.

7.2 Convergence of series: ratio and root tests

Let's start with a triviality: if the terms of a series do not go to zero then the series can't possibly converge. Believe it or not, this gets its own theorem, test and example on pages 588–589 in Section 10.2 of the textbook. If the terms do go to zero, then the series still may not converge if the terms get small too slowly. For example, $\sum_{n=1}^{\infty} 1/\sqrt{n}$ does not converge (use the integral test). One case where we know the terms get small fast enough is when they decrease like a geometric series. If $r < 1$ and $a_n \leq Kr^n$ for any constant K, then $\sum_{m=1}^{\infty} a_n$ will converge. (Again, note, for convergence issues, mutliplying by a constant doesn't affect anything.)

It's not always easy to test whether $a_n \leq Kr^n$ and it's not necessary either. If $\sqrt[n]{a_n}$ has a limit, $r < 1$, then a_n is enough like r^n to guarantee convergence. The following statement of this is simplified somewhat from the form in Section 10.5, page 608.

Root test: Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|} = r < 1$. Then $\sum_{n=1}^{\infty} a_n$ converges.

If you're having trouble computing the limit of $|a_n|^{1/n}$ you can always try the ratio test.

Ratio test: Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r < 1$. Then $|a_n|^{1/n} \to r$ and hence the series converges.

In both cases, if $r > 1$ then the terms get big, hence the series can't possibly converge, but if $r = 1$ you don't know anything. Let's try these tests on some series.

Example: Does $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converge?

Root test:

$$\begin{aligned} \lim_{n\to\infty} \left(\frac{n}{2^n}\right)^{1/n} &= \frac{\lim_{n\to\infty} n^{1/n}}{\lim_{n\to\infty} (2^n)^{1/n}} \\ &= \frac{1}{2}. \end{aligned}$$

Because $1/2 < 1$ we conclude that this series converges.

Ratio test: This is a little easier because it does not require evaluating $\lim_{n\to\infty} n^{1/n}$.

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n} \cdot \frac{1}{2}$$

which has the obvious limit $\frac{1}{2}$. Again we conclude that $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges.

Example: Does $\sum_{n=1}^{\infty} \frac{5^n}{n!}$ converge?
Ratio test: Let $a_n = 5^n/n!$. Then $a_{n+1}/a_n = 5/(n+1)$. Clearly $\lim_{n\to\infty} 5/(n+1) = 0$, so the ratio test tells us that this series goes to zero plenty fast for convergence to occur: all that was needed was a limit less than 1, and we managed to get 0.

7.3 Power series

One very important class of infinite series are the **power series** which are series that has a variable x occurring in a very specific way: it has the form $\sum_{n=1}^{\infty} a_n x^n$. If you plug in a real number for x then you get a series that you can try to sum. As it is, it is a function of x, except that for some values of x it may be undefined.

Let me say that again: THIS SERIES IS A FUNCTION OF x. Think this over till it makes sense.

If you can evaluate $\lim_{n\to\infty} |a_{n+1}/a_n|$ and it is equal to a real number r, then $\lim_{n=\infty} |a_{n+1}x^{n+1}/(a_n x^n)|$ will equal $r \cdot |x|$. So you'll get convergence if $|rx| < 1$ and not when $|rx| > 1$. When $|rx| = 1$, you won't know without further examination.

Similarly if you can evaluate $\lim_{n\to\infty} |a_n|^{1/n}$ and you get r, again $\lim_{n\to\infty} |a_n x^n|^{1/n}$ will be $r|x|$. Again the series will converge when $|x| < 1/r$ and diverge when $|x| > 1/r$.

So that's really all you need to know about power series. Here are some examples.

Example: For which x does the series $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n x^n$ converge? The coefficient $(2/3)^n$ is an easy candidate for either the ratio or the root test, resulting in a limt of $r = 2/3$. Therefore the sum converges when $(2/3)|x| < 1$, that is, $|x| < 3/2$ and diverges when $(2/3)|x| > 1$, that is, $x > 3/2$. What about when $|x| = 3/2$? We get no information from the ratio or root tests. The series, for $x = 3/2$, is $\sum_{n=1}^{\infty} 1$, while the series for $x = -3/2$ is $\sum_{n=1}^{\infty} (-1)^n$. In neither case does the term go to zero, so in neither case does the series converge.

Example: For which x does the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converge? Taking $a_n = 1/n!$, we see that $a_{n+1}/a_n \to 0$, so the limiting ratio (and therefore root) is 0. This means we get convergence if $0 \cdot |x| < 1$, in other words, for all real x. Another way of saying this is that factorials grow way faster than any power, or equivalently $x^n = o(n!)$ for any x, which means the terms of the series go to zero quite rapidly.

Example: For which x does the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ converge? Letting $a_n = 1/n$, both ratio and root test result in a limit of $r = 1$. For example, $a_{n+1}/a_n = n/(n+1)$

which clearly converges to 1. Therefore the series converges for $|x| < 1$ and diverges for $|x| > 1$. When $x = 1$ we get the famous **harmonic series** $1+1/2+1/3+1/4+\cdots$. This is one you should commit to memory if you haven't already: by the integral test, because $\int_1^\infty (1/x)\,dx$ diverges, so does the harmonic series. When $x = -1$ we get the **alternating harmonic series** $1 - 1/2 + 1/3 = 1/4 + \cdots$. This converges by the alternating series test, so we see that the power series $\sum_{n=1}^\infty x^n/n$ converges exactly when $x \in [-1, 1)$.

Taylor series

The Taylor series is just the Taylor polynomial with $n = \infty$.

Definition of Taylor series: Let a be any real number and let f be a function that is *smooth* at the point a, meaning it can be differentiated infinitely often. The **Taylor series** for f about the point a is the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \,.$$

For any particular value of x this series may or may not converge. The Taylor series with $a = 0$ is called the **MacLaurin series**:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k \,.$$

Example: What it the Taylor series for the function $f(x) = 1/(1-2x)$ at $x = 0$ and for which values of x does it converge? The easiest way to compute this is by composition. Remember computing the Taylor polynomials for $1/(1-x)$ and getting $P_n(x) = 1+x+x^2+\cdots+x^n$? Evidently the Taylor series for $1/(1-x)$ is $1+x+x^2+\cdots$. Substituting $2x$ for x we find that the Taylor series for $1/(1-2x)$ is

$$1 + (2x) + (2x)^2 + (2x)^3 + \cdots \quad = \quad 1 + 2x + 4x^2 + 8x^3 + \cdots \,.$$

This converges when $|2x| < 1$, thus $|x| < 1/2$. If you graph it, you will see why the series might have problems converging beyond that.