6

$$\begin{aligned}
\tanh' x &= \frac{d}{dx}\frac{e^x - e^{-x}}{e^x + e^{-x}} \\
&= \frac{(e^x + e^{-x})(e^x - (-e^{-x})) - (e^x - e^{-x})(e^x + (-e^{-x}))}{(e^x + e^{-x})^2} \\
&= \frac{(e^x + e^{-x})(e^x - (-e^{-x})) - (e^x - e^{-x})(e^x + (-e^{-x}))}{(e^x + e^{-x})^2} \\
&= \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} \\
&= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} \\
&= 1 - \frac{(e^x - e^{-x})^2}{(e^x + e^{-x})^2} \\
&= 1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 \\
&= 1 - (\tanh x)^2.
\end{aligned}$$

Now we plug in $\tanh^{-1} x$:

$$\tanh'(\tanh^{-1}(x)) = 1 - (\tanh(\tanh^{-1} x))^2 = 1 - x^2.$$

Using the derivative rule for inverses,

$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{\tanh'(\tanh^{-1}(x))} = \frac{1}{1 - x^2}.$$

7

a

The units of $Q'(p)$ are $\frac{\text{units of Q}}{\text{units of p}}$, so widgets/dollar (assuming price is in dollars). Typically $Q'(p)$ will be negative: as price goes up, we expect to sell fewer units.[1]

[1] In fact, this is so common that products where $Q'(p)$ is positive have a special name: they're known as *Giffen goods*.

b

Elasticity is *unitless*: the units are $\frac{\text{dollars}}{\text{widgets}}\frac{\text{widgets}}{\text{dollars}}$, and the top and bottom cancel. (This is part of why elasticity is useful: the derivative depends on what currency you use, but the elasticity doesn't.)

c

$\ln Q(p) = \ln 3p^{-1/2} = \frac{-1}{2}\ln 3p = \frac{-1}{2}\ln 3 + \frac{-1}{2}\ln p$. The derivative is

$$[\ln Q(p)]' = 0 + \frac{-1}{2p} = \frac{-1}{2p}.$$

d

$E(p) = \frac{p}{Q(p)}Q'(p) = p(\ln Q(p))' = p\frac{-1}{2p} = \frac{-1}{2}$.