Notation for intersections and projections, and how projections give us vanishing points and lines

Notation for elements, subset and intersections:

Distinguishing between the element symbol $\in$ and the subset symbol $\subset$ is not important. Because a couple of people asked, I am writing a complete explanation for clarity.

Lines and planes are defined to be sets of points. So it's always correct to write $P \in k$ or $P \in \alpha$ when P is in the line k or the plane α. One could also write $P \subset k$, though technically if you want to go that route you should say $\{P\} \subseteq k$: the set containing only P is a subset of k.

To say that the line k lies in the plane α, we would normally say $k \subseteq \alpha$, because literally both k and α are sets of points and one set contains the other. But we won't be unhappy with you if you say $k \in \alpha$. The inaccuracy does not create ambiguity.

Intersections of planes, lines, etc. can be written two ways, with an intersection symbol or a dot. These are synonomous.

$$\begin{aligned} \text{if } k \text{ and } m \text{ are lines then } k \cdot m &= k \cap m \,; \\ \text{if } k \text{ is a line and } \alpha \text{ is a plane then } k \cdot \alpha &= k \cap \alpha \,. \end{aligned}$$

Notation for projection that's NOT in the book:

Let O be any ordinary point of $\mathbb{R}^3$ and let ω' be an ordinary plane not containing O. Then:

- $\Pi_{O,\omega'}$ denotes the function "projection from the point O to the plane ω'."
- The **domain** of $\Pi_{O,\omega'}$ is all points of $\mathbb{R}^3$ except for O. Its *range* is the extended plane $\omega'_* = \omega' + \ell_{[[\omega']]}$. Most of the time I will drop the $*$ and just write ω'.
- Its **mathematical definition** is simple: $\Pi_{O,\omega'}(P) = OP \cdot \omega'$. In other words, it maps any point P to the place the line OP intersects the plane ω'. This is always exactly one point, because our assumptions imply that OP is not in the plane ω'; Theorem E4 in homework problem E5.5 says that OP and ω' intersect in exactly one point (possibly an ideal point).

The function $\Pi_{O,\omega'}$ is a mesh map if we define it for lines by $\Pi_{O,\omega'}(k) = \{\Pi_{O,\omega'}(P) : P \in k\}$. For those of you not fluent in set notation, that means that if k is a line, $\Pi_{O,\omega'}(k)$ is the set you get by applying $\Pi_{O,\omega'}$ to every point in k. This will always be a line, so $\Pi_{O,\omega'}$ defines a valid mesh map.

Notation for mesh maps that IS in the book:

The book prefers just using the symbol $'$ (the prime symbol) instead of naming a function. On one hand this is easy: if A, B and C are points and k and m are lines, then A', B', C', k', m' denote their images under the function $'$. The bad part is that the notation doesn't specify O or ω', so you might not know to *which* projection the notation refers.

Properties of projections:

They are mesh maps, so they map points to points and lines to lines. Note that as a mesh map, the domain excludes any line through O, because the point O was already excluded.

Also, always remember, they map three-dimensional space (the "real world") to two-dimensional space (the "picture plane" or "canvas"). Using the extended spaces $\mathbb{R}^3$ and $\mathbb{R}^2$ makes the definitions the same whether or not a line is parallel to the picture plane. The projection map is not one to one. Many real world points get mapped to the same point on the canvas. That's why an object in a drawing can obscure another object.

Applying the definition to an ideal point $P_{[[k]]}$ we see that

$$\Pi_{O,\omega'}(P_{[[k]]}) = OP_{[[k]]} \cdot \omega' \,.$$

Because the line $OP_{[[k]]}$ is the line through O parallel to k, we see that the projection maps the ideal point of a line k to the intersection of this parallel to k through O with the plane ω'. This is exactly how we defined the *vanishing point* of the line k. Summing this up:

Theorem (vanishing points)**.** *Projecting onto a plane maps the* **ideal point** *of a real-life line to the* **vanishing point** *of this line in the canvas.*

We didn't mention what happens when k is parallel to ω'. Extended space doesn't care – there is still a unique point $m \cdot \omega'$ where m is the line through O parallel to k, hence still parallel to ω'. But now this point is the ideal point $P_{[[m]]}$ which is the same as $P_[[k]]$. We could say this as, "Vanishing points of lines parallel to the picture plane are ideal points of the canvas, not ordinary points.

What about planes? They map to the whole canvas, but in a tricky way which consumed almost two days of your time in Chapter 5. If α is a plane in $\mathbb{R}^3$, not containing O of course, let β be the plane parallel to α through O and let m be the line $\beta \cdot \omega'$. It's easy to check that no ordinary point of α can map to any point of m under the projection $\Pi_{O,\omega'}$.

Theorem (vanishing lines)**.** *Projecting onto a plane maps the* **ideal line** *of a plane α to the* **vanishing line** *of α defined by $\beta \cdot \omega'$, where β is the plane through O parallel to α.*

One final remark about the notation. I have been using O for the viewing point of the projection and ω' for the picture plane (synonym for canvas) but one could use any point and any plane with any names: $\Pi_{X,\alpha}$ projects from X onto α, and so forth.

Sorry about the extra reading, but I think this should go a long way toward clarifying what a vanishing line is. It explains why we were allowed to make inferences such as the inference that all the vanishing points of lines within the rightmost wall of the house were on a vertical line in the canvas. This was because the vanishing line of the wall is a vertical line in the canvas, and all the vanishing points of lines in the wall are therefore in said vertical line.

Practice exercise #1 (refer to Figure 5.0 on page 63 of the textbook, reproduced below):

(a) The line ℓ_1 is the vanishing line for what real world plane, call it α?

(b) Give a verbal description of a point in the plane α that maps to P_2 in the picture.

(c) Name any point and lines you want via verbal description so that you can give an appropriate mathematical name for this point, then write the fact that this point maps to P_2 as an equation using the Π notation.

(d) Does any ordinary point of α map to P_2 in this projection? Describe such a point or argue why it doesn't exist.

(e) Does any ordinary point of $\mathbb{R}^3$ map to P_2 in this projection? Describe such a point or argue why it doesn't exist.

(f) Extend the last sideways line in the sidewalk (colored red in the picture) until it crosses ℓ_1. Call the intersection point X. Is X the image of a point of interest in the real world? If so, what point? If not, why not?