A couple things you should notice about this solution. First, I'm including a lot of steps that are fairly small, and many of them seem obvious. This is intentional, and you should be doing the same. Your solutions shouldn't just demonstrate that you know how to solve the problem, they should actually walk through all the steps of solving it. You should be practicing this with easier problems--like problem 1 from this in-class work--because it will make it easier to solve harder problems.

Second, notice that I'm putting down all the facts we need in order. Again, this will make it easier to solve difficult problems: if you write down the facts that you know, often things become apparent which weren't clear when those same facts were floating around unorganized in your head.

a

1. We name the function $f(x) = x^4 + 4x^2 - 10x + 1$.
   Note: a lot of groups were leaving out this step, and just talking about a function f without giving any clue what it was or where it came from. In some of these problems there's an obvious function which we're likely to call f, but we still need to name it so that someone reading our solution can be sure what it is.

2. $f(0) = 1 > 0$

3. $f(1) = -4 < 0$

4. $f(2) = 13 > 0$

5. f is continuous everywhere because it is a polynomial, and all polynomials are continuous everywhere.
   Note: I don't want to be really picky about this on exams, but on work that isn't time pressured it's important to be thorough, which means *always* checking that the assumptions of our theorems are satisfied when we use them.

6. Since $f(0) > 0 > f(1)$, by IVT there must be a value c in the interval $(0, 1)$ so that $f(c) = 0$.
   Note: many groups were really reluctant to include a few words to explain what they were doing. When we use a theorem, we

`need to mention it by name. This is something I` *`will`* `be picky about on exams.`

7. Since $f(1) < 0 < f(2)$, by IVT, there must be a value d in the interval $(1, 2)$ so that $f(d) = 0$

8. Saying that $f(c) = 0$ and $f(d) = 0$ is the same as saying that c and d are roots of the polynomial $x^4 + 4x^2 - 10x + 1$

9. Since c and d are in disjoint intervals, we know $c \neq d$

10. Therefore c and d are two distinct roots of $x^4 + 4x^2 - 10x + 1$.

b

1. Suppose there were more than two roots. Then there would be at least three roots.

2. We could pick three of these roots and name the smallest of them a, the middle one b, and the largest one c.

3. Since a, b, and c are roots, $f(a) = 0$, $f(b) = 0$, and $f(c) = 0$.

4. Because we picked a, b, c in order, $a < b < c$.
   `Note: Notice that in this step and the previous one, we take some facts we had in words and turn them into formulas. In general, when we have facts expressed in words, we should take the time to actually write down what they mean in equations, because it often makes further steps clearer.`

5. Since f is a polynomial, it is continuous and differentiable everywhere.

6. Since $f(a) = f(b)$, by Rolle's Theorem there must be a value d in the interval (a, b) such that $f'(d) = 0$.
   `Note: you can choose any name you want in place of` d`, but of course it has to be a new name that you haven't already used for something else.`

7. Since $f(b) = f(c)$, by Rolle's Theorem there must be a value e in the interval (b, c) such that $f'(e) = 0$.

8. Since $d < b < e$, also $d < e$.

9. Since f' is also a polynomial, it is also continuous and differentiable everywhere.

10. Since $f'(d) = f'(e)$, there must be an h in the interval (d, e) such that $f''(h) = 0$.

11. We know that $f'(x) = 4x^3 + 8x - 10$ and $f''(x) = 12x^2 + 8$.

12. For every x, $12x^2 \geq 0$ (because x^2 is a square number) and $8 > 0$, so $f''(x) > 0$.

13. This is a contraduction: $f''(h) = 0$ but $f''(h) > 0$. So our assumption, that there were more than two roots, was wrong.

14. Therefore there are at most two roots.