**Solution to Exercises 2 and 3 in Chapter 5, with justification**

**Answer to Exercise 5.2 (no justification yet):** All three pairwise intersections, $\alpha\cdot\beta, \alpha\cdot\gamma$ and $\beta\cdot\gamma$, are parallel.

For the proof, first I am copying in the instructor manual solution to Exercise 5.2. See if you can spot a gap.

Ⓔ 5.2. The lines $\alpha\cdot\beta$, $\beta\cdot\gamma$, and $\gamma\cdot\alpha$ are parallel to one another. Choose a line that vanishes to the common intersection point; then the three planes must all be parallel to that line. For example, in IM-Figure 5.1, the roof of the house, the side wall of the house, and the ground are all parallel to the line along the top of the roof. Notice that these three planes might or might not intersect in a common line, but they are all parallel to a common line. Therefore, their pairwise intersections are also parallel to that line.

The gap I see is that there are a lot of different lines parallel to any plane, and the fact that two planes have a parallel line in common does not to me obviously imply that their intersection is parallel to that line. Here is a proof of that fact.

LEMMA: if a line $k \subset \mathbb{R}^3$ is parallel to two distinct planes $\alpha$ and $\beta$, then either $\alpha$ and $\beta$ are parallel to each other or $k$ is parallel to the intersection $m$ of $\alpha$ and $\beta$.

To me, the easiest proof is the one that uses extended space, so I will give this one and leave it as an exercise for those who want to prove this lemma without extended space.

PROOF OF LEMMA: Let $k_*, \alpha_*$ and $\beta_*$ be the extended versions of $k, \alpha$ and $\beta$ in $\mathbb{E}^3$. We can assume $\alpha$ and $\beta$ are not parallel, hence in $E^3$, $\alpha\cdot\beta$ has ordinary points and is therefore some line $m_*$ where $m$ is an ordinary line. The definition of which ideal points are contained in which ideal lines, the ideal point $P_{[[k]]}$ is in both the ideal lines $\ell_{[[\alpha]]}$ and $\ell_{[[\beta]]}$. Because $\ell_{[[\alpha]]} \subset \alpha_*$ and $\ell_{[[\beta]]} \subset \beta_*$, we conclude that $P_{[[k]]} \in m_*$. There is only one ideal point $P_{[[m]]}$ in $m_*$, hence $P_{[[k]]} = P_{[[m]]}$. By definition of ideal points, this means that $k \parallel m$, which is what we needed to prove.

That does it for Exercise 5.2, but I'm not done, because I first came up with a different proof. So next, I will give the proof (also requiring the lemma) that I first thought of. This is mostly to illustrate that there are often many proofs and that you might want to go with one you think is most understandable to others (I am guessing this one is not the one you'd pick).

**Alternate Proof:**

First, let's explicate the given information. The term "vanishing line" means the collection of vanishing points of a plane in $\mathbb{R}^3$, under projection from a point $O$ to a picture plane, call it $\omega'$. The vanishing point of a line $k$ was originally defined as the intersection of $\omega'$ with a line $m$ parallel to $k$ through the viewing point of the projection. The vanishing line of $\alpha$, is therefore the intersection of $\omega'$ with an entire plane $\alpha''$ parallel to $\alpha$ passing through $O$. Similarly, the vanishing lines of $\beta$ and $\gamma$ are respectively $\beta'' \cdot \omega'$ and $\gamma'' \cdot \omega'$, for planes $\beta''$ and $\gamma''$ passing through $O$ and respectively parallel to $\beta$ and $\gamma$.

We can therefore describe $\alpha''$ as the unique plane containing $\ell_2$ and $O$. Similarly $\ell_3$ and $O$ define $\beta''$ and $\ell_4$ and $O$ define $\gamma''$. We can then compute that $\alpha'' \cap \beta'' = \alpha'' \cap \gamma'' = \beta'' \cap \gamma'' = OP$. We conclude that the three planes $\alpha, \beta$ and $\gamma$ are all planes in $\mathbb{R}^3$ parallel to a single line $OP$.

From the lemma, we conclude that $\alpha \cdot \beta, \alpha \cdot \gamma$ and $\beta \cdot \gamma$ are all parallel to a single line, $OP$, answering the question posed in the problem.

**Answer to Exercise 5.3:** The pairwise intersections meet in a single ordinary point.

Proof:

Let's first rule out that all three pairwise intersections are the same line. That's impossible because then, projecting, the vanishing point of that line would have to be on all three vanishing lines, but the three vanishing lines don't have a common intersection.

Next, let's rule out that two of the planes are parallel. If that were true, they would have the same vanishing line. (Think: why?)

Last, let's rule out that two planes intersect in a line parallel to the third plane. Then the vanishing point for this line would be a point in the vanishing line for the third plane, meaning there would be a point in all three vanishing lines, which we know is not true.

We conclude, because there are only a few possibilities for how planes intersect (fact of ordinary geometry of $\mathbb{R}^3$) that the three planes intersect in a point of $\mathbb{R}^3$, i.e., an ordinary point.