Theorem 1 (Exercise 1.12)**.**

1. *If a set of lines in $\mathbb{R}^3$ is parallel to each other and also parallel to the picture plane, and their projections are not points or empty, then the projections are parallel lines.*

2. *If a set of lines in $\mathbb{R}^3$ is parallel to each other and not parallel to the picture plane, then their projections are a set of lines through a common point called the* **vanishing point** *for that set of lines. That point is missing from each of the lines, unless you "close" the image by including the image of the point at infinity, a point that is in $\mathbb{R}^3$ but not in $\mathbb{R}^3$.*

PROOF: 1. Let ℓ and m be two such lines. Two easy cases: if $\ell = m$; then their images are the same, hence parallel; if either line contains the viewing position then its image is at most one point. Assuming neither of these occurs, let α be the unique plane containing ℓ and the viewing position. One more easy case: if α is parallel to the viewing position then ℓ has no image. Assuming this is not the case then α intersects the viewing plane in a line k. The lines k and ℓ are coplanar but do not intersect, hence ℓ is parallel to k. The same reasoning shows that m is parallel to its image in the picture plane. Because being parallel is an equivalence relation, the two images, being parallel to lines that are parallel to each other, must themselves be parallel.

2. Let ℓ be one of the lines. Denote the viewing position by O, the line through O parallel to ℓ by m and the vanishing point by V. Because ℓ is parallel to m, when B is on ℓ, the points O, V, B are never collinear, meaning that V can never be the projection of a point of ℓ. On the other hand, as a point B on ℓ moves farther from V in the direction that crosses the picture plane, the angle $\angle BOV$ becomes closer and closer to zero. Therefore, the image of B gets closer and closer to V. Because the projection of ℓ, which we know is a line minus a point, gets arbitrarily close to V but does not hit V, it must be a line through V, minus V itself. Because ℓ could have been any one of the lines, they all have this property. $\square$

Bonus question: is the converse to (1) true? That is, if the projections of two lines to the picture plane are parallel lines (possibly missing a point) then do the lines have to be parallel in real life? Homework problem P3.1 might give you some help with this.

Theorem 2 (proportional change)**.** *If a plane ω is parallel to the picture plane, then, given a fixed viewing position O, there is a constant multiplying factor c such that projecting to the picture plane multiplies the length of every line segment ℓ in $\mathbb{R}^3$ by the constant c.*

PROOF: The constant c is the ratio of the distance from O to the viewing plane to the distance from O to ω. This is shown using several similar triangles. Let $\overline{AB}$ be a line segment in ω, with projection $\overline{A_*B_*}$ in the picture plane. Let P_* be the *viewing target*, where a perpendicular from O to the viewing plane meets the viewing plane and let P be the intersection of this same perpendicular $\overline{OP_*}$ with ω. Then $\triangle OAP$ is similar to $\triangle OA_*P_*$, hence $\overline{OA_*} : \overline{OA} = \overline{OP_*} : \overline{OP}$. The same is true with B and B_* in place of A and A_*. This is the ratio known as c. Applying this to the similar triangles $\triangle OA_*B_*$ and $\triangle OAB$ shows that also the ratio between $\overline{A_*B_*}$ and $\overline{AB}$ is equal to c. $\square$

Bonus question: where did we use the fact that ω was parallel to the viewing plane?