MATH 105 Course Packet

Theory of Arithmetic
for Pre-service Teachers

Contents

Math 105 (5 Credits)
Fundamental Math Concepts for Teachers, I: Arithmetic
Autumn 2002

Prerequisite:	Proficiency at the level of Math 104
Instructor:	Robin PEMANTLE
Office:	636 Math Tower, 292-1849, pemantle@math.ohio-state.edu
Office hours:	M 2:30, Tu 11:30, and by appointment
Co-Instructor:	Vic FERDINAND
Office:	436 Math Building (connected to Math Tower), 688-3126, ferdinand@math.ohio-state.edu
Office Hours:	daily 2:30–3:20 and by appointment

Text: The only text is the course packet available at the University Bookstore. Please buy the packet ASAP. You may return any other text you have bought, as only the packet will be needed.

Grading: Your grade in this course will be based on:

> Attendance and participation 15%. Being late or leaving early counts as half an absence. Each of the 48 days (10 weeks plus three days minus 1 midterm, 3 holidays and one cancellation day) 1/3 of 1%. That means anyone who misses at most 3 days can get 100% of the grade for attendance, assuming their participation is satisfactory. Because we allow you 3 absences without penalty, we will not have make-ups.
>
> Written work (45%) including problem writeups, group writeups, reflections and quizzes. For your problem writeups, you will be graded roughly as follows. Half the grade is for a reasonable approach: you must be trying to solve the right problem, you must have some ideas as to how to do this, you must give a clear explanation of any methods or reasoning that you use. The other half is for the

solution, which includes not only the right answer but also an explanation of why and, when possible, a proof. If you are not satisfied with your score on a homework assignment, we will accept a re-do no later than two class meetings after the assignment is handed back, for up to 90% of the original credit. Any work that is illegible or is not grammatically correct will not be graded. More detailed specifications for your written work can be found in your course packet.

A midterm exam (15%) will be given in class on Tuesday October 29.

A final exam (25%) will be given during the scheduled final exam period, Monday December 9, 3:30–5:18 PM.

Please bring your binder to class every day, along with any completed work you have accumulated.

You will notice there is usually NO ROOM on the printed worksheets for working out an answer. Please do your work on blank paper. Three-hole paper will be more convenient for preserving in your binder, though if you don't mind some inconvenience, you can use regular paper and punch it afterwards.

The course materials are divided into Worksheets (Sections 1–4) and Readings (Sections 5 and 6). Most of the readings in Section 5 come along with self-check problems. Whenever one of these readings is assigned, **the self-check problems are assigned also**. When you find them easy, quick answers will suffice. When you find them difficult, they are to be tackled, alone or with classmates, so that you come to class prepared to ask questions and with at least provisional answers. We will check at the beginning of class that you have answers or other written work on any assigned self-check questions. Do not expect to read the readings at your normal reading pace – expect rather to read them at the pace you read math texts, and allow time for stopping to do the problems. The readings in Section 6 are textual and may be read at a regular pace, with no self-check problems.

1 Problem-solving and abstract thinking

Quoted from the movie "Life of Brian", scene 19

BRIAN:	Look. You've got it all wrong. You don't need to follow me. You don't need to follow anybody! You've got to think for yourselves. You're all individuals!
FOLLOWERS:	Yes, we're all individuals!
BRIAN:	You're all different!
FOLLOWERS:	Yes, we are all different!
BRIAN:	You've all got to work it out for yourselves!
FOLLOWERS:	Yes! We've got to work it out for ourselves!
BRIAN:	Exactly!
FOLLOWERS:	Tell us more!

1.1 Poison

Reading: Paradigm for abstraction and generalization.

A Game

Number of Players: Two teams of one or more players

Equipment: Counters

Rules:
1. Begin with 10 counters.
2. Teams take turns playing.
3. When it is your team's turn, take away 1 or 2 counters.
4. The team that takes the last (POISONED) counter loses.
5. Toss a coin to decide which team goes first the first time you play POISON; after that, alternate which team begins the game.

Instructions: Play the game a few times. As you play, remember or keep a record of how you played the game; that is, how many counters did you pick up? Why? Who won the game?

After you have played the game a few times, reflect on your game records and see if your group can devise a winning strategy. Once you think you have one, first test it out (against other players), and then see if you can make it work when the number of initial counters is changed from ten.

Question: Suppose a game of POISON begins with 431 counters. Do you want to play first or second? What should be your strategy (that is, how should you play)?

1.2 Time to weigh the hippos

Martha is the chief hippopotamus caretaker at the Wild Animal Park in San Diego, California. She has just arrived at the cargo dock in order to pick up four members of the endangered species *hippopotamus mathematicus* that were recently rescued from African poachers. Before the animals are released by the harbormaster, Martha must weigh them. BUT the only scale big enough to weigh a hippo is the truck scale that doesn t weigh anything lighter than 300 kilograms (kg); this is more than each of the hippos weighs. Martha is puzzled for a few minutes, then gets the idea of weighing the hippos in pairs, thinking that if she gets the mass of every possible pair, she can later figure out the masses of the individual hippos. She weighs the hippos pair-by-pair and gets 312 kg, 356 kg, 378 kg, 444 kg, and 466 kg. When she tries to weigh the heaviest pair of hippos, the scale breaks. Alas!

1. What was the mass of the last pair of hippos who broke the scale?

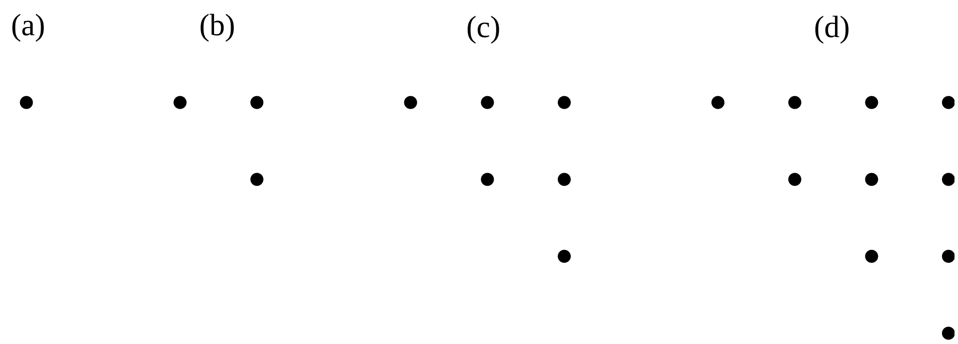
2. What are the masses of the individual hippos?

NOTE: Be sure to write down explicitly any assumptions you make, and be able to explain your reasoning at each step.

1.3 Dots and patterns

Reading: Introduction to sets

1. Look at these triangular arrays of dots.



Count the number of dots in each triangle. How many dots would there be in the next (undrawn) triangle?

Fill in the table.

n	number of dots
1	1
2	3
3	6
4	10
5	
6	
7	

Do you see a pattern which relates the number of dots in the n^{th} array to the number n? If so, describe the pattern.

You may have noticed the pattern in the following table.

n	number of dots	
1	1	$1 = 1$
2	3	$1 + 2 = 3$
3	6	$1 + 2 + 3 = 6$
4	10	$1 + 2 + 3 + 4 = 10$
5		
6		
7		

Check to see whether this pattern holds in the rest of the table.

Questions:

(a) What is the 100^{th} triangular number, i.e., the number of dots in the 100^{th} array?

(b) Write, in terms of n, a formula for the number of dots in the n^{th} array.

2. This is a related counting problem.

   (a) Let X be the set $\{a, b\}$. List all the two element subsets of X. The number of two element subsets of X is ____.

   (b) Suppose that X is the set $\{a, b, c\}$. List all the two element subsets of X. The number of such subsets is ____.

   (c) Suppose that X is the set $\{a, b, c, d\}$. List all the two element subsets of X. The number of such subsets is ____.

   Do you see a relationship between the numbers in this problem and the "triangular numbers" of Problem 1? Try to explain, in a complete sentence, what relationship you are observing.

   How many two letter subsets do you think could be formed if the entire English alphabet were available?

1.4 Glicks, Glucks and Dr. Seuss

Reading: Introduction to propositional logic

1. Read the following true statement about **Glicks** and indicate TRUE, FALSE, or CAN'T TELL for each of the conclusions listed.

   All **Glick** numbers greater than 20 are even.

STATEMENT	TRUE?	FALSE?	CAN'T TELL?
a. No odd number greater than 20 is a Glick.			
b. If an even number is greater than 20, then it is a Glick.			
c. No odd number is a Glick.			
d. No number less than 20 can be a Glick.			
e. All even numbers are Glicks.			
f. 33 is not a Glick.			

2. Read the true following statement about *Glucks* and indicate TRUE, FALSE, or CAN'T TELL for each of the conclusions listed.

If a number is a multiple of 6 or is divisible by 7, then it is a **Gluck**.

STATEMENT	TRUE?	FALSE?	CAN'T TELL?
a. 17 is a Gluck.			
b. 42 is a Gluck.			
c. 38 is not a Gluck.			
d. No even number is a Gluck.			
e. No positive number less than 5 is a Gluck.			
f. All Glucks are divisible by 42.			

3. Ben and Nancy have a daughter Tessa. They read to Tessa a lot. The following statements are all true.

   - All of the books by Dr. Seuss which they own are among Tessa's favorites.
   - If they read a book to Tessa yesterday, then it was one of her favorites.
   - They own *Green Eggs and Ham* by Dr. Seuss and *Jamberry* by Bruce Degen.

   Classify each of the following as TRUE, FALSE or CAN'T TELL:

   (a) *Green Eggs and Ham* is one of Tessa's favorites.

   (b) *Jamberry* is one of Tessa's favorites.

   (c) They read *Green Eggs and Ham* to Tessa yesterday.

   (d) They own all of Tessa's favorites.

   (e) They own all the books by Dr. Seuss.

1.5 Gotta hand it to you

This summer, the town of Hicksville celebrated the Fourth of July by having a parade. One of the floats held most of the business people and their families and friends. At the end of the parade, all the people on the float shook hands with each other.

Then the Judge arrived suddenly from the County Seat, and shook hands just with the people he knew, which were mostly the biggest of the bigwigs. Altogether, there were $1,625$ handshakes. How many people did the Judge know? How many people were on the float? Be sure to explain your approach(es) to this problem – there could be more than one!

1.6 Banned book survey

Reading: Counting revisited

As part of the Banned Books Week commemorations, the Madison Public Library survey their patrons on what previously banned books they had read. Fifty-three percent had read *Huckleberry Finn*; 42% had read the Bobbsey twins books; and 36% had read *Lady Chatterly's Lover*. To get more detailed information, the library staff did a follow-up survey and found that 25% of their patrons had read both *Huckleberry Finn* and the Bobbsey twins books, while 14% had read the Bobbsey twins books and *Lady Chatterly's Lover*, and only 10% had read *Huckleberry Finn* and *Lady Chatterly's Lover*. A reporter tried to figure out from these data how many of the people surveyed had read all three books. It turned out that 12% of those surveyed had not read any of the books. Did the reporter need this last piece of information to answer her question? How many had people had read all of the books?

1.7 Chickens, rabbits, temperature, driving, ages and headaches!

1. In the barnyard there are some chickens and some rabbits. There are 50 heads and 120 legs on these chickens and rabbits. How many chickens and how many rabbits are there in the barnyard?

2. How hot is an oven whose temperature in degrees Fahrenheit is double the temperature in degrees Celsius?

3. You can drive from Columbus to Chicago in 6 hours. If you drive 13 MPH faster, you can make the drive in 5 hours. How fast do you drive when it takes you 6 hours to complete the trip?

4. Donna is now twice as old as Jalen was when Donna was 30 years older that Jalen is now. How old are they now?

5. Invent a problem whose main point is the assignment of algebraic variables to quantities. Make it at least as hard as the chickens and rabbits problem but no harder than the problem about driving to Chicago. Now explain how to solve your problem. How did you go about assigning variables? What was the hardest part? Trade problems with one other person in your group, solve theirs, and answer the same questions about how you did it and what was difficult. State how your answers compare with those of the problem's author.

1.8 Comparing without counting

1. One day an argument arose between two four-year olds as to who had the most Lego pieces. They each counted; one reported having "a million and five" while the other reported "a thousandy hundred". Needless to say, a recount revealed that neither could reliably count high enough to count the Legos belonging to either of them. How can they settle their argument without the intervention of adults?

2. Suppose that X is the set $\{a, b, c\}$ and Y is the set $\{a, b, c, d\}$.

   (a) List all the subsets of X.

   (b) List all the subsets of Y that are not subsets of X.

   (c) Explain how one can tell, without counting, that the number of subsets in (a) is equal to that in (b).

   (d) What can you say about the relationship between the number of subsets of X and the number of subsets of Y.

   (e) If you knew that a set X had 32 subsets, how many subsets do you think the set $X \cup \{t\}$ would have if t is not an element of X? ___ What about if t is an element of X? ___

1.9 Word Problem Review

1. Right now my oldest child is four times as old as my youngest child. Next year, my oldest child will be three times as old as my youngest child. How old are they right now?

2. The GPA for a class is supposed to be 3.5. It turns out that 10% of the students don't take the final, so they flunk. To the remainder of the students, Professor Nice is only willing to give A's and B's. What portion of the class should get A's?

3. Diana has to drive 600 miles every week. Sometimes she has to take the kids places in the van which gets only 10 miles to the gallon, but the rest of teh time she drives a PT Cruiser which gets 20 miles to the gallon. If she wants to average at least 15 miles to the gallon, how many miles a week must she drive in the Cruiser?

4. Last year, Republicans had a 2 to 1 majority over Democrats in the Indiana State Senate. In the recent election, the Democrats gained 5 seats and now hold a 1 seat majority. How many of each are in the Senate?

2 Whole numbers and their properties

Quoted from "Surreal Numbers" By D. E. Knuth

In the beginning, everything was void, and J. H. W. H. Conway began to create numbers. Conway said, "Let there be two rules which bring forth all numbers large and small.... And the first number was created ... and Conway called this number "zero".

2.1 All in the timing

Reading: Factors and prime numbers

1. Which of these numbers is divisible by 7?

$$98; 384; 168; 32768; 84; 144$$

   Explain how you can tell this from the prime factorization. Now do the same for divisibility by 24.

2. True or false:

$$7^{29} = 13^{22}\,?$$

3. Find all the factors of these three numbers: 12, 75, 98. Count the factors of each one. What do you notice? Can you explain it?

4. Can you find a number smaller than 100 with exactly one factor? Exactly two factors? three factors? What is the smallest number n for which no number smaller than 100 has exactly n factors?

5. What number less than 1,000 has the most factors?

6. Jupiter orbits the sun once every 12 years. Saturn orbits the sun once every 30 years. They were both in the constellation Virgo in 1981. When will they next both be in Virgo? When were they next in the same position as each other?

7. The U.S. Census Clock in Washington, D.C. has signs with flashing lights to indicate gains and losses in population. Here are the time periods of these flashes in seconds: birth, 10; death, 16; immigrant, 81; emigrant, 900. In other words, every 10 seconds there is a birth, and so on.

   (a) If the immigrant and emigrant signs lit up at the same time, how long will it be before they light up simultaneously again?

   (b) What is the increase in population during a one-hour period?

2.2 Counting Factors

1. Write down a number with exactly two factors. Write down another one. Describe, in a word or phrase, what numbers have exactly two factors.

2. Can you write down a number with 16 factors? With 32 factors? Suppose a number Q is the product of n different primes; how many factors does Q have?

3. Write down a number with exactly three factors. Write down another one. Describe, in a word or phrase, what numbers have exactly three factors.

4. Write down all the factors of 12 (see "All in the Timing" problem 3). Now write down all the factors of $p^2 \cdot q$ where p and q are unspecified prime numbers.

5. Can you write down a number with exactly 9 factors? How about a number with exactly 27 factors?

6. State a rule to determine the number of factors of a number Q if you are given the prime factorization of Q. You may use mathematical notation of your choosing or stick to a purely verbal description.

7. Find the flaw in this proof that 24 has exactly 12 factors.

   > $24 = 4 \times 6$. The number 4 has three factors: 1, 2, 4. The number 6 has four factors: 1, 2, 3, 6. A factor of 24 is any factor of 4 times any factor of 6. The number of ways of choosing a factor of 4 and then a factor of 6 is $3 \times 4 = 12$, so there are 12 factors of 24.

2.3 Stamps

Suppose you have an unlimited supply of 4 cent stamps and 9 cent stamps. What amounts *CAN'T* you make with these stamps? What amounts can't you make with a supply of 9 cent stamps and 21 cent stamps? Can you make any generalizations about what can and cannot be made with a supply of a cent stamps and b cent stamps?

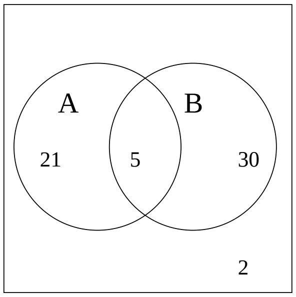
2.4 Describing sets of numbers

Reading: Sets of people

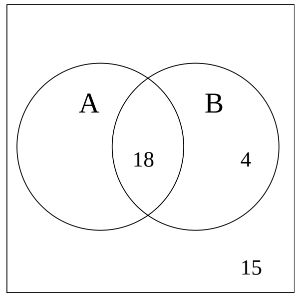
In each of the diagrams on the next page, identify the sets A and B by selecting the choices that best describe the sets.

Choices for sets A and B
(a) Multiples of 2
(b) Odd numbers
(c) Prime numbers
(d) Divisors of 24
(e) Smaller than 10
(f) Multiples of 3
(g) Larger than 10
(h) Multiples of 6
(i) Multiples of 5
(j) Divisors of 15

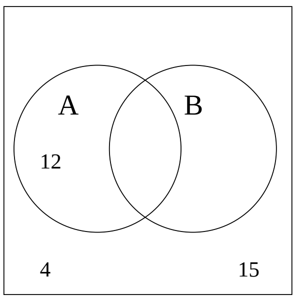
1.


A
B
21
5
30
2

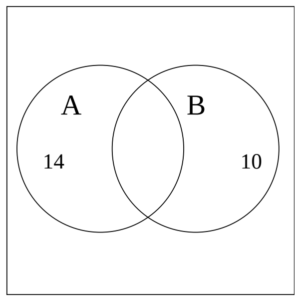
2.


A
B
18
4
15

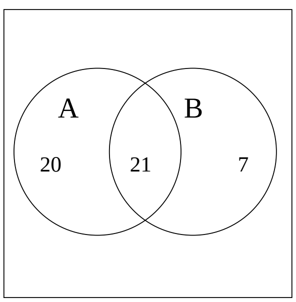
3.


A
B
12
4
15

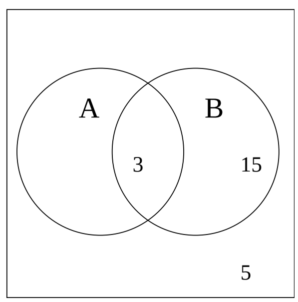
4.


A
B
14
10

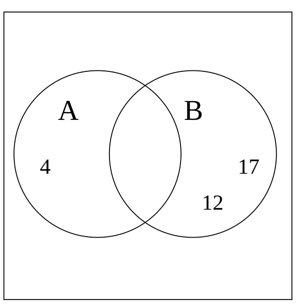
5.


A
B
20
21
7

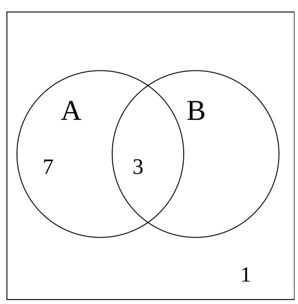
6.


A
B
3
15
5

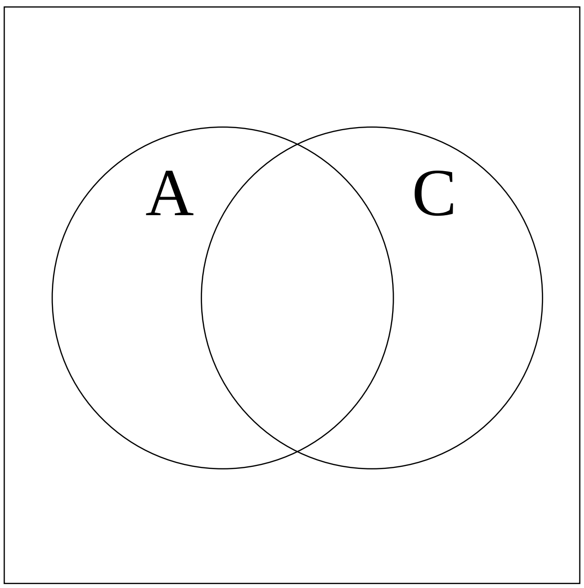
7.


A
B
4
17
12

8.


A
B
7
3
1

9. Now choose two different sets from the list above (a different pair from the eight already used), and place three or four numbers in appropriate places in the Venn diagram below so that they uniquely define your sets A and C among the choices given.

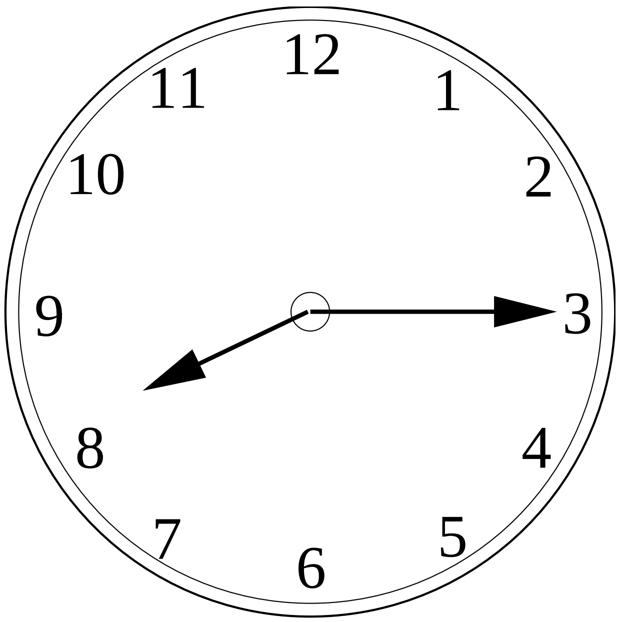


2.5 The locker problem

The lockers at Martin Luther King Middle School are numbered from 1 to 100. (It's a small school.) On Sunday morning the janitor opens all of the lockers. Then another janitor comes through and closes every second locker (that is, those numbered 2, 4, 6, ...). Then the first janitor goes through and changes every third locker: if it is open s/he closes it and if it is closed s/he opens it. Then the other janitor changes every fourth locker, and so on. At the last stage, one of the janitors changes every 100th locker (just the last locker, in other words). Which lockers are open at the end of all of this opening and closing of locker doors?

2.6 Clock arithmetic

Reading: Introduction to operations



1. Come up with a definition for a new operation $\oplus$ which will be addition in "clock arithmetic", under which the set $\{1, 2, 3, \ldots, 12\}$ is *closed* (see the first property on the reading "Properties of operations").

2. Explain how to compute $3 \oplus 5$ and $7 \oplus 10$ using your definition. Then try to state your definition in terms unambiguous enough that a computer could follow them. Does your definition work for adding more than two numbers at a time?

3. Define clock multiplication. Is it easier to solve equations such as $x \oplus 9 = 7$ in clock addition, or $x \odot 9 = 7$ in clock multiplication? What is the major difference here?

2.7 Tarzan

Reading: Properties of operations

1. You are in a checkout line, and when you get to the front, you find that your cashier is a friendly man in a loincloth. He seems to be bent on being as helpful as can be, but talks as if he had been raised in the jungle. He has been taught to run the register, but appears to have the mathematical sophistication of a small child. Some of his statements crack you up because the math they lack is so ingrained in you. Try to figure out which mathematical properties of addition prevent normal cashiers from making the following statements.

   (a) I'm sorry, I can't sell you both of those items together. Although the individual price of each is OK, there's no knowing whether the total will be a whole number of cents.

   (b) This box has nothing in it, but when I ring it up, your total may change.

   (c) Would you like me to ring up your ice cream before or after your cake? The order could make a big difference in the total, you know.

   (d) Do you want me to ring up subtotals so you can keep track of what you're spending? Well, the way I choose to group the items in subtotals will affect your total cost.

2. What property of addition is Tarzan using when he comes up with an inspired method for calculating $8 + 5$, saying "$8 + 2$ is 10, plus 3 more makes 13"?

3. (a) Is the set of natural numbers with the number 5 removed closed under addition?

   (b) Is the set of natural numbers with the number 0 removed closed under addition?

   (c) Are the even numbers closed under addition? The odd numbers? The perfect squares?

   (d) Find the smallest set containing the number 3 that is closed under addition.

   (e) Find the smallest set containing the number 1 that is closed under addition.

   (f) Find the smallest set containing the number 0 that is closed under addition.

   (g) Repeat the last three problems, but with subtraction instead of addition.

2.8 Negative numbers and the question "Why?"

Reading: Modeling addition, subtraction, multiplication and division.

One of the questions most often asked of math teachers is "Why do two negatives make a positive?" Ultimately, there is no one right answer to this, since it depends on what we mean by negative numbers and how we define operations on them. In this worksheet, we'll explore several approaches to negative numbers and several ways to answer this common question.

The vector approach. Here, negative numbers are thought of as representing physical quantities that have, somehow, an opposite direction or sense to the corresponding positive quantity. For example, on a number line, if positive numbers represent distance to the right of zero, negative numbers represent distance to the left. Another common example is *velocity*, by which is meant speed, together with the direction of travel. Someone's speed eastward on I-70 in MPH might be represented by a quantity x, in which case a negative value of x represents speed westward. A third example is profit: a negative measurement of profit indicates a loss. Time may be viewed in this way: one may measure time from a reference event (e.g., from the birth of Christ), in which case a negative time stamp indicates a time prior to the reference event. It is hard to view quantity in this way (what is negative five cheerios?), but difference in quantity may be negative, e.g., if I have negative five cheerios more than you, then I have five fewer.

From point of view of negative numbers as physical quantities, the properties of negative numbers are determined by how they behave in story problems. For each of the following story problems, define a set of variables representing the given quantities, using negative numbers when appropriate and write equations representing information given or required. The object in this exercise is to clarify whether quantities are positive or negative, and whether they are being added or subtracted, so that we may see how physical sense determines the correct way to handle various combinations, such as a positive number plus a negative number, a negative number minus a negative number, and so forth.

1. In a game of Chutes and Ladders, Sally gets to move 5 steps forward, but then falls down a chute that moves her 17 steps backwards. What is her net movement in this turn?

2. Lucy, the first human fossil, was born around the year 600, 000 BC. A wave of primate migration was supposed to have taken place 2.5 million years prior to that. Roughly what year did the migration take place?
3. Eugene is learning to drive a stick shift in San Francisco. Attempting to start uphill, he finds himself rolling backwards at 6.8 MPH. Trying not to panic, he applies the brakes, reducing his backward speed by 4.9 MPH. What is his speed now?
4. Pacific Stereo promises customers a rebate of $30 on all complete stereo systems, but the rebate is $13 dollars less if you don't have a printed coupon. What is the cost of a $149 system if you don't have a coupon?

For each story above, state what laws involving negative numbers are implied.

The charges model. A model related to the vector model, but a little more abstract, is the charges model. In this model, positive and negative numbers are represented by charges of opposite type, which cancel when combined. In this model, any number has many representations, for example, the number +4 may be represented by 4 positive charges, or by 7 positive charges and 3 negative charges, and so forth. (Don't try to represent 4 as 3 positive charges and −1 negative charges, since in this model there is no natural definition for the quantity −1 that is not circular.)

1. Represent the problems below with red chips for negative, black chips for positives, and use cancellation to arrive at the correct sum.

$$\begin{aligned} 2+3 &= \_\_\_ \\ -2+-5 &= \_\_\_ \\ -4+1 &= \_\_\_ \end{aligned}$$

2. If we wanted to subtract -2 from 4, we would need to take away two red chips from our representation of 4. This time, we'd better pick a representation of 4 that has at least two red chips! Pick such a representation, then take away the two red chips and read off from this what $4 - (-2)$ is. Fill in the blanks to indicate what you did.

start: red chips	start: black chips	end: red chips	end: black chips	result

   Now use this technique to solve

$$\begin{aligned} -4-(-3) &= \_\_\_ \\ 1-3 &= \_\_\_ \\ -3-(-5) &= \_\_\_ \end{aligned}$$

The formal model. Properties of negative numbers may be derived from the postulate that they should obey exactly the same rules as positive numbers, that is, the commutative laws for addition and multiplication, the associative law, the distributive law, as well as rules about the sum or difference of equals being equal. In this model,

- negative numbers are *defined* to be additive inverses of positive numbers, that is, $-x$ is defined as the solution to the equation $x + (-x) = 0$;
- subtraction is defined as the inverse operation to addition, so by definition, $x - y$ is the number z satisfying $z + y = x$.

While this kind of formal interpretation is neither appropriate nor satisfying for small children, it is increasingly germane as one continues into higher mathematics, where objects become more abstract and definitions and laws are the only concrete pieces of knowledge we have. For your own education, you must learn to work with at least the basic formal rules.

1. Use these definitions and these properties to give a line-by-line justification of each of the following derivations.

(a) To show that $x + (-y)$ is equal to $x - y$:

$$\begin{aligned} [x + (-y)] + y &= x + [(-y) + y] \\ &= x + 0 \\ &= x\,, \end{aligned}$$

so the quantity $x + (-y)$ is a number which you can add y to to get x, therefore $x + (-y)$ is the number $x - y$.

(b) To show that $x \cdot (-y)$ is equal to $-(x \cdot y)$,

$$\begin{aligned} x \cdot y + x \cdot (-y) &= x \cdot [y + (-y)] \\ &= x \cdot 0 \\ &= 0 \end{aligned}$$

so the quantity $x \cdot (-y)$ is the quantity z for which $x \cdot y + z = 0$, so by definition it is the quantity $-(x \cdot y)$.

2. Solve the equation

$$(-x) - 3 = -7$$

for the variable x. Then explain a step by step procedure to solve all equation involving a single subtraction and possibly some negative numbers, and justify that it works.

2.9 "Why", part II

Pick one of these story problems and write an explanation of why the product of two negative numbers is a positive number, using the chosen story as an aid. Then say why you chose the story that you did, and whether you could have used one of the others.

- Sheila's farm is rectangular, with one edge running 1.5 miles North from the center of the valley, and the other edge running 2.25 miles East from the center of the valley. Bruce's farm is also rectangular, with one edge running 4 1/3 miles South from the center of the valley and the other edge running 3/4 of a mile West from the center of the valley. What are the areas of the two farms?
- Jane is the CEO of a large investment company, whose earnings for the 2002 fiscal year were $-15,000,000. That is, the company lost fifteen million dollars. She thinks up a slogan for investors: "Our earnings were ten million more than our competitor's!" What were the earnings of Jane's major competitor?
- For my Forestry lab, I am measuring the speed at which minnows travel downstream. One athletic minnow passes me at a speed of -0.14 feet per second downstream. How far downstream will it be from me in a minute? How far downstream was it from me 1 2/3 minutes ago?

2.10 Stupid number tricks

Think of a positive integer (it will be easier for you if it's not too big). Add 3. Multiply by 4. Add your original number. Subtract 11. Double the result. Cross out the last digit. Do you have your original number back again? Was the crossed out digit a 2? Explain how I knew.

2.11 Tarzan II

Tarzan is back at the cash register. This time it is more complicated, since he is just learning how to enter sales tax, and what's more, he has added in some items that you didn't want to buy. Explain what properties of operations would prevent normal cashiers from making his statements.

1. Would you like me to tax each item separately, or should I just compute tax on your total? I am sure one of these options will be much better for you.

2. This item gets a yellow tag – that means you get half off. Would you like me to compute the discount before I ring up the tax, or would you rather I add the tax and then give you a discount on the total including tax?

3. I'm so sorry I rang up $100 worth of items you don't want. Please wait while I subtract each one separately to see by how much your total will be reduced.

3 Place value

QUOTED FROM THE SONG "NEW MATH", BY TOM LEHRER

Now that actually is not the answer that I had in mind, because the book that I got this problem out of wants you to do it in base eight. But don't panic. Base eight is just like base ten really – if you're missing two fingers.

3.1 Throwing yourself off base

Reading: Introduction to bases

1. Old English money was in a combination of bases: there were 12 pence in a shilling and 20 shillings in a pound. How would you add this?

	1	pound	7	shillings	5 pence
+	3	pounds	18	shillings	9 pence

2. How many three-digit numbers are there in
   (a) base ten?
   (b) base four?
   (c) base X? (assume X is a whole number, at least 2)

3. (a) how can you tell if a number in base ten is even?
   (b) how can you tell if a number in base four is even?
   (c) how can you tell if a number in base seven is even?
   (d) how can you tell if a number in base seven is divisible by seven?
   (e) what generalizations can you find about divisibility tests?

4. Write a formula for the value of the number ABC in base X.

3.2 Arithmetic in other bases

If you haven't done so already, discuss whether (and how) the four basic arithmetic algorithms with which you're familiar extend to other bases. Then try to work out the problems below, *without* converting to base ten and back. Work directly in the given bases, and think about what you're doing.

NOTE: The character X represents ten in base eleven; in hexadecimal, the characters A, B, C, D, E, F represent the numbers ten through fifteen.

1. (a) base six

$$\begin{array}{rrrr} 4 & 0 & 4 & 3 \\ + & 3 & 1 & 3 \\ \hline \end{array}$$

(b) binary (base two)

$$\begin{array}{rrrrrr} 1 & 1 & 0 & 0 & 1 & 1 \\ + & 1 & 1 & 1 & 0 & 1 \\ \hline \end{array}$$

(c) hexadecimal (base sixteen)

$$\begin{array}{rrrr} & 3 & A & E \\ + & B & 0 & 5 \\ \hline \end{array}$$

2. (a) base eight

$$\begin{array}{rrrr} 2 & 6 & 1 & 3 \\ - & 7 & 0 & 4 \\ \hline \end{array}$$

(b) binary (base two)

$$\begin{array}{rrrrrr} 1 & 1 & 0 & 0 & 1 & 1 \\ - & 1 & 1 & 1 & 0 & 1 \\ \hline \end{array}$$

(c) base eleven

$$\begin{array}{rrrr} 2 & 3 & X & 5 \\ - & 6 & 0 & 6 \\ \hline \end{array}$$

3. (a) ternary (base three)

$$\begin{array}{rrrr} 2 & 0 & 2 & 1 \\ \times & & 1 & 2 \\ \hline \end{array}$$

(b) base nine

$$\begin{array}{rrr} & 8 & 8 \\ \times & 4 & 6 \\ \hline \end{array}$$

4. (a) heximal (base six) (b) binary (base two)

$$1\ 3\ \overline{)\ 5\ \ 4\ \ 3\ \ 0\ }$$

$$1\ 1\ 0\ \overline{)\ 1\ \ 0\ \ 1\ \ 1\ \ 0\ \ 1\ }$$

5. Can you say what was difficult about these exercises?

3.3 Justify!

Give a complete justification for why the standard addition algorithm works. To save some work, we will only consider adding two 2-digit numbers. You will probably need to use algebra (e.g., "let the first number have digits a and b, ...").

3.4 A multiplication problem

Some sixth grade teachers noticed that several of their students were making the same mistake in multiplying large numbers. In trying to compute

$$\begin{array}{cccc} & 1 & 2 & 3 \\ \times & 6 & 4 & 5 \\ \hline \end{array}$$

the students seemed to be forgetting to "move the numbers" (i.e., the partial products) over on each line. They were doing this:

$$\begin{array}{cccc} & 1 & 2 & 3 \\ \times & 6 & 4 & 5 \\ \hline & 6 & 1 & 5 \\ & 4 & 9 & 2 \\ & 7 & 3 & 8 \\ \hline 1 & 8 & 4 & 5 \end{array}$$

instead of this:

$$\begin{array}{ccccc} & & 1 & 2 & 3 \\ & \times & 6 & 4 & 5 \\ \hline & & 6 & 1 & 5 \\ & 4 & 9 & 2 & \\ 7 & 3 & 8 & & \\ \hline 7 & 9 & 3 & 3 & 5 \end{array}$$

While these teachers agreed that this was a problem, they did not agree on what to do about it. What would you do if you were teaching sixth grade and you noticed that several of your students were doing this?

3.5 A base four lesson

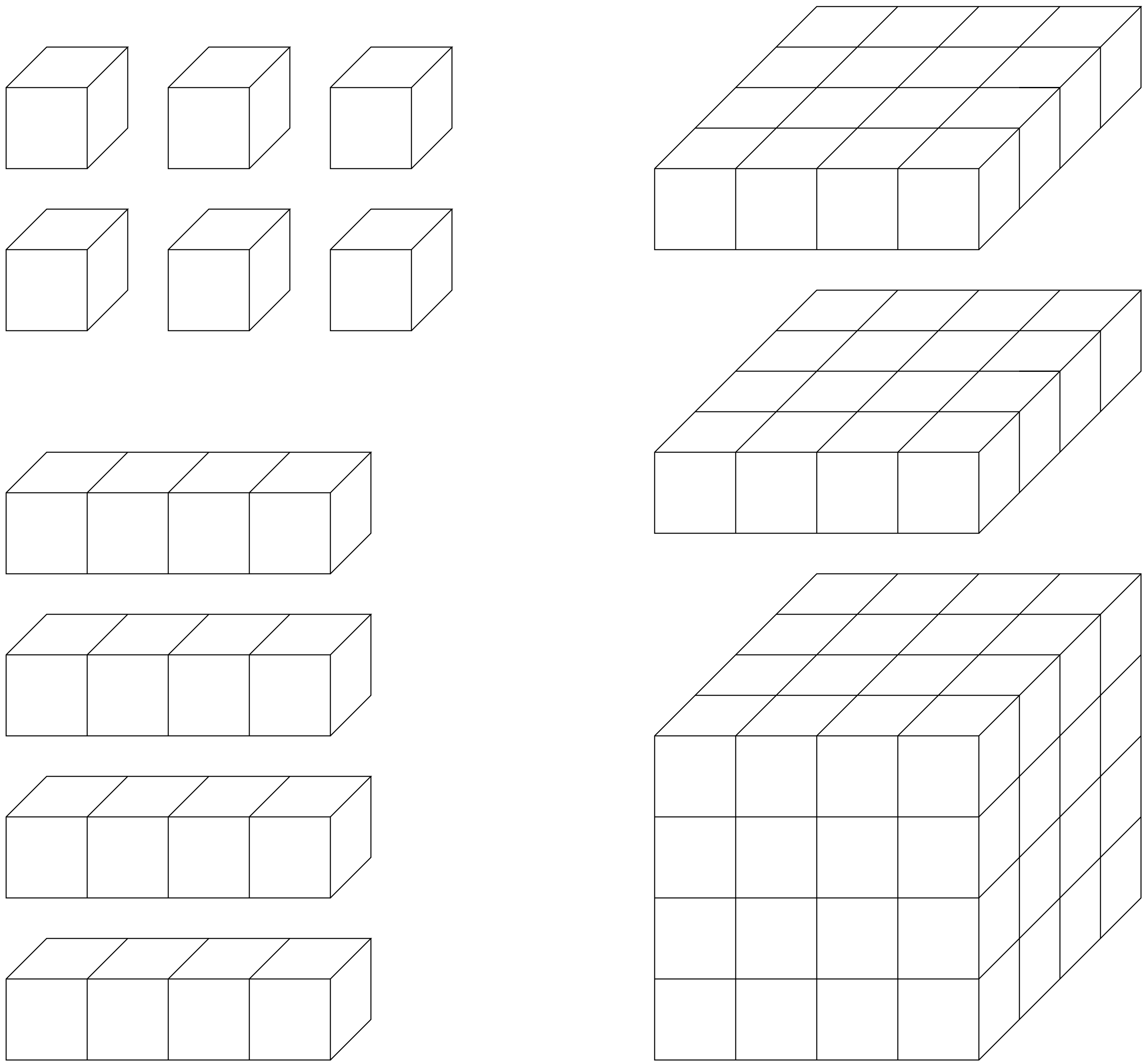
Representing numbers in different number bases and operating on (for example, adding) those numbers in different number bases can help your students to understand better ways of using different representations, and to see that (correct) algorithms for addition, subtraction, multiplication, and division work the same no matter what number base is being used.

You are to plan a lesson (not necessarily completable in one day) that makes use of base 4 blocks like those shown below (or other appropriate hands-on material). Your goals are:

1. Students will be able to represent numbers in base 4 and base ten and to translate from one of these number bases to the other one.

2. Students will be able to add two base 4 numerals correctly.

You may assume that students already know how to add in base ten. Be creative! Remember your target audience. Also be sure to specify grade level.



3.6 Big and little

Reading: exponentiation and scientific notation

1. Approximately how long would it take you to count to a billion? To a thousand?

2. Try to come up with reasonable estimates for the following quantities. Express your answer in scientific notation. The choice of units is up to you, but you must say what they are and how you came up with your estimate.

   (a) How much your hair grows each second

   (b) How many blades of grass in the oval

   (c) The time it takes an airplane's shadow to cross the Shoe

3. Give some advice as to how to estimate quantities such as the ones in the first problem.

4. What do you get (to 2 digits of precision) when you add

$$5.7 \times 10^{16} + 2.8 \times 10^{11}\,?$$

5. What do you get (to 2 digits of precision) when you multiply

$$(5.7 \times 10^{16}) \cdot (2.8 \times 10^{11})\,?$$

6. Rufus and Dufus are studying for a chemistry exam. They share the task of memorizing Avagadro's number. Rufus remembers that it's $6.023 \times 10^{\text{something}}$ and Dufus remembers that it's ($\text{something} \times 10^{23}$). Sure enough, a problem comes up in which they need to use Avagadro's number. Unfortunately, they are not sitting together. Who will do better on this problem, and why?

3.7 Negative place value

1. (a) What number does 32.1 represent in base four. Why?
   (b) What number does 32.1 represent in base seven. Why?
   (c) What number does $AB.C$ represent in base X?
   (d) What number does $AB.CD$ represent in base X?
   (e) How would you write the base ten number $9\frac{1}{2}$ in base four?
   (f) How would you write the base ten number $9\frac{1}{2}$ in base seven?

2. Most children first guess that raising a number to the power 0 should yield 0. According to the previous problem, what is the more logical assignment of a value to 10^0? To 7^0?

3. A student is upset because of a seeming inconsistency in her math paper. She wrote $.4 \times .6 = .24$ which was marked correct, but $.2 \times .3 = .6$ was marked incorrect and the alternative of .06 was written in. What can you say to this student to convince her that this is not arbitrary?

4 Division and fractions

Fear and loathing in mathematics

The ancient Greeks thought that all numbers ought to be representable as fractions. Hippasus, legend has, was the first to discover that $\sqrt{2}$ could not be represented as a fraction. His comrades designated such numbers irrational, and threw Hippasus overboard.

In the 1800's, mathematicians such as Karl Weierstrass were inventing new functions so bizarre as to shock much of the mathematics community. Hermite and his pupil Poincaré in particular described Weierstrass' new creations as 'deplorable evil'! 'I turn away with fear and horror from this lamentable plague,' he said.

4.1 Ratio and proportion problems

Below are a bunch of ratio and proportion problems suitable for elementary school kids. For each one, solve it in at least two different ways. You should be sure to clearly explain each method, why it works and why it makes sense. Avoid using standard algebraic methods (e.g. cross-multiply and divide), unless you can make sense of them. Focus on the *reasoning* necessary to solve each problem.

1. You're having a party, and have ordered pizza for everyone to share. There are 12 people at the party, and you ordered a bunch of pizzas. Three more people show up a little late, but before the pizza arrives. How much less pizza does each person get than you originally planned, now that the extra people are there?

2. Joan used exactly 15 cans of paint to paint 18 chairs. How many chairs can she paint with 25 cans?

3. John and Mary are making lemonade. John used 10 cups of lemon juice and 5 teaspoons of sugar. Mary used 7 cups of lemon juice and 3 teaspoons of sugar. Whose lemonade will be sweeter, or will they taste the same? If they don't taste the same, how can John adjust his recipe to make it taste the same as Mary's?

Look back over your work. How did your problem solving strategies change from problem to problem? Why did they change? How does this inform how you will teach fractions?

4.2 The sense of it

Mathematically, you can add, subtract, multiply or divide any numbers and the result makes sense. But when these numbers represent physical quantities, their meaning depends as well on the units of measure, and not all operations make sense. Your job on this worksheet is to make up story problems illustrating various operations and combinations of units. The story problems should be as short as possible while giving a clear physical scenario; they may be at a very elementary level.

1. Make up a problem in which two units of distance are added together. Were they the same or different units of distance? What was the resulting unit?

2. Make up a problem in which the unit "number of people" is multiplied by a unit of time. What is the resulting unit? Can you describe what it means?

3. Make up a problem in which kilograms are divided by kilograms. What is the resulting unit and what are the properties of this unit?

4. Make up a problem in which miles are divided by miles per hour. What is the resulting unit? Does this suggest any useful advice for solving this general kind of story problem?

5. Make up a problem in which dollars per day are subtracted from acres. What is the resulting unit? Does this suggest any useful advice for solving story problems?

4.3 The cider press and the condominium

A cider press can squeeze out one third of the juice remaining in an apple or remaining in apple pulp that is put in the press. How many times must you run an apple through to get out 4/5 of the juice it contains?

In an adult condominium complex, $\frac{2}{3}$ of the men are married to $\frac{3}{5}$ of the women. What portion of the residents are married? (Assume men are married only to women, and vice versa, and that married residents' spouses are also residents.)

4.4 Law and order

Readings: Models of fractions.

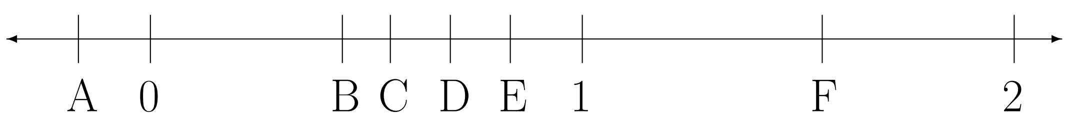
1. Suppose that $a > 1$, $0 < b < 1$, and $0 < c < 2$. Fill in each box with $<$, $=$, $>$, or NMI (Need More Information).

   (a) $a \cdot b \; \square \; a$ (b) $b \cdot c \; \square \; a$ (c) $a \cdot b \cdot c \; \square \; b$

   (d) $a + b \; \square \; a$ (e) $a + c \; \square \; a$ (f) $a + c \; \square \; b$

   (g) $\frac{c}{c} \; \square \; c$ (h) $b^2 \; \square \; b$

2. Pictured below is a number line with some identified points on it. Use this number line to answer the questions in (a)–(e).



   (a) If the numbers represented by the points D and E are multiplied, what point on the number line best represents this product?

   (b) If the numbers represented by the points C and D are divided, what point on the number line best represents the quotient?

   (c) If the numbers represented by the points B and F are multiplied, what point on the number line best represents the product?

   (d) Suppose 20 is multiplied by the number represented by E on the number line. Estimate the product.

   (e) Suppose 20 is divided by the number represented by E on the number line. Estimate the quotient.

4.5 Lynna's arithmetic

The following is Lynna's work to solve a certain arithmetic problem.

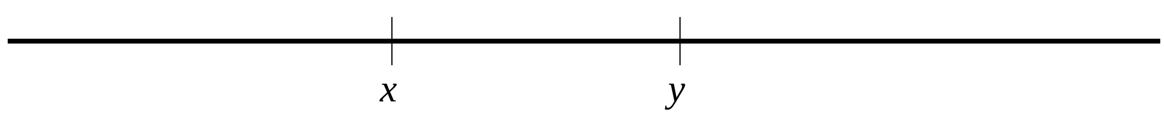
$$
\begin{array}{rr}
544 & \\
\underline{170} & 10 \\
374 & \\
\underline{170} & 10 \\
204 & \\
\underline{170} & 10 \\
34 & \\
\underline{34} & \underline{2} \\
0 & 32
\end{array}
$$

1. What arithmetic problem is Lynna solving?
2. Is her work correct? If so, then justify it. If not, then correct it.
3. Write a word problem which illustrates this arithmetic problem. Does your word problem still make sense if 544 is replaced by $544\frac{2}{3}$ and 17 is replaced by $16\frac{4}{7}$? If not, then write one that does.

4.6 Visual operations

1. Shown below are two quantities, x and y. Can you either draw $x + y$ or write down a number this is approximately equal to? Now do the same for $x - y$, $x \cdot y$ and $x \div y$.

2. Shown on the numberline below are quantities x and y. Can you indicate where $x+y$ is on the number line? What if someone indicates where 0 is? Can you indicate where $x \cdot y$ is on the number line? What if someone indicates where 0 is?.



4.7 Fractions!

1. Draw a picture that you think provides the best visual explanation of why $\dfrac{15}{24} = \dfrac{5}{8}$.

2. What is the best way to tell which of two fractions is greater? State a theorem about this, starting with "To tell whether $\frac{a}{b}$ is greater or less than $\frac{c}{d}, \ldots$" Then prove your theorem as best you can.

3. Multiply $1\frac{1}{2}$ by $8,604,587\frac{13}{18}$ and express the result in scientific notation to 3 digits of precision. Explain how you can do this in your head.

4. Explain carefully how to add any two fractions, including mixed fractions and improper fractions. You must justify each step.

5. Suppose you have to multiply $4\frac{1}{10}$ by $20\frac{1}{4}$ in your head. It is a daunting task, but upon reflection, not impossible. Explain what you think is the best way to go about this.

6. You have 50 inches of rope, from which you need to make a bunch of pieces each $3\frac{3}{4}$ inches long. How many can you make? Explain what fraction operation(s) you used, and why the computation involved in this operation was correct.

7. Your students are having trouble with the problem

$$26 \div \frac{1}{2} = ?$$

   Make up a story problem to help them understand the abstract math problem. Then explain carefully what they need to do to find the answer, and why this procedure works.

8. Compute the result of the fraction multiplication:

$$\frac{a}{b} \times c\frac{d}{e}\,.$$

4.8 Decimals

1. Express the following fractions exactly as decimals, using the repeating decimal notation when necessary. You may skip whichever you think is the single hardest fraction to compute a decimal expansion for. Which was hardest and which (several) were easiest?
$$\frac{2}{9}, \frac{13}{23}, \frac{28}{100}, \frac{4}{5}, \frac{1}{7}, \frac{19679}{1000}, \frac{11}{16}, \frac{1}{2}, \frac{17}{25}.$$

2. Find a rule to express a terminating decimal as a fraction.

3. State a theorem about which fractions become terminating decimals. Give as much justification as you can.

4. Can $\frac{5}{17}$ be written as a repeating decimal? How do you know? What about $\frac{351}{487}$?

5. Find a rule to express a repeating decimal as a fraction.

4.9 Percentages

1. Write a mathematical equation to describe what it means to discount by 35%. You must clearly state the interpretation of any variables. What is the fewest mathematical operations you can use in such a definition?

2. Write a mathematical equation to describe what it means to increase a quantity by x percent. You must clearly state the interpretation of any variables. Does your definition work if x is a negative number?

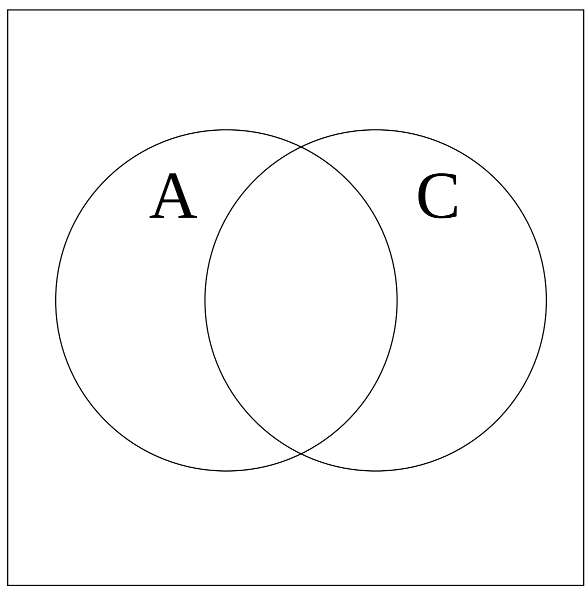
5 Readings and tutorials

5.1 Introduction to sets

Remember that a *set* is a collection of objects, like colors, numbers, people, pizza toppings, etc. These objects are called *elements* of the set, which is sometimes written like this: $A = \{$red, orange, yellow, pepperoni$\}$. Here are a few bits of set notation you need to be able to recognize and use:

- $\bigcup$ UNION: a joining together of two sets to form a larger set. For example, if $A = \{$red, orange, yellow, pepperoni$\}$ and $B = \{1$, 2, pepperoni$\}$, then $A \cup B = \{$red, orange, yellow, pepperoni, 1, 2$\}$. A union is a logical OR: in order to be a member of $A \cup B$, you must be a member of either A OR B (or both).
- $\bigcap$ INTERSECTION: an extraction of the elements that two sets have in common. For example, with sets A and B as above, the set $A \cap B$ would contain the single element pepperoni. An intersection is a logical AND: in order to be a member of $A \cap B$, you must be a member of both A AND B.
- $\bar{}$ COMPLEMENT: the set which contains everything which is *not* in a given set. $\overline{A}$ "complements" A because together they contain every possible element — $A \cup \overline{A} =$ the universe. Here we have to be careful to specify what "the universe" is. For example, taking our set A as above, if our universe is the set of all words in the dictionary, then $\overline{A}$ is huge. But if we are only considering the "universe" of primary colors and pizza toppings, then we might say $\overline{A} = \{$ green, blue, indigo, violet, cheese, hamburger, mushrooms, onions $\}$. A complement is a logical NOT: in order to be a member of $\overline{A}$, you must NOT be a member of A.

1. Pictured below is what is called a *Venn diagram*. This is a drawing which allows us to visualize how elements relate to sets. The circle on the left, labeled "A", represents set A; the one on the right represents set C. The circles are enclosed in a rectangle which represents the universe of things we are considering. The way we use it is to put (i.e., write) every element where it belongs with regard to which sets it is a member of. An object which belongs to neither set should go outside the circles, inside the rectangle; an object which belongs to both sets should go in the area which is inside both circles.



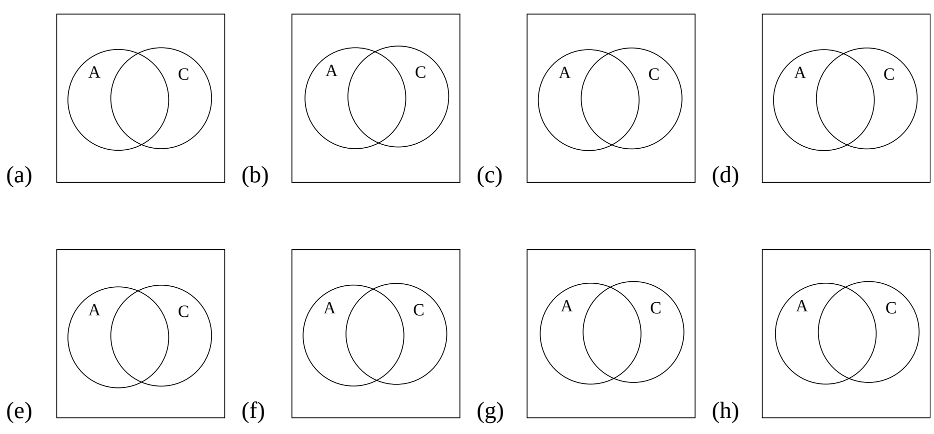
Now try putting all the objects in the following universe in their place in the diagram, using the set definitions below: blue, cheese, green, hamburger, indigo, mushrooms, onions, orange, pepperoni, red, violet, yellow.

$$\begin{aligned} A &= \{ \text{ red, orange, yellow, pepperoni } \} \\ C &= \{ \text{ orange, onions, blue, cheese, yellow, mushrooms } \} \end{aligned}$$

2. Now write down the members of the following sets, in the way A and C are defined above.

   (a) $A \cup C$
   (b) $A \cap C$
   (c) $\overline{C}$
   (d) $A \cap \overline{C}$
   (e) $\overline{A \cup C}$
   (f) $\overline{A} \cup \overline{C}$
   (g) $A \cap \overline{A}$
   (h) $\overline{A} \cap \overline{C}$

3. Now shade the regions of the small Venn diagrams below which correspond to the sets in question 2.



4. Two of the eight sets above should be identical. This is no coincidence — it is in fact a law, one of two called De Morgan's Laws. Write an equation saying that these two sets are equal. Then obtain the other of De Morgan's Laws by making another copy of the first equation in which all unions (∪) become intersections (∩), and vice versa.

5.2 Sets of people

1. Write each of the following sets as some combination (union, intersection, and/or complement) of simpler sets. Be sure to state very carefully which sets you're using, particularly when you are using complements.

   (a) the set of left-handed, near-sighted mathematicians with beards

   (b) the set of education students who have not taken Math 105, Math 106, nor Math 107

   (c) the set of students at this university who are named Amy or Dawn

   (d) the set of adults who are under five feet tall or over six feet tall

   (e) the set of people who have been President but not Vice President

2. Now let us restrict our attention to a math class with the following roster of students:

   - Jennifer Ahrendt
   - Barbara Allen
   - Carrie Ann Baerman
   - Elise Black
   - Allison Camp
   - Bill Cooper
   - Chris Danes
   - Joy Smith

Let A be the set of people in the class with a name that starts with A, and likewise let B be the set of people in the class with a name that starts with B.

(a) Who is in each of the following sets? (A Venn diagram may help.) Can you give a description in words of each of these sets without mentioning any specific names?

$\overline{A} \cup B$ $\qquad$ $\overline{(A \cap B)}$ $\qquad$ $\overline{A} \cap \overline{B}$ $\qquad$ $\overline{(\overline{A} \cup \overline{B})}$

(b) How would you use set notation to describe the following sets in terms of A and B?

People with a name that starts with A or a name that starts with B, but not both

People with a name that starts with A but without a name that starts with B

(c) Now let C be the set of class members who have a name that starts with C. Try to create a Venn diagram showing how the class members are placed with respect to sets A, B and C.

5.3 Introduction to propositional logic

What does math have to do with logic, and why should we care about using math to speak about logic, instead of plain English? Have a look at the following two quotes:

Nothing is better than eternal happiness.
A ham sandwich is better than nothing.
Conclusion: A ham sandwich is better than eternal happiness.

Every dog is an animal.
An animal is in my front yard.
Conclusion: Every dog is in my front yard.

The point of formal logic is to make mathematically precise the meanings of the terms *and*, *or*, *not* and *if*. The symbols that stand for these terms are:

$$\wedge = \text{"and"}, \quad \vee = \text{"or"}, \quad \neg = \text{"not"}, \quad \ldots \to \ldots = \text{"if} \ldots, \text{then} \ldots\text{"}.$$

Here are a couple of examples of these terms in common language; for each example say what you think the answer is to the question that follows it, and whether it is ambiguous.

(A) You will send in your tax return by April 15 or pay a $50 late fee.

• Is it possible that both will happen: send in your tax bill by April 15 and pay a late fee?

(B) You will turn in your homework on time or receive an F on the assignment.

• Is it possible that both will happen: turn in your homework on time and receive an F on the assignment?

(C) Your lunch comes with soup and cole slaw or fries.

• Do you have to choose between getting fries and getting soup & cole slaw, or do you get soup for sure and a choice between fries and cole slaw?

Traditionally the letters $p, q, r, s, \ldots$ are used for the names of variables representing *propositions* — a proposition is a statement like "you received an F on the assignment" or "your lunch comes with soup", which can have one of two truth values: True or False. Let's say for example that

$p =$ "You turn in your homework on time"
$q =$ "You receive an F on the assignment"

The proposition in example (B) is a compound proposition which we write as

$$p \vee q.$$

• Sometimes you need parentheses to make a compound proposition unambiguous. In the space below, write the compound proposition in example (C) by assigning $p =$ "you get soup", $q =$ "you get cole slaw", $r =$ "you get fries" and then using only the symbols $p, q, r, \wedge, \vee, \neg, \rightarrow,$ (and).

Now try each of these. Make sure you state which simple propositions you assign to be p, q and r.

(D) Either I'm going crazy or the bank made a mistake and they owe me ten dollars.

(E) Either Gore or Daschle will get the Democratic nomination, and Bush will get the Republican nomination.

(F) I'm not going home for Thanksgiving or for Christmas.

(G) If I study hard and the test is fair, then I'll pass the course.

(H) I'm going to take either Math 221 or Physics 201 but not both.

(I) It's not true that if you work 60 hours a week you'll get a raise.

Whether a compound proposition is true or false depends on whether each of the simple propositions it is built from is true or false. If you don't know whether each of the simple propositions is true or false, you can list all possible cases; this is called making a *truth table*. Let's take the compound proposition $p \vee q$ from example (B). How many possibilities are there for whether p and q are true or false? Make a table with a row for each possibility. The first possibility, namely that they're both true, has already been filled in.

p	q
T	T

Now put another column with $p \vee q$ at the top, and fill in each row with the correct truth value for the compound proposition $p \vee q$. The top row is filled in for you, indicating that when p and q are both true then $p \vee q$ is *by definition* true. The definitions may be intuitive to you, but if not, look them up on the definition page.

p	q	$p \vee q$
T	T	T

Now let's make a truth table for example (D). Let's say p = "I'm going crazy", q = "The bank made a mistake" and r = "The bank owes me ten dollars". The compound statement is then $p \vee (q \wedge r)$. There are going to be quite a few rows in the truth table, since there are quite a few possibilities for whether p, q and r are true or false. To help keep track of things, we'll make a column for the proposition $q \wedge r$ as well as for the whole proposition $p \vee (q \wedge r)$.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Does the truth table match the intuitive interpretation? Perhaps I am going crazy and the bank made no mistake and owes me no money. In this case we would say the sentence is true. This corresponds to the third line of the table. How about if I'm not crazy, the bank made a mistake, and it owes me ten dollars: this corresponds to the fifth line of the table and again we'd say the sentence was true. How about if I'm not crazy but the bank made no mistake and doesn't owe me money: then we'd say the sentence was false (last line of the truth table).

• Make a truth table for each of the propositions (E) – (I).

Definition Page

The proposition $p \wedge q$ is true only if p and q are both true. The proposition $p \vee q$ is true as long as at least one of p or q is true; the only case it is false is when p and q are both false. The proposition $\neg p$ (pronounced "not p") is true when p is false and false when p is true. The proposition $p \to q$ (pronounced "if p then q" or "p implies q") is true unless p is true and q is false. The truth tables for these four terms are as follows.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	$\neg p$
T	F
F	T

Whether or not these make intuitive sense to you, these are the definitions! To most people, the definitions of $\wedge$ and $\neg$ make sense, as does the definition of $\vee$ once you accept the principle that $p \vee q$ always allows both p and q to be true. The definition of $\to$ seems counterintuitive to some people. For example, if p = "Madison is in Minnesota" and q = "There are 105 senators", most people would probably argue that the statement $p \to q$ "If Madison is in Minnesota then there are 105 senators" is either false or nonsense. Mathematically, however, it is defined to be true, since both p and q are false. If you want all the definitions to seem more intuitive, try thinking of a contract. If you promise someone you will do $p \vee q$ and you do both, they certainly can't sue you for breach of contract. How about if you promise someone "if you do X then I will do Y". Under what conditions can they sue you for breach of contract?

5.4 Counting revisited

Especially if you may work with young kids, it pays to stop and think about counting. Did you know that most kids can count to ten or beyond before they can reliably count more than two objects? The reason is that learning the sequence of numbers from 1 up to 10 is much easier than learning their meaning. To count a set of objects, you have to match them up with numbers, starting at 1 and going up, without skipping or repeating an object or number. To interpret the last number s a count of the objects also requires faith that this will not change if you do it over again, and that all collections objects with this count have a "sameness" which we call the "size" of the collection. To get a glimpse of another idea hidden in the task ofcounting, wait till you try Problem 1 on the worksheet "Comparing without counting".

Advanced counting techniques have to do with arithmetical operations such as addition and multiplication. A basic cognitive model for addition is that it tells you how to count two sets together if you already know how to count each one: I have 5 apples, you have 3, together we have 8. But what about this problem? 7 kids are wearing green jackets, 8 have purple jackets and 10 have gold jackets; you have to say how many jackets in total but beware: 3 of the jackets are irridescent and look simultaneously green, purple and gold. (Answer this as a self-check problem.) If you can solve this with 7, 8, 10 and 3 replaced by x, y, z and w then you understand a useful principle of counting! (Do this too as a self-check problem.) When there are just two colors of jackets involved, this principle is called *inclusion-exclusion.* (Self-check: what is the formula in that case?)

The most advanced counting technique we'll deal with for now has to do with *Cartesian Products*. You can choose a song on a juke box by pressing a letter from A to M and a number from 1 to 8. How many songs does this represent? Though you don't need to know it at this level of formality, a mathematician would say that the set of possible pairs you can press to select a song is the Cartesian product of the set of letters up to M and the set of numbers up to 8[1]. The Cartesian product is written this way:

$$\{A, B, C, D, E, F, G, H, I, J, K, L, M\} \times \{1, 2, 3, 4, 5, 6, 7, 8\} .$$

Self-check: how many possible pairs are there? What is the Cartesian product of the sets $\{green, purple, gold\}$ and $\{Yes, No\}$ – can you write it in set notation? (To be clear: the Cartesian product of two sets is a set.) How many pairs does it contain?

[1]A mathematician would write each pair in parentheses separated by a comma, for example $(E, 7)$ but we'll usually be informal and write E7.

5.5 Models of addition, subtraction, multiplication and division

Many people, when they think of "elementary school math", equate it with "arithmetic": addition, subtraction, multiplication, and division. Although this is a restrictive viewpoint (see the NCTM Principles and Standards), these four operations still are a significant part of the curriculum. But what should we mean when discussing these four operations? Doesn't addition just mean "combine two groups and count the result", subtraction "take part of a group away and count what's left", multiplication "combine like groups and count the result", and division "share equally among all sets and count how many in each set"? This is a common perspective among many people because of the restrictive curriculum they experienced. However, these four operations can model more types of problems than what this perspective gives and children's actions when presented these models may be surprising.

Addition and Subtraction

There are five distinct categories of models one can use addition and subtraction:

1. Change Add-To (the usual "Fred has 4 apples and Sara gave him 3 more. How many apples does Fred have altogether now?"),
2. Change Take-Away ("Fred had 7 apples. He gave 3 to Sara. How many apples does Fred have left?"),
3. Part-Part Whole ("Fred has 4 apples and 3 oranges. How much fruit does he have altogether?" – note the change in the units for each quantity here),
4. Equalize ("Fred has 7 apples. Sara has 3 apples. How many more does Sara have to buy to have as many as Fred?"), and
5. Compare ("Fred has 7 apples. Sara has 3 apples. How many more apples does Fred have than Sara?").

Note: Most Math for Elementary Teachers texts define addition as "combine" (equivalent to Change Add-To and Part-Part Whole) and subtraction as Change Take-Away, Compare, and "missing addend" (equivalent to Equalize).

Each of these five categories can be further broken up into three different problems each, depending on what is unknown (e.g., a Change Take-Away problem with an unknown change would be "Fred had 7 apples. He gave some to Sara. Now he has 3 apples. How many apples did he give to Sara.?"). You can view examples of the resultant 15 types in the first two tables below.

These tables are meant as a reference: it would be a waste of time to read them in their entirety. The important idea here is not to memorize these categories (and sub-categories), but to be aware that there are many conceptually (and physically) different ways in which the operations called "addition" and "subtraction" can be modeled. Note that in each category's row, both addition and subtraction are used. Also note that the role of the "equals" sign is subtly different across categories because of the different kinds of actions that are taken. For example, when one says "13-5=8", one could mean "13 take-away 5 becomes 8" (a physical action on one set requiring a result) or "comparing 13 and 5 is the same as 8" (a static action of comparison of two distinct sets).

CHANGE-ADD-TO, with:	...unknown outcome	... unknown change	... unknown start
	Alexi had 5 candies. Barb gave him 3 more. How many candies does he have altogether now?	Alexi had 5 candies. Barb gave him 3 more. Now he has 8 altogether. How many candies did Barb give him?	Alexi had some candies. Barb gave him 3 more. Now he has 8 altogether. How many candies did he start with?
CHANGE-TAKE-AWAY, with:	...unknown outcome	... unknown change	... unknown start
	Alexi had 8 candies. He gave 5 to Barb. How many candies does he have left?	Alexi had 8 candies. He gave some to Barb. Now he has 3 left. How many candies did he give to Barb?	Alexi had some candies. He gave 5 to Barb. Now he has 3 left. How many candies did he start with?
PART-PART-WHOLE, with:	...unknown outcome	... unknown change	... unknown start
	Alexi had 5 fireballs and 3 lollipops. How much candy did he have altogether?	Alexi had 5 fireballs and some lollipops. He had 8 candies altogether. How many were lollipops?	Alexi had some fireballs and 3 lollipops. He had 8 candies altogether. How many were fireballs?

EQUALIZE, with:	...unknown difference	... unknown second part	... unknown first part
	Alexi had 8 candies. Barb had 5. How many more does Barb have to buy to have as many as Alexi?	Alexi had 8 candies. Barb had to get 3 more to have the same number as Alexi. How many candies did Barb start with?	Alexi had some candies. Barb, who had 5 candies, had to get 3 more to have the same number as Alexi. How many candies did Alexi have?
COMPARE, with:	...unknown difference	... unknown second part	... unknown first part
	Alexi had 8 candies. Barb had 5. How many more candies did Alexi have than Barb?	Alexi had 8 candies. He had 3 more than Barb. How many candies did Barb have?	Alexi had some candies. He had 3 more than Barb, who had 5. How many candies did Alexi have?

Excerpted from "Representation of addition and subtraction word problems", by T. Carpenter, J. Moser & H. Bebout (1980), appearing in *Journal for Research in Mathematics Education*, **19**, 345–357.

Many might say that all of this is an academic exercise because for each of these problems, one either "adds" or "subtracts" to get the answer. However, for children who are exposed to these problems (editorial comment: a rarity in most schools and texts), this is not the case at first exposure (even if they are "told" to either add or subtract). To the students, these are 15 different problems. To see this, let's take a step back to earlier in childhood.

According to current research, people develop knowledge through reflection on experiences they have or that have been provided for them. In particular, children develop notions of "number" through watching people count and their own attempts to do likewise. In virtually all cultures, children go through several stages in learning how to count. This sequence includes learning the verbal sequence, learning to point, learning to make exactly one point to each object, and learning that the last word in the sequence is the number of objects in the set (cardinality). Again, all of this is not taught to the child, but is learned upon the child reflecting on counting experiences over a number of years.

In the same manner, children initially deal with the 15 addition and subtraction situations as distinct from each other and model each situation (with objects) exactly as it is written. It is over a long period of experiencing (and reflecting on) these problems that children begin to make mental connections between them (e.g., combining and taking away are inverse actions). This period usually lasts through some time in third grade. One cannot tell the children "just add in this situation and subtract in that situation" (etc.) unless the goal is for the children to memorize the categories without meaning (editorial comment: something children are asked to do all too often with "word problems" from this point forward!!).

During this period of experiences, children also develop strategies for quickly obtaining an answer. Many of these are continuations of the counting strategies they developed earlier. For example, in the traditional "Combine" category, when putting a set of 4 objects with a set of 5 objects, a child initially puts the two sets together and counts all the objects: "1-2-3-4-5-6-7-8-9". Later, s/he begins to count on by beginning at 4 and saying: "5-6-7-8-9" (keeping track on his/her fingers if there are no objects). Eventually, the child sees that when 4 objects are placed with 5 more of the same object, the result is always 9 objects. Further, through experiencing and reflecting on the other categories, the student begins to recognize the triple (4-5-9) is interconnected by two operations, addition and subtraction, that model all the categories (Note: The development of knowledge in the Combine category has been researched more often than the others. Thus, we know more details about it.). Eventually, they find relationships between the triples and use those relationships to get others (e.g., 7+6 must be 13 because 6+6 is 12).

Also, children begin to see more particular features such as the commutative property of addition. Children notice fairly quickly that "combining one" is the same as counting. Thus, "six combined with one" must result in seven. However, "one combined with six" is different and more difficult because one needs to count up by six from one. It does not occur to the child (even if told) that the "one" object could be switched over to the other side for quite awhile. The actual idea of commutativity (without the ugly terminology!) in general takes time and reflection beyond the "plus one" situation.

Thus, children do not approach these arithmetic problems by asking (as adults do), "Do I add or subtract here?". Instead, if given the opportunity, they develop within themselves the concepts and interrelationships of addition and subtraction.

Multiplication

As with addition, many people consider only one kind of situation when they think of multiplication: repeated addition (i.e., the combining of like-sized sets). However, multiplication models other situations as well that are not presented as often in the traditional elementary school curriculum. Less is known about children's development of these multiplicative concepts when compared to that of additive concepts. However, one can probably expect that, because there are different models of multiplication problems, children should be provided with plenty of experiences with these models in order to develop a full grasp of the distinctions and relationships between them. The models are:

1. Repeated Addition or Equivalent Sets (The usual "three kids have five cookies each")
2. Comparison ("Julie has three times the cookies Jim has. If Jim has 5 cookies, how many cookies does Julie have?")
3. Cartesian Product ("Jim has 3 different pairs of pants and 5 different shirts. How many combinations of pants and shirts are possible for him to wear?")
4. Area (or array if whole numbers): ("How much space is in a rectangle that is 3 inches by 5 inches?" or "How many cells are in an array that has 3 rows and 5 columns?")

As with the relationship between addition and subtraction, by changing the role of the unknown quantity in each of these problems, both multiplication and division are represented here. For simplicity's sake, I will not deal with division here. However, you may see a detailed table of multiplication and division models in the next table below (note that the four categories are further split up by the author's choice of applications).

One important difference between these four categories that was not present between the additive categories (with the exception of Part-Part-Whole) is the different units attached to the quantities involved:

1. Repeated Addition: 3 *kids* times 5 *cookies per kid* is 15 *cookies.*
2. Comparison: 3 (no unit) times 5 *cookies* is 15 *cookies.*
3. Cartesian Product: 3 *pants* times 5 *shirts* is 15 *pants-shirts* (or combinations).
4. Area: 3 *inches* by 5 *inches* is 15 *square inches* (or $inches^2$).

Often, children have difficulty with units in a multiplication problem because (a) It usually was not an issue in addition (3 oranges plus 5 oranges is 8 oranges) and/or (b) the school's curriculum (or textbook) does not address the issue. Of particular confusion is the use of the "per quantity" or "rate" used in the Repeated Addition model. This concept that combines two units to make a new and somewhat artificial one is difficult because one is asked to now pay attention to two things at once. This difficulty is exacerbated later when addressing the concept of "slope".

		PARTITIVE	QUOTITIVE
Class	Multiplication problem	Division (by multiplier)	Division (by multiplicand)
Equal groups	3 children each have 4 oranges. How many oranges do they have altogether?	12 oranges are shared equally among 3 children. How many does each get?	If you have 12 oranges, how many children can you give 4 oranges?
Equal measures	3 children each have 4.2 liters of orange juice. How much orange juice do they have together?	12.6 liters of orange jice is shared equally among 3 children. How much does each get?	If you have 12.6 liters of oranged juice, to how many children can you give 4.2 liters?
Rate	A boat moves at a steady speed of 4.3 meters per second. How far does it move in 3.3 seconds?	A boat moves 13.9 meters in 3.3 seconds. What is its average speed in meters per second?	How long does it take a boat to move 13.9 meters at a speed of 4.2 meters per second?
Measure conversion	An inch is about 2.54 centimeters. About how long in 3.1 inches in centimeters?	3.1 inches is about 7.81 centimeters. About how many centimeters are there in an inch?	An inch is about 2.54 centimeters. About how long in inches is 7.84 centimeters?
Multiplicative comparison	Iron is 0.88 times as heavy as copper. If a piece of copper weighs 4.2 kg, how much does a piece of iron of the same size weigh?	Iron is 0.88 times as heavy as copper. If a piece of iron weighs 3.7 kg, how much does a piece of copper of the same size weigh?	If equally sized pieces of iron and copper weigh 3.7 kg and 4.2 kg respectively, how heavy is iron relative to copper?
Part/whole	A college passed the top 3/5 of its students in an exam. If 80 students took the exam, how many passed?	A college passed the top 3/5 of its students in an exam. If 48 passed, how many students took the exam?	A college passed the top 48 out of 80 students who took an exam. What fraction passed?
Multiplicative change	A piece of elastic can be stretched to 3.3 times its original length. What is the length of a piece 4.2 meters long, when fully stretched?	A piece of elastic can be stretched to 3.3 times its original length. When fully stretched it is 13.9 meters long. What was its original length?	A piece of elastic 4.2 meters long can be stretched to a maximum of 13.9 meters. By what factor is it lengthened?
Cartesian product	If there are 3 routes from A to B, and 4 routes from B to C, how many different ways are there of going from A to C via B?	If there are 12 different routes from A to C via B, and 3 routes from A to B, how many routes are there from B to C?	
Rectangular area	What is the area of a rectangle 3.3 meters long by 4.2 meters wide?	If the area of a rectangle is $13.9\,m^2$, and the length is 3.3 m., what is the width?	
Product of measures	If a heater uses 3.3 kilowatts of electricity for 4.2 hours, how many kilowatt-hours is that?	A heater uses energy at the rate of 3.3 kilowatts. For how long can it run before it uses 13.9 kilowatt-hours?	

Excerpted from "Multiplication and division as models of situations" by B. Greer, appearing in *Handbook of Research on Mathematics Teaching and Learning*, D. A. Grouws (Ed.), a publication of the National Council of Teachers of Mathematics: Reston, VA (1992).

Division

Division, like the other three operations, is treated in the traditional curriculum as modeling only one situation (sharing). However, like in the case of subtraction's connection to addition, it can model the "reverse" of all the multiplication models, depending on what number will play the role of unknown. And, because the two factors of a product often have different units, there are two different division problems corresponding to (almost) each multiplication model. These two models are (with examples reversing the repeated addition multiplication model):

1. Partitive (also called "sharing"): (the usual "Sally has 15 cookies. Each of her 3 friends share the cookies evenly. How many cookies does each friend get?")
2. Quotitive (also called "measurement"): ("Sally has 15 cookies. She gives 3 cookies to each of her friends. How many friends get 3 cookies each?").

Note: Examples of these two models featuring the other models of multiplication can be found on the table on the previous page. Notice the difference in the treatment of the units in each model from the repeated addition point of view.

1. Partitive: 15 cookies divided by 3 friends is 5 cookies per friend. ($3\times ? = 15$)
2. Quotitive: 15 cookies divided by 3 cookies per friend is 5 friends. ($? \times 3 = 15$)

The traditional curriculum often gets so caught up with the "long division" algorithm that one often loses the perspective of division as "repeated subtraction". Note that in each of the stories, if a child had never heard of division, he or she could still solve the problem by repeatedly passing out cookies until there is not enough to give to her friends in the manner prescribed. In the partitive model, a child will usually pass out one cookie to each friend at a time (a total of 3 cookies passed out), check to see if s/he has enough to do so again (at least 3 cookies), then repeat the process until there are less than 3 cookies left (then count how many each friend has). In the quotitive model, the child will pass out 3 cookies to a friend, check to see if there at least three cookies left, then pass out three more to another friend, and repeat the process until there are less than 3 cookies left (then count how many friends have cookies). In each case, 3 is being repeatedly subtracted from 15 until there is less than 3 left – similar to how 3's were repeatedly added to obtain 15 in multiplication.

Note that this idea of "repeated subtraction" brings the possibility of a "remainder"- the number of cookies left over that could not be passed out. For example, if we started with 17 cookies instead of 15, there would come a point in the process when the child could not pass out a total of 3 cookies anymore. That is when each friend had 5 cookies (or 5 friends had cookies), if we put the 17 cookies back together again, we would see we had 3 sets of 5 (or 5 sets of 3) with 2 cookies left over (note the unit of the remainder is cookies in each model, while the quotient is never cookies). Thus, we have $17 = 3 \times 5 + 2$ (or $5 \times 3 + 2$). Thus, we essentially have a conjecture that for every pair of whole numbers a and b (b not zero), there is a unique whole number q (the quotient) and whole number r (the remainder, which must be less than b because it's when you can't pass out any more cookies!) such that $a = b \times q + r$.

An educational note: Often, the traditional curriculum emphasizes "quotient and remainder" so much that one only thinks of 17 divided by 3 to be "5 remainder 2". Not only does this give the impression that the 5 and 2 have the same units (they don't), but also that it is the only practical answer to the problem. Can you think of "real-life" situations where you might better consider 17 divided by 3 to be only 5? Only 6?

One final note: Division has the strange characteristic that "you can't divide by zero. It's undefined". However, in school, children are often seen memorizing that 15 divided by 0 is undefined while 0 divided by 15 is zero. In the process, because they are only memorizing without using meaning, the two often get mixed up. From an "inverse of multiplication" perspective, it is easy to see why the results exist: $15 \div 0 = ?$ is the same as saying $? \times 0 = 15$. Since any whole (or real) number times zero is zero (why?), no whole number can fill the question mark. For $0 \div 15 = ?$, it is the same as saying $? \times 15 = 0$. Zero easily fits the question mark here. Thus one is undefined and the other is zero.

However, the terms "undefined" and "zero" often become equivalent to a child when the meaning goes away. Here, the quotitive model of division can come to the rescue in giving a new perspective to the problem. If one puts $0 \div 15$ and $15 \div 0$ into quotitive style story problems, the meaning comes a little clearer. For $0 \div 15$, we can say we have 0 cookie and want to pass out 15 cookies to each friend. How many friends get 15 cookies each? Here, because you don't have any cookies, you can't even get started passing out any. Thus 0 friends (a small number) get 15 cookies each. As for $15 \div 0$, if we start with 15 cookies and pass out 0 cookies to each friend, how many friends get 0 cookies each? Here, you can pass out 0 cookies to as many friends that you want (an infinite number!) and you will never run out of cookies. Thus, $15 \div 0$ can be considered infinite (a large "number"!). Thus an infinite number of friends get zero cookies each. (Note: Don't actually try this, because you might wind up with zero friends!!). Thus $0 \div 15$ is as small as you can get and $15 \div 0$ is a large as you can get.

Thought question: What about $0 \div 0$? Hint: Try the “inverse of multiplication” idea.

5.6 Factors and prime numbers

1. You may have heard the terms *prime* and *composite* before. These words talk about how numbers can be broken down in terms of *factors*. Factors are numbers (usually whole numbers) which divide evenly into a given number. For example, the factors of 6 are 1, 2, 3 and 6, because we can write $1 \times 6 = 6$ and $2 \times 3 = 6$. The factors of 49 are 1, 7 and 49 (we don't list 7 twice). A number is *prime* if its only factors are 1 and itself. The first prime number is 2 (we don't count 1). A number is *composite* if it is not prime (again, we usually do not classify 1 in either category).

The notation $3|6$ is sometimes used to say that 3 divides evenly into 6. For some numbers, it is pretty easy to check whether they divide evenly into another number.

1. List as many of these divisibility tests as you know.
2. What is the first composite number?
3. Are there any other even primes besides 2?
4. List the first ten prime numbers.
5. If p is a prime number, what are the factors of p^4?

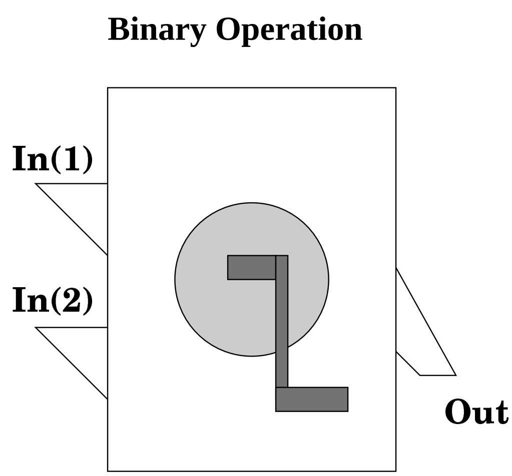
2. The *greatest common divisor*, or GCD, of two numbers, is the largest factor they share in common. GCD(12,18)=6. The *least common multiple*, or LCM, of two numbers is the smallest number which is a multiple of both numbers. LCM(12,18)=36. Finally, two numbers are said to be *relatively prime* if their GCD is 1 – in other words, they have no common factors (except 1), so relative to each other, they appear to be prime. The numbers 12 and 18 are clearly not relatively prime, but 10 and 21 are relatively prime even though neither number individually is prime.

- Can you find two numbers whose GCD is 10? Whose LCM is 10?
- Can you find a relationship between two numbers a and b and $\text{GCD}(a, b)$ and $\text{LCM}(a, b)$?
- Can you find two numbers, each larger than 100, which are relatively prime?

◦ These definitions can be extended to larger sets of numbers. What would GCD(4,8,10) be? GCD(4,8,11)? What about LCM(4,8,10)? LCM(4,8,11)? Is either of these sets of numbers relatively prime?

5.7 Introduction to operations

An *operation* is a way to combine some elements of a set to get another element of the set. You are already familiar with many operations: addition, subtraction, multiplication and division are all operations. They are called *binary operations* because they operate on two elements of a set. Taking a square root is called a *unary* operation because you only operate on one number to get the square root. You can think of an operation like a machine, with a certain number of slots for inputs and a slot for the output. You put the inputs into the appropriate slots, turn a crank, and out comes the output.



Operations can be defined on any kind of set, not just numbers. Let's consider the following set S: $\{\square, \bigcirc, \triangle, \star, \heartsuit\}$. Now let's define an operation $\bowtie$ on S by making the output of $a \bowtie b$ equal to whichever of the symbols a and b in S has more (sharp) corners. Thus $\square \bowtie \bigcirc = \square$ since $\square$ has four sharp corners while $\bigcirc$ has none.

Another way to represent a binary operation on a set of finite size is to make a table showing all the possible outcomes. The elements of S run down the left side and across the top. The fact that $\Box \bowtie \bigcirc = \Box$ has been filled in on the table. Complete the table by filling in the rest of the spaces.

$\bowtie$	$\bigcirc$	$\heartsuit$	$\triangle$	$\Box$	$\star$
$\bigcirc$					
$\heartsuit$					
$\triangle$					
$\Box$	$\Box$				
$\star$					

Now see if you can work backward to figure out the algebraic rule for the operation $\odot$ defined on the set $\{0, 1, 2, 3, 4\}$ by the table at right.

$\odot$	0	1	2	3	4
0	0	-2	-4	-6	-8
1	3	1	-1	-3	-5
2	6	4	2	0	-2
3	9	7	5	3	1
4	12	10	8	6	4

5.8 Properties of operations

- CLOSURE: A set S is *closed under* an operation iff (if and only if) all the operation's outputs are elements of S. The set S on the last page is closed under $\bowtie$ because $\bowtie$ing two elements of S always produces an element of S. If, however, we define the operation $\uplus$ on the digits 0 through 9 by $a \uplus b = \max(a, b) + 1$, the set of digits 0 through 9 is not closed under $\uplus$ because $0 \uplus 9 = 9 + 1$ is not in that set.

  ∘ Is the set $\{0,\ 1,\ 2,\ 3,\ 4\}$ closed under the $\odot$ operation defined in the previous reading? If not, can you think of a set which *is* closed under $\odot$?

- COMMUTATIVE: An operation is *commutative* iff the order of the two things being operated on doesn't matter. For example, $\bowtie$ (from the previous reading) is commutative because $a \bowtie b = b \bowtie a$ no matter what a and b are.

  ∘ Is there a visual clue in the $\bowtie$ table that lets you know it's commutative?

  ∘ Is $\uplus$ as defined in the previous paragraph commutative?

  ∘ Can you think of an operation that is not commutative?

- ASSOCIATIVE: An operation is *associative* iff when you apply the operation repeatedly, it doesn't matter where you put the parentheses. For example, addition on the natural numbers is associative, which is illustrated by the fact that $1 + (9 + 7) = (1 + 9) + 7$.

  ∘ Is the $\bowtie$ operation associative?

  ∘ Is associativity something that you can see from looking at a table?

  ∘ Can you make up an operation which is associative but *not* commutative? (It doesn't have to involve numbers.)

- IDENTITY: An operation has an *identity* element if one of the elements in the set makes the operation leave all other elements alone. This must be true regardless of whether the identity is the first or second of the numbers being operated on. Addition on the natural numbers has an identity element, namely 0, because $0 + a = a$ and $a+0 = a$, no matter what a is. However, the $\odot$ operation defined on the previous page has no identity element, no single element which makes $\odot$ leave the other operand (thing being operated on) alone, no matter what it is.

  ◦ Does the $\bowtie$ operation defined previously have an identity element? If so, what is it?

  ◦ Does the $\uplus$ operation defined above have an identity element? If so, what is it?

  ◦ Can you tell from looking at a table whether an operation has an identity element?

  ◦ Is it possible for an operation to have *two* (or more) identity elements?

- INVERSES: If an operation has an identity, then it may also have *inverses*. The inverse of an element a is something which gets operated on along with a in order to get the identity element. Addition on the integers has inverses, e.g., the inverse of 42 is -42, because adding 42 and -42 gets you 0, the additive identity. Multiplication on the natural numbers does not have inverses, because the multiplicative inverse of 2 is $\frac{1}{2}$, which is not a natural number. Of course, if an operation does not have an identity element, it cannot have inverses.

  ◦ Does the $\bowtie$ operation have inverses?

  ◦ Can you think of a set on which the $\odot$ operation has not only an identity element but inverses?

5.9 Introduction to bases

As you may know, our normal way of representing numbers is called "base ten", because it uses ten digits. We count from 0 to 9, and then we write 10, representing 1 group of ten and 0 leftover units. The next number, 11, represents 1 group of ten and 1 unit. This can continue indefinitely; the number 43,507 represents 4 groups of ten thousand, 3 groups of one thousand, 5 groups of one hundred, 0 groups of ten, and 7 units.

1. Perhaps the best way to get a feel for other bases is to practice grouping. Take fifty three chips. Those can be grouped into 5 groups of ten and 3 units. That's why this number is written 53 (base ten). But they could also be grouped into 1 group of thirty six (which is six times six), 2 groups of six, and 5 units. So we could write this number as 125 (base six), or 125_{six}. By grouping your counters, write this number in the bases two through twelve:

Base two:	Base six: 125	Base ten: 53
Base three:	Base seven:	Base eleven:
Base four:	Base eight:	Base twelve:
Base five:	Base nine:	

2. Now that you are good at grouping, it's fairly easy to add numbers. Take thirty-seven and fifteen, and put both in base ten groupings (we'll start by adding in base ten). Now, start by adding the units. There are 7 units and 5 units, for a total of twelve. This is too many for our base, so we take a group of ten and move them over. Now we add the tens. There are $3 + 1 + 1$ groups of tens, for a total of 5. This number is less than our base, so we leave it. We now have 5 groups of ten and 2 units, so $37 + 15 = 52$ in base ten.

Once you understand this idea, do the same (adding thirty-seven and fifteen) in the following bases:

(a) base eight (b) base four (c) base two

5.10 Exponentiation and scientific notation

Just how important is this topic for PK–4 and Grade 4–8 certification? The standards (for PK–2) say that **"children must use a variety of methods and tools to compute, including ... mental computation, estimation, ...** . For grades 3–5, it is also expected that teachers be proficient in exponentiation (see the MAA recommendations). More important than either of these, however, are the notions of a logarithmic scale, and of order of magnitude. These are very much in the province of a young child. No formal computations of exponentials need arise in the discussion of these concepts, yet a teacher who does not understand exponentiation will be ill-prepared to handle these discussions adequately.

1. Exponents. Exponents stand for repeated multiplication. For example, 2^3 stands for $2 \times 2 \times 2$ and is read "2 to the power (of) 3"; likewise, 3^9 is read "3 to the power 9" and stands for

$$3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3\ ,.$$

This doesn't tell you how to compute powers that are negative numbers or fractions; as was the case with multiplication, we will have to see what makes logical sense there.

Exponents may be used to count things. Remember how many subsets there are of a set with n elements (the "Comparing without counting" worksheet)? How many license plates are possible in a state where each plate is six letters (no numbers)?

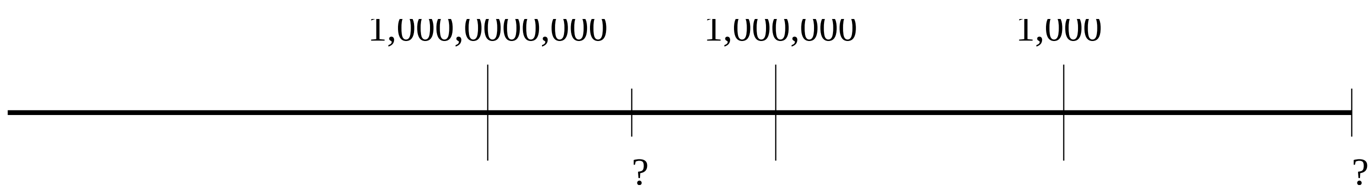
Exponents obey certain laws. You might have learned these, but if not, or if you're a little rusty, use trial and error on these to see which are valid laws and which aren't.

$$\begin{aligned}
y^a \times y^b &= y^{a \cdot b} \\
y^a \times y^b &= y^{a+b} \\
(x+y)^2 &= x^2 + y^2 \\
(x^y)^a &= x^{(y^a)} \\
x^a \cdot y^a &= (x \cdot y)^a \\
x^y &= y^x
\end{aligned}$$

Exercise: pick one of the above laws that is true and justify it, using laws such as the commutative and associative laws (but no circular reasoning!).

2. Logarithmic scales. Some physical measurements tend to range from very small to very large. A regular number line isn't very good for depicting these. If you try to put intensities of earthquakes, for example, on a regular number line, then any chart with enough room to show the San Francisco earthquake of the year 1906 will have almost all other recorded earthquakes scrunched into the leftmost millimeter of the paper. There are many examples of this: geological age charts, deciBel measures of sound intensities, and so forth.

The logarithmic scale was invented to depict these in a rational and standardized way. On a logarithmic time scale, for instance, the marks for a thousand, a million, and a billion years ago are evenly spaced.



The guiding principle for these number lines is multiplication instead of addition. Each equal spacing represents multiplication by an equal factor. The spaces between big marks in the picture are each a factor of $1,000$. What if we wanted little marks representing a factor of 10? How many little marks should there be between each big mark?

What number shall we write in halfway between a million and a billion years ago? It can't be the additive halfway point, namely $500,500,000$ years ago, since if you answered the last question, you saw that this number should appear somewhere in the leftmost third of the interval between a billion and a million years ago. What value would you put there?

What about at the end of the number line? The distance from $1,000$ to the end is supposed to be the same as the distance from $1,000,000$ to $1,000$. What time goes there? Not the present! (Where does the present go?)

See if you can relate the logarithmic number line to the mathematical operation of exponentiation. In the above diagram, each half inch to the left is a factor of 10. Thus x half inches to the left represents 10^x years ago. What about x half inches to the right? What about if x is not a whole number, say $x = 1/2$ (one quarter-inch to the left): how many years ago is that, and what does that tell you about fractional exponents.

3. Scientific notation. If you're working with very large numbers, it is a pain to write down all 34 digits, and what's more, it's hard to see how many digits and thereby know how large the number is. The same goes for a very small number with a lot of zeros between the decimal point and the first nonzero digit. Instead, people (not just scientists) use scientific notation. This mean writing a number as a power of 10 times something (usually) between 1 and 10. For example, $123,987,282,343$ becomes 1.24×10^{11}. [Do you see where the 11 comes from?] Why did we round off there? If we didn't round off at all, we wouldn't have saved any space (though it still would be clearer roughly how big the number was). But the difference between 1.23×10^{11} and 1.24×10^{11} is probably too small for us to care about (even though it is actually a billion). For instance, no one would notice if the population of the US increased by $100,000$ (which it does every day) even though $100,000$ is a large crowd in any one place. We say that 1.24×10^{11} has "three digits of precision", though this is not the most accurate measure of precision, since a number between 9.97×10^{11} and 9.98×10^{11} has a much smaller relative uncertainty than a number between 1.11×10^{11} and 1.12×10^{11}. [How much smaller?]

Try converting the following numbers into or out of scientific notation.

- 314159265
- 6.023×10^{23}
- .0000000000147

Can you come up with a law for multiplying numbers in scientific notation?

5.11 Models of fractions

The number system of fractions developed from the need to represent an amount between two whole numbers. Many societies rejected this idea, particularly the Greeks (who would write the ratio "2 girls for every 3 boys" rather than say "2/5 of the class are girls") and the Romans (who just divided things into smaller units so they could still discuss whole number quantities). The problem these societies had stemmed from the notion of expressing one quantity using two numbers (i.e., 3/4 is one number, not two!). The notation we use today was developed by the Hindus.

There are four mathematical roles that a fraction (say 3/4) can take. The first two are how many people traditionally consider fractions, while the last two are fairly well known but often misinterpreted and misused by the general public of the United States

Here are the four models:

1. **Measure.** "Joey grew 3/4 of an inch last month."
2. **Quotient.** "Four children want to share 3 cakes equally. How much pie does each person get?" This model points out that fractions are the smallest extension of the whole numbers closed under division (except by zero).
3. **Operator.** "At a recent meeting, 3/4 of the participants were women; if there were 12 people at the meeting, how many were women?" This extends the notion of whole number multiplication (e.g., whereas before we talked about 5 sets of 12, we can talk about 3/4 of a set of 12 or even 93/17 of a set of 12.
4. **Ratio.** "At a recent meeting, 3/4 of the participants are women." Ratio is the same the Operator interpretation, but we don't know (or don't need to know) the actual amount of the total. We only care about what portion of the total. Other notations used for this model are "ratio notation" (the ratio of women to men is 3 : 1) and percentage (where the denominator is 100): 75/100 or 75% of the participants are women).

Note that there are commonalities between the models:

1. Each case can be interpreted in part-whole relationships.

2. Something can be partitioned into parts of equal size. The denominator tells you the number of parts the "something" is divided into and the numerator tells you how many of those parts to consider.

3. The numerator and denominator tell you about the relative sizes of the parts with respect to the unit in a multiplicative way. For example, the 1 and 2 in "1/2" indicate that there are *twice as many* total parts the unit has been split up into than the number of parts we are considering (rather than *1 more* part than the number of parts we are considering). Therefore, 1/2 is equivalent to 4/8 rather than 7/8.

6 Supplement

- Abstraction paradigm
- Problem-solving help
- Guidelines for Math 105 write-ups
- General guidelines for writing math
- NCTM Standards
- Textbook excerpts

6.1 Paradigm for abstraction and generalization

The teacher is responsible not only for transmitting basic computational skills, but also for giving the students a solid foothold in abstract thinking and generalization. Everyone known that mathematics is the foundation for science and engineering, but many have noted that the ability for abstraction also provides the foundation for the study of law, economics, computer-related disciplines, and to some extent, music and the graphic arts.

In Math 105, just as in elementary school classrooms, learning will take place on several levels. Usually the motion is from the specific to the general, as indicated by the following sequence:

1. Solve the specific problem
2. Observe a pattern
3. State the observation in English
4. State the observation in mathematical language
5. Justify the observation: prove that the pattern continues
6. Generalize if possible; determine the scope of validity

At first, I will take you explicitly through these steps. As the quarter progresses and it gets tiresome to address these steps explicitly in every situation, I will leave that up to you, with the expectation that a complete solution to a problem ought to contain all these steps.

In the elementary school classroom, unlike in Math 105, it is not usually appropriate or necessary to discuss these steps. Instead, we hope that by learning to think along these lines until it is second nature, you will more easily and automatically convey to your own students the sense that all the steps, not just the first one, are important, and that even as young children, they have the power to generalize and thereby to discover their own new mathematics.

6.2 A Short Problem Solving Self-Help Checklist

1. **Understand the problem**. It is foolish to answer a question you do not understand. It goes without saying that you need to understand all the terminology. In addition, you need to know what is given. Make a sketch, if appropriate, to see that all the given data makes sense. Also, be sure you understand what you are supposed to find or determine. A good way to check this is to ask yourself whether you could verify the answer if someone gave it to you. For a true/false or a prove/disprove question, ask what would constitute a counterexample. You should be especially aware of whether you are trying to show that something always holds, or whether you are you are being asked to find a specific case where something holds. Another question to ask yourself in this phase is whether the data really determine the unknown. Is there really a solution? More than one solution?

2. **Devising a plan**. If a road to the solution occurs to you naturally, you don't need this checklist. Assuming you find yourself somewhat stuck, here are a few general procedures you can follow.

   (a) Try a few examples. Trial and error is the single best problem-solving method. If you are supposed to find the relation between price and profit in a business application, try listing a few pairs of values and looking for a pattern. If you are supposed to find all triples of whole numbers summing to 25, start listing them. If there are variables in the problem, replace them with numbers and see how the problem goes then.

   (b) Try the problem with a smaller number. If this is easier, it might give you insight into the original problem or, if you try several smaller numbers, might give you an idea of a pattern that exists.

   (c) Work from both ends. When you cannot deduce any more from what is given, work backwards from the goal, for example: if you are supposed to find the area of something, perhaps you see that part of it is a square with known side, so you just need the area of the rest; perhaps after you do this a few times, you will see that what remains is a familiar shape which you had not recognized was embedded in the problem before.

   (d) Can you solve a special case? Perhaps if you assume that one of the three runners is moving at speed zero (staying still), you can solve it. Perhaps if you solve an analogous problem with two runners instead of three, you will gain insight into the three-runner problem you are actually trying to solve.

(e) Use physical intuition. The ancients always described mathematical quantities in geometric terms, and still today this is where much of our intuition and ability to understand abstract mathematical concepts comes from. See if you can imagine people lining up to shake each other's hand, or a cashier making change, or how big five thirteenths of a round cake would look.

3. **Carrying out the Plan**. If you are using variables, make sure you understand what they mean. If you get stuck, ask yourself whether you need any more variables. Once you have an idea you think is right, and are trying to prove it, examine why you believe it. Probe at it: why can't you find a number divisible by 12 but not by 4 or 6? How would that contradict what you know about divisibility? Try to recall facts you already know, so you don't have to solve every problem from scratch. Sometimes the contrapositive of an assertion is easier to think about (you'll learn about this ink the thirds week). In order to have a better sense of what assertions require proof, try to think of situations where a similar-sounding assertion might be wrong.

4. **Checking It Over**. You should always ask yourself whether the answer is plausible. In addition, you should ask whether you used all the data. If you did not, perhaps you made a mistake, or perhaps the problem did not require all the data; in this case, you should try to understand why some of the data was irrelevant, and mention this in your report. If you used some algebra you are not sure of, test it out on a calculator with some numerical examples. If you used variables, make sure you believe any equations you wrote relating them. If you are able to solve a problem more general than the one stated, please by all means include this in your writeup.

6.3 Problem Report Tips

- Speak out during large group discussions to get other groups' ideas. Make them explain their ideas to you, so you can explain them clearly in your report. It is assumed you already do this in your own small group!
- Address every question: even if you do not know the answer, at least acknowledge that and perhaps guess at it or suggest a strategy for finding it.
- Problem reports will ideally have one or two key sentences which hit upon exactly why the answer you provide is true. These do not replace a complete argument, but it aids the reader enormously, so you are less likely to lose points for clarity or coherence when you include this kind of guiding sentence. For example, if your assignment is to analyze the game of Tic-Tac-Toe, you might say:

  > Go first and choose the center. On your next two turns, choose two adjacent squares, so that you have two ways to win and your opponent can do no better than to tie.

- If you cannot explain a complete solution, then (1) acknowledge this, (2) pinpoint the difficulties, and (3) solve a simpler version if you can, or a special case.
- Ideas that you or your group had that did not lead to the solution may be worth reporting. I don't want a complete list of your thought processes, but a dead-end that leads to new understanding is worth pointing out.
- Write a conclusion that actually draw a conclusion, rather than simply rewriting your introduction. The overall flow should be:
  - Here's what I'm going to tell you about
  - Content
  - This is what we can come away with

  and not

  - Here's what I'm going to tell you about
  - Content
  - Here's what I just told you

- Use paragraphs, one for each idea. It's not a rule, but usually you should have more than one paragraph per page.

- A good report is exhaustive but not verbose, and could be understood by a friend who is not in this class.

6.4 Guide to writing math

A Guide to Writing in Mathematics Classes

Dr. Annalisa Crannell

Franklin & Marshall College

altered in places by R.P. for use in Math 105

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1. Why Should You Have To Write Papers In A Math Class?

For students in Math 105-106, we hope the answer to this question is obvious! But here is what Dr. Annalisa Crannell has to say for students in any college level math course.

For most of your life so far, the only kind of writing you've done in math classes has been on homeworks and tests, and for most of your life you've explained your work to people that know more mathematics than you do (that is, to your teachers). But soon, this will change.

With each additional mathematics course you take, you further distance yourself from the average person on the street. You may feel like the mathematics you can do is simple and obvious (doesn't everybody know what a function is?), but you can be sure that other people find it bewilderingly complex. It becomes increasingly important, therefore, that you can explain what you're doing to others that might be interested: your parents, your boss, the media.

Nor are mathematics and writing far-removed from one another. Professional mathematicians spend most of their time writing: communicating with colleagues, applying for grants, publishing papers, writing memos and syllabi. Writing well is extremely important to mathematicians, since poor writers have a hard time getting published, getting attention from the Deans, and obtaining funding. It is ironic but true that most mathematicians spend more time writing than they spend doing math.

But most of all, one of the simplest reasons for writing in a math class is that writing helps you to learn mathematics better. By explaining a difficult concept to other people, you end up explaining it to yourself.

2. How is Mathematical Writing Different from What You've Done So Far?

A good mathematical essay has a fairly standard format. We tend to start solving a problem by first explaining what the problem is, often trying to convince others that it's an interesting or worthwhile problem to solve. On your homeworks, you've usually just said, "9(a)" and then plunged ahead; but in your formal writing, you'll have to take much greater pains.

After stating what the problem is, we usually then state the answer, even before we show how we got it. Sometimes we even state the answer right along with the problem. It's uncommon, although not so uncommon as to be exceptional, to read a math paper in which the answer is left for the very end. Explaining the solution and then the answer is usually reserved for cases where the solution technique is even more interesting than the answer, or when the writers want to leave the readers in suspense. But if the solution is messy or boring, then it's typically best to hook the readers with the answer before they get bogged down in details.

Another difference is that when you do your homework, it is important to show exactly how you got your answer. However, when you write to a non-mathematician, sometimes it's better to show why your answer works, with just a brief explanation as to how you got it. For example, compare:

Homework mathematics: To solve for x when $3x^2 - 21x + 30 = 0$, we use the quadratic formula:

$$\begin{aligned} x &= \frac{21 \pm \sqrt{21^2 - 4 \times 3 \times 30}}{2 \times 3} \\ &= \frac{21 \pm \sqrt{9}}{6} \\ &= 5 \text{ or } 2 \end{aligned}$$

and so either $x = 5$ or $x = 2$.

More formal mathematics: To solve for x when $3x^2 - 21x + 30 = 0$, we used the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where a, b and c are the coefficients, in this case, 3, −21 and 30. We found that either $x = 5$ or $x = 2$. It's easy to see that these are the right answers, because

$$3 \times 5^2 - 21 \times 5 + 30 = 75 - 105 + 30 = 0$$

and also

$$3 \times 2^2 - 21 \times 2 + 30 = 12 - 42 + 30 = 0\,.$$

The difference is that, in the first example, you're trying to convince someone who knows a lot of math that you, too, know what you're doing (and if you don't, to get partial credit). In the second example, you're trying to show someone who may or may not be good at math that you got the right answer.

Math is difficult enough that the writing around it should be simple. "Beautiful" math papers are the ones that are the easiest to read: clear explanations, uncluttered expositions on the page, well-organized presentation. For that reason, mathematical writing is not a creative endeavor the same way that, say, poetry is: you shouldn't be spending a lot of time looking for the perfect word, but rather should be developing the most clear exposition. Unlike humanities students, mathematicians don't have to worry about over-using "trite" phrases in mathematics. In fact, at the end of this booklet are a list of trite but useful phrases that you may want to use in your papers, either in this class or in the future.

This guide, together with the checklist below, should serve as a reference while you write. If you can master these basic areas, your writing may not be spectacular, but it should be clear and easy to readwhich is the goal of mathematical writing, after all.

Checklist:

Does this paper ...
1. Clearly (re)state the problem to be solved?
2. State the answer in a complete sentence which stands on its own?
3. Clearly state the assumptions which underlie the formulas?
4. Provide a paragraph which explains how the problem will be approached?
5. Clearly label diagrams, tables, graphs, or other other visual representations of the math (if these are indeed used)?
6. Define all variables used?
7. Explain how each formula is derived, or where it can be found?
8. Give acknowledgement where it is due?
In this paper ...
9. Are the spelling, grammar, and punctuation correct?
10. is the mathematics correct?
11. Did the writer solve the question that was originally asked?

3. Following the Checklist

1. Restating

Do not assume that the reader knows what you're talking about. (The person you're writing to might be out on vacation, for example, or have a weak memory). You don't have to restate every detail, but you should explain enough so that someone who's never seen the assignment can read your paper and understand what's going on, without any further explanation from you. Outline the problem carefully.

2. Statement of answer

If you can avoid variables in your answer, do so; otherwise, remind the reader what they stand for. If your answer is at the end of the paper and you've made any significant assumptions, restate them, too. Do not assume that the reader has actually read every word and remembers it all (do you?).

3. Assumptions

For example, what physical assumptions do you have to make? (No friction, no air resistance? That something is lying on its side, or far away from everything else?) Do you assume that any values are whole numers, or positive numbers, or that one quantity is greater than another? Sometimes things are so straightforward that there are no assumptions, but not often.

4. Approach

It's not polite to plunge into mathematics without first warning your reader. Carefully outline the steps you're going to take, giving some explanation of why you're taking that approach. It's nice to refer back to this paragraph once you're deep in the thick of your calculations.

5. Label graphics

In math, even more than in literature, a picture is worth a thousand words, especially if it's well labeled.

Label all axes, with words, if you use a graph. Give diagrams a title describing what they represent. It should be clear from the picture what any variables in the diagram should represent. The whole idea is to make everything as clear and self-explanatory as possible.

6. Define variables

(a) Even if you label your diagram (and you should), you should still explain in words what your variables are.

(b) If there's a quantity you use only a few times, see if you can get away with not assigning it a variable.

Some examples:

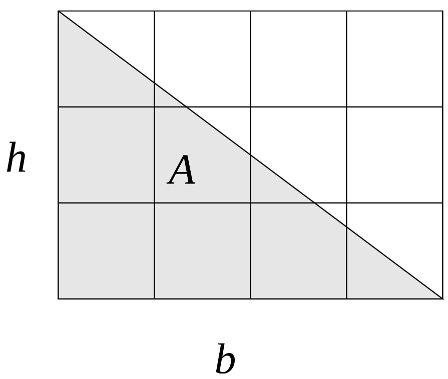

Figure 1: diagram of the triangle

(each square is 1" by 1")

easy to read: We see that the area of the triangle will be one half of the product of its height and base, that is, the area of the triangle is $(1/2) \times 3 \times 4 = 6$ square inches.

hard to read: We see that $A = (1/2)hb$, where A stands for the area of the triangle, b stands for the base of the triangle, and h stands for the height of the triangle, and so $A = (1/2) \times 3 \times 4 = 6$ square inches.

easy to read: Elementary physics tells us that the velocity of a falling body is proportional to the amount of time it has already spent falling. Therefore, the longer it falls, the faster it goes.

hard to read: Elementary physics tells us that $vt = g(t - t_0)$, where vt is the velocity of the falling object at time t, g is gravity, and t is the time at which the object is released. Therefore as t increases, so does vt, i.e., as time increases, so does velocity.

I hope that you'll agree that the first example of each pair is much easier to read.

(c)The more specific you are, the better. State the units of measurement. When you can use words like "of", "from", "above", etc., do so. For example:

complete: We get the equation $d = rt$, where d is the distance from Sam's car to her home (in miles), r is the speed at which she's traveling (measured in miles per hour), and t is the number of hours she's been on the road.

imcomplete: We get the equation $d = rt$, where d is the distance, r is the rate, and t is the time.

7. Explaining and deriving

Don't pull formulas out of a hat, and don't use variables which you don't define. Either derive the formula yourself in the paper, or explain exactly where you found it, so other people can find it, too.

Put important or long formulas on a line of their own, and then center them; it makes them much easier to read. Compare these two versions:

The total number of infected cells in a honeycomb with n layers is

$$1 + 2 + 3 + \cdots n = \frac{n(n+1)}{2},$$

therefore, there are $100(101)/2 = 5050$ infected cells in a honeycomb with 100 layers.

The total number of infected cells in a honeycomb with n layers is $1 + 2 + 3 + \cdots n = n(n+1)/2$, therefore, there are $100(101)/2 = 5050$ infected cells in a honeycomb with 100 layers.

8. Acknowledging sources

For example, I think Plagiarism is almost certainly the greatest sin in academiasome fiction writers make plagiarism a motive for murder. It's extremely important to acknowledge where your inspiration, your proofreading, and your support came from. (For example, I thank Mark Stanton, a high school mathematics teacher in New York City, for catching a spelling mistake in the previous sentence.) In particular, you should cite: any book you look at, any compuational or graphical software which helped you understand or solve the problem, any student you talk to (whether in this class or not), any professor you talk to (including and especially me, because I'll catch you if you leave me out). The more specific you are, the better.

9. Grammar

(a) It may surprise you that it is on spelling and grammar that people tend to lose most of their points on their mathematics papers. Please spell-check and proofread your work for grammar mistakes. Better yet, ask a friend to read your paper. Mathematicians are generally not petty, but neither are we amused by sloppy or careless writing.

(b) Mathematical formulas are like clauses or sentences: they need proper punctuation, too. Put periods at the end of a computation if the computation ends the sentence; use commas if it doesn't. An example follows.

> If Dr. Crannell's caffeine level varies proportionally with time, we see that
>
> $$C_t = kt\,,$$
>
> where C_t is her caffeine level t minutes after 7:35 AM and k is a constant of proportionality. We can solve to show that $k = 202$, and therefore her caffeine level by 11:02 AM ($t = 207$) is
>
> $$\begin{aligned} C_{207} &= (202)(207) \\ &= 41,814\,. \end{aligned}$$
>
> In other words, she's mightily buzzed.

(c) Do not confuse mathematical symbols for English words (= and # are especially common examples of this). The symbol "=" is used only in mathematical formulas, not in sentences:

> **correct:** We let V stand for the volume of a single mug and n represent the number of mugs. Then the formula for the total amount of root beer we can pour, R, is $R = nV$.
>
> **incorrect:** We let V = volume of a single mug and n = the # of mugs. Then the formula for the total amount of root beer $R = nV$
>
> **incorrect:** We let V stand for the volume of the mug and n represent the number of mugs. Then the formula for the total amount of root beer we can pour, R, is R is nV.

(d) Do, however, use equal signs when you state formulas or equations, because mathematical sentences need subjects and verbs, too.

correct: Then the formula for the total amount of root beer we can pour is $R = nV$.

incorrect: Then the formula for the total amount of root beer we can pour is nV.

4. Good Phrases to Use in Math Papers

- Therefore (also: so, hence, accordingly, thus, it follows that, we see that, from this we get, then)
- I am assuming that (also: assuming, where, M stands for; in more formal mathematics: let, given, M represents)
- show (also: demonstrate, prove, explain why, find)
- This formula can be found on page 9-743 of Discovering Calculus by Levine and Rosenstein.
- (see the formula above). (also: (see *), this tells us that . . .)
- if (also: whenever, provided that, when)
- notice that (also: note that, notice, recall)
- since (also: because)

5. Helpful Hints for the Computer

1. Under the Tools menu, pick Preferences and then check the box that says "smart quotes" (as opposed to "dumb" quotes). It'll also give you nice apostrophes: it won't be long 'til you need those.
2. Underlining is the poor writer's italics. If you're on the computer, don't underline without permission from the dean.
3. Instead of merely typing the minus sign, which is really a hyphen, type "option-minus"; also, add spaces: Compare "5-3=2" with "5 - 3 = 2". If you want an even longer dash, type "option-shift-minus" all at once.
4. Remember, you can always leave a blank space and put in symbols by hand!

6.5 Excerpts from the NCTM's "Principles and Standards for School Mathematics (2000)"

The National Council of Teachers of Mathematics published a set of Standards in 1989, revised in 2000. The standards describe in detail what mathematics school children should learn, and in what depth; auxiliary documents illustrate the Standards in great detail. The Standards are quite lengthy. What follows are excerpts that have to do with some of what you will see emphasized in Math 105. Note specifically the emphasis on reasoning and proofs starting at the very beginning, in prekindergarten. Also note the early introduction of algebra and abstraction, and the attention to justifying basic computational skills, e.g., the Grade 3–5 Standards for understanding the relationships of the operations:

> identify and use relationships between operations, such as division as the inverse of multiplication, to solve problems;
>
> understand and use properties of operations, such as the distributivity of multiplication over addition.

On What Environments Teachers Need to create in classrooms (p. 18)

Teachers establish and nurture an environment conducive to learning mathematics through the decisions they make, the conversations they orchestrate, and the physical setting they create. Teachers' actions are what encourage students to think, question, solve problems, and discuss their ideas, strategies, and solutions. The teacher is responsible for creating an intellectual environment where serious mathematical thinking is the norm. More than just a physical setting with desks, bulletin boards, and posters, the classroom environment communicates subtle messages about what is valued in learning and doing mathematics. Are students' discussion and collaboration encouraged? Are students expected to justify their thinking? If students are to learn to make conjectures, experiment with various approaches to solving problems, construct mathematical arguments and respond to others' arguments, then creating an environment that fosters these kinds of activities is essential.

In effective teaching, worthwhile mathematical tasks are used to introduce important mathematical ideas and to engage and challenge students intellectually. Well-chosen tasks can pique students' curiosity and draw them into mathematics. The tasks may be connected to the real-world experiences of students, or they may arise in contexts that are purely mathematical. Regardless of the context, worthwhile tasks should be intriguing, with a level of challenge that invites speculation and hard work. Such tasks often can be approached in more than one way, such as using an arithmetic counting approach, drawing a geometric diagram and enumerating possibilities, or using algebraic equations, which makes the tasks accessible to students with varied prior knowledge and experience.

On the Need to Develop Autonomous Learners and how they're developed through engaging in tough tasks (p. 21)

A major goal of school mathematics programs is to create autonomous learners, and learning with understanding supports this goal. Students learn more and learn better when they can take control of their learning by defining their goals and monitoring their progress. When challenged with appropriately chosen tasks, students become confident in their ability to tackle difficult problems, eager to figure things out on their own, flexible in exploring mathematical ideas and trying alternative solution paths, and willing to persevere. Effective learners recognize the importance of reflecting on their thinking and learning from their mistakes. Students should view the difficulty of complex mathematical investigations as a worthwhile challenge rather than as an excuse to give up. Even when a mathematical task is difficult, it can be engaging and rewarding. When students work hard to solve a difficult problem or to understand a complex idea, they experience a very special feeling of accomplishment, which in turn leads to a willingness to continue and extend their engagement with mathematics.

On Problem Solving's Pervasive Role in the Curriculum (p. 54)

Instructional programs from prekindergarten through grade 12 should enable all students to:

- build new mathematical knowledge through problem solving;
- solve problems that arise in mathematics and in other contexts;
- apply and adapt a variety of appropriate strategies to solve problems;
- monitor and reflect on the process of mathematical problem solving.

Problem solving means engaging in a task for which the solution method is not known in advance. In order to find a solution, students must draw on their knowledge, and through this process, they will often develop new mathematical understandings. Solving problems is not only a goal of learning mathematics but also a major means of doing so. Students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking. By learning problem solving in mathematics, students should acquire ways of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations that will serve them well outside the mathematics classroom. In everyday life and in the workplace, being a good problem solver can lead to great advantages. Problem solving is an integral part of all mathematics learning, and so it should not be an isolated part of the mathematics program. Problem solving in mathematics should involve all the five content areas described in these Standards. The contexts of the problems can vary from familiar experiences involving students' lives or the school day to applications involving the sciences or the world of work. Good problems will integrate multiple topics and will involve significant mathematics.

On that Students should be expected to develop many problem-solving strategies over the years in school (p. 53)

Of the many descriptions of problem-solving strategies, some of the best known can be found in the work of Pólya (1957). Frequently cited strategies include using diagrams, looking for patterns, listing all possibilities, trying special values or cases, working backward, guessing and checking, creating an equivalent problem, and creating a simpler problem. An obvious question is, How should these strategies be taught? Should they receive explicit attention, and how should they be integrated with the mathematics curriculum? As with any other component of the mathematical tool kit, strategies must receive instructional attention if students are expected to learn them. In the lower grades, teachers can help children express, categorize, and compare their strategies. Opportunities to use strategies must be embedded naturally in the curriculum across the content areas. By the time students reach the middle grades, they should be skilled at recognizing when various strategies are appropriate to use and should be capable of deciding when and how to use them. By high school, students should have access to a wide range of strategies, be able to decide which one to use, and be able to adapt and invent strategies.

On the Importance of Reasoning and Proofs (p. 56)

Instructional programs from prekindergarten through grade 12 should enable all students to:

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof.

Mathematical reasoning and proof offer powerful ways of developing and expressing insights about a wide range of phenomena. People who reason and think analytically tend to note patterns, structure, or regularities in both real-world situations and symbolic objects; they ask if those patterns are accidental or if they occur for a reason; and they conjecture and prove. Ultimately, a mathematical proof is a formal way of expressing particular kinds of reasoning and justification. Being able to reason is essential to understanding mathematics. By developing ideas, exploring phenomena, justifying results, and using mathematical conjectures in all content areas and-with different expectations of sophistication-at all grade levels, students should see and expect that mathematics makes sense. Building on the considerable reasoning skills that children bring to school, teachers can help students learn what mathematical reasoning entails. By the end of secondary school, students should be able to understand and produce mathematical proofs-arguments consisting of logically rigorous deductions of conclusions from hypotheses-and should appreciate the value of such arguments.

On the role of conjecturing (p. 57)

Doing mathematics involves discovery. Conjecture-that is, informed guessing-is a major pathway to discovery. Teachers and researchers agree that students can learn to make, refine, and test conjectures in elementary school. Beginning in the earliest years, teachers can help students learn to make conjectures by asking questions: What do you think will happen next? What is the pattern? Is this true always? Sometimes? Simple shifts in how tasks are posed can help students learn to conjecture. Instead of saying, "Show that the mean of a set of data doubles when all the values in the data set are doubled," a teacher might ask, "Suppose all the values of a sample are doubled. What change, if any, is there in the mean of the sample? Why?" High school students using dynamic geometry software could be asked to make observations about the figure formed by joining the midpoints of successive sides of a parallelogram and attempt to prove them. To make conjectures, students need multiple opportunities and rich, engaging contexts for learning. Young children will express their conjectures and describe their thinking in their own words and often explore them using concrete materials and examples. Students at all grade levels should learn to investigate their conjectures using concrete materials, calculators and other tools, and increasingly through the grades, mathematical representations and symbols. They also need to learn to work with other students to formulate and explore their conjectures and to listen to and understand conjectures and explanations offered by classmates.

On the vital importance of communicating one's mathematical thoughts (verbal and written) (p. 60)

Instructional programs from prekindergarten through grade 12 should enable all students to

- organize and consolidate their mathematical thinking through communication;
- communicate their mathematical thinking coherently and clearly to peers, teachers, and others;
- analyze and evaluate the mathematical thinking and strategies of others;
- use the language of mathematics to express mathematical ideas precisely.

Communication is an essential part of mathematics and mathematics education. It is a way of sharing ideas and clarifying understanding. Through communication, ideas become objects of reflection, refinement, discussion, and amendment. The communication process also helps build meaning and permanence for ideas and makes them public. When students are challenged to think and reason about mathematics and to communicate the results of their thinking to others orally or in writing, they learn to be clear and convincing. Listening to others' explanations gives students opportunities to develop their own understandings. Conversations in which mathematical ideas are explored from multiple perspectives help the participants sharpen their thinking and make connections. Students who are involved in discussions in which they justify solutions-especially in the face of disagreement-will gain better mathematical understanding as they work to convince their peers about differing points of view (Hatano and Inagaki 1991). Such activity also helps students develop a language for expressing mathematical ideas and an appreciation of the need for precision in that language. Students who have opportunities, encouragement, and support for speaking, writing, reading, and listening in mathematics classes reap dual benefits: they communicate to learn mathematics, and they learn to communicate mathematically. Students need to work with mathematical tasks that are worthwhile topics of discussion. Procedural tasks for which students are expected to have well-developed algorithmic approaches are usually not good candidates for such discourse. Interesting problems that "go somewhere" mathematically can often be catalysts for rich conversations.

On gaining insight as a result of consolidating their thinking (p. 60)

Students gain insights into their thinking when they present their methods for solving problems, when they justify their reasoning to a classmate or teacher, or when they formulate a question about something that is puzzling to them. Communication can support students' learning of new mathematical concepts as they act out a situation, draw, use objects, give verbal accounts and explanations, use diagrams, write, and use mathematical symbols. Misconceptions can be identified and addressed. A side benefit is that it reminds students that they share responsibility with the teacher for the learning that occurs in the lesson (Silver, Kilpatrick, and Schlesinger 1990). Reflection and communication are intertwined processes in mathematics learning. With explicit attention and planning by teachers, communication for the purposes of reflection can become a natural part of mathematics learning. Children in the early grades, for example, can learn to explain their answers and describe their strategies. Young students can be asked to "think out loud," and thoughtful questions posed by a teacher or classmate can provoke them to reexamine their reasoning. With experience, students will gain proficiency in organizing and recording their thinking.Writing in mathematics can also help students consolidate their thinking because it requires them to reflect on their work and clarify their thoughts about the ideas developed in the lesson. Later, they may find it helpful to reread the record of their own thoughts.

On the need for teachers to develop a mathematical community of argument and the development of communication skills (written and verbal) through the years (p. 62)

n order for a mathematical result to be recognized as correct, the proposed proof must be accepted by the community of professional mathematicians. Students need opportunities to test their ideas on the basis of shared knowledge in the mathematical community of the classroom to see whether they can be understood and if they are sufficiently convincing.

When such ideas are worked out in public, students can profit from being part of the discussion, and the teacher can monitor their learning (Lampert 1990). Learning what is acceptable as evidence in mathematics should be an instructional goal from prekindergarten through grade 12.

To support classroom discourse effectively, teachers must build a community in which students will feel free to express their ideas. Students in the lower grades need help from teachers in order to share mathematical ideas with one another in ways that are clear enough for other students to understand. In these grades, learning to see things from other people's perspectives is a challenge for students. Starting in grades 3-5, students should gradually take more responsibility for participating in whole-class discussions and responding to one another directly. They should become better at listening, paraphrasing, questioning, and interpreting others' ideas. For some students, participation in class discussions is a challenge. For example, students in the middle grades are often reluctant to stand out in any way during group interactions. Despite this fact, teachers can succeed in creating communication-rich environments in middle-grades mathematics classrooms. By the time students graduate from high school, they should have internalized standards of dialogue and argument so that they always aim to present clear and complete arguments and work to clarify and complete them when they fall short. Modeling and carefully posed questions can help clarify age-appropriate expectations for student work. Written communication should be nurtured in a similar fashion. Students begin school with few writing skills. In the primary grades, they may rely on other means, such as drawing pictures, to communicate. Gradually they will also write words and sentences. In grades 3-5, students can work on sequencing ideas and adding details, and their writing should become more elaborate. In the middle grades, they should become more explicit about basing their writing on a sense of audience and purpose. For some purposes it will be appropriate for students to describe their thinking informally, using ordinary language and sketches, but they should also learn to communicate in more-formal mathematical ways, using conventional mathematical terminology, through the middle grades and into high school. By the end of the high school years, students should be able to write well-constructed mathematical arguments using formal vocabulary. Examining and discussing both exemplary and problematic pieces of mathematical writing can be beneficial at all levels. Since written assessments of students' mathematical knowledge are becoming increasingly prevalent, students will need practice responding to typical assessment prompts. The process of learning to write mathematically is similar to that of learning to write in any genre. Practice, with guidance, is important. So is attention to the specifics of mathematical argument, including the use and special meanings of mathematical language and the representations and standards of explanation and proof. As students practice communication, they should express themselves increasingly clearly and coherently. They should also acquire and recognize conventional mathematical styles of dialogue and argument. Through the grades, their arguments should become more complete and should draw directly on the shared knowledge in the classroom. Over time, students should become more aware of, and responsive to, their audience as they explain their ideas in mathematics class. They should learn to be aware of whether they are convincing and whether others can understand them. As students mature, their communication should reflect an increasing array of ways to justify their procedures and results. In the lower grades, providing empirical evidence or a few examples may be enough. Later, short deductive chains of reasoning based on previously accepted facts should become expected. In the middle grades and high school, explanations should become more mathematically rigorous and students should increasingly state in their supporting arguments the mathematical properties they used.

On what students particularly should learn about Number and Operations

- **PreK–2**

  Instructional programs from prekindergarten through grade 12 should enable all students to

  1. Understand numbers, ways of representing numbers, relationships among numbers, and number systems
  2. Understand meanings of operations and how they relate to one another
  3. Compute fluently and make reasonable estimates

  Correspondingly:

  In prekindergarten through grade 2 all students should

  1. count with understanding and recognize "how many" in sets of objects;
     use multiple models to develop initial understandings of place value and the base-ten number system;
     develop understanding of the relative position and magnitude of whole numbers and of ordinal and cardinal numbers and their connections;
     develop a sense of whole numbers and represent and use them in flexible ways, including relating, composing, and decomposing numbers;
     connect number words and numerals to the quantities they represent, using various physical models and representations;
     understand and represent commonly used fractions, such as 1/4, 1/3, and 1/2.

2. understand various meanings of addition and subtraction of whole numbers and the relationship between the two operations;
   understand the effects of adding and subtracting whole numbers;
   understand situations that entail multiplication and division, such as equal groupings of objects and sharing equally.
3. develop and use strategies for whole-number computations, with a focus on addition and subtraction;
   develop fluency with basic number combinations for addition and subtraction;
   use a variety of methods and tools to compute, including objects, mental computation, estimation, paper and pencil, and calculators.

The concepts and skills related to number and operations are a major emphasis of mathematics instruction in prekindergarten through grade 2. Over this span, the small child who holds up two fingers in response to the question "How many is two?" grows to become the second grader who solves sophisticated problems using multidigit computation strategies. In these years, children's understanding of number develops significantly. Children come to school with rich and varied informal knowledge of number (Baroody 1992; Fuson 1988; Gelman 1994). During the early years teachers must help students strengthen their sense of number, moving from the initial development of basic counting techniques to more-sophisticated understandings of the size of numbers, number relationships, patterns, operations, and place value. Students' work with numbers should be connected to their work with other mathematics topics. For example, computational fluency (having and using efficient and accurate methods for computing) can both enable and be enabled by students' investigations of data; a knowledge of patterns supports the development of skip-counting and algebraic thinking; and experiences with shape, space, and number help students develop estimation skills related to quantity and size. As they work with numbers, students should develop efficient and accurate strategies that they understand, whether they are learning the basic addition and subtraction number combinations or computing with multidigit numbers. They should explore numbers into the hundreds and solve problems with a particular focus on two-digit numbers. Although good judgment must be used about which numbers are important for students of a certain age to work with, teachers should be careful not to underestimate what young students can learn about number. Students are often surprisingly adept when they encounter numbers, even large numbers, in problem contexts. Therefore, teachers should regularly encourage students to demonstrate and deepen their understanding of numbers and operations by solving interesting, contextualized problems and by discussing the representations and strategies they use.

- **Grades 3–5**

  Instructional programs from prekindergarten through grade 12 should enable all students to

  1. Understand numbers, ways of representing numbers, relationships among numbers, and number systems
  2. Understand meanings of operations and how they relate to one another
  3. Compute fluently and make reasonable estimates

  Correspondingly,

  In grades 3-5 all students should

  1. understand the place-value structure of the base-ten number system and be able to represent and compare whole numbers and decimals;
     recognize equivalent representations for the same number and generate them by decomposing and composing numbers;
     develop understanding of fractions as parts of unit wholes, as parts of a collection, as locations on number lines, and as divisions of whole numbers;
     use models, benchmarks, and equivalent forms to judge the size of fractions;
     recognize and generate equivalent forms of commonly used fractions, decimals, and percents;
     explore numbers less than 0 by extending the number line and through familiar applications;
     describe classes of numbers according to characteristics such as the nature of their factors.
  2. understand various meanings of multiplication and division;
     understand the effects of multiplying and dividing whole numbers;
     identify and use relationships between operations, such as division as the inverse of multiplication, to solve problems;
     understand and use properties of operations, such as the distributivity of multiplication over addition.
  3. develop fluency with basic number combinations for multiplication and division and use these combinations to mentally compute related problems, such as $30 \div 50$;
     develop fluency in adding, subtracting, multiplying, and dividing whole numbers;

develop and use strategies to estimate the results of whole-number computations and to judge the reasonableness of such results;
develop and use strategies to estimate computations involving fractions and decimals in situations relevant to students' experience;
use visual models, benchmarks, and equivalent forms to add and subtract commonly used fractions and decimals;
select appropriate methods and tools for computing with whole numbers from among mental computation, estimation, calculators, and paper and pencil according to the context and nature of the computation and use the selected method or tools.

In grades 3-5, students' development of number sense should continue, with a focus on multiplication and division. Their understanding of the meanings of these operations should grow deeper as they encounter a range of representations and problem situations, learn about the properties of these operations, and develop fluency in whole-number computation. An understanding of the base-ten number system should be extended through continued work with larger numbers as well as with decimals. Through the study of various meanings and models of fractions-how fractions are related to each other and to the unit whole and how they are represented-students can gain facility in comparing fractions, often by using benchmarks such as 1/2 or 1. They also should consider numbers less than zero through familiar models such as a thermometer or a number line. When students leave grade 5, they should be able to solve problems involving whole-number computation and should recognize that each operation will help them solve many different types of problems. They should be able to solve many problems mentally, to estimate a reasonable result for a problem, to efficiently recall or derive the basic number combinations for each operation, and to compute fluently with multidigit whole numbers. They should understand the equivalence of fractions, decimals, and percents and the information each type of representation conveys. With these understandings and skills, they should be able to develop strategies for computing with familiar fractions and decimals.

On what students should learn about algebra and functions

- ## PreK–2

  Instructional programs from prekindergarten through grade 12 should enable all students to

1. Understand patterns, relations, and functions
2. Represent and analyze mathematical situations and structures using algebraic symbols
3. Use mathematical models to represent and understand quantitative relationships
4. Analyze change in various contexts

Correspondingly,

In prekindergarten through grade 2 all students should

1. sort, classify, and order objects by size, number, and other properties; recognize, describe, and extend patterns such as sequences of sounds and shapes or simple numeric patterns and translate from one representation to another; analyze how both repeating and growing patterns are generated.
2. illustrate general principles and properties of operations, such as commutativity, using specific numbers;
3. use concrete, pictorial, and verbal representations to develop an understanding of invented and conventional symbolic notations.
4. model situations that involve the addition and subtraction of whole numbers, using objects, pictures, and symbols.
5. describe qualitative change, such as a student's growing taller; describe quantitative change, such as a student's growing two inches in one year.

Algebraic concepts can evolve and continue to develop during prekindergarten through grade 2. They will be manifested through work with classification, patterns and relations, operations with whole numbers, explorations of function, and step-by-step processes. Although the concepts discussed in this Standard are algebraic, this does not mean that students in the early grades are going to deal with the symbolism often taught in a traditional high school algebra course. Even before formal schooling, children develop beginning concepts related to patterns, functions, and algebra. They learn repetitive songs, rhythmic chants, and predictive poems that are based on repeating and growing patterns. The recognition, comparison, and analysis of patterns are important components of a student's intellectual development. When students notice that operations seem to have particular properties, they are beginning to think algebraically. For example, they realize that changing the order in which two numbers are added does not change the result or that adding zero to a number leaves that number unchanged. Students' observations and discussions of how quantities relate to one another lead to initial experiences with function relationships, and their representations of mathematical situations using concrete objects, pictures, and symbols are the beginnings of mathematical modeling. Many of the step-by-step processes that students use form the basis of understanding iteration and recursion.

- **Grades 3–5**

  Instructional programs from prekindergarten through grade 12 should enable all students to

  1. Understand patterns, relations, and functions
  2. Represent and analyze mathematical situations and structures using algebraic symbols
  3. Use mathematical models to represent and understand quantitative relationships
  4. Analyze change in various contexts

  Correspondingly,

  In grades 3-5 all students should

  1. describe, extend, and make generalizations about geometric and numeric patterns;
     represent and analyze patterns and functions, using words, tables, and graphs.
  2. identify such properties as commutativity, associativity, and distributivity and use them to compute with whole numbers;
     represent the idea of a variable as an unknown quantity using a letter or a symbol;
     express mathematical relationships using equations.
  3. model problem situations with objects and use representations such as graphs, tables, and equations to draw conclusions.
  4. investigate how a change in one variable relates to a change in a second variable;
     identify and describe situations with constant or varying rates of change and compare them.

Although algebra is a word that has not commonly been heard in grades 3-5 classrooms, the mathematical investigations and conversations of students in these grades frequently include elements of algebraic reasoning. These experiences and conversations provide rich contexts for advancing mathematical understanding and are also an important precursor to the more formalized study of algebra in the middle and secondary grades. In grades 3-5, algebraic ideas should emerge and be investigated as students:

- identify or build numerical and geometric patterns;
- describe patterns verbally and represent them with tables or symbols;
- look for and apply relationships between varying quantities to make predictions;
- make and explain generalizations that seem to always work in particular situations;
- use graphs to describe patterns and make predictions;
- explore number properties;
- use invented notation, standard symbols, and variables to express a pattern, generalization, or situation.

6.6 Excerpts from The MAA recommendations on the mathematical preparation of teachers

This second set of recommendations has to do with teachers should learn. This naturally builds on what school children should learn. The recommendations for teacher preparation are excerpted from "The Mathematical Education of Teachers," which is a joint publication of the Mathematical Association of America and the American Mathematical Society. "The Mathematical Education of Teachers" appeared in print as Volume 11 of the Conference Board of the Mathematical Sciences Issues in Mathematics Education and is available online at http://www.maa.org/cbms/.

Overall curriculum and instruction

Recommendation 1. Prospective teachers need mathematics courses that develop a deep understanding of the mathematics they will teach.

Recommendation 2. Although the quality of mathematical preparation is more important than the quantity, the following amount of mathematics coursework for prospective teachers is recommended.

Prospective elementary grade teachers should be required to take at least 9 semester-hours on fundamental ideas of elementary school mathematics.

Recommendation 3. Courses on fundamental ideas of school mathematics should focus on a thorough development of basic mathematical ideas. All courses designed for prospective teachers should develop careful reasoning and mathematical "common sense" in analyzing conceptual relationships and in solving problems. Attention to the broad and flexible applicability of basic ideas and modes of reasoning is preferable to superficial coverage of many topics. Prospective teachers should learn mathematics in a coherent fashion that emphasizes the interconnections among theory, procedures, and applications. They should learn how basic mathematical ideas combine to form the framework on which specific mathematics lessons are built. For example, the ideas of number and function, along with algebraic and graphical representation of information, form the basis of most high school algebra and trigonometry.

Recommendation 4. Along with building mathematical knowledge, mathematics courses for prospective teachers should develop the habits of mind of a mathematical thinker and demonstrate flexible, interactive styles of teaching. Mathematics is not only about numbers and shapes, but also about patterns of all types. In searching for patterns, mathematical thinkers look for attributes like linearity, periodicity, continuity, randomness, and symmetry. They take actions like representing, experimenting, modeling, classifying, visualizing, computing, and proving. Teachers need to learn to ask good mathematical questions, as well as find solutions, and to look at problems from multiple points of view. Most of all, prospective teachers need to learn how to learn mathematics.

How we ought to teach pre-service teachers

Those who prepare prospective teachers need to recognize how intellectually rich elementary-level mathematics is. At the same time, they cannot assume that these aspiring teachers have ever been exposed to evidence that this is so. Indeed, among the obstacles to improved learning at the elementary level, not the least is that many teachers were convinced by their own schooling that mathematics is a succession of disparate facts, definitions, and computational procedures to be memorized piecemeal. As a consequence, they are ill-equipped to offer a different, more thoughtful kind of mathematics instruction to their students.

Yet, it is possible to break this cycle. College students with weak mathematics backgrounds can rekindle their own powers of mathematical thought. In fact, the first priority of preservice mathematics programs must be to help prospective elementary teachers do so: with classroom experiences in which their ideas for solving problems are elicited and taken seriously, their sound reasoning affirmed, and their missteps challenged in ways that help them make sense of their errors. Teachers able to cultivate good problem-solving skills among their students must, themselves, be problem solvers, aware that confusion and frustration are not signals to stop thinking, confident that with persistence they can work through to the satisfactions of new insight. They will have learned to notice patterns and think about whether and why these hold, posing their own questions and knowing what sorts of answers make sense. Developing these new mathematical habits means learning how to continue learning.

The key to turning even poorly prepared prospective elementary teachers into mathematical thinkers is to work from what they do know the mathematical ideas they hold, the skills they possess, and the contexts in which these are understood so they can move from where they are to where they need to go. For their instructors, this requires learning to understand how their students think. The disciplinary habits of abstraction and deductive demonstration, characteristic of the way professional mathematicians present their work, have little to do with the ways each of us initially enters the world of mathematics, that is, experientially, building our concepts from action. And this is where mathematics courses for elementary school teachers must begin, first helping teachers make meaning for the mathematical objects under study meaning that often was not present in their own elementary educations and only then moving on to higher orders of generality and rigor.

The medium through which this ambitious agenda can be realized is the very mathematics these elementary teachers are responsible for first and foremost, and still the heart of elementary content, number and operations; then, geometry, early algebraic thinking, and data, all of which are receiving increased emphasis in the elementary school curriculum.

This is not to say that prospective teachers will be learning the mathematics as if they were nine-year-olds. The understanding required of them includes acquiring a rich network of concepts extending into the content of higher grades; a strong facility in making, following, and assessing mathematical argument; and a wide array of mathematical strategies.

What prospective elementary teachers need to experience with respect to number and operation

To be prepared to teach arithmetic for understanding, elementary teachers, themselves, need to understand:

- A large repertoire of interpretations of addition, subtraction, multiplication and division, and of ways they can be applied.
- Place value: how place value permits efficient representation of whole numbers and finite decimals; that the value of each place is ten times larger than the value of the next place to the right; implications of this for ordering numbers, estimation, and approximation; the relative magnitude of numbers.
- Multidigit calculations, including standard algorithms, "mental math," and non-standard methods commonly created by students: the reasoning behind the procedures, how the base-10 structure of number is used in these calculations.
- Concepts of integers and rationals: what integers and rationals (represented as fractions and decimals) are; a sense of their relative size; how operations on whole numbers extend to integers and rational numbers; and the behavior of units under the operations.

The study of number and operations provides opportunities for prospective teachers to create meaning for what many had only committed to memory but never really understood. It should begin by placing the mathematics in everyday contexts—e.g., comparing, joining, separating, sharing, and counting quantities that arise in one's daily activities—and working with a variety of representations—e.g., number lines, area diagrams, and arrangements of physical objects. Instead of solving word problems by looking for "key words" or applying other superficial strategies, prospective teachers should learn to consider the actions the problems might posit. Learning to recognize that a single situation can be modeled by different operations opens up discussion of how the operations are related.

Future teachers must understand the conceptual underpinnings of the conventional computation algorithms as well as alternative procedures such as those commonly generated by children, themselves. This process might begin by having teachers perform multidigit calculations mentally, without the aid of pencil and paper, to help loosen the hold of the belief that there is just one correct way to solve any mathematics problem. As they become aware and then pursue their own ideas, they will recognize, often for the first time, that they do, indeed, have mathematical ideas worth following. Similar exercises can be used to help teachers see how decimal notation allows for approximation of numbers by "round numbers" (multiples of powers of 10), facilitating mental arithmetic and approximate solutions.

Although most teachers are able to identify the ones place, the tens place, etc. and write numbers in expanded notation, they often lack understanding of core ideas related to place value. For example, future teachers should understand: how place value permits efficient representation of large numbers; how the operations of addition, multiplication, and exponentiation are used in representing numbers as "polynomials in 10"; and how decimal notation allows one to quickly determine which of two numbers is larger. Furthermore, they should be familiar with the notion of "order of magnitude."

Having developed a variety of models of whole number operations, teachers are ready to consider how these ideas extend to integers and rational numbers. First they must develop an understanding of what these numbers are. For integers, this means recognizing that numbers now represent both magnitude and direction. And though most teachers know at least one interpretation of a fraction, they must learn many interpretations: as part of a whole, as an expression of division, as a point on the number line, as a rate, or as an operator. Teachers may have learned rules for comparing fractions, but now, equipped with a choice of representations, they can develop flexibility in determining relative size.

As with whole-number operations, placing operations with fractions in everyday contexts helps give meaning to algorithms hitherto regarded as mechanical devices. (Many college students see fractions only as pairs of natural numbers plugged into arithmetic procedures; hence, to them, adding two fractions is simply a computation with four integers.) Teachers must recognize that some generalizations often made by children about whole-number operations, e.g., a product is always larger than its factors (except when a factor is 0 or 1) and a quotient is always smaller than its dividend (unless the divisor is 1) no longer hold, and that the very meanings of multiplication and division must be extended beyond those derived from whole-number operations.

The idea of "unit" that the same object can be represented by fractions of different values, depending on the reference whole is central to work with fractions. In addition and subtraction, all the quantities refer to the same unit, but do not in multiplication and division.

Another area to be explored is the extension of place-value notation from whole numbers to finite decimals. Teachers must come to see that any real number can be approximated arbitrarily closely by a finite decimal, and they must recognize that the rules for calculating with decimals are essentially the same as those for whole numbers. Explorations of decimals lend themselves to work with calculators particularly well.

As with all of the content described in this document, the topics enumerated are not to be taught as discrete bits of mathematics. Always, the power comes from connection using the concepts and skills flexibly, recognizing them from a variety of perspectives as they are embedded in different contexts.

Some excerpts as well from the expectations for middle school teachers:

Teachers should understand how decimals extend the place value work from the earlier grades. They should be able to convert easily among fractions, decimals, and percents. They should understand why only repeating decimals can be converted to fractions, and why non-repeating decimals are not rational, thus leading to a discussion of irrational numbers. Their knowledge of positive rational numbers can then be extended to a study of negative rational numbers. Although prospective teachers will have some familiarity with operational properties, the rational number system is usually their first encounter with a field. Teachers should be able to develop, for example, Venn diagrams to represent the hierarchy of the different types of numbers: whole, integer, rational, irrational, and real, and how they are related.

Mental computation and estimation can lead to better number sense. Most middle grades students and some prospective teachers, when asked to use mental computation, will attempt to mentally undertake the pencil-and-paper algorithm with which they are familiar rather than use number properties to their advantage (e.g., using the distributive property to find 7×28 or the associative property to think of 7×28 as $7 \times 7 \times 4$, or 49×4). To help prospective teachers develop "rational number sense," tasks can be designed that include mentally ordering a set of rational numbers (e.g., 0.23, 5/8, 51%, and 1/4) using knowledge of number size; estimating the outcomes of rational number operations (e.g., $7/8 + 9/10$ must be a little less than 2 because each fraction is a little less than 1), and recognizing wrong answers (e.g., $2/3 \div 1/2$ cannot be less than 1 because there is more than one 1/2 in 2/3). Developing flexibility in working with numbers will take time, even for prospective teachers, because most have never been asked to think about numbers in these ways.

Basic number theory has a valuable role to play in contemporary middle grades mathematics and should have a role in courses designed for middle grades teachers. They should experience conjecturing and justifying conjectures about even and odd numbers and about prime and composite numbers. They should have a good grasp of the Prime Factorization Theorem and how it extends to algebra learning. The difficulty of finding the greatest common factor of two numbers can lead students to an appreciation of the efficiency of the Euclidean Algorithm.

Prospective teachers need to attach meaning to very large numbers that they see daily. Developing benchmarks for large numbers (e.g., calculating one's share of the national debt) can lead to a better sense of what these numbers mean. Examples of very small numbers can be found in middle grades science. The difficulty of writing, expressing, and calculating with very large numbers and very small numbers will lead prospective teachers to appreciate the structure and sophistication of scientific notation.

Finally, experiences using ratios as a means of comparison can lead prospective teachers to think about situations that are proportional in nature. For example, when prospective teachers are asked to compare the steepness of two ramps, some do so by comparing the differences between the heights and depths of the ramps rather than by comparing the ratios of these two quantities. A problem such as this one can lead to finding slopes of lines in coordinate systems and understanding what the slope means. Percents are, of course, ratios, and need to be presented as such.

What prospective elementary teachers should know about algebra and functions

Although the study of algebra and functions generally begins at the upper-middle- or high-school levels, some core concepts and practices are accessible much earlier. If teachers are to cultivate the development of these ideas in their elementary classrooms, they, themselves, must understand those concepts and practices, including:

- Representing and justifying general arithmetic claims, using a variety of representations, algebraic notation among them; understanding different forms of argument and learning to devise deductive arguments.
- The power of algebraic notation: developing skill in using algebraic notation to represent calculation, express identities, and solve problems.
- Field axioms: recognizing commutativity, associativity, distributivity, identities, and inverses as properties of operations on a given domain; seeing computation algorithms as applications of particular axioms; appreciating that a small set of rules governs all of arithmetic.
- Functions: being able to read and create graphs of functions, formulas (in closed and recursive forms), and tables; studying the characteristics of particular classes of functions on integers.

Algebraic notation is an efficient means for representing properties of operations and relationships among them. In the elementary grades, well before they encounter that notation, children who are encouraged to recognize and articulate generalizations will become familiar with the sorts of ideas they will later express algebraically. In order to support children's learning in this realm, teachers first must do this work for themselves. Thus, they must come to recognize the centrality of generalization as a mathematical activity. In the context of number theory explorations (e.g., odd and even numbers, square numbers, factors), they can look for patterns, offer conjectures, and develop arguments for the generalizations they identify. And the arguments they propose become occasions for investigating different forms of justification. If, in this work, teachers learn to use a variety of modes of representation, including conventional algebraic symbols, the algebra they once experienced as the manipulation of opaque symbols can be invested with meaning.

Particularly instructive in work on word problems are comparisons of solution procedures using a variety of representations, illustrating how algebraic strategies mirror the actions modeled by other methods. As teachers become more confident of their skill in using algebra, they come to appreciate the advantages of its economy as against the cumbersomeness of other modes of representation, such as blocks or diagrams.

Although initially teachers' work in number and operations must be grounded experientially, now they are equipped to return to the study of computation, this time to appreciate the algorithms on whole numbers, integers, or rationals as applications of commutativity, associativity, distributivity, identities, and (when it holds) inverses, the small set of rules governing all of arithmetic.

Especially important for teachers is recognition of how young children's work with patterns can be related to the concept of function for example, that labeling the terms or units of a pattern by the natural numbers creates a function. As they pursue the study of functions, teachers learn to move fluently among descriptions of situations, tables of values, graphs, and formulas. And as they explore, they become familiar with certain elementary functions on integers: linear, quadratic, and exponential. They also learn to work with functions defined by physical phenomena, say, distance traveled by a runner over time, growth of a plant over time, or the times of sunrise and sunset over a year.

6.7 Excerpts from elementary school mathematics textbooks

The elementary school math curriculum is forever changing. The content of Math 105-106 is designed to give you the knowledge and abstract thinking tools you will need as an educator. While this content is not restricted to topics you will teach, you may be surprised how many of the seemingly new or advanced topics you come across in this course actually do appear in elementary level math textbooks.

To illustrate this, we have included pages copied from mathematics textbooks currently in use in Columbus area public schools. All the excerpts are from grades no higher than grade 5. The designations handwritten at the bottom of the copied pages are the grade level and the code EM for the textbook series *Everyday Mathematics*, AW for the Addison Wesley series, and IM for *Invitation to Mathematics*.

Note in particular that

- Exponentials and Scientific Notation are explicitly addressed
- Logic and set theory (Venn diagrams) enter as early as grade 3
- Not only are teachers expected to understand alternative algorithms, but these are shown to children as early as grade 3
- Functions appear as early as grade 1
- By grade 4, children are not only using algebra but talking about functions in algebraic terms
- Models for negative numbers are explicitly addressed