MATH 106 Course Packet

Theory of Geometry
for Pre-service Teachers

Contents

Math 106 (5 Credits)
Fundamental Math Concepts for Teachers, II: geometry
Winter 2003

Prerequisite:	Continuation of Math 105
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Text: The only text is the course packet available at the University Bookstore. It is packaged with other equipment that you will need (compass, protractor, graph paper, and so forth); please buy the packet ASAP.

Grading: Your grade in this course will be based on:

Attendance and participation 15%. Being late or leaving early counts as half an absence. Each of the 48 days (10 weeks minus 1 midterm and 1 holiday) day counts as 1/3 of 1%. That means anyone who misses at most 3 days can get 100% of the grade for attendance, assuming their participation is satisfactory. Because we allow you 3 absences without penalty, we will not have make-ups.

Written work (45%) including problem writeups, group writeups, reflections and quizzes. For your problem writeups, you will be on four aspects:

Your understanding of the question, your explanation, your justification, and your answer. The first and last of these are worth a few points each, but most of the points are split between your explanation and justification. Explanation includes clarity and grammar as well as mathematical detail about what you did, what variables you used, and so forth. The level of justification we require will vary, but you should try to provide as complete a proof as you can. A grading rubric is posted on the course website. Right now it is a general rubric, but specific rubrics for each problem will eventually be posted.

If you are not satisfied with your score on a homework assignment, we will accept a re-do no later than two class meetings after the assignment is handed back, for up to 85% of the original credit. Any work that is illegible or is not grammatically correct will not be graded. More detailed specifications for your written work can be found in your course packet.

A midterm exam (15%) will be given in class on Tuesday February 11.

A final exam (25%) will be given during the scheduled final exam period, Wednesday March 19 from 3:30 PM to 5:18 PM.

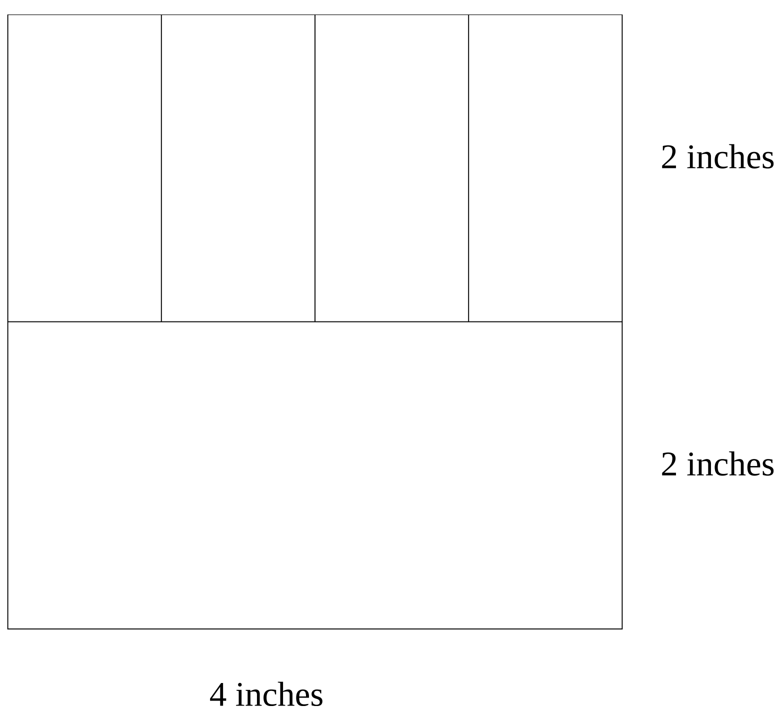
Please bring your binder to class every day, along with your drawing equipment and the completed work you have accumulated.

1 Problem-solving introduction and basic terms

1.1 Photo Layouts

Billie has just bought a digital camera and is planning to put together her photos into a scrapbook with square pages. She will be able to put photos of different sizes together, but needs all the photos to be the same shape or else she will lose picture quality when uploading the images. Her photo software does allow her to rotate the photos 90° if necessary, and to expand or shrink by any factor. Her goal is to have the photos on a page fit exactly, so they completely cover the page and don't overlap. She also refuses to put more than four on any page.

Find all the aspect ratios she could choose for her photos so that some number of photos of that shape, but no more than four, will exactly fit on a square page. Once you have found these, which among them is closest to the aspect ratio of a standard $3\frac{1}{2} \times 5$ photograph? Shown below is an example of how, if she were willing to fit five photos on a page, she could use the aspect ratio 2.



1.2 Reflection on Teacher Preparation

First, we want to know honestly what you feel when you read about the NCTM and MAA recommendations. Do you agree or disagree? (Be specific.) Does this kind of mathematical learning intrigue you? Does it intimidate you? Secondly, try to describe how your own mathematical education up to this point compares to the ones recommended in the documents quoted.

1.3 A Short Problem Solving Self-Help Checklist

1. **Understand the problem**. It is foolish to answer a question you do not understand. It goes without saying that you need to understand all the terminology. In addition, you need to know what is given. Make a sketch, if appropriate, to see that all the given data makes sense. Also, be sure you understand what you are supposed to find or determine. A good way to check this is to ask yourself whether you could verify the answer if someone gave it to you. For a true/false or a prove/disprove question, ask what would constitute a counterexample. You should be especially aware of whether you are trying to show that something always holds, or whether you are you are being asked to find a specific case where something holds. Another question to ask yourself in this phase is whether the data really determine the unknown. Is there really a solution? More than one solution?

2. **Devising a plan**. If a road to the solution occurs to you naturally, you don't need this checklist. Assuming you find yourself somewhat stuck, here are a few general procedures you can follow.

   (a) Try a few examples. Trial and error is the single best problem-solving method. If you are supposed to find the relation between coordinates in a before and after picture, try listing a few pairs of points and looking for a pattern. If you are supposed to find all semi-regular polyhedra, start listing them. If there are variables in the problem, replace them with numbers and see how the problem goes then.

   (b) Work from both ends. When you cannot deduce any more from what is given, work backwards from the goal, for example: if you are supposed to find the area of something, perhaps you see that part of it is a square with known side, so you just need the area of the rest; perhaps after you do this a few times, you will see that what remains is a familiar shape which you had not recognized was embedded in the problem before.

   (c) Can you solve a special case? Perhaps if you assume that a triangle is isosceles, or that a certain angle is a right angle, you can solve it. Perhaps if you find an analogous two-dimensional problem and solve that, you will gain insight into the three-dimensional problem you are actually trying to solve.

(d) Use physical intuition – after all the main reason for learning geometry is to develop and harness physical intuition. See if you can imagine the line segments in a geometry problem as made of steel or wood, the vertices as flexible joints, the two-dimensional components as sheets of stiff material with weights proportional to their areas,

3. **Carrying out the Plan**. If you are using variables, make sure you understand what they mean. If you get stuck, ask yourself whether you need any more variables. Once you have an idea you think is right, and are trying to prove it, examine why you believe it. Probe at it: why can't this line segment be longer than that other one? How would that contradict the data given? Try to recall facts and theorems you already know, so you don't have to prove every little bit from scratch. In order to have a better sense of what assertions require proof, try to think of situations where a similar-sounding assertion might be wrong. The reason we wait until the worksheet "False from True" to have you rigorously prove what you discover about the Pythagorean Theorem is that we want you first to see similar arguments that have flaws, then to prove those flaws do not arise in your argument.

4. **Checking It Over**. You should always ask yourself whether the answer is plausible. In addition, you should ask whether you used all the data. If you did not, perhaps you made a mistake, or perhaps the problem did not require all the data; in this case, you should try to understand why some of the data was irrelevant, and mention this in your report. If you used some algebra you are not sure of, test it out on a calculator with some numerical examples. If you used variables, make sure you believe any equations you wrote relating them. If you are able to solve a problem more general than the one stated, please by all means include this in your writeup.

1.4 Problem Report Tips

- Speak out during large group discussions to get other groups' ideas. Make them explain their ideas to you, so you can explain them clearly in your report. It is assumed you already do this in your own small group!
- Address every question: even if you do not know the answer, at least acknowledge that and perhaps guess at it or suggest a strategy for finding it.
- Problem reports will ideally have one or two key sentences which hit upon exactly why the answer you provide is true. These do not replace a complete argument, but it aids the reader enormously, so you are less likely to lose points for clarity or coherence when you include this kind of guiding sentence. For example, if your assignment is to analyze the game of Tic-Tac-Toe, you might say:

  > Go first and choose the center. On your next two turns, choose two adjacent squares, so that you have two ways to win and your opponent can do no better than to tie.

- If you cannot explain a complete solution, then (1) acknowledge this, (2) pinpoint the difficulties, and (3) solve a simpler version if you can, or a special case.
- Ideas that you or your group had that did not lead to the solution may be worth reporting. I don't want a complete list of your thought processes, but a dead-end that leads to new understanding is worth pointing out.
- Write a conclusion that actually draw a conclusion, rather than simply rewriting your introduction. The overall flow should be:

  - Here's what I'm going to tell you about
  - Content
  - This is what we can come away with

  and not

  - Here's what I'm going to tell you about
  - Content
  - Here's what I just told you

- Use paragraphs, one for each idea. It's not a rule, but usually you should have more than one paragraph per page.
- A good report is exhaustive but not verbose, and could be understood by a friend who is not in this class.

1.5 Basic Terms

The list of geometric vocabulary on the next few pages is mean to serve as a common base for class discussions. Some of the terms you may already know; these you may ignore. Others you may have some idea of but lack a precise definition; for these, you will want to look at the definitions. Some you will never have heard of; after reading the definitions you may want to discuss these in class.

I will tell you in class which terms you need to learn, a day or two in advance of when we will be using them. So there is no need to go over the whole list now. Rather, this list is meant to serve as a reference for you. Many of the definitions you need may also be found in Supplements A and C, and in the other materials at the end of this packet. Chapter 12 of Burger/Musser/Peterson, which many of you are familiar with, contains some definitions as well.

In addition to the basic terms, there are a few standard notations I would like to point out. If A and B are points, then we use $\overline{AB}$ for the line segment between them. Sometimes this is abbreviated as AB. We use $\overrightarrow{AB}$ for the ray from A through B and $\overleftrightarrow{AB}$ for the line through A and B. We use $\triangle ABC$ for the triangle with vertices A, B and C, and $\angle ABC$ for the angle made by the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$. We use the symbol $\cong$ for congruence. Although some people maintain a distinction between congruence and equality, in this class you will be permitted to say that two line segments with the same lengths are equal (for instance, $AB = CD$) or that two angles with the same measure are equal (for instance, $\angle ABC = \angle DBC$).

VOCABULARY

- acute angle: an angle that is less than 90°. (Euclid first defined it as an angle that lay inside a right angle.)
- adjacent angle: two angles are adjacent if the two rays that bound the second angle include one of the rays that bound the first angle and one ray that does not bound nor lay inside the first angle.

- altitude (of a triangle)

- angle: One definition is that it is the region of a plane between two rays emanating from a common point. Another, possibly preferable because it avoids making the angle contain the whole region, is that an angle is made up of two rays emanating from a common point, and that its measure is the extent to which one ray must be rotated to get the other way, viewing a rotation all the way around and back to start as $360°$ or 2π radians.

- angle bisector: the ray in the interior of an angle dividing it into two equal parts.

- area: the space contained within the boundary of a two-dimensional region. This may be measured as the number of square units, such as squares with a given side length, that fit in the region to be measured.

- aspect ratio: Webster's dictionary says, "a ratio of one dimension to another, as the ratio of the width of a television image to its height." In this class, only rectangles will be said to have aspect ratios.

- axiom: a basic fact, usually held to be self evident, which is assumed without proof.

- axis of rotation: *see definition of rotation in three-dimensional space.*

- bisect: to divide into two equal parts.

- center of rotational symmetry (of a plane figure): if the figure has a rotational symmetry, its center is called a center of rotational symmetry for the figure.

- centroid: the three medians of a triangle are concurrent; the point where they meet is called the centroid of the triangle. If you cut the triangle out of a uniform material and balanced it on a needle, the point in the interior touching the needle would be the centroid. All polygons have centroids, where they would balance, but we will not give a general definition. The centroid of a regular polygon coincides with the center of the circumscribed circle. The centroid of any polygon having rotational symmetry coincides with the center of rotation.

- circle: a plane figure consisting of all points equidistant from a given point.

- circumscribed circle: given a polygon, if there is a circle passing through all the vertices of the polygon and containing the polygon in its interior, this circle is called the circumscribed circle.

- collinear points: a set of points all lying on some line. Thus any two points are collinear, but only certain sets of three points are.

- complementary angles: Euclid defined these as adjacent angles that togther form a right angle. Once we have defined degree measure, we may define complementary angles as any angles summing to $90°$.

- composition of rigid motions: to do one rigid motion, then another, is to compose them, and the resulting rigid motion is called the composition.

- cone: a finite cone is defined like a pyramid, except that any plane figure may be used instead of a polygon. A cone is called a *circular* cone if the plane figure is a circle; the common usage of the word cone usually assumes it is a circular cone. An infinite cone is the set of points in space that lie on any ray passing through a given circle and emanating from a given pont not in the plane of the circle.

- congruent: two objects are congruent if their points can be placed in one-to-one correspondence so that the distance between two points of one object is always the same as the distance between the corresponding two points of the other. One way to test this is to see if one object can be picked up and placed on the other so the two match perfectly. For two-dimensional objects, this is a good test, but for three-dimensional objects, one ust be willing to pass an object through parts of another to do this test, and one must allow mirror reflection (which in two dimensions is just turning the object over). Once we have defined rigid motions, we may say that two objects are congruent if there is a rigid motion transforming one into the other.

- convex: a plane figure is convex if any line segment whose endpoints are contained in the figure is wholly contained in the figure. The same definition may be used to define convexity of an object in space.

- coplanar: a set of objects is said to be coplanar if there is one plane containing them all.

- cylinder: the technical definition is more general than the one in common usage. A cylinder is any three dimensional object consisting of two congruent bases and the lines connecting corresponding points of the two bases, provided the two bases are subject to the conditions of the two bases in the definition of a prism, though not necessarily polygons. The cylinder is a right cylinder if one base lies directly above the other. The common usage of the term "cylinder" is to mean a right circular cylinder.

- decagon: a polygon with ten sides.
- dihedral angle:
- dilation: a dilation by a factor of x with center P is a transformation mapping each point Q to a point on the ray $\overrightarrow{PQ}$ that is precisely x times as far from P as was the original point Q. So for example, a dilation by a factor of two doubles all lengths.
- distance: the distance between any two points may be defined as the length of a straight line between them. Conceptually, it may be preferable to think of distance as defined even when a straight line cannot be drawn between the points, but we will not give a definition of this kind since it requires coordinate geometry.
- dodecagon: a polygon with twelve sides.
- dodecahedron: a regular dodecahedron is a regular polyhedron with twelve pentagonal faces.
- edge: *see the definition of polygon.*
- equiangular: a polygon is equiangular if all interior angles are equal.
- equilateral: a polygon is equilateral if all edges are equal.
- face: *see the definition of polyhedron.*
- flip: synonym for reflection in two dimensions.
- flip-glide: a rigid motion which is a flip composed with a translation along the axis of the flip.
- hexagon: a polygon with six sides.
- hypercube: a four-dimensional figure analogous to a cube.

- hyperoctahedron: a four-dimensional figure analogous to an octahedron.

- icosahedron: a regular icosahedron is a regular polyhedron with twenty triangular faces.

- inscribed circle: given a polygon, if there is a circle tangent to all the edges of the polygon and contained in the polygon's interior, this circle is called the inscribed circle.

- inversion through a point (for 3-D figures): an inversion through the point P is a rigid motion moving each point x to the point y such that P is the midpoint of $\overline{xy}$. Informally, each point is moved through P and out the other side an equal distance.

- isosceles: a triangle is said to be isosceles if two of its sides are equal.

- length: the length of a line segment is the number of standard units (e.g., inches, meters) that can be placed end to end along the line segment. The length of a curved path is the length of the line segment gotten by "straightening out" the curve as if it were a piece of string. A less physical definition is that length may be measured along the curve by placing dots at equal distances. The dots actually measure straight line distances, but as the spacing between the dots gets smaller, the total of these spacings represents more and more accurately length along the curve.

- line: this is taken by Euclid to be one of the undefined terms. Informally, we mean the set of points lying on a straight line, infinite in extent.

- line of symmetry (of a plane figure): a line such that reflection across the line preserves the object (is a symmetry of the object).

- line segment: the finite portion of a line between two points; the points are called the *endpoints* of the line segment.

- median (of a triangle): a line segment from one vertex to the midpoint of the opposite side.

- obtuse angle: an angle that is greater than $90°$. (Euclid first defined it as an angle that contained a right angle in its interior.)

- octagon: a polygon with eight sides.

- octahedron: a regular octahedron is a regular polyhedron with faces that are equilateral triangles.

- order of a rotational symmetry: if P is a center of rotational symmetry of some object, then there is a minimal angle of rotation possible before the object is mapped to itself. The order of rotational symmetry about P is the number of times the symmetry of minimal angle must be composed before each point is moved back to where it was initially. An analogous definition holds for the order of a rotational symmetry about an axis in three-dimensional space.

- orientation-preserving: a rigid motion of the plane is orientation-preserving if it can be done by composing translations and rotations. Informally, one must be able to carry it out without "picking up the figure and flipping it over". A rigid motion of space is orientation-preserving if it does not mirror-reverse objects. Informally, again, it can be carried out without the object being "picked up into four-dimensional space and flipped over".

- orientation-reversing: oppostie of orientation-preserving.

- parallel lines: coplanar lines that do not intersect.

- parallelogram: a quadrilateral whose opposite (nonintersecting) sides are parallel.

- pentagon: a polygon with five sides.

- perpendicular bisector: the perpendicular bisector of a line segment is the line passing through the midpoint of the segment making a right angle with the segment.

- perpendicular lines: lines making a right angle where they cross.

- plane figure: anything that can be drawn in a bounded (finite) region in a plane.

- point: this is taken by Euclid to be one of the undefined terms. Informally we mean a dot representing a single point in a plane or in space. Euclid called it "that which has no length width or breadth".

- polygon: a polygon can be defined in terms of its edges, or in terms of the region it covers. Definition 1 (in terms of edges): a polygon must have at least 3 edges (sides); if it has n edges, it is called an n-gon. The edges must form a closed chain, meaning that there are n points, call them $X_1, \ldots, X_n$, and the edges are the line segments $\overline{X_1X_2}, \overline{X_2, X_3}, \ldots, \overline{X_{n-1}X_n}, \overline{X_nX_1}$. A polygon is a plane figure, so all the points $X_1, \ldots, X_n$, which are called the *vertices* of the polygon, must lie in one plane. Furthermore, no two of the vertices can be equal, and no two of the edges may cross – the only point they may share is their common endpoint. Definition 2 (in terms of the region covered): a polygon is a region in a plane, whose boundary consists of some number n, at least 3, of line segments, meeting the criteria in Definition 1.

- polyhedron: this term is very difficult to define precisely. We will settle for an imprecise definition. A polyhedron is a finite figure in space, such as a cube or prism, whose boundary is made of polygons, called its *faces*. The rules for the way faces are allowed to intersect are not commonly agreed upon, but faces must meet each other at their edges and vertices only, and in the end, the polyhedron must have an inside and an outside. The plural is polyhedra.

- postulate: *same as axiom.*

- prism: To construct a prism, begin with two parallel, unequal planes. We will think of these planes as defining the horizontal direction, so we may talk of the "upper" and "lower" plane. First, the *bases* of the prism must be two congruent polygons, one in each plane. The upper polygon may lie directly above the lower polygon (in which case the prism is said to be a *right prism*), or may be moved from this position to another in the upper plane as long as it is not rotated. The remaining sides are gotten as follows. For each edge $\overline{XY}$ of the lower base, there corresponds an edge $\overline{X'Y'}$ of the upper base. The rules by which this is constructed ensure that all four points X, X', Y and Y' lie in one plane, so they form a quadrilateral $XYX'Y'$ which is then taken to be a face of the prism. A *triangular* prism is a prism whose bases are triangles, and this kind of phrasing can be used for other base shapes as well.

- pyramid: Given a polygon and a point not in the plane of the polygon, the pyramid they form is the set of all points on line segments one of whose endpoints is the given point and the other of whose endpoints is a point in the polygon. The given point is said to be the *apex* of the pyramid.

- quadrilateral: a polygon with four sides.

- ray: a half line, including the point at which it starts; the ray is said to emanate from this point.

- rectangle: a quadrilateral all of whose angles are right angles.
- reflection: a reflection across a line l in two dimensions is a rigid motion moving each point P to a point P' on the opposite side of l so that the line l is the perpendicular bisector of $\overline{PP'}$. A reflection across the plane S in three dimensions is a rigid motion moving each point P to a point P' so that again P and P' are the sme distance from the plane, and $\overline{PP'}$ is perpendicular to the plane.
- regular polygon: a polygon that is both equilateral and equiangular.
- regular polyhedron: a polyhedron all of whose faces are congruent regular polygons, and all of whose vertices are alike in the sense that any vertex may be moved to any other by a rigid motion that preserves the polyhedron.
- regular tesselation: a tesselation all of whose faces are congruent regular polygons, and all of whose vertices are alike in the sense that any vertex may be moved to any other by a rigid motion
- rhombus: an equilateral parallelogram.
- right angle: Euclid defined this as an angle such that if you place an equal angle next to it, the resulting sum of angles forms a straight line. Once we have discussed degree measure, we can simply say a right angle is an angle of $90°$, though you should remember what is special about it from Euclid's definition.
- rigid motion: a transformation of the plane or space that moves each point to another point in such a way that the distance between any two points is the same afterward as it was before. Informally, all shapes and sizes are preserved. A synonym is *isometry*.
- rotation: in the plane, a rotation centered about a point P with angle θ in the clockwise direction is a rigid motion that moves each point x to the point y such that $Px = Py$ and $\angle yPx$ is equal to θ in the clockwise direction. The point P is called the *center* of the rotation. In three-dimensional space, a rotation has a line for a center. The rotation by angle θ in the clockwise direction with axis l is a rigid motion, which in each plane perpendicular to l is a rotation around the point p where l intersects the plane.
- rotational symmetry (of a plane figure): a rotation preserving the figure.
- scalene: a triangle is said to be scalene if no two sides are equal.

- semiregular polyhedron: a polyhedron in which all faces are regular polygons and all vertices are alike in the sense that any vertex may be moved to any other by a rigid motion.

- semiregular tesselation:a tesselation in which all faces are regular polygons and all vertices are alike in the sense that any vertex may be moved to any other by a rigid motion.

- similar: two figures are similar if all corresponding lengths are proportional. In Euclidean geometry, the only lengths we can really talk about are distances between pairs of points, so it boils down to: two Euclidean figures are similar if the distance between any pair of points is always the same multiple of the distance between the corresponding pair of points. Note that the notion of similarity, like tht of congruence, requires a correspondence between points of the two figures.

- simplex: the generalization of a triangle into any dimension. An n-dimensional simplex is obtained from an $n-1$-dimensional simplex by making a finite cone over it with some new apex.

- sphere: the set of all points in space at a given distance from a fixed point; the fixed opint is called the *center* of the sphere.

- square: a regular quadrilateral.

- supplementary angles: Euclid defined these as adjacent angles that together form a straight line. Once we have defined angle measure, we may refer to two angles as supplementary if angles equal to them, if placed so as to be adjacent, would form a straight line; more simply we might say angles that sum to $180°$.

- symmetry (of a plane figure): a rigid motion that preserves the plane figure, meaning that all points of the plane figure get mapped to points of the plane figure.

- tesselation: a subdivision of the plane into plane figures. Sometimes it is meant that the figures be polygons and that each two polygons meet at one common edge, at one common vertex, or not at all.

- tetrahedron: a regular tetrahedron is a regular polyhedron with triangular faces.

- theorem: a fact someone has proved, usually by applying deductive rules starting with axioms.
- torus: the set of points in space all at an equal distance, s, from a fixed circle of radius $r > s$.
- translation: a rigid motion in which all points move by the same amount in the same direction.
- transversal: a line that cuts across two given lines or rays.
- trapezoid: a quadrilateral having two parallel sides.
- triangle: a polygon with three sides.
- trisect: to divide into three equal parts.
- vertex: a vertex of a polygon is any endpoint of one of its edges. a vertex of a polyhedron is any vertex of one of its faces. The plural of vertex is *vertices*.
- vertical angles: when two lines cross, they form four angles; each pair that are not adjacent are called vertical (thus an angle cannot be vertical, only a pair of angles can be).
- volume: the amount of three-dimensional space contained in an object, measured in units of cubic length, such as identical cubes of a given size. As a practical matter, given a physical bounded space, its volume may be measured by filling it with water and then measuring the amount of water (say in a measuring cup); this relies on the fact that water is incompressible, so its volume cannot change as it is poured in and out of various objects.

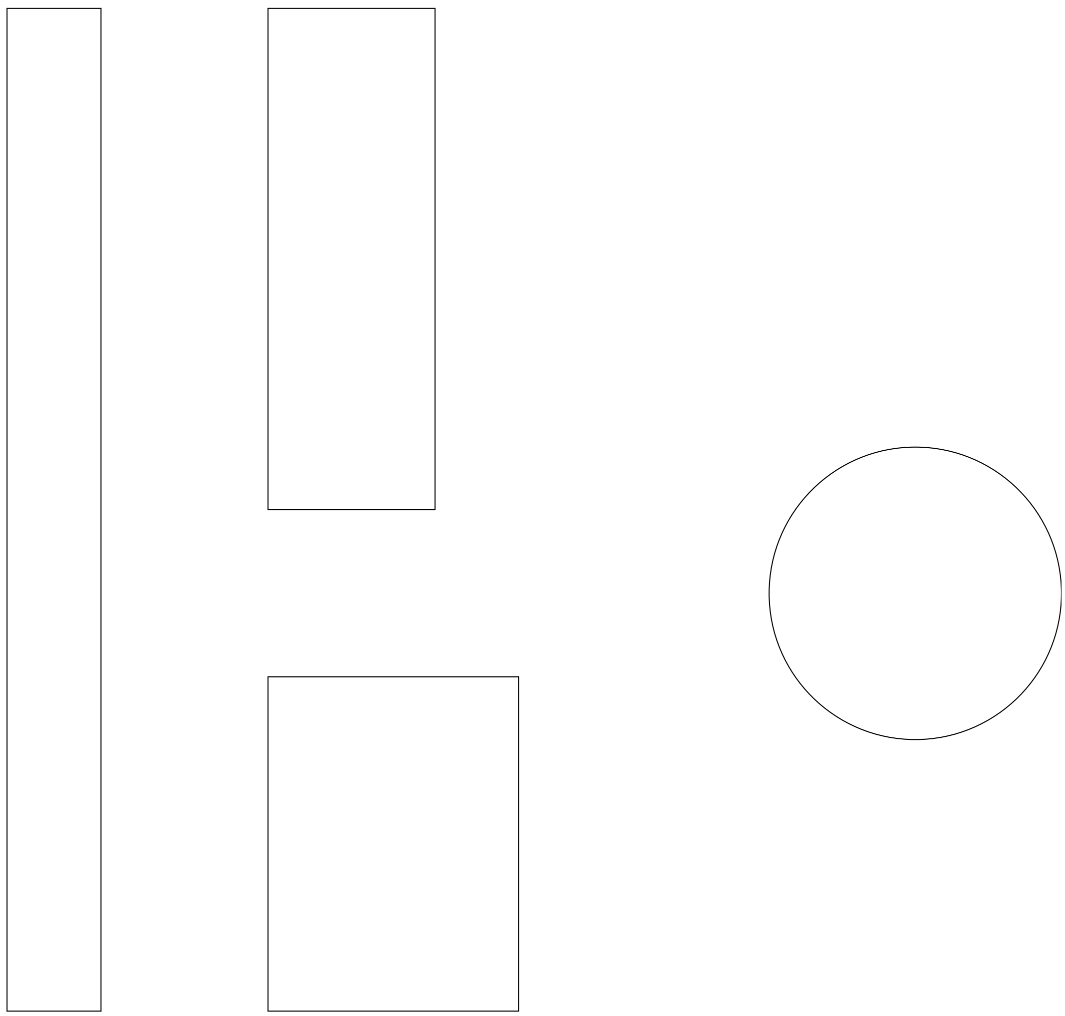
2 Area

2.1 Pizzas

If a 12-inch pizza costs \$5, how much should a 14-inch pizza cost?

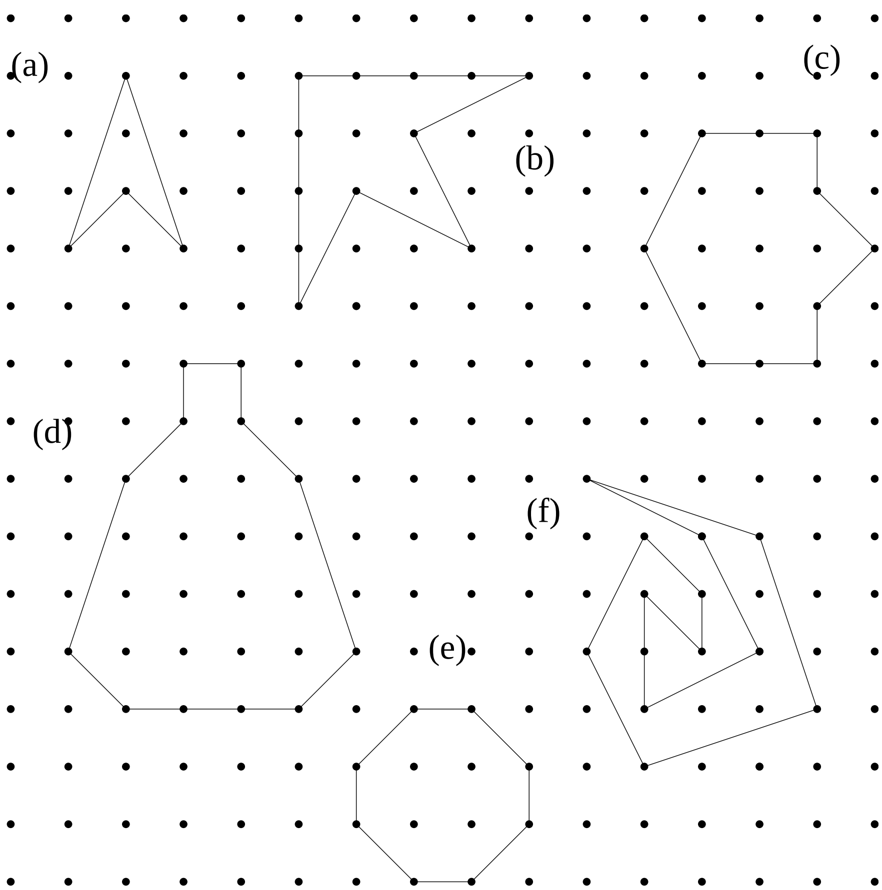
2.2 Paper Shapes

Cut out the four shapes on the next page. Determine which of the four shapes covers the most space. Discuss what concepts are necessary in order to do this problem.



2.3 Geoboard Areas

Find the perimeters and areas of the figures below.



2.4 Measuring a Sector

Get out a piece of graph paper. Choose a vertex near the lower-left corner of the page and call it A. Choose another vertex on the vertical line through A, about halfway up; call it B. Using your compass centered at A, draw an arc through B, extending down toward the bottom right corner of the page. Draw a 45° diagonal line from A up and to the right; where this line crosses the arc, call it C. The figure with two straight sides and a curved side, with vertices at A, B and C is called a sector. Measure as accurately as you can the area of this sector.

2.5 Area Formulae

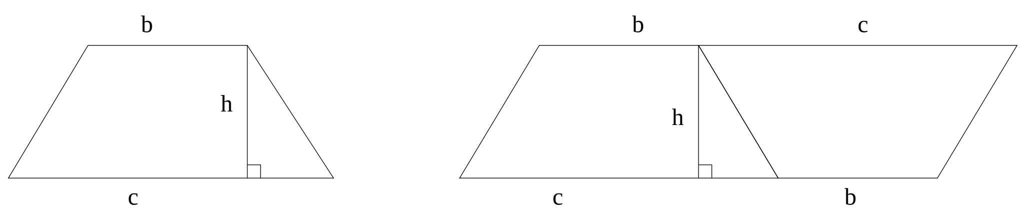
Gather as many area formulae as you can. A prize will be awarded to the person with the greatest number of essentially different formulae. You must be able to explain the meaning of all formulae you turn in, but do not need to know how to derive them.

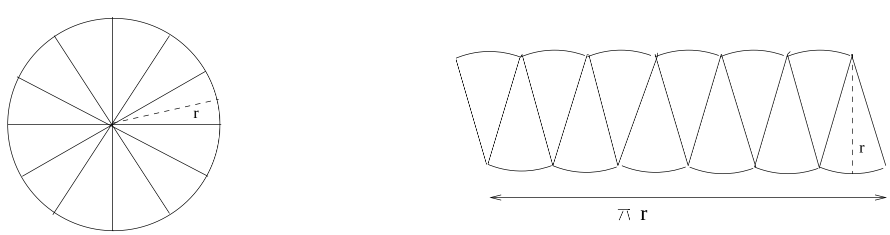
Accompanying reflection on formulae

What role do you think formulae play in the definition of area? Can you think of any ways in which knowing formulae inhibits understanding of area? Can you think of any ways in which formulae make it easier to explain the concept of area? Should people who are going to teach the concept of area know a lot of area formulae?

2.6 Picture Proofs

Write an explanation for each of these “proofs without words”.





Here are three questions that you must be able to answer about each part of the Picture Proof worksheet that you have correctly solved.

1. What statement does the picture purport to prove?

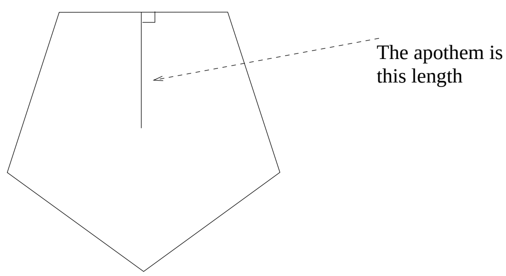
2. What re-arrangement of the picture proves it, and why?

3. What might you need to justify about how the pieces are supposed to fit together?

Go back and make sure you know the answers to these three questions for each of the three picture proofs.

2.7 The Apothem

The *apothem* of a regular polygon is defined to be the shortest distance from the center of the polygon (i.e., center of a circumscribed circle) to an edge.



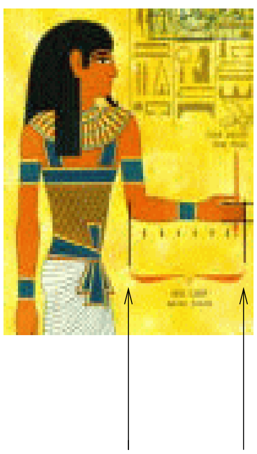
1. There is a nice relation between the apothem, perimeter and area for a regular polygon. See if you can find it.

2. What does this say about the area of a circle?

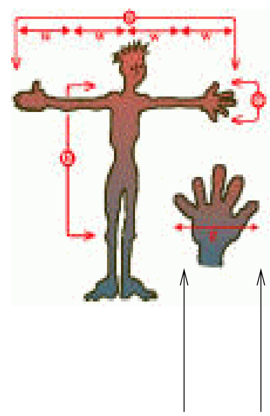
3 Length, area and volume

3.1 Need for Standard Units

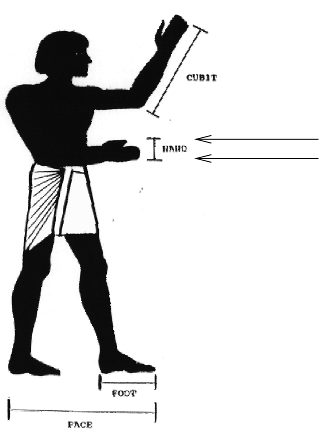
The cubit was a measure of length based on body parts: roughly the lenngth of the lower arm and outstretched fingers. The span was the distance between the tip of the thumb and pinky with fingers spread apart, and the hand (roughly half a span) was the width of the four fingers of one hand when the fingers are not spread apart.



A cubit



A span



A hand

A basic step in the measurement process is the choice of a unit of measure. The following investigation may help the reader capture the spirit of this importance and help establish criteria for choosing units.

> "And there went out a champion out of the camp of the Philistines, named Goliath of Gath, whose height was six cubits and a span" (I Samuel 17:4)

1. Using your cubit and span, cut a string that is as long as Goliath was tall. Compare your string length with the strings of others in your group. Discuss any difficulties that might arise from choosing and using units determined by each person's own body.

2. Now, invent an original unit for measuring length, based on part of the body, an object in the environment, etc. Invent some smaller and larger related units and make a ruler using your unit.

(a) Measure some things.

(b) Communicate with another group about your unit and compare it as accurately as possible with their unit. If you measure with your unit and they measure with their unit, would you have any difficulty communicating about your measurements? Try it and discuss any difficulties that might arise.

(c) Make a list of any difficulties that might arise if each nation had a local unit of measure different from others.

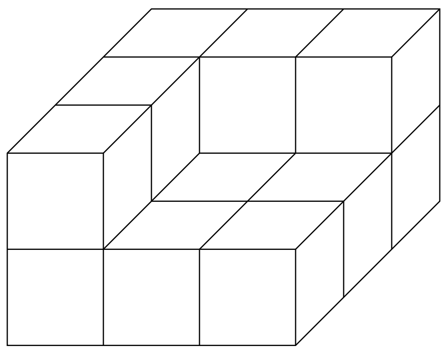
3.2 Units of Measurement

1. Find eight different units of length and state how they are related (you can relate all to the first one – you don't need to relate each pair).

2. Find three different units of area, only one of which may start with the word "square". State how they are related.

3. Find three different units of volume, only one of which may start with the work "cubic", and state how they are related.

3.3 Surface Area



1. What is the surface area of the above configuration (there are 14 blocks)?

2. What is its volume?

3. Can the 14 blocks be arranged so that the surface area is less? More?

4. What are the maximum and minimum possible surface areas of the possible configurations?

3.4 Length, Area and Volume

1. Come up with a definition for the length of a curve. You may discuss how length can be measured, but the definition should be of an ideal concept, not dependent physical measurement.

2. Do the same for the definition of area of a plane figure.

3. Do the same for the definition of volume of a solid.

4. Describe three objects or mathematical figures, the first essentially one-dimensional, the second two-dimensional, and the third three-dimensional. Which of these objects can be said to have a “length” associated with it?

3.5 Volume: Eureka!

1. Name at least two ways to measure the volume of something.

2. When you double a cube's sides, what happens to its volume?

3. When you double its volume, what happens to its sides?

4. For what kind of an object is it true that

$$\text{Volume} = \text{Area} \times \text{height?}$$

5. When the Greek scientist Archimedes sat down in his bathtub, he exclaimed, "Eureka!" What method of measuring volume had he just discovered? (It became known as the Archimedian principle.)

3.6 Volume: Prisms and Cones

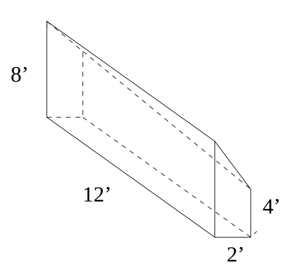
Many volume formulae are just extensions of area formulae we already know:

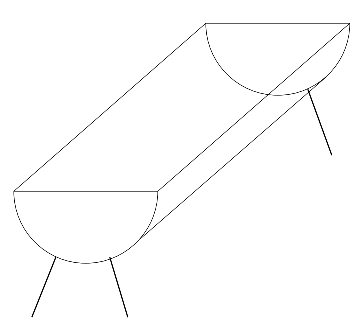
2-D OBJECT	**AREA**	**3-D OBJECT**	**VOLUME**
rectangle	base x height	prism	base area x height
triangle	1/2 base x height	pyramid	1/3 base area x height

1. Build a prism (or cylinder) and pyramid (or cone) of the same base and height, and verify the above formulae by using the two objects as containers (upside-down with the base left open), to see how many pyramids of water (or sugar or sand) it takes to fill the prism.

2. The closet in my living room has an odd shape because my roof slants down. The closet is eight feet tall in front but only 4 feet tall in back. It is 2 feet deep and 12 feet long. How many cubic feet of storage space are there in the closet?

3. How many cubic feet of water does a semi-cylindrical trough hold which is 10 feet long and 1 foot deep?





3.7 The Length of a Square

1. Take a good look at my supply of string. Now draw a 2-inch by 2-inch square. Estimate what length of string can be placed in the square, packed back and forth like an intestine, so that it fills up the square but does not go outside it or pile on top of itself.

2. Cut yourself some string, perform the experiment, and write down the answer. In what units should your answer be stated?

3. Is this problem about length, area, volume, or something different?

3.8 Perimeter

1. Suppose a rectangle has a perimeter of 36 units. Draw several rectangles showing some possibilities.

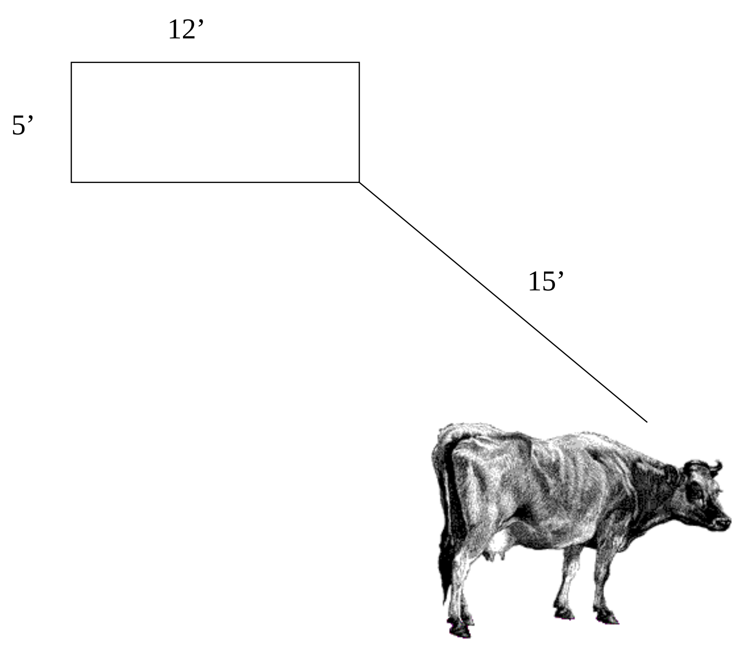
2. Make a graph plotting the area of a rectangle whose perimeter is 36 versus its width. What width gave the maximum area? What gave the minimum area?

3. Now suppose a rectangle has a fixed area of 100 square units. What is the smallest perimeter that such a rectangle can have? Can you make a graph that will convince me?

4. What do you think is the geometric figure with area 1 that has the smallest perimeter? Could Pick's formula help you to think about this?

3.9 The Hungry Cow

A farmer tied her cow at the corner of a small barn with plenty of water and left for a 5-day trip. When she returned, the cow had eaten all the grass it could reach and had become sick. Before prescribing medicine for the cow, the veterinarian wants to know how much grass the cow has eaten (the grass in the pasture was 3 inches high). Compute the square footage of reachable grass to answer the vet's question.



4 Scaling

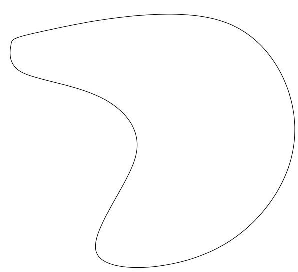
4.1 Scaling worksheet

1. Suppose a rectangle has perimeter L. Now you make a magnification of this by a factor of 3. What is the perimeter of the new rectangle? Prove your answer.

2. Suppose the area of the original rectangle in problem 1 was X. What is the new area? Prove your answer.

3. What would happen in problems 1 and 2 if instead of a rectangle you had a triangle?

4. What if you had a zany shape such as this, instead of a rectangle or triangle: if the original area is X, what is the new area? Can you prove it?



5. What happens to the volume of any object when you magnify by a factor of 5? By a factor of k?

6. My lawn is an irregular shape covering 428 square feet. How many square inches does it cover?

7. Recall the definition of a span from the worksheet entitled "Need for Standard Units". How many spans tall are you? How many spans tall do you think a giant would be? How about a toddler? Assume that each individual is measured with their own hands.

8. Suppose you place a metal band around the earth, so it fits snugly around the equator. Then someone adds 1 foot to the band, smoothly, so it lies the same amount off the ground everywhere. How far off the ground will it lie?

4.2 Changing units

An object has a volume of 100 cc and a surface area of 150 square centimeters. An inch is roughly 2.54 centimeters. What will the area and volume turn out to be when measured respectively in square inches and cubic inches? By what scale would the object need to be expanded or shrunk to have a volume of 100 cubic inches? What would the surface area be then?

4.3 Reflection on Units

You overhear the following conversation among your students. How do you assess their views and what do you tell them?

> Student A: Miles are 1.6 times kilometers, so to get the number of miles, multiply the number of kilometers by 1.6.
>
> Student B: But miles are bigger, so the length measured in miles should come out smaller.
>
> Student C: No, the length is a physical quantity and won't change regardless of the units in which you measure it.

5 Angle

5.1 Units of Angle

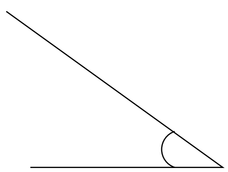
1. When angles must be measured more accurately than to the nearest degree, the traditional subdivision is into minutes, which are 1/60 of a degree, and seconds, which are 1/60 of a minute. Does the earth rotate on its axis more or less than 1 minute or arc in 1 minute of time?

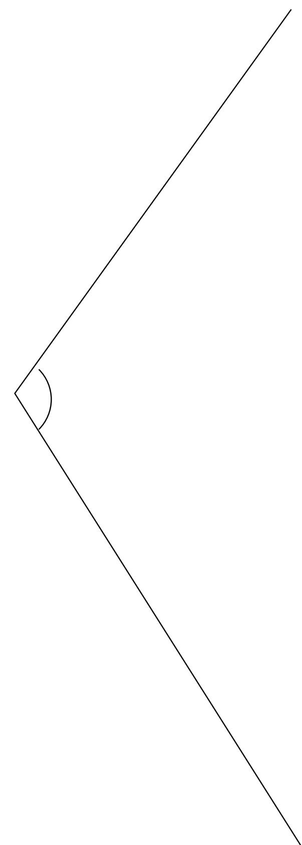
2. Merriam-Webster's Collegiate Dictionary defines a radian as "a unit of plane angular measurement that is equal to the angle at the center of a circle subtended by an arc equal in length to the radius"; there is also a definition in your coursepack. Approximately how many degrees is this, and how can you tell for sure?

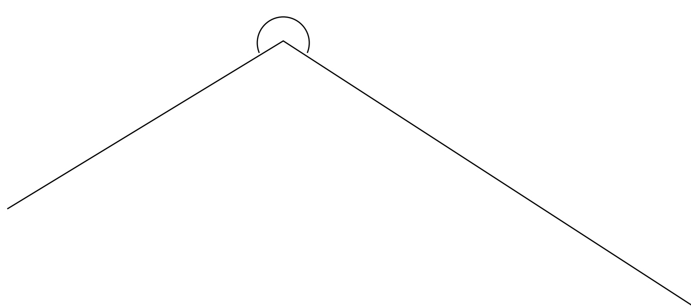
3. Suppose you join two equilateral triangle with a hinge. How would you define the angle between them?

5.2 Angle Measurement

1. Use a protractor to measure, as accurately as possible, the angles drawn below.







2. If you use the bottom edge of the protractor to align an angle, rather than the pinhole, are angles overestimated or underestimated?

3. Suppose the lines are drawn in very thick pencil. How would you obtain an accurate measurement in that case?

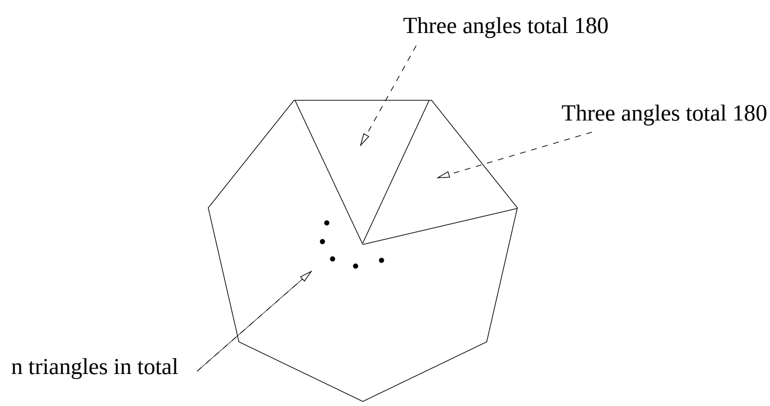
4. Insert a thread through your protractor's pinhole. Tie a paperclip to the end so that the thread cannot come out of the pinhole. Cut the thread to the length of a foot or so and tie a heavier object to the other end. Hold the protractor so that the plane of the protractor is perpendicular to the plane of the floor, and the flat edge is at the top, at an angle between horizontal and vertical. Look at the number where the thread crosses the curved part of the protractor. What physical angle does this measure?

5.3 Angles of a Polygon

A student is curious about the sum of the interior angles of an n-sided polygon. After some thought, she comes up with the formula

$$\text{Total } = n \times (180^\circ) \, .$$

Her derivation is accompanied by a figure, reproduced below. How convincing is her derivation? Is it correct? What tips would you give her on turning the derivation into a proof?



5.4 Trisection

Draw an angle. Now draw a line cutting across both rays of the angle. Using a ruler to measure, carefully divide the line segment between the rays into thirds. Do the rays from the vertex of the angle through the one-third and two-thirds marks on the line segment trisect the angle? Does your answer depend on whether the transversal is drawn so that the figure is symmetric, that is, it intersects the two rays at equal distances from the vertex of the angle?

5.5 Olentangy River

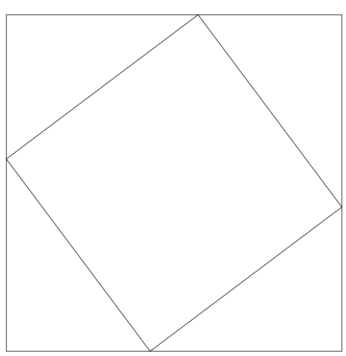
You need to measure the Olentangy River but you are not allowed to cross it. I will allow you any tools you wish, and you can look across the river, but no person nor tool can cross it. Devise a way to measure the width of the river (assume the river is straight).

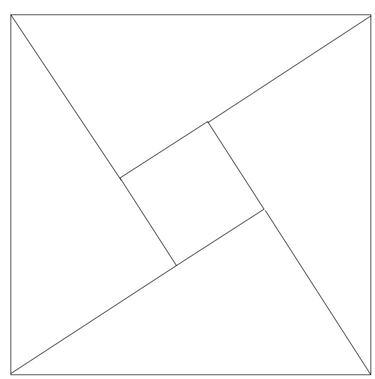
6 Deductive reasoning

6.1 Pythagorean Theorem

What does the Pythagorean Theorem state?

Can you use one of the pictures below to find a proof of the Pythagorean Theorem? (Each picture leads to a slightly different proof.)





6.2 Rigidity

1. Use informal reasoning to answer the questions below. In the first set of questions, you must determine each time whether the information determines the exact shape and size of the triangle (i.e., in which cases can you draw a triangle guaranteed to be congruent to $\triangle ABC$?).

   (a) $AB = 4$ and $BC = 5$.

   (b) $AB = 8$, $AC = 12$ and $\angle ABC = 45°$.

   (c) $\angle CAB = 25°$, $\angle ABC = 75°$ and $\angle BCA = 80°$.

   (d) $\angle ABC = 30°$, $BC = 10$ and $CA = 7$.

   (e) $BC = 7, CA = 8$ and $AB = 9$.

   (f) I tell you two of the sides and all three angles, but don't say which is which.

   (g) $\angle ABC = 60°$, $BC = 10$ and $CA = 3$.

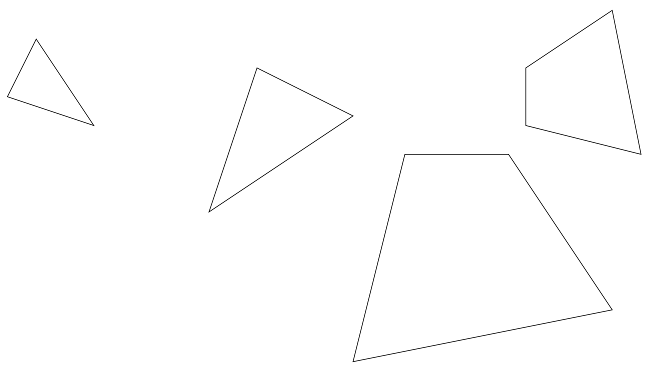
2. I draw a circle of radius 2 inches on a piece of paper and ask you to draw on it five points A, B, C, D and E so that: CD bisects AB perpendicularly, BE bisects CD, and AE is 1 inch. Does this determine the figure up to congruence (i.e., if two students follow my instructions will their figures always be congruent)?

3. My neighbor asks me why I put a diagonal support on the gate between our yards. What should I tell him?

6.3 Similarities

1. Someone hands you a drawing of two triangles. Suppose they make the claim that one is a scale copy of the other. What is the least number of measurements you must make in order to verify this claim?

2. Answer the same question for two quadrilaterals instead of two triangles.

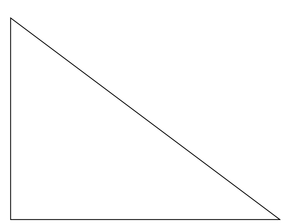


3. Shown below is a right triangle with sides of 3, 4 and 5 units.

   (a) Draw an altitude from the vertex at the right angle to the opposite side (the hypotenuse).

   (b) Determine the precise length of this altitude.

   (c) The altitude divides the hypotenuse into two line segments. Determine precise lengths of these two segments.



6.4 Applying Postulates and Theorems

1. Look over the postulates and theorems listed in the supplement called "Postulates and Theorems". (You may wish to read the FAQ sheet as well.) Then go back to the Rigidity worksheet and see whether you can use the postulates and theorems to answer those questions.

2. Can you explain why the ASA and SAS criteria appear as theorems (or postulates – see the discussion on page 151), while SSA does not appear at all on the list of postulates and theorems?

6.5 Proofs

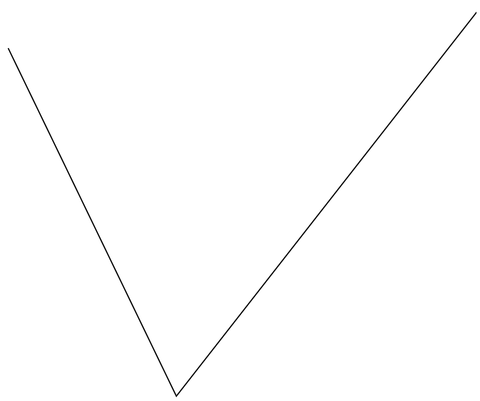
1. Prove or disprove: if both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

2. Draw a square $ABCD$. Draw the line segments connecting the midpoints of each pair of neighboring sides. Label the new quadrilateral $EFGH$. Prove or disprove: $EFGH$ is also a square.

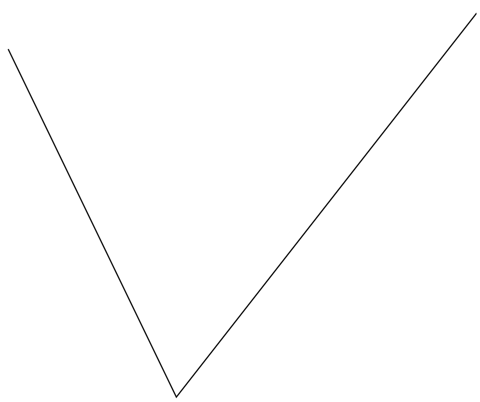
6.6 First Constructions

Throughout the section on constructions, it will be assumed that you may use only a compass and straightedge and that you must try to prove that your construction works.

1. Given the following angle, figure out how to construct a copy of it.

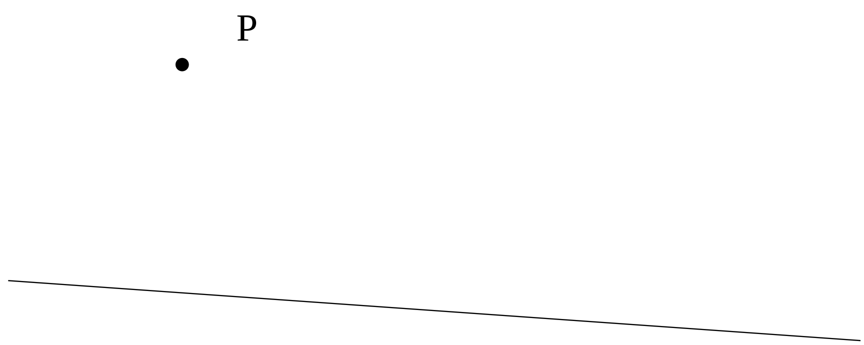


2. Given the following angle, figure out how to construct its bisector.

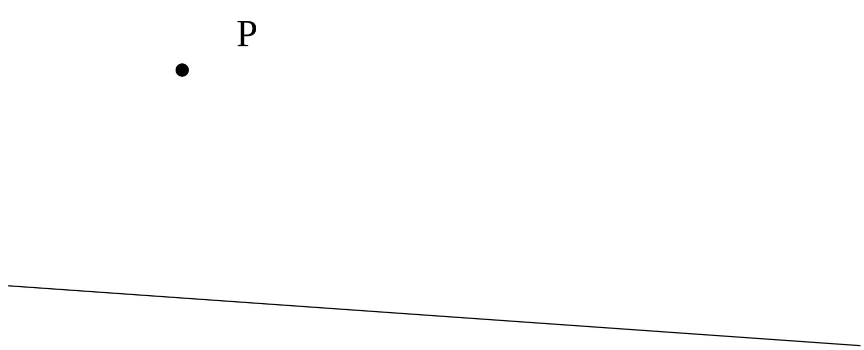


3. Figure out how to bisect a line segment.

4. Figure out how to construct a perpendicular to a given line, through a given point, P.



5. Figure out how to construct a line through the given point, Q, that is parallel to the given line, P.

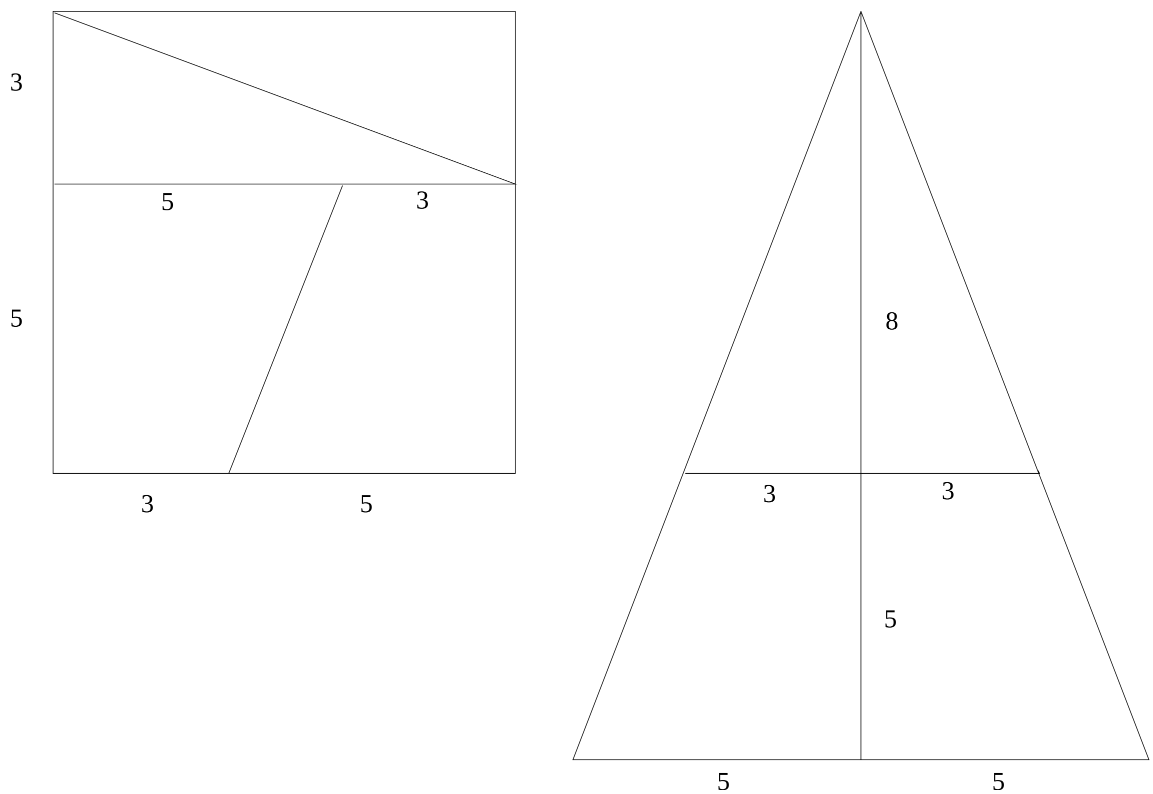


6.7 Circle Construction

Draw any three non-collinear points on a piece of paper. Can you construct a circle containing all three on its circumference?

6.8 False Proofs from True

1. The figure below shows a proof that $8 \times 8 = 5 \times 13$. What is wrong with the proof.



2. Go back to your proof of the Pythagorean Theorem from the Pythagorean Theorem worksheet. How do you know that the same flaws are not present there as in the above "picture proof"? Improve your argument until it is a real proof of the Pythagorean Theorem, using Euclid's Postulates, any theorems we have accepted, and any algebra you need.

6.9 Postulates and Theorems

Euclid's Postulates (The axioms for Euclidean Geometry)

1. A straight line can be drawn from any point to any other point.
2. A line segment can be extended to produce a straight line.
3. A circle may be described with any center and distance.
4. All right angles are equal (congruent) to one another.
5. Through a given point not on a line, there can be drawn only one line parallel to the given line.

Some common notions:

- Things equal to the same thing are also equal to one another.
- If equals are added to equals, the wholes are equal.
- If equals are subtracted from equals, the remainders are equal.
- Things which coincide with one another are equal.
- The whole is greater than the part.

Basic Theorems These follow immediately from the common notions:

- Supplements of congruent angles are congruent.
- Complements of congruent angles are congruent.
- If two lines are parallel to the same line, they are parallel to each other.
- Corresponding parts of congruent triangles are congruent.

Euclid's first three theorems were about some basic constructions, such as being able to copy a length, even though the compass was supposed to be a "collapsing compass". The next theorem is Euclid's fourth proposition (theorem) but some later thought the proof was flawed and added it as a postulate; the subsequent three theorems were proved shortly after (the last being the 26th proposition).

- *Side-Angle-Side Congruence Theorem* (SAS): If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the triangles are congruent.
- *Angle-Side-Angle Congruence Theorem* (ASA): If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then the triangles are congruent.
- *Side-Side-Side Congruence Theorem* (SSS): If three sides of one triangle are congruent to three sides of another triangle, then the triangles are congruent.
- *Angle-Angle-Side Congruence Theorem* (AAS): If two angles and the side opposite one of them in one triangle are congruent to two angles and the side opposite one of them in another triangle, then the triangles are congruent.
- (AA Theorem) If two angles of one triangle are congruent to two angles of another triangle, then the two triangles are similar.
- (Similarity preserves angles) If two triangles are similar, then corresponding angles are congruent.

These are some more of the earliest theorems:

- Vertical angles formed by intersecting lines are congruent (Euclid's 15th proposition).
- If two lines are cut by a transversal and alternate interior angles (respectively: alternate exterior angles, or corresponding angles), then the lines are parallel.
- If two parallel lines are cut by a transversal, then alternate interior angles are congruent, alternate exterior angles are congruent, and corresponding angles are congruent.
- If a triangle is isosceles, then the angles opposite congruent sides are congruent (Euclid's 5th proposition.)
- If two angles of a triangle are congruent, then the triangle is isosceles (Euclid's 6th proposition).
- Pythagorean Theorem: If a right triangle has legs of length a and b and hypotenuse of c, then $c^2 = a^2 + b^2$.
- Sum of angles: The interior angles of a triangle sum to a straight angle.

6.10 Postulates and Theorems FAQ

Q: I know a postulate is supposed to be different from a theorem, but I can't see any systematic difference. Is there one?

A: Yes and no. Many theorems are delicate and complicated truths that you would never perceive immediately to be true (as one is supposed to with postulates). However, the theorems listed on the previous sheet were the most basic ones, that could indeed have been taken as postulates. Why weren't they? People felt is was such a big deal to assume something without proof, that they wanted to have as few postulates as possible. Once they saw they could prove some of the basic facts from others, they threw them off the postulate list. In fact, much effort was spent (and a whole branch of mathematics developed) trying to prove the parallel postulate.

Q: I don't understand why these rules of deductive reasoning make it any easier to tell when something has been proved.

A: Nowadays, there are tighter and more sophisticated formal systems for proving geometric facts, but they are all quite difficult and they eliminate much of the mathematical intuition we are trying to foster here. The Euclidean system, however, is pretty darned good, in that it is very hard to construct a proof, using the Euclidean rules, that has a flaw. Most such situations involve a diagram where something is drawn between two other things when it really can't be. Aside from this type of ambiguity, deciding when something has been proved by Euclidean rules is something a computer can do (and hence so can you).

Q: What do the postulates and theorems have to do with straightedges and compasses?

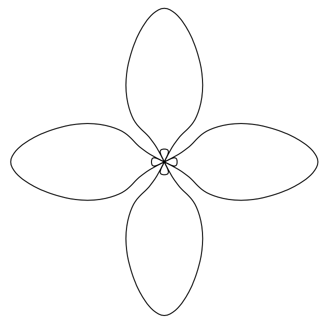
A: You could use the Euclidean postulates and theorems without ever referring to straightedges and compasses. The straightedge and compass are “safe” tools, in that anything they can do is guaranteed by the postulates to be do-able. You might ask why you would want such a guarantee, since anything any instrument can do is by definition do-able. One answer is that the Greeks were very cautious about asserting that a drawing procedure did something that could be stated and understood, so they wished their instruments only to be able to do what they could prove theorems about. This is why some of the theorems explicitly address straightedge and compass constructions.

7 Symmetry and rigid motions

7.1 Symmetry of Planar Figures

1. What are all the symmetries of the following objects?

(a) A

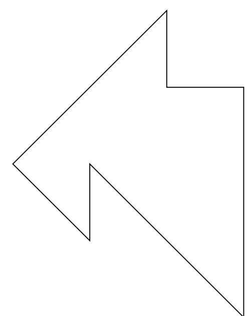


(c)

(b)

... B B B B B B B B ...

(d)



2. For each of the following, sketch the figure and indicate its lines of reflectional symmetry and centers of rotational symmetry.

    (a) an equilateral triangle
    (b) an isosceles triangle
    (c) a scalene triangle
    (d) a square
    (e) a rectangle
    (f) a parallelogram
    (g) a regular hexagon
    (h) a circle

3. Can you find a quadrilateral that has reflectional symmetry but no rotational symmetry? What about the other way around?

7.2 Rigid Motions

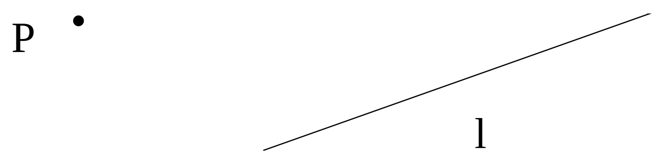
1. Translation:

   (a) The translation of a figure by an arrow a units to the right and b units up is given by the function

   $$T(x, y) = ?$$

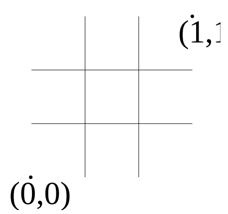
   What about the translation a units to the right and b units down?

   (b) Given a line segment, l, and given a point, P, construct the translation of P by the segment l in one direction. (You may quote the result of any previous construction you have done.)



   (c) Is it possible to divide an isosceles triangle into two pieces that differ only by a translation? Can this be done with a parallelogram?

   (d) A Tic-Tac-Toe board is drawn inside the unit square whose corners are $(0, 0), (0, 1), (1, 0)$ and $(1, 1)$, as shown. What is the name of the translation that moves the upper-left square of the Tic-Tac-Toe board to the lower-right one?

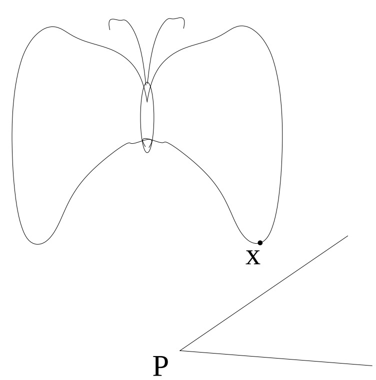


2. Rotation:

(a) A 90° clockwise rotation around the origin is given by the function

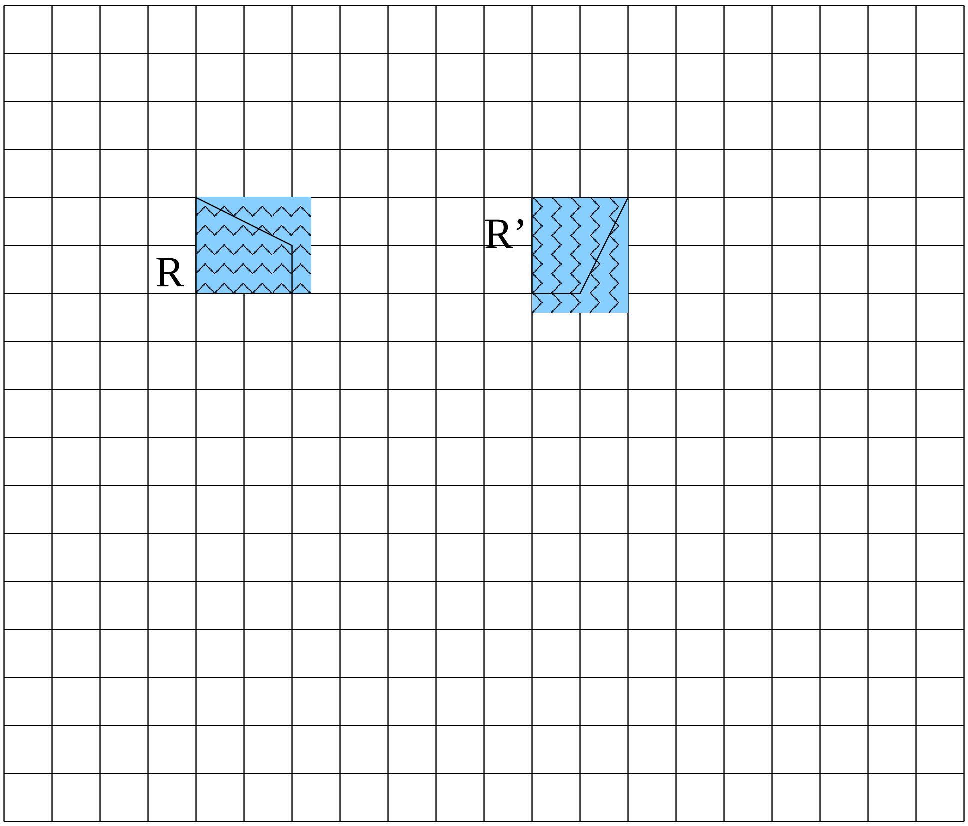
$$T(x, y) = ?$$

(b) An angle is drawn with vertex P. You job is to construct a rotation by that angle with P as the center of rotation. Specifically, use your straightedge and compass to locate where the tip x of the butterfly's wing should go after the rotation.



(c) Can an isosceles right triangle be cut into two pieces that differ only by a rotation? Can this be done with an equilateral triangle?

(d) Find the center of the rotation that sends the trapezoid R to the trapezoid R' in the figure on the next page.

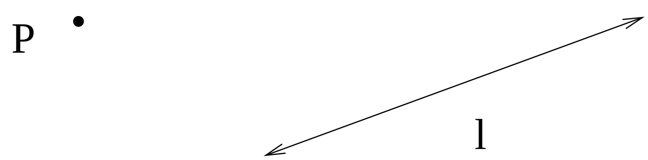

R
R'

3. Reflection:

(a) Reflection across the diagonal line $y = x$ is given by the function

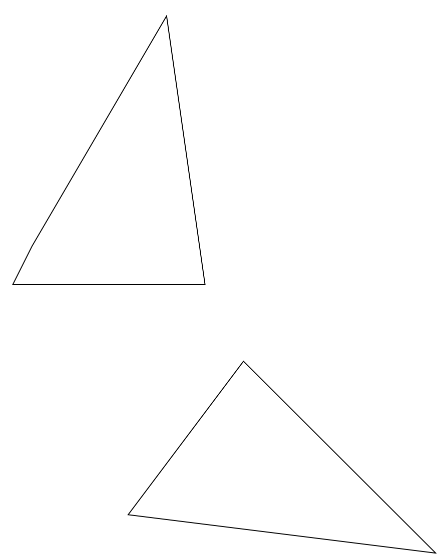
$$T(x, y) = ?$$

(b) Given the line l and the point P, construct the reflection of P across l.



(c) Can an isosceles triangle be cut into two pieces that differ only by a reflection? Can this be done with a parallelogram?

(d) In the drawing below, the triangle $\triangle A'B'C'$ is gotten from $\triangle ABC$ by a single reflection. Construct the line across which the triangle was reflected.



7.3 What symmetries are possible?

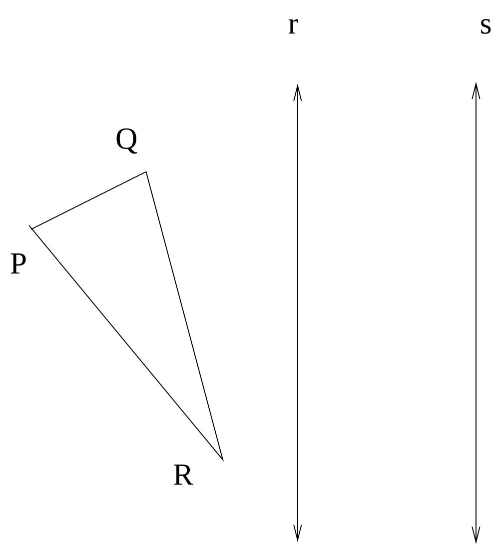
1. Draw a plane figure with two different lines of symmetry and no rotational symmetry.

2. Draw a plane figure with two different rotational symmetries and no line of symmetry.

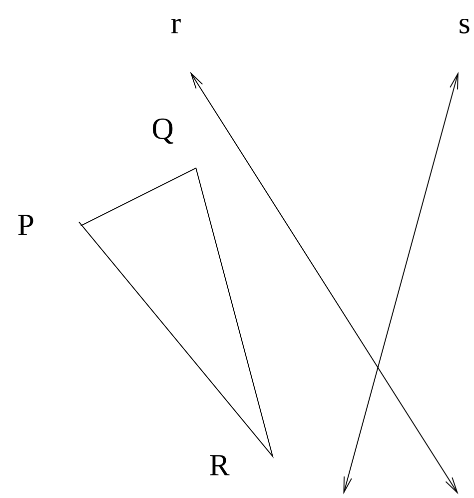
3. Draw a plane figure with two different centers of rotational symmetry.

7.4 Composing Rigid Motions

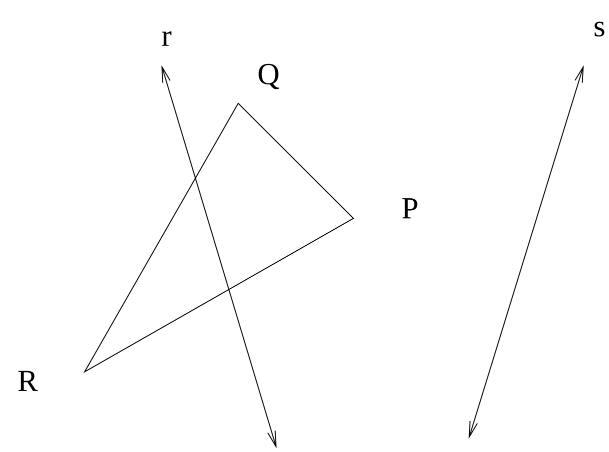
For each situation, 1 through 3, carefully carry out the two reflections, first across line r then across line s. You may use straightedge and compass, or different tools if you prefer. Then, in each case, describe a single rigid motion that has the same effect as the two together that you have just done. Tell as much about the single motion as you can, and try to form a generalization about the situation. You may need to try more examples of your own in order to test and formalize your generalizations.



r
s
Q
P
R



r
s
Q
P
R

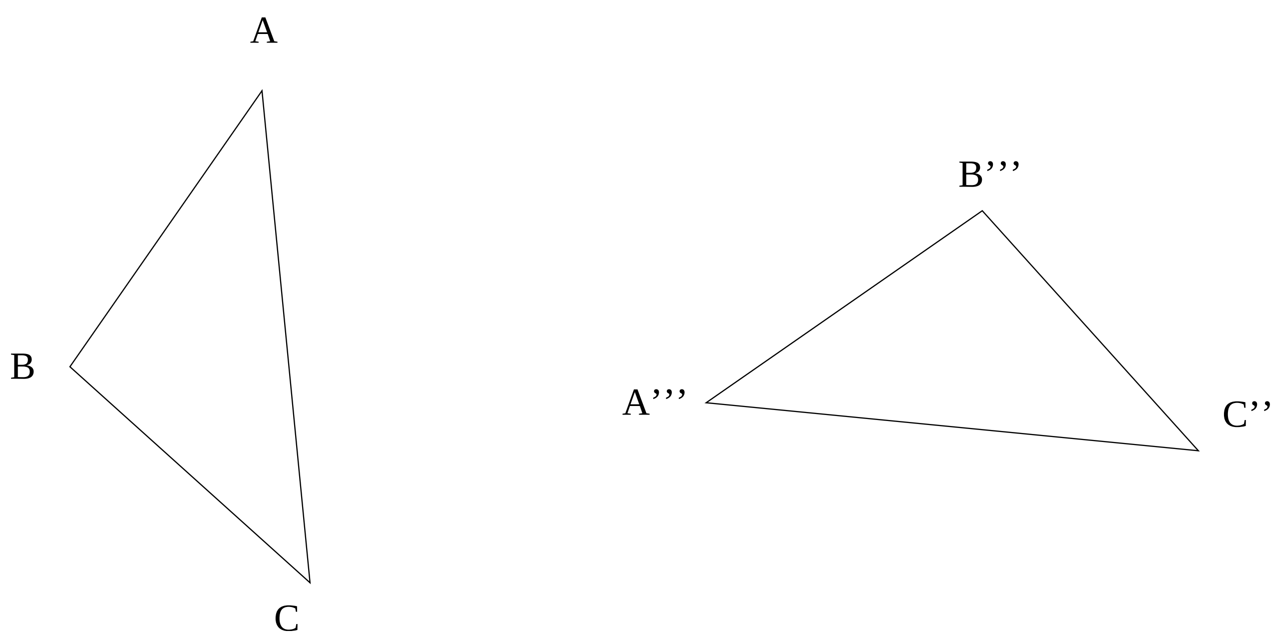


r
s
Q
P
R

7.5 Three Flips

It turns out that any rigid motion in the plane is equivalent to a series of three or fewer reflections. If I have in mind a rigid motion, and you want to compose it from three reflections, here is one way to do it (there are many ways).

> Draw a triangle; in the figure, your triangle is $\triangle ABC$. Ask me to perform my rigid motion and see where your triangle gets sent; in the figure, I sent it to $\triangle A'''B'''C'''$.
>
> Find the line l_1 that reflects A to A'''. Finish reflecting $\triangle ABC$ across l_1 and call this $\triangle A'B'C'$. Of course $A' = A'''$.
>
> Now find the line l_2 that reflects B' to B'''. Finish reflecting $A'B'C'$ across l_2 and call the result $A''B''C''$.
>
> Finally, find the line l_3 that reflects C'' to C'''. The lines l_1, l_2 and l_3 are the three lines of reflection you seek.



7.6 Three Flips, continued

1. After reflecting in l_1 brings A to the position A''', why does this point stay in position and not get moved by the reflections in l_2 and l_3?

2. Suppose my original rigid motion had been orientation-preserving rather than orientation-reversing as in the example. What would have happened after the second reflection? Why?

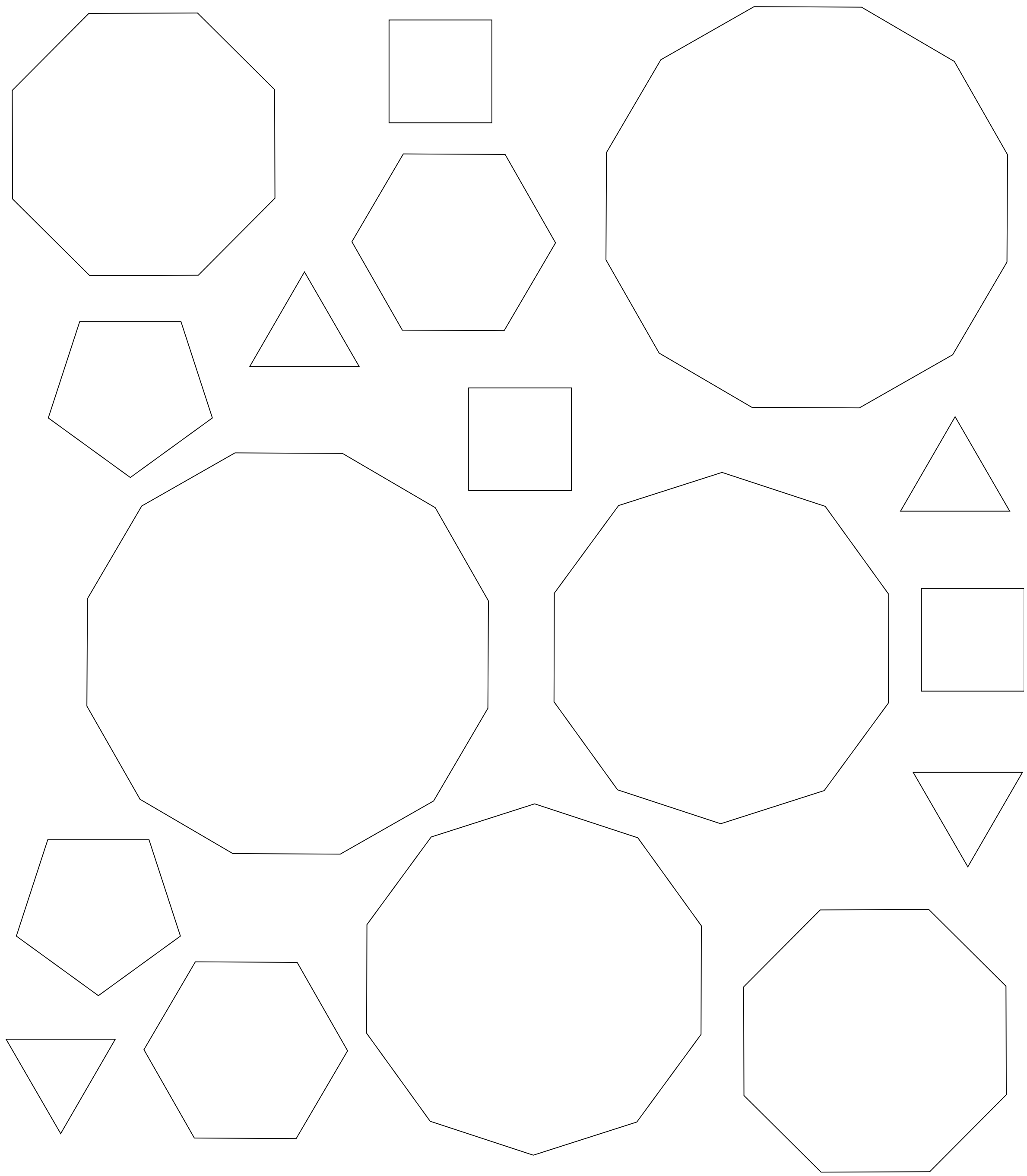
3. Prove or disprove: If you do 3 reflections, and the result is not a reflection, then it is a glide-reflection, that is, a reflection followed by a translation along the same line.

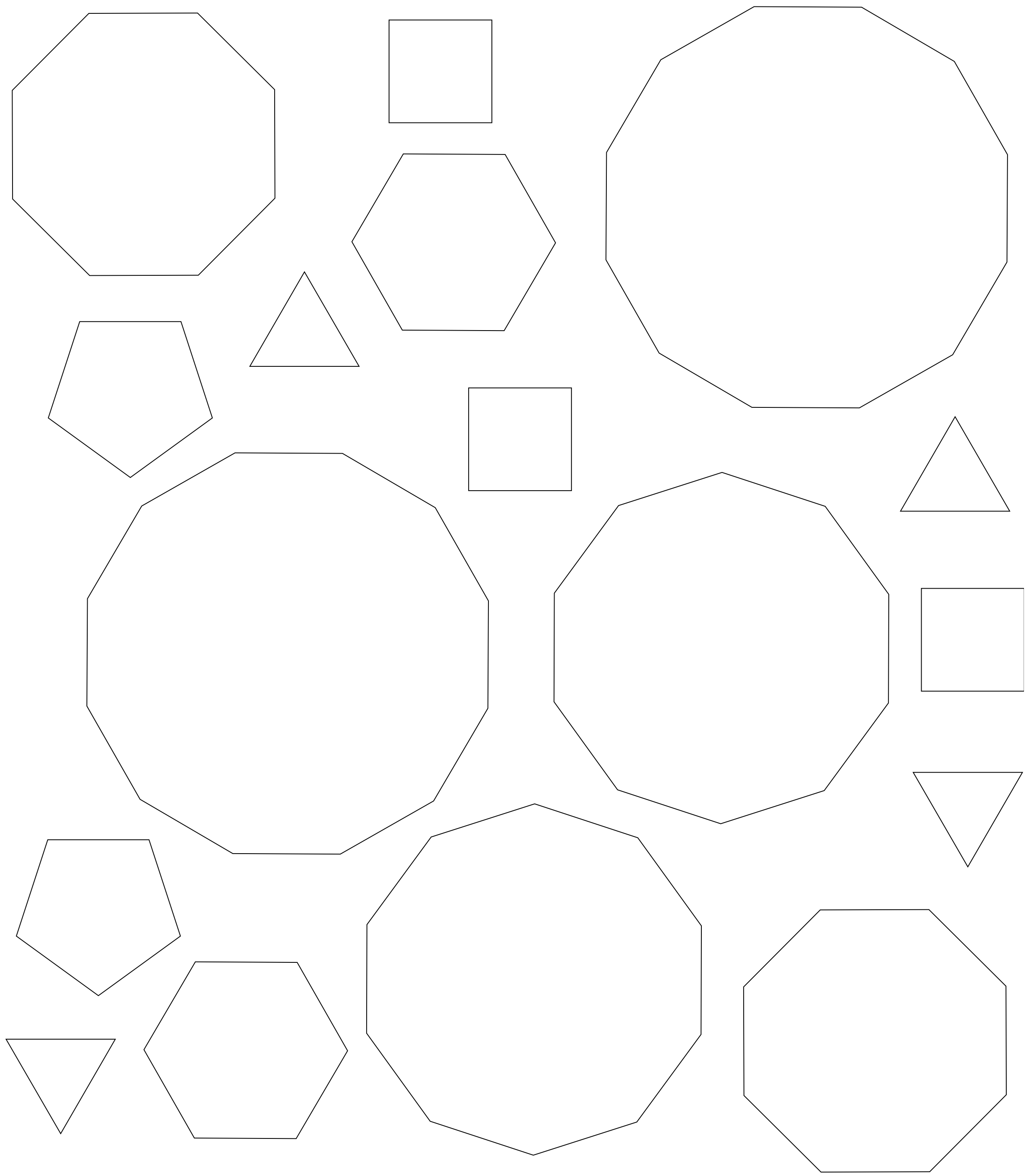
7.7 Tesselations of the Plane

Cut out the regular polygons provided on the following pages for use with the questions below. You may wish to trace them onto your tagboard and cut them out of that.

1. Find (and describe) all the regular tesselations of the plane. (Note that septagons, nonagons, and 11-gons are not provided. Does this bother you?)
2. Find (and describe) all the semi-regular tesselation of the plane. You will need to develop a good system for denoting these.
3. Which triangles will tesselate the plane? (Try all kinds.)
4. Which quadrilaterals will tesselate the plane? (Try all kinds.)

Be sure you can defend your answers to all these questions!





7.8 Symmetries of Solids

Find all the planes of reflectional symmetry and axes of rotational symmetry (with orders) of the following polyhedra. You may wish to use physical models. It is also a good idea to be systematic, so you don't leave any out.

1. a cube

2. a tetrahedron

3. an equilateral triangular prism

4. a pentagonal pyramid (on a regular pentagon)

5. an octahedron

8 Supplements

8.1 Excerpts from the NCTM Standards, 2000

8.2 Excerpts from the MAA recommendations

8.3 Supplemental readings